Feynman-Kac Formula and Restoration of High ISO Images*

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Abstract. In this paper we explore the problem of reconstruction of RGB images with additive Gaussian noise. In order to solve this problem we use Feynman-Kac formula and non local means algorithm. Expressing the problem in stochastic terms allows us to adapt to anisotropic diffusion the concept of similarity patches used in non local means. This novel look on the reconstruction is fruitful, gives encouraging results and can be successfully applied to denoising of high ISO images.

1 Introduction

Let D be a closed rectangular in $\mathbf{R}^2$, $u : D \to \mathbf{R}^n$ be an original image and $u_0 : D \to \mathbf{R}^n$ be the observed image of the form $u_0 = u + \eta$, where η stands for a white Gaussian noise (added independently to all coordinates with standard deviation ρ). We assume that u and u_0 are appropriately regular. We are given u_0, the problem is to reconstruct u. This is a typical example of an inverse problem [2].

Various techniques were proposed to tackle this inverse problem. One may quote the linear filtering, DCT [23], wavelets theory [11], variational methods [19], stochastic modelling [12, 18] and methods driven by nonlinear diffusion equation [8, 17, 21, 22]. In another class, one could include methods that take advantage of the non-local similarity of patches in the image. Among the most famous, we can name non local means (in short NL-means) [5, 6], BM3D [9, 10, 13], NL-Bayes [14] and K-SVD [1, 15].

In the paper [4] the authors considered the problem of the reconstruction of grey levels images using anisotropic diffusion expressed in stochastic terms. This representation allows them to adapt to the reconstruction process the idea of patches similarity using in NL-means algorithm. This novel look on the reconstruction problem was fruitful and gave very good results for gray images. In this paper we generalise the results from [4] to colour images and apply proposed metod to denoising of high ISO images taken from digital cameras.

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2 Feynman-Kac Formula

In order to express the anisotropic diffusion equation $\dfrac{\partial u}{\partial t} = Au$ in stochastic terms, where Au is some diffusion operator, one needs to use the Feynman-Kac formula [16].

Theorem 1 (Feynman-Kac Formula). *Let $u_0 \in C_0^2(\mathbf{R}^n)$ (continuously twice differentiable with compact support) then the function $u(t,x)$ defined by*

$$u(t,x) = \mathbf{E}\left[u_0(X_t)\right] \tag{1}$$

satisfies the diffusion equation $\dfrac{\partial u}{\partial t} = Au$, where X is some stochastic process driven by the operator Au and vice versa this operator determines a stochastic process X.

We do not want to focus on the relationship between Au and X but for us the important information is that the anisotropic diffusion can be expressed in the form of the expected value of some stochastic process.

3 Non Local Means Algorithm

In this section we cite results from [5–7] and for precise definitions and deeper discusion about NL-means algorithm we refer the reader to these articles.

Let $v = \{u_0(i) | i \in I = \mathbf{Z}^2 \cap D\}$ be a discrete noisy image and $\{w(i,j)\}$ be the weights that depend on the similarity between the pixels i and j and satisfy the usual conditions $0 \leq w(i,j) \leq 1$ and $\sum_j w(i,j) = 1$. The reconstructed value $NL(v)(i)$ for a pixel i is defined as a weighted average of all pixels in the image

$$NL(v)(i) = \sum_{j \in I} w(N_i, N_j) v(j).$$

The weight $w(N_i, N_j)$ depends on the similarity of the intensity gray level or colour vectors of neighbourhoods N_i, N_j centred at pixels i and j and can be defined by $w(N_i, N_j) = \frac{1}{Z(i)} \exp\left(-\frac{d(\mathcal{N}_i, \mathcal{N}_j)}{s^2}\right)$ where $Z(i)$ is the normalising factor $Z(i) = \sum_j \exp\left(-\frac{d(\mathcal{N}_i, \mathcal{N}_j)}{s^2}\right)$ and $d(\mathcal{N}_i, \mathcal{N}_j)$ is some measure of distance between intensity gray level or colours vectors of similarity windows. The number s is a parameter that controls the decay of the exponential function.

In [7] the authors proposed to use the following weight function for RGB images:

$$w(B_{i,r}, B_{j,r}) = \exp\left(-\frac{\max\left(\frac{\|B_{i,r} - B_{j,r}\|^2}{3(2r+1)} - 2\rho^2, 0\right)}{s^2}\right),$$

where $B_{i,r}$ means a neighbourhood of a size $(2r+1) \times (2r+1)$ RGB pixels centred at i and $\|B_{i,r} - B_{j,r}\|$ is the Euclidean distance between $B_{i,r}$ and $B_{j,r}$.

Fig. 1. Original test images: 512 × 512 a) Peppers b) Lenna

4 Image Reconstruction of Colour Images Based on Feynman-Kac Formula and Non Local Means

Note that in the case of numerical scheme the Feynman-Kac formula can be written as

$$u(t,x) = \frac{1}{M}\sum_{i=1}^{M} u_0(X_t(\omega_i)),$$

where M is a number of iterations of Monte Carlo method. In particular, for the terminal time $t = T$, for which we get the reconstructed image,

$$u(x) = u(T,x) = \frac{1}{M}\sum_{i=1}^{M} u_0(X_T(\omega_i)).$$

Now we need to construct a stochastic process X driven by a geometry of a colour image. Unfortunately, we can not use the gradient function as it was for grey levels images [4]. And therefore we propose to use the following stochastic process X being a particular case of a general model taken from [3].

Let

$$\begin{aligned} &X_0 = x, \\ &H_k = \Pi_D\left(X_{k-1} + h\cdot(\Phi_{0,1},\Phi_{0,1})\right) \\ &X_k = \begin{cases} H_k, & \text{if } \Theta, \\ X_{k-1}, & \text{elsewhere,} \end{cases} \qquad\qquad k = 1,2,\dots,\tau_m, \end{aligned}$$

where $\Pi_D(x)$ denotes a projection of x on the set D and $\Phi_{0,1}$ is a random number generator from the normal distribution with mean 0 and standard deviation 1. By Θ we mean the condition

$$\|(G_\delta * u_0)(H_k) - (G_\delta * u_0)(X_{k-1})\| \le 0.8\cdot\rho$$

where G_δ is 3×3 Gaussian mask and by τ_m

$$\tau_m = \min\{k; k \ge m \,\text{and}\; \Theta \text{ is true } m \text{ times}\}.$$

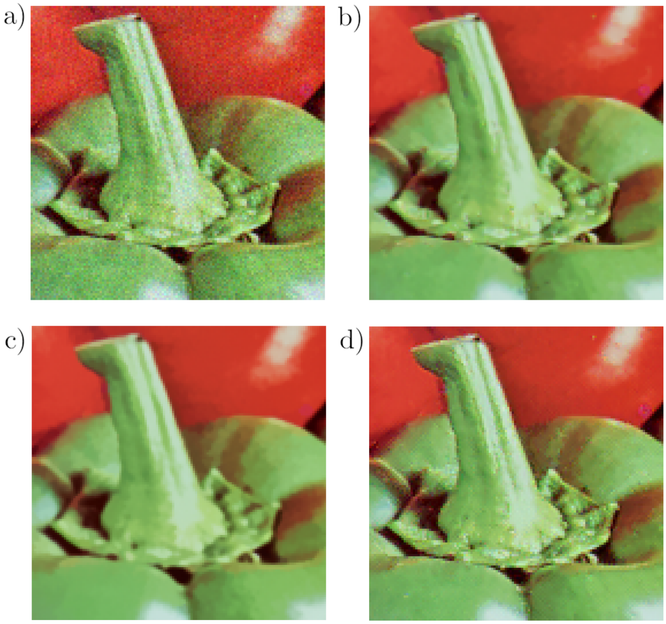

Fig. 2. a) Noisy image $\rho = 10$ b) New method c) Anisotropic diffusion d) NL-means

The interpretation of the process X is the following. First, note that the stochastic process $\tilde{X}$, where $\tilde{X}_0 = x$ and $\tilde{X}_k = \tilde{X}_{k-1} + h \cdot (\Phi_{0,1}, \Phi_{0,1})$ is a discrete approximation of 2-dimensional Wiener process with time step parameter equals h. Modification of this Wiener process by adding the condition Θ ensures that the process X will move to the homogeneous areas. The above construction of X has two important advantages. Firstly, we do not use gradient function or its equivalence for colour images. Secondly, since we can use large value of time step parameter, the process X can be simulated fast (see details in [3]).

Since the process X is considered on the random interval with the terminal time τ_m, the model of stochastic anisotropic diffusion based on Feynman-Kac formula has the following form:

$$u(x) = \sum_{i=1}^{M} \frac{1}{M} u_0(X_{\tau_m(\omega_i)}(\omega_i)), \tag{2}$$

which means that each pixel $u_0(X_{\tau_m(\omega_i)}(\omega_i))$ is weighted with the same value $\frac{1}{M}$. But since pixels have different colours we may consider them with different weights depending on their neighbourhood. We follow NL-means algorithm and propose to think of weights that depend on patches similarity. Finally, we can introduce a new method of the image restoration based on Feynman-Kac formula

$$u(x) = \frac{1}{Z} \sum_{i=1}^{M} u_0(X_{\tau_m(\omega_i)}(\omega_i)) w(B_{x,r}, B_{X_{\tau_m(\omega_i)}(\omega_i),r}). \tag{3}$$

The meaning of the parameters in the new method is the same as in original approaches [3, 7]. Very good results we can obtain with $(M, m, h) = (50, 10, 4)$ for which the time of the reconstruction is comparable to NL-means.

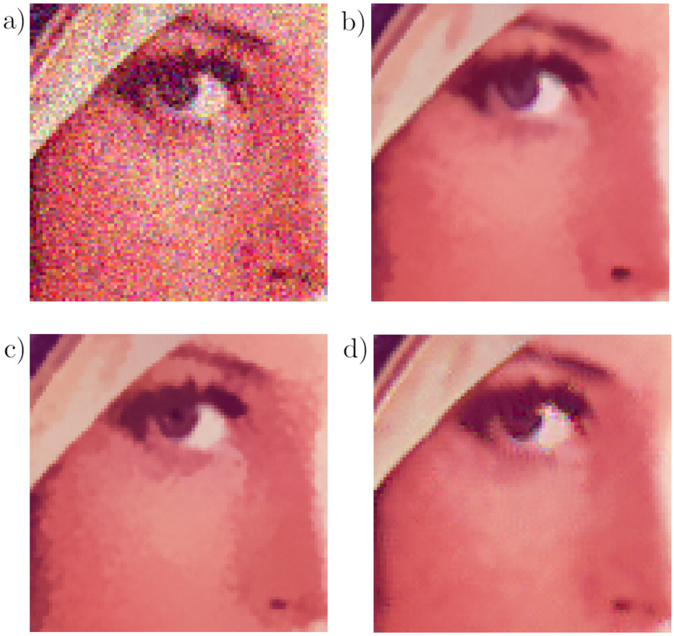

Fig. 3. a) Noisy image $\rho = 30$ b) New method c) Anisotropic diffusion d) NL-means

5 Experimental Results

Some measures of quality for our evaluation experiments regarding new method, non local means algorithm and anisotropic stochastic diffusion are presented in Table 1, Table 2, Fig. 2 and Fig. 3. The results refer to RGB colour images *Lenna* and *Peppers* corrupted (independent all channels) with the Gaussian noise with standard deviation ρ. Noisy images have been reconstructed with using vector analysis in RGB space. The maximum values of Peak Signal to Noise Ratio (in short PSNR) and Structural SIMilarity (in short SSIM) index obtained using tested methods are given in tables. Parameters of SSIM were set to the default values as recommended by [20].

The analysis of the measures of image quality shows that the new method performs better. Moreover, when comparing the figures one can observe that the image created by the new method is visually more pleasant. The reason for this

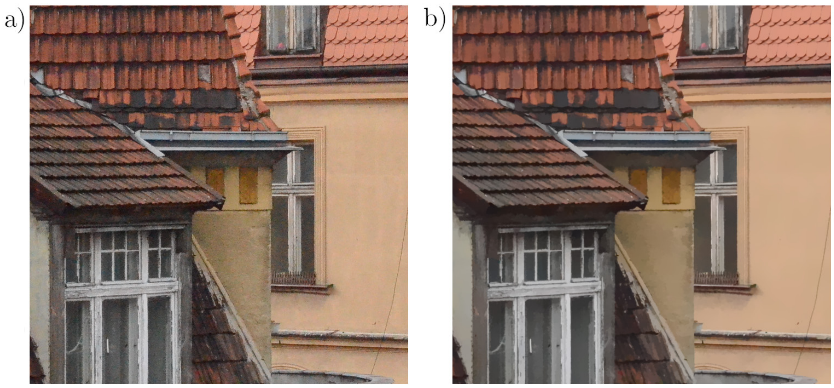

Fig. 4. a) Input high ISO image b) Result of the reconstruction using new method

is that the NL-means approach shows clear evidence of a halo of noise effect around the edges whereas anisotropic diffusion smooth details too much and show of a block image.

The type of high ISO sensor noise produced by a typical digital camera sensor can be modelled as an additive white Gaussian distribution with zero mean and a standard deviation proportional to the value of ISO. In figures Fig. 4., Fig. 5. we see images taken at high ISO value and the result of reconstruction using the new algorithm.

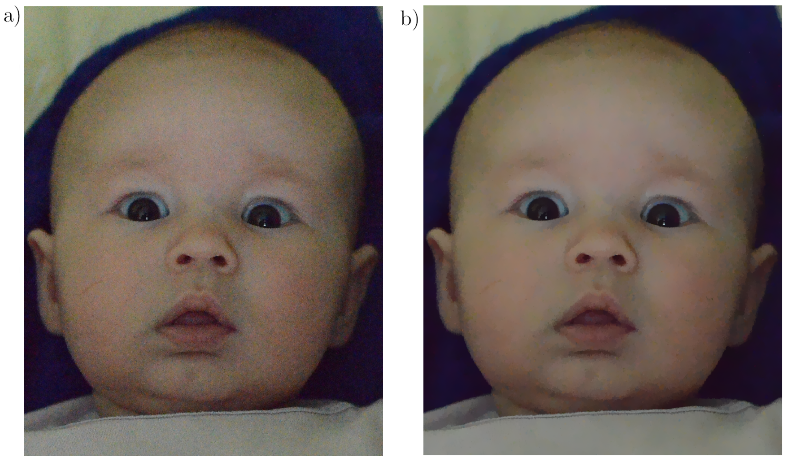

Fig. 5. a) Input high ISO image b) Result of the reconstruction using new method

Table 1. Maximum values of PSNR

Image	Noise ρ	NL-means algorithm	Stoch. anisotropic diffusion	New method
Peppers	10	32.9404	32.3572	**33.1289**
	20	30.2984	30.6057	**31.0984**
	30	27.3031	29.2140	**29.8178**
	40	26.6428	28.1988	**28.6991**
	50	25.9941	27.2549	**27.8611**
	60	25.5353	26.5094	**27.0802**
Lenna	10	34.0127	33.0964	**34.3550**
	20	31.5780	31.1301	**31.8386**
	30	29.6376	29.6041	**30.2871**
	40	28.6433	28.5068	**29.0773**
	50	27.7377	27.5843	**28.1308**
	60	27.0094	26.7234	**27.3205**

Table 2. Maximum values of SSIM

Image	Noise ρ	NL-means algorithm	Stoch. anisotropic diffusion	New method
Peppers	10	0.9536	0.9470	**0.9548**
	20	0.9194	0.9146	**0.9232**
	30	0.8898	0.8866	**0.8994**
	40	0.8645	0.8645	**0.8759**
	50	0.8383	0.8447	**0.8564**
	60	0.8284	0.8229	**0.8362**
Lenna	10	**0.9588**	0.9467	0.9573
	20	0.9224	0.9115	**0.9232**
	30	0.8929	0.8790	**0.8930**
	40	0.8640	0.8528	**0.8648**
	50	0.8377	0.8289	**0.8414**
	60	0.8174	0.8050	**0.8176**

6 Conclusion

In this paper we proposed a new method of digital image denoising. Applying the Feynman-Kac formula to express anisotropic diffusion allows us to adapt the idea from non local means. The new method can be used successfully to reconstruction of high ISO images by giving what is the best from anisotropic diffusion and non local means method.

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