Nonlinear Hyperbolic Systems of Conservation Laws and Related Applications

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1 Introduction

In this chapter, we present an overview of evolution equations defined by nonlinear hyperbolic conservation laws. This overview is not comprehensive rather it is a bird's eye view of the ideas that govern the analysis of nonlinear hyperbolic conservation equations. Various mathematical concepts as well as some ideas related to their discrete algorithms will be presented. Thus basic ideas of viscous regularisation, entropy, monotonicity, total variation bounds and the Riemann problem are discussed. These are also the underlying ideas in the development of discrete solutions as well as discrete theory. Finite volume methods such as the relaxation schemes will also be introduced. This is not a recommendation for the reader to use the method but rather a bias due to the author's previous work in the field. Some examples of applications of the nonlinear conservation laws in networks as well as optimal control leading to stabilisation of the system of interacting equations will be presented.

By a nonlinear conservation law in one space dimension, we imply a first-order partial differential equation (PDE) of the form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0. \tag{1}$$

In this equation ρ represents a conserved quantity, for example, the mass density. The variables t and x are independent variables representing time and space,

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respectively, and u is a flow velocity variable. The term $(\rho u)(x,t)$ is the mass flux of the conserved quantity through a cross-section of the flow domain in the normal direction at a point x and t. Equation (1) is popularly referred to as a continuity equation [50].

Examples of such equations include the density of gas flowing in a pipe considering that significant changes in the flow are one-dimensional, the continuity equation represents the flow of density of a non-viscous (inviscid) gas in which the flux represents a flow rate of gas through the cross-section of the pipe at a point x and t. To get a closed relation between ρ and u, one needs a constitutive relation or additional conservation laws. If u is known, the equation is referred to as an advection equation. Another popular example is the traffic flow model on a section of a single lane highway. Assume that on a highway section the car density is defined by ρ, cars are a conserved quantity giving Eq. (1). To close the equation, there is need for a relation between ρ and u, for example [50]

$$u = u(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$$

i.e. the denser the traffic, the slower the cars will move. Here $u_{\max}$ is the maximum velocity and $\rho_{\max}$ is the maximum traffic density, i.e. when the cars are so-to-say bumper-to-bumper. This defines the flux as

$$f(\rho) = \rho u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right).$$

The two equations discussed are prototypes of hyperbolic scalar conservation laws in one space dimension.

A popular protoype for a system of conservation laws is the inviscid Euler Equations of gas dynamics model, a system of hyperbolic conservation laws. In this case density (mass), momentum and energy are conserved. Let the flow of a gas be defined by its density ρ, velocity v, energy E and pressure p. To model the complete system, there is need for additional conservation laws (i.e. a system). The model takes the form:

$$\rho_t + (\rho v)_x = 0; \quad \text{conservation of mass,} \tag{2a}$$

$$(\rho v)_t + (\rho v^2 + p)_x = 0; \quad \text{conservation of momentum,} \tag{2b}$$

$$E_t + (v(E + p))_x = 0; \quad \text{conservation of energy.} \tag{2c}$$

The subscripts in this case represent partial differentiation with respect to the variable in the subscript. To close the system there is need for constitutive relations. Assume T is the absolute temperature of a gas then:

$$p = \rho T; \quad \text{the ideal gas law,}$$

$$E = \frac{\rho v^2}{2} + c_v \rho T; \quad \text{representing kinetic and thermal (internal) energy.}$$

In this case a system of three equations for three conserved quantities is obtained. To close the system a relation involving variable p which depends on ρ, v, E is assumed based on empirical considerations. In this case the hyperbolic conservation laws can be written in the familiar form with

$$u = \begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix} \quad \text{and} \quad F(u) = \begin{pmatrix} \rho v \\ \rho v^2 + p(u) \\ v(E + p(u)) \end{pmatrix}$$

giving a hyperbolic system of conservation laws in one space dimension:

$$\partial_t u + \partial_x f(u) = 0. \tag{3}$$

A further simplification of the above system would be the isothermal Euler equations of gas dynamics: assume temperature, T, is constant then

$$p = a^2 \rho$$

where a is the speed of sound in the gas. Hence we obtain a system that takes the form:

$$\begin{aligned} \rho_t + (\rho v)_x &= 0; \ \text{conservation of mass,} \\ (\rho v)_t + (\rho v^2 + a^2 \rho)_x &= 0; \ \text{conservation of momentum.} \end{aligned}$$

The flow is referred to as isothermal flow.

The above are but a few examples of conservation laws. In general the Cauchy problem for the hyperbolic conservation laws consists of Eq. (3) with initial conditions

$$u(x, 0) = u_0(x). \tag{4}$$

In this chapter, a lot of discussion will centre around the Riemann problem which is (3) with initial conditions of the form:

$$u(x, 0) = \begin{cases} u_L, & \text{for } x < 0; \\ u_R, & \text{for } x \geq 0. \end{cases} \tag{5}$$

The solution for a nonlinear Riemann problem can either be a shock propagating, a contact discontinuity or a rarefaction wave.

In this introductory review, we give an overview of the mathematical methods used in investigating the well-posedness and constructing of solutions of the scalar nonlinear conservation laws (1). This discussion will be extended to systems in

one space dimension (3). This will be undertaken in Sect. 2. For further reading, we recommend [46, 63, 71]. For the theory of numerical methods, the reader may consult [33, 50, 64]. A brief introduction of numerical methods based on the Relaxation schemes will be presented in Sect. 3. Applications of these methods to networked flow or evolution equations can be found in 4. The interested reader is also encouraged to consult the following references: [62] for a classical treatment of systems; [48,59] for more recent aspects of nonlinear hyperbolic conservation laws; [31] for conservation laws on networks; [14,30,42] for careful studies and advances in the theory of low-order numerical approximation of hyperbolic conservation laws.

2 Mathematical Modelling and Analysis with Hyperbolic Conservation Laws

In this section the basic formulation of conservation laws will be discussed. These are derived from the basic physical principles of conservation, for example, conservation of mass or momentum, as alluded to in Sect. 1. Examples of conservations laws have been presented. Such equations are also popularly referred to as evolution equations. To be more specific, the special case of hyperbolic systems of conservation laws will be discussed in detail. These are first-order hyperbolic partial differential equations (PDE). They also play a prominent role in modelling flow and transport processes. It needs to be noted that interesting cases of this class of equations are usually nonlinear. It can be said that due to their nonlinear nature these models present special difficulties: formation of shocks (jump discontinuities) even though the initial data is continuous. As a consequence a great deal about their mathematical structure is not yet known. A popular alternative to approximate the solutions of these models is to apply numerical approximations which are also able to resolve such jump discontinuities.

We would like to point out that systems of conservation laws in a single space variable have been well-studied. For example, Euler equations of compressible flow, the inviscid gas dynamics equations in Eq. (2), are an important example of a hyperbolic system of conservation laws. The second popularly discussed model is the linear wave equation. In such phenomena signals propagate with finite speed. Singularities propagate along characteristics, such singularities arise spontaneously, leading to formation of shocks or jump discontinuities. Therefore, in nonlinear cases, time is not reversible as for linear equations thus future and past are different [46]. There is loss of information as time moves forward, which can be interpreted as an increase in entropy. Basic existence theory of solutions of hyperbolic conservation laws in single space variables was discussed in [32]. In general, apart from isolated results, no comparable theory exists for more space dimensions.

In this chapter we will, therefore, discuss the case of one spatial dimensional problems. In general, we seek a weak solution $u : \mathbb{R} \times [0, T] \to \mathbb{R}^k$ to the Cauchy problem, see also Eq. (3):

$$\partial_t u + \partial_x f(u) = 0; \quad u(x, 0) = u_0(x)$$

with $f : \mathbb{R}^k \to \mathbb{R}^k$, which is a system of k conservation laws in $\mathbb{R} \times [0, T]$ with flux functions f. In this case T represents the final time. This system can also be written in quasilinear form as:

$$\partial_t u + A(u)\partial_x u = 0; \quad u(x, 0) = u_0(x). \tag{6}$$

where

$$A(u) = \left(\frac{\partial f_i}{\partial u_j}(u)\right)_{1 \le i,j \le k}$$

is the Jacobian matrix of $f = (f_1, f_2, \ldots, f_k)^T$ i.e. $A(u) = D_u f$.

For smooth solutions, the solutions for Eqs. (3) and (6) are equivalent. In cases where it is not possible to obtain classical or smooth solutions, we seek weak solutions which will be discussed in Sect. 2.1. For basic knowledge on conservation laws, we recommend the following literature [26,33,46,52,63]. The peculiarities of the conservation laws include:

1. the evolution of shock discontinuities which require weak (in the distributional sense) solutions of (3).
2. non-uniqueness of weak solutions of (3).
3. identification of unique 'physically relevant' weak solutions of (3). In general we seek a solution, $u = u(x, t)$, which can be defined as a viscosity limit solution, $u = \lim_{\epsilon \to 0} u^\epsilon$ such that [15,26,33]

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon \partial_x (\nu \partial_x u^\epsilon), \qquad \epsilon \nu > 0.$$

4. The entropy condition which requires that for all convex entropy functions, $\eta(u)$, the following holds:

$$\eta(u)_t + \psi(u)_x \le 0.$$

The viscosity limit solution is somehow related to the entropy solution, u.

At this point, we would like to give a formal definition of a hyperbolic system in Definition 1.

Definition 1 Consider the quasilinear system in Eq. (6). The system is referred to as (strictly) hyperbolic if and only if the matrix $A(u)$ has only real (pairwise disjoint) eigenvalues (characteristic speeds) and the eigenvectors form a basis of $\mathbb{R}^k$.

Examples such as the inviscid Euler equations of gas dynamics have been given in Sect. 1. For the one-dimensional scalar conservation laws, the equation is always strictly hyperbolic. For Eq. (6) one needs to investigate the matrix $A(u)$. Below we give some examples of hyperbolic conservation laws.

One-Dimensional Isentropic Gas Dyamics Equations Consider a one-dimensional isentropic gas dynamics model in Lagrangian coordinates [33]:

$$\begin{aligned} \frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} &= 0; \\ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} p(v) &= 0; \end{aligned}$$

v is specific volume, u is the velocity, and the pressure $p(v)$. For a polytropic isentropic ideal gas: $p(v) = \alpha v^{-\gamma}$ for some constant $\alpha = \alpha(s) > 0$ (depending on entropy), $\gamma > 1$. The above system in quasilinear form is of the form:

$$\frac{\partial w}{\partial t} + A(w)\frac{\partial w}{\partial x} = 0;$$

where $A(w)$ is the Jacobian matrix with two real distinct eigenvalues

$$\lambda_1 = -\sqrt{(-p'(v))} < \lambda_2 = \sqrt{(-p'(v))}$$

i.e. $p'(v) < 0$. Hence the system is strictly hyperbolic provided $p'(v) < 0$. This is also referred to as the p-system. The p-system is the simplest nontrivial example of a nonlinear system of conservation laws.

Nonlinear Wave Equation Consider any nonlinear wave equation [33]

$$\frac{\partial^2 g}{\partial t^2} - \frac{\partial}{\partial x}\Big(\sigma(\frac{\partial g}{\partial x})\Big) = 0.$$

This equation can be re-written in the form of a p-system as follows: set

$$u = \frac{\partial g}{\partial t}, \quad v = \frac{\partial g}{\partial x}, \quad p(v) = -\sigma(v).$$

Isothermal Flow Consider the isothermal Euler equation [33, 50]:

$$u = \begin{pmatrix} \rho \\ q \end{pmatrix}; \qquad f(u) = \begin{pmatrix} q \\ \frac{q^2}{\rho} + a^2\rho \end{pmatrix}; \qquad A(u) = \begin{pmatrix} 0 & 1 \\ -\frac{q^2}{\rho^2} + a^2 & \frac{2q}{\rho} \end{pmatrix} \tag{7}$$

where the momentum $q = \rho v$. The matrix has eigenvalues

$$\lambda_1 = v + a; \qquad \lambda_2 = v - a$$

where a is the speed of sound in the flow.

2.1 Weak Solutions

In this section, the idea of a weak solution will be presented. Two one-dimensional examples will be presented: the Burgers equation with a smooth initial condition and periodic boundary conditions and secondly the inviscid Euler equations of gas dynamics with jump initial conditions. The first example demonstrates the evolution of a jump discontinuity even though the initial conditions are continuous. The latter example will introduce the three types of solutions expected: propagating shocks, rarefaction waves as well as contact discontinuities.

2.1.1 Burgers Equation

Consider the Burgers equation with the given initial condition:

$$\partial_t u + \partial_x\left(\frac{u^2}{2}\right) = 0; \quad u(x,0) = 0.5 + \sin(x), \quad x \in [0, 2\pi].$$

According to Fig. 1, as time evolves the solution develops steep gradients which eventually become a discontinuity which propagates with time. Such discontinuous

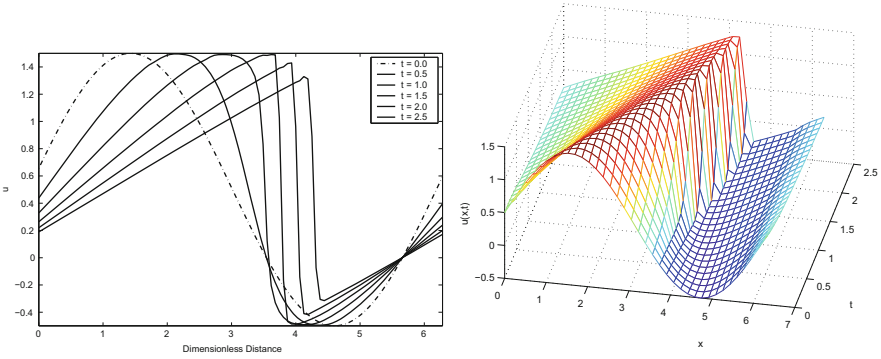

Fig. 1 Evolution of the solution for the Burgers equation at different times up to $T = 2.5$ to demonstrate the development of a jump discontinuity

solutions do not satisfy the PDE in the classical sense at all points. Therefore, there is need for a weaker definition of a solution of the PDE.

2.1.2 Sod's Problem

In this example, a nonlinear system of inviscid Euler Equations of gas dynamics, Eq. (2), is considered. The system is considered subject to the following initial conditions, a Riemann problem, with boundary conditions.

$$u(x,0) = \begin{cases} u_L, \ x < 0; \\ u_R, \ x > 0; \end{cases} \text{ and transparent boundary conditions,}$$

where $u_L = (1, 0, 2.5)^T$ and $u_R = (0.125, 0, 0.25)^T$. This can be considered as a case of a gas in a long pipe in which the two halves of the pipe are initially separated by a membrane. The gas on the left side of the membrane is subjected to a higher pressure (density) and as the experiment starts, the membrane is broken. In the solution in Fig. 2 in which different numerical schemes were used to resolve the solution, it can be seen that on the left hand side of the pipe a rarefaction wave (smooth solution) moving leftwards has developed. Around the position $x = 6$ a contact discontinuity has developed in which there is a jump in density but not in pressure and velocity. Lastly around $x = 8$ one observes a shock wave travelling to the right. This is a jump discontinuity.

The two examples demonstrate that there is need to re-think what we mean by a solution. Thus we seek weak solutions rather than classical ones. The idea of weak solutions will be discussed below. Before we discuss weak solutions, we would

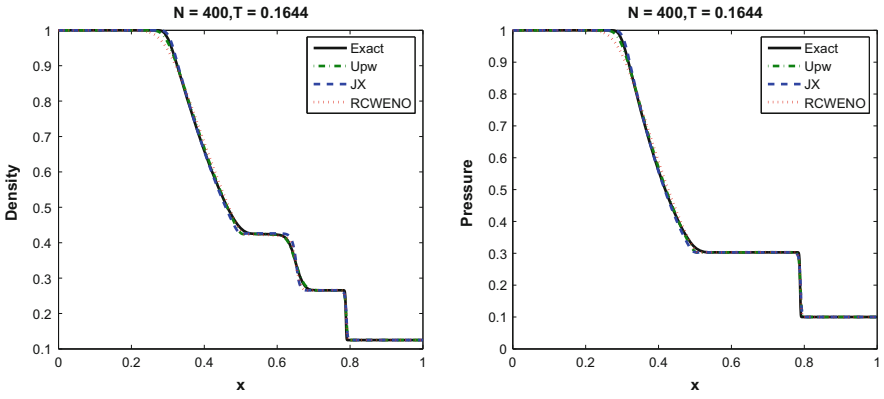

Fig. 2 Solution of the Sod's problem at time $t = 0.1644$ obtained using three methods: the upwind scheme (Upw), the second-order relaxation scheme (JX) and the third order relaxation scheme (RCWENO) [39] based on central weighted essentially non-oscillatory (CWENO) interpolation [44] compared with the exact solution (Exact)

like to demonstrate how discontinuities actually develop by using the method of characteristics in the next section.

2.2 Method of Characteristics

In this section a Cauchy problem for scalar conservation laws is considered:

$$u_t + f(u)_x = 0; \qquad u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}, \qquad f : \mathbb{R} \to \mathbb{R}, \tag{8}$$

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}. \tag{9}$$

To solve the problem, the method of characteristics will be discussed. For simplicity, consider an advection equation:

$$u_t + a u_x = 0, \qquad a \in \mathbb{R}, \tag{10}$$

or the Burgers Equation:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0. \tag{11}$$

The solution of the Cauchy problem for the advection equation is:

$$u(x, t) = u_0(x - at),$$

i.e. the initial data travels unchanged with velocity a. The left side of Eq. (10) can be interpreted as a directional derivative with:

$$\frac{du}{dt} = 0, \qquad \frac{dx}{dt} = a.$$

The equation $\frac{dx}{dt} = a, x(0) = x_0$ is the characteristic line which cuts through x_0 in the (x, t)-plane, i.e.

$$x = x_0 + at.$$

It can be shown that along the characteristics the solution is constant [33,50]. Hence suppose (x, t) is such that $x = x_0 + at$ holds, then

$$u(x, t) = u_0(x_0) = u_0(x - at)$$

is constant.

On the other hand Eq. (8) can be written in the form

$$u_t + f'(u)u_x = 0$$

for smooth solution u. The term $f'(u)$ is referred to as the characteristic velocity, the velocity with which information propagates. To find characteristics one needs to solve:

$$x'(t) = f'(u(x(t), t)), \qquad x(0) = x_0.$$

Along the solution $x(t)$, $u(x, t)$ is constant, i.e.:

$$\frac{d}{dt}u(x(t), t) = \frac{\partial u(x(t), t)}{\partial t} + \frac{\partial}{\partial x}u(x(t), t)x'(t) = u_t + f'(u)u_x = 0.$$

Therefore, $x'(t)$ is constant since $f'(u(x(t), t)$ is constant. Thus characteristics are straight lines given by:

$$x = x(t) = x_0 + f'(u(x_0, 0))t = x_0 + f'(u_0(x_0))t. \tag{12}$$

If the above can be solved for x_0 for all (x, t), then the solution of the conservation law takes the form:

$$u(x, t) = u(x_0, 0) = u_0(x_0)$$

where x_0 is implicitly given as:

$$x = x_0 + f'(u_0(x_0))t.$$

Unfortunately, Eq. (12) above cannot always be uniquely solved—the characteristics can intersect after some time. For example, in the case of the Burgers equation $f(u) = \frac{u^2}{2}$. Let the initial value be given by $u_0(x) = -x$. Then the characteristics are

$$x = x(t) = x_0 - x_0 t$$

and this gives

$$t = \frac{x_0 - x}{x_0} = 1 - \frac{x}{x_0}$$

and also

$$u(x, t) = u_0(x_0) = u_0\left(\frac{x}{1 - t}\right).$$

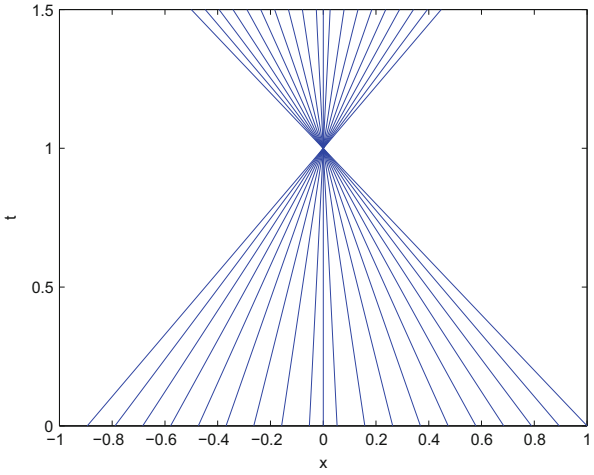

Fig. 3 Characteristics showing intersection resulting in shock development

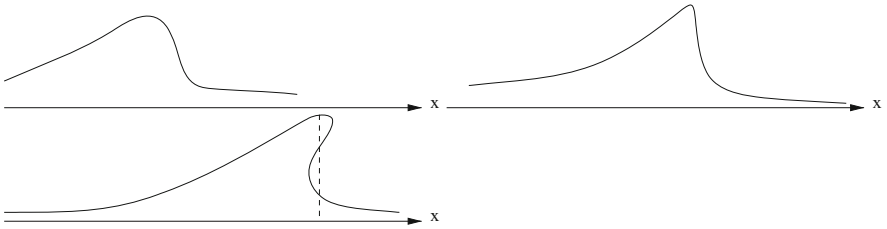

Fig. 4 Evolution of waves—development of a multi-valued function

The characteristics intersect at the point $(x,t) = (0,1)$. In this case there is no unique solution for (x,t) anymore, see Figs. 3 and 4

In these figures, it can be seen that one attains a multivalued solution of the equation. Physically, for example, the density of a gas cannot be multivalued. In this case the discontinuity propagates as a shock. This can now be mathematically accurately treated with the help of weak (non-differentiable) solutions. In this case we seek solutions in L^1.

Definition 2 A function $u = u(x,t)$ is called a weak solution of the Cauchy–Problem, Eq. (8), if

$$\int_{\mathbb{R}} \int_0^{\infty} [u\phi_t + f(u)\phi_x]\, dt\, dx + \int_{\mathbb{R}} u_0(x)\, \phi(x,0)\, dx = 0, \quad \forall \phi \in C_0^1(\mathbb{R} \times \mathbb{R}).$$

If u is a smooth solution of the equation, then

$$\int_{\mathbb{R}} \int_0^{\infty} [\phi u_t + \phi f(u)_x]\, dt\, dx = 0, \quad \forall \phi \in C_0^1(\mathbb{R} \times \mathbb{R}). \tag{13}$$

Integration by parts gives

$$\int_{\mathbb{R}} \left[-\int_0^{\infty} \phi_t u dt + \phi u \Big|_0^{\infty} \right] dx + \int_0^{\infty} \left[-\int_{\mathbb{R}} \phi_x f(u) dx + \phi f(u) \Big|_{-\infty}^{\infty} \right] = 0.$$

So smooth solutions are also weak solutions. Indeed, if the Riemann problem in Eq. (5) is considered, a weak solution is, for example, the shock wave

$$u(x,t) = \begin{cases} u_l\,, & x < st \\ u_r\,, & x > st, \end{cases}$$

where s is the shock speed:

To find s, the shock speed, the Rankine–Hugoniot conditions

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r} =: \frac{[f]}{[u]}$$

are applied [33, 50].

In addition the differential equation can be integrated to obtain:

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} \Big\{ \partial_t u(x,t) + \partial_x f(u(x,t)) \Big\}\, dx\, dt = 0; \quad u(x,0) = u_0(x)$$

and further giving the following integral forms:

1. $\displaystyle\int_{x_1}^{x_2} u(x,t_2)dx = \int_{x_1}^{x_2} u(x,t_1)dx + \int_{t_1}^{t_2} f(u(x_1,t))dt - \int_{t_1}^{t_2} f(u(x_2,t))dt$
2. $\displaystyle\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)dx = f(u(x_1,t)) - f(u(x_2,t)).$

These are more difficult to work with than the differential equation. As a consequence, the "weak" form of the PDE is introduced. The weak form allows discontinuous solutions and is easier to work with. This form is also fundamental in development and analysis of numerical methods. Integrating (13) gives:

$$\int_0^{\infty} \int_{-\infty}^{\infty} \Big\{ \phi_t u(x,t) + \phi_x f(u(x,t)) \Big\} dxdt = \int_{-\infty}^{\infty} \phi(x,0) u(x,0) dx$$

$$\forall \phi \in C_0^1(\mathbb{R} \times \mathbb{R})$$

which is integration over a bounded domain. This approach introduces a new problem: non-uniqueness of solutions with same initial data—hence a criterion for finding the physically relevant solution is necessary. As an example, consider Burgers equation

$$f(u) = \frac{u^2}{2}, \quad s = \frac{1}{2} \cdot \frac{u_l^2 - u_r^2}{u_l - u_r} = \frac{1}{2}(u_l + u_r). \tag{14}$$

If the characteristics run into the shock, then it is called stable otherwise unstable. The mean value theorem implies:

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r} = f'(\xi), \quad \xi \in (u_l, u_r) \text{ or } \xi \in (u_r, u_l).$$

If f is convex, i.e. f' is monotone increasing, then

$$u_l > u_r \Rightarrow f'(u_l) > f'(\xi) > f'(u_r),$$

one has a stable shock, see Fig. 5, if on the other hand

$$u_l < u_r \Rightarrow f'(u_l) < f'(\xi) < f'(u_r),$$

then one has an unstable shock [33, 46, 50].

Another weak solution for $u_l < u_r$ is given by the so called rarefaction wave

$$u(x,t) = \begin{cases} u_l, & \frac{x}{t} \le f'(u_l) \\ v(\frac{x}{t}), & f'(u_l) \le \frac{x}{t} \le f'(u_r) \\ u_r, & \frac{x}{t} \ge f'(u_r) \end{cases}$$

with $f'(v(\xi)) = \xi$ (Figs. 6 and 7).

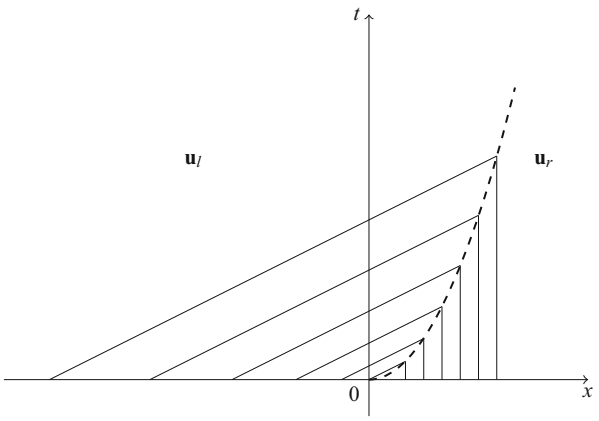

Fig. 5 Characteristics for $u_l > u_r$

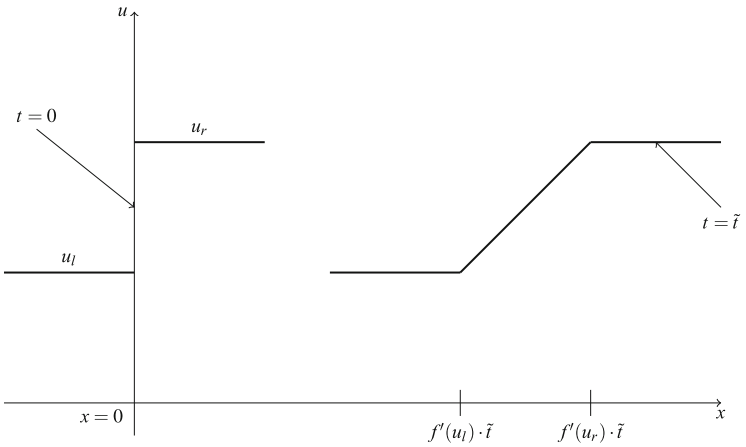

Fig. 6 A rarefaction wave

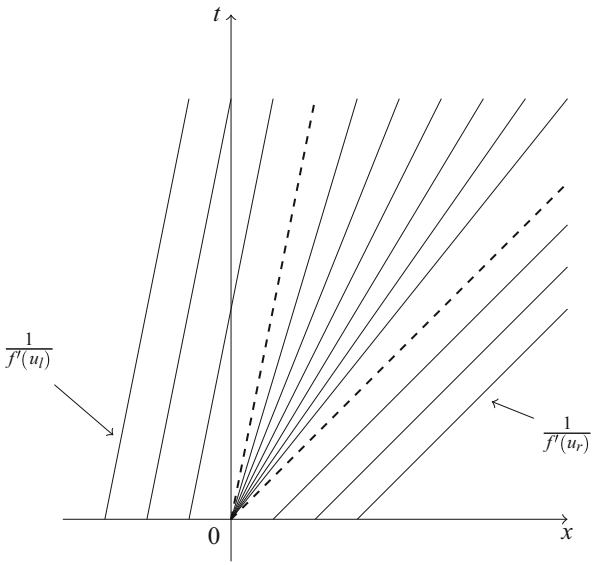

Fig. 7 Characteristics for rarefaction wave

For the Burgers equation ($f'(u) = u, v(\xi) = \xi$) then

$$u(x,t) = \begin{cases} u_l, & x \le u_l t \\ \frac{x}{t}, & u_l \le \frac{x}{t} \le u_r \\ u_r, & x \ge u_r t \end{cases},$$

i.e. for $u_l < u_r$, we obtain two weak solutions, not a unique solution. But only one can be physically correct. Supplement the PDE by additional "jump conditions" that are satisfied across discontinuities. Rankine-Hugoniot conditions give the shock speed, s. Unfortunately, even if equations have the same smooth solutions, it does not necessarily mean that their weak solutions will remain the same in cases where a discontinuity develops. For example, multiplying the Burgers equation (11) by $2u$ gives

$$\partial_t(u^2) + \partial_x\left(\frac{2}{3}u^3\right) = 0; \tag{15}$$

which is also a conservation law for u^2. Equation (11) and (15) possess the same smooth solution. Considering a Riemann Problem with $u_l > u_r$, one can observe that Eq. (15) has shocks travelling at speed

$$s_2 = \frac{[\frac{2}{3}u^3]}{[u^2]} = \frac{2}{3}\left(\frac{u_r^3 - u_l^3}{u_r^2 - u_l^2}\right).$$

A comparison of the shock speeds for Eqs. (11) and (15) shows that

$$s_2 - s = \frac{1}{6}\frac{(u_l - u_r)^2}{(u_l + u_r)}$$

hence $s_2 \neq s$ when $u_l \neq u_r$. Hence, we obtain different discontinuous solutions [50]. Also note that to obtain equation (15), careful manipulation of the smooth solution had to be undertaken. To obtain a unique solution, it is possible to use viscosity regularisation or to supplement the conservation law with an entropy condition. In the following section, we will discuss the entropy condition which is a concept that finds its roots in thermodynamics.

2.3 *Energy and Entropy*

In this section a discussion of entropy will be undertaken. Entropy is usually at the centre of defining solutions of hyperbolic conservation laws in the weak sense (non-classical solutions). It is usually used to identify weak solutions that are of physical relevance. The entropy used as such is referred to as mathematical entropy. Before the concept of entropy is discussed in subsequent sections, a short discussion of the physical entropy will be undertaken.

In this section the presentation in [38] will be followed closely. It is known that the energy content of a system is measured by its internal energy per unit mass e. In a fluid, the total energy considered in the conservation of energy equation is the sum of its internal energy and its kinetic energy per unit mass:

$$E = e + \frac{v^2}{2}$$

where E denotes the total energy. Using the first law of thermodynamics the sources of variation of the total energy is due to work of forces acting on the system as well as heat transmitted to the system. Hence two fluxes are considered: the convective flux F_C:

$$F_C = \rho v(e + \frac{v^2}{2}) = \rho v E$$

as well as the diffusive flux F_D:

$$F_D = -\gamma \rho \kappa \nabla e$$

where κ is the thermal diffusivity coefficient and is defined empirically while γ is the ratio of specific heat coefficients under constant pressure and constant volume:

$$\gamma = c_p / c_v.$$

It can be noted that the diffusion flux represents the diffusion of heat in a medium at rest due to molecular conduction. Under Fourier's law of heat conduction it can be written as

$$F_D = -k \nabla T$$

where T is the absolute temperature and k is the thermal conductivity coefficient.

Furthermore, one can also consider energy variations due to two sources: the volume sources which are due to work of the volume forces f_e and heat source other than conduction (radiation, chemical reactions) denoted by q_H giving $Q_v = \rho f_e \cdot v + q_H$. There are also surface energy sources due to work done by internal shear stresses acting on the surface of the volume $Q_s = \sigma \cdot v = -pv + \tau \cdot v$ where p is the pressure, τ are the viscous shear stresses. Accounting for all the energy contributions, the conservation of energy takes the form:

$$\frac{\partial}{\partial t}(\rho E) + \nabla \cdot (\rho v E) = \nabla \cdot (k \nabla T) + \nabla \cdot (\sigma \cdot v) + W_t + q_H$$

where W_t is the work of the external volume forces $W_t = \rho f_e \cdot v$.

From the equation above, one can obtain an equation of internal energy of the form:

$$\frac{\partial}{\partial t}(\rho e) + \nabla \cdot (\rho v h) = (v \cdot \nabla) p + \varepsilon_v + q_H + \nabla \cdot (k \nabla T) \tag{16}$$

where $h = e + \frac{p}{\rho}$ and ε_v is the dissipation term. For details the reader may consult [38]. In addition the dissipation term ε_v takes the form

$$\varepsilon_v = (\tau \cdot \nabla) \cdot v = \frac{1}{2\mu}(\tau \otimes \tau^T) = \tau_{ij}\frac{\partial v_i}{\partial x_j}$$

where μ is the dynamic viscosity.

Introducing the continuity equation, Eq. (1), in Eq. (16) above, one obtains:

$$\rho\frac{de}{dt} = -p(\nabla \cdot v) + \varepsilon_v + \nabla \cdot (k\nabla T) + q_H \tag{17}$$

where: $\frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla$ is the material or convective derivative, $p(\nabla \cdot v)$ is the reversible work of the pressure forces (vanishes in incompressible flows). Other terms are considered as heat additions: ε_v dissipation term—acting as an irreversible heat source; $F_C = -k\nabla T$ is flux due to heat conduction (Fourier's Law); q_H are heat sources as discussed above.

To clarify the entropy contribution, the entropy s per unit mass will be introduced through the thermodynamic relation

$$T\ ds = de + p\ d\left(\frac{1}{\rho}\right) = dh - \frac{dp}{\rho} \tag{18}$$

where h is the enthalpy. Thus it is now possible to separate reversible and irreversible heat additions using:

$$T\ ds = dq + dq'$$

where dq is a reversible heat addition and dq' is an irreversible heat addition. From the Second Principle of Thermodynamics: $dq' \geq 0$, i.e. in adiabatic flow ($dq = 0$). Hence entropy will always increase. Introducing Eq. (18) in (17) the following equation is obtained:

$$\rho T\frac{ds}{dt} = \varepsilon_v + \nabla \cdot (k\nabla T) + q_H \tag{19}$$

where the last two terms are reversible heat addition by conduction and other sources. For $q_H = 0$ and $k = 0$ the non-negative dissipation term ε_v is a non-reversible heat source. Equation (19) is the entropy equation of the flow. In conclusion it can be noted that this equation is important but not independent from the energy equation. Therefore, either this equation or the energy conservation equation needs to be added to the mass and momentum conservation equations. In addition entropy can not be classified as a 'conservative' quantity in the sense discussed above.

Thus mathematical expression of second principle of thermodynamics for an adiabatic flow without heat conduction or heat sources can be expressed as:

$$\rho T \frac{ds}{dt} = \varepsilon_v;$$

$$\rho T \Big(\frac{\partial s}{\partial t} + v \cdot \nabla s \Big) = \varepsilon_v.$$

The viscous dissipation is positive i.e. $\varepsilon_v \geq 0$ which implies that solutions of the inviscid Euler equations, which also physically imply a vanishing viscosity limit, need to satisfy the following entropy condition

$$\rho T \Big(\frac{\partial s}{\partial t} + v \cdot \nabla s \Big) \geq 0.$$

In other words any solution of Euler equations with a physical meaning satisfies the entropy condition above. Therefore, entropy is used as an additional condition, where necessary, to exclude non-physical solutions for uniqueness.

To conclude, it must be noted also that non-viscous, non-heat-conducting fluid, flow in the limit of vanishing viscosity admits both continuous and discontinuous solutions. Therefore, entropy variations for continuous flow variations give the entropy equation:

$$\rho T \Big(\frac{\partial s}{\partial t} + v \cdot \nabla s \Big) = 0$$

i.e. entropy constant along flow path. In absence of discontinuities, inviscid Euler equation describes isentropic flows where the value of entropy only varies from one flow path to another. From the inviscid Euler equations classical solutions also satisfy the entropy equation which can be derived as:

$$\frac{\partial(\rho S)}{\partial t} + \frac{\partial(v \rho S)}{\partial x} = 0$$

where S is the specific entropy. For a strict convex entropy function, for example, of the form $U = -\rho S$ [33] it can be shown that the limit of the viscosity solution will also satisfy, in the weak sense, the entropy inequality

$$\frac{\partial(\rho S)}{\partial t} + \frac{\partial(v \rho S)}{\partial x} \leq 0.$$

This is an instance of the Clausius-Duhem inequality [26].

Nature has its own way of solving the same problems. Equations are models of reality and some physical effects are ignored. For example, fluid flow always has viscous effects: strong near discontinuity i.e. a discontinuity can be thought of as a thin region with very steep gradients. Thus, one needs further conditions, in order to eliminate the non-physical weak solutions. Such a simple additional assumption is the entropy-condition [33, 46, 50, 63]. We observe, geometrically, for the Burgers equation that

- for convex f and $u_l > u_r$ we need to get a stable shock.
- For $u_l < u_r$ the shock solution disappears. In this case the rarefaction wave is the correct physical solution.

For convex f the entropy conditions discussed above are sufficient, but we need a more general solution, in which case we appeal to viscosity regularisation:

1. introduce a diffusive term in equation: consider linear advection equation:

$$\partial_t u + a\partial_x u = 0; \quad u(x,0) = u_0(x) \quad \text{gives } u(x,t) = u_0(x - at),$$

2. to obtain an advection-diffusion equation:

$$\partial_t u + a\partial_x u = Du_{xx}; \quad u(x,0) = u_0(x) \quad \text{where the flux is } f = f(u, u_x).$$

   The parabolic, conservation law, always has a smooth solution for $t > 0$ even if u is discontinuous. As the reader may notice, for $D \ll 1$, it is a good approximation for a linear advection equation.
3. Letting $D \to 0$—gives a "vanishing viscosity" method which is useful for analysis, in general, but practically not optimal [50].

For practical purposes, the entropy condition has been well accepted. Let η be a convex function and let ψ exist, for which we have for all smooth solutions u of the conservation law:

$$\eta(u)_t + \psi(u)_x = 0,$$

i.e. $\eta'(u)u_t + \psi'(u)u_x = 0$. For smooth solutions $u_t + f'(u)u_x = 0$ or $\eta'(u)u_t + \eta'(u)f'(u)u_x = 0$ giving

$$\eta'(u)f'(u) = \psi'(u). \tag{20}$$

Equation (20) needs to be solvable for the case $n \geq 2$. Such functions (η, ψ) are called entropy–entropy flux pairs. The entropy condition is sufficient for existence and uniqueness of weak solutions. The function $u(x,t)$ is the entropy solution if, for all convex entropy functions, $\eta(u)$, and corresponding entropy fluxes, $\psi(u)$, the inequality

$$\eta(u)_t + \psi(u)_x \leq 0,$$

is satisfied in the weak sense: i.e.

$$\int_{\mathbb{R}} \int_0^\infty (\phi_t \eta(u) + \phi_x \psi(u))\, dt\, dx + \int_{\mathbb{R}} \phi(x,0)\eta(u)(x,0)\, dx \leq 0 \qquad \forall \phi \in C_0^1(\mathbb{R} \times \mathbb{R}), \quad \phi \geq 0. \tag{21}$$

Now we are able to define what we mean by an entropy solution:

Definition 3 Let u be a weak solution of the conservation law. Moreover, suppose u fulfills (21) for any entropy–entropy flux pairs (η, ψ), then the function u is called an entropy solution.

For scalar conservation laws every convex function η leads to an entropy, i.e. there exist infinite many entropies. It is also interesting to note that for multi-dimensional systems, entropy weak solutions are not unique. Recent results show that the usual concept of entropy solutions is inadequate for uniqueness of compressible and incompressible Euler equations [18, 27, 29].

At this point it is convenient to introduce more function spaces that are useful in analysing nonlinear conservation laws. These spaces are also well exploited in designing numerical schemes for approximating solutions for these equations. Here the notions of monotonicity, total variation bounds are significant.

Definition 4 Let $u \in L^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ be open. Then the total variation of u is defined by

$$\mathrm{TV}(u) = \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\Omega} |u(x+\varepsilon) - u(x)|\, dx.$$

The space of bounded variation is

$$\mathrm{BV}(\Omega) := \{u \in L^\infty(\Omega) : \ \mathrm{TV}(u) < \infty\}.$$

If $u' \in L^1(\Omega)$ holds, then

$$\mathrm{TV}(u) = \int_{\Omega} |u'|\, dx.$$

Theorem 1 (Kruskov[43]) *The scalar Cauchy-Problem*

$$\begin{aligned} u_t + (f(u))_x &= 0, \qquad f \in C^1(\mathbb{R}); \\ u(x,0) &= u_0(x), \quad u_0 \in L^\infty(\mathbb{R}) \end{aligned}$$

has a unique entropy solution $u \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$, having the following properties:

(i) $\|u(\cdot,t)\|_{L^\infty} \le \|u_0(\cdot)\|_{L^\infty}, \quad t \in \mathbb{R}^+;$
(ii) $u_0 \ge v_0 \ \Rightarrow \ u(\cdot,t) \ge v(\cdot,t), \quad t \in \mathbb{R}^+;$
(iii) $u_0 \in BV(\mathbb{R}) \ \Rightarrow \ u(\cdot,t) \in BV(\mathbb{R})$ *and* $TV(u(\cdot,t)) \le TV(u_0);$
(iv) $u_0 \in L^1(\mathbb{R}) \ \Rightarrow \ \int_{\mathbb{R}} u(x,t)\, dx = \int_{\mathbb{R}} u_0(x)\, dx, \quad t \in \mathbb{R}^+;$

(i)–(iv) are called L^∞-stability, monotonicity, TV-stability, conservativity.

The theorem can be extended to several dimensions $x \in \mathbb{R}^d$, $d > 1$. The theorem cannot be extended to the general case of systems ($k > 1$). Until now, there is no general proposition proved for the system case.

At this point some general remarks are in order [67]:

1. The family of admissible entropies in the scalar case consists of all convex functions, and the envelop of this family leads to Kruzkov's entropy pairs[43]

$$\eta(u;c) = |u-c|, \qquad \psi(u,c) = sgn(u-c)(f(u)-f(c)), \qquad c \in \mathbb{R}. \tag{22}$$

2. L^1-contraction. If u^1, u^2 are two entropy solutions of the scalar conservation law, then

$$\|u^2(\cdot,t) - u^1(\cdot,t)\|_{L^1(x)} \leq \|u_0^2(\cdot) - u_0^1(\cdot)\|_{L^1(x)}$$

This implies that the entropy solution operator related to scalar conservation laws in L^1 is non-expansive or contractive and by Crandall-Tartar lemma [24], it is monotone

$$u_0^2(\cdot) \geq u_0^1(\cdot) \Rightarrow u^2(\cdot,t) \geq u^1(\cdot,t)$$

3. Semi-group: Let $\{S_t, t \geq 0\}$ be a one-parameter family of operators which form a semi-group of constant-preserving, monotone operators. S_t preserves constants if

$$S_t[u \equiv \text{Const}] = \text{Const} \tag{23}$$

and its monotonicity implies

$$u_0^2(\cdot) \geq u_0^1(\cdot) \Rightarrow u^2(\cdot,t) \geq u^1(\cdot,t), \qquad \forall t \geq 0. \tag{24}$$

Assuming $\{S_t\}$ satisfies basic semi-group relations:

$$S_{t+s} = S_t S_s, \qquad S_0 = I,$$

and it has an infinitesimal generator,

$$\partial_x A(u) = \lim_{\Delta t \downarrow 0} (\Delta t)^{-1}(S_t(u) - u).$$

Thus $S_t u_0 = u(t)$ may be identified as the solution of the abstract Cauchy problem

$$u_t + \partial_x f(u(t)) = 0 \tag{25}$$

subject to initial conditions $u(0) = u_0$. For an L^1-setup for quasilinear evolution equations, the reader may consult [23]. In addition monotone, constant

preserving solution operators of the Cauchy problem (25) are uniquely identified by the following entropy condition [67].

Theorem 2 (Kruzkov's Entropy Condition) *Assume* $\{u(t), t \geq 0\}$ *is a family of solutions for the Cauchy problem* (25) *which is constant-preserving,* (23)*, and satisfies the monotonicity condition* (24)*. Then the following entropy inequality holds:*

$$\partial_t |u(t) - c| + \partial_x \{sgn(u - c)(f(u) - f(c))\} \leq 0, \qquad \forall c.$$

Hence one observes that monotonicity and constant preserving properties recover Kruzkov entropy pairs.

2.4 Analysis of Systems

At this point we would like to delve into the ideas that guide the construction of solutions for one-dimensional systems. The building block for such constructions is the Riemann problem discussed in Sect. 2.1. We will firstly consider linear problems and then close with a brief discussion of the nonlinear systems. In this case we will only concentrate on the strict hyperbolic case.

Consider systems of conservation laws in one-space-dimension in (3) with $x \in \mathbb{R}, \quad u, f \in \mathbb{R}^k, \quad f = (f_1, \ldots, f_k)$. In quasilinear form, Eq. (6),

$$A(u) = f'(u) = \frac{\partial f^{(i)}(u)}{\partial u_j}, \quad 1 \leq i, j \leq k.$$

Hence the linear systems are denoted as

$$u_t + Au_x = 0, \quad A \in \mathbb{R}^{k \times k}, \quad u(x, 0) = u_0(x).$$

For a hyperbolic equation A is diagonalisable with eigenvalues $\lambda_1, \ldots, \lambda_k$ and eigenvectors $r_1, \ldots, r_k$. Let $R = (r_1 | \ldots | r_k), \quad AR = RD, \quad A = RDR^{-1}$. Using this, we can diagonalise the system as follows: Let $v = R^{-1}u$ (characteristic variables)

$$Rv_t + RDR^{-1}Rv_x = 0 \quad \text{or} \quad v_t + Dv_x = 0,$$

since R is constant. We obtain k scalar problems for $(v_1, \ldots, v_p, \ldots, v_k)$ with solutions

$$v_p(x, t) = v_p(x - \lambda_p t, 0).$$

Given $v(x,0) = R^{-1}u_0(x)$, we obtain

$$u(x,t) = Rv(x,t) = \sum_{p=1}^{k} v_p(x,t)r_p = \sum_{p=1}^{k} v_p(x-\lambda_p t, 0)r_p.$$

The curves $x = x_0 + \lambda_p t$ are called characteristics of the p-th family ($x_p'(t) = \lambda_p$). The characteristics of the p-th family are given by $x_p(t)$ and take the form

$$x_p'(t) = \lambda_p, \quad x_p(0) = x_0, \quad p = 1, \ldots, k.$$

Note that in general problems depend on u and are strongly coupled due to $R = R(u)$. In terms of the Riemann problem, we consider systems of conservation laws in one-space-dimension:

$$u_t + Au_x = 0,$$

where A is a constant matrix and initial conditions as given in Eq. (4). Assume we can express the initial conditions as

$$u_L = \sum_{p=1}^{k} \alpha_p r_p, \quad u_R = \sum_{p=1}^{k} \beta_p r_p, \quad v_p(x,0) = \begin{cases} \alpha_p, & x < 0; \\ \beta_p, & x > 0. \end{cases}$$

Now

$$v_p(x,t) = \begin{cases} \alpha_p, & x - \lambda_p t < 0; \\ \beta_p, & x - \lambda_p t > 0. \end{cases}$$

Hence

$$u(x,t) = \sum_{p=1}^{k_0} \beta_p r_p + \sum_{p=k_0+1}^{k} \alpha_p r_p.$$

where k_0 is the minimal value of p with $x - \lambda_p t > 0$, ($\lambda_1 \leq \cdots \leq \lambda_k$). State u can also be expressed in terms of jump discontinuities as follows:

$$\begin{aligned} u(x,t) &= \sum_{p=1}^{k} \alpha_p r_p + \sum_{p=1}^{k_0} \beta_p r_p - \sum_{p=1}^{k_0} \alpha_p r_p \\ &= u_L + \sum_{p=1}^{k_0} (\beta_p - \alpha_p) r_p \\ &= u_R - \sum_{p=k_0+1}^{k} (\beta_p - \alpha_p) r_p \end{aligned}$$

and

$$u_R - u_L = \sum_{p=1}^{k} (\beta_p - \alpha_p) r_p$$

Hence the solution of a Riemann problem can be considered as a splitting of difference $u_R - u_L$ in a sum of jumps, which move with velocity λ_p in the direction r_p in the phase space. For $k = 2$ a phase plot can be used to determine the intermediate states between u_L and u_R. If u_m is the intermediate state:

$$u_R - u_L = (\beta_1 - \alpha_1) r_1 + (\beta_2 - \alpha_2) r_2 = u_m - u_L + u_R - u_m$$

The jump $u_m - u_L$ moves with velocity λ_1 in the direction r_1 and the jump $u_R - u_m$ with λ_2 in the direction r_2. Now $\lambda_1 \leq \lambda_2$, which implies velocity of $u_m - u_L$ is smaller, thus this must be the first jump (recall, entropy conditions).

In the general nonlinear case, we consider Eq. (3). Rewriting the equation in quasilinear form, we obtain:

$$u_t + A(u) u_x = 0$$

for which $A(u)$ has eigenvalues $\lambda_1(u), \ldots, \lambda_k(u)$ and eigenvectors $r_1(u), \ldots, r_k(u)$. In this case hyperbolicity needs to be understood in the sense that $A(u)$ has a complete real eigensystem. For the ensuing discussion strict hyperbolicity is assumed, i.e. distinct eigenvalues $\lambda_i(u) \neq \lambda_j(u)$ for $i \neq j$.

We first start with introducing the Lax entropy conditions:

Definition 5 (Lax Entropy Condition) A discontinuity with left state u_L and right state u_R, moving with speed s for a conservation law with a convex flux function is entropy satisfying if

$$f'(u_L) > s > f'(u_R).$$

This implies that the characteristics are ingoing into the shock as time evolves. We also note that if f is convex then the correct weak solution is the limit as $\epsilon \to 0$ of the viscous problem if and only if the Lax entropy condition holds [63].

Definition 6 The p-th characteristic field is genuinely nonlinear, if

$$\nabla \lambda_p(u) \cdot r_p(u) \neq 0, \qquad \forall u$$

or it is linear degenerate, if

$$\nabla \lambda_p(u) \cdot r_p(u) = 0, \qquad \forall u.$$

At this point we observe that in the linear case all fields are linear degenerate.

The building block for constructing solutions in the one-dimensional case is the solution of the Riemann problem. A solution for (3) is sought subject to initial conditions (4). The solution is composed of k simple waves. Each of these waves is associated with one eigenpair $(\lambda_p(u), r_p(u)), \quad 1 \leq p \leq k$. The simple waves come in three forms as pointed out in the Sod's problem: if the p-th field is genuinely nonlinear, the waves are either p-shock or p-rarefaction waves. On the other hand if the p-th field is linearly degenerate, we obtain a contact wave. These simple waves are centred and depend on $\xi = \frac{x}{t}$. This is how the simple waves are determined:

1. A p-shock discontinuity of the form

$$u(\xi) = \begin{cases} u_L, & \xi < s \\ u_R, & \xi > s. \end{cases}$$

   As usual s denotes the shock speed which can be determined by the Rankine-Hugoniot condition and to be entropy satisfying $\lambda_p(u_L) > s > \lambda_p(u_R)$.
2. A p-rarefaction wave, $u(\xi)$, directed along the p-eigenvector, $\dot{u}(\xi) = r_p(u(\xi))$. The eigenvector is normalised such that $r_p \cdot \nabla\lambda_p \equiv 1$. This ensures that the gap between $\lambda_p(u_L) < \lambda_p(u_R)$ is filled with a fan of the form

$$\lambda_p(u(\xi)) = \begin{cases} \lambda_p(u_L), & \xi < \lambda_p(u_L) \\ \xi, & \lambda_p(u_L) < \xi < \lambda_p(u_R) \\ \lambda_p(u_R), & \lambda_p(u_R) < \xi. \end{cases}$$

3. A p-contact discontinuity of the form

$$u(\xi) = \begin{cases} u_L, & \xi < s \\ u_R, & \xi > s. \end{cases}$$

   As usual s denotes the shock speed which can be determined by the Rankine-Hugoniot condition such that $\lambda_p(u_L) = s = \lambda_p(u_R)$.

The admissibility of systems has been discussed in [46]. The theorem below summarises the admissibility of systems:

Theorem 3 (Lax Solution of Riemann Problems [15]) *The strictly hyperbolic admissible system (3), subject to Riemann initial data (4) with $u_L - u_R$ sufficiently small, admits a weak entropy solution, which consists of shock-, rarefaction- and contact-waves.*

For more discussion on the solution of Riemann problems, the reader may consult [16]. An extension to generalised Riemann problems subject to piecewise-linear initial data can be found in [13, 49].

In conclusion to this section we would like to briefly mention Glimm's theorem [32] which is celebrated as essential for existence theorems which are designed for general one-dimensional systems, see also [53, 60]

Theorem 4 ([67]) *There exists a weak entropy solution,* $u(\cdot, t) \in L^{\infty}[BV \cap L^{\infty}(\mathbb{R}), [0, T]]$*, of strictly hyperbolic systems* (3)*, subject to initial conditions with sufficiently small variation,* $\|u_o(\cdot)\|_{BV \cap L^{\infty}(\mathbb{R})} \leq \epsilon$.

In the next section, we will consider numerical approximations for solutions of nonlinear hyperbolic equations. The relaxation scheme will be considered. Its simplicity and ease of implementation has been a motivation for presenting this scheme.

3 Numerical Approximation

In this section numerical approximations of solutions of nonlinear hyperbolic conservation laws will be presented. We will start by presenting the simplest and basic scheme in the finite volume framework. In the latter part of the section derivation of the relaxation scheme will be presented and numerical results on inviscid Euler gas dynamics equations will be presented.

3.1 Design Ideas of Numerical Schemes: The Godunov Upwind Scheme and the Lax-Friedrichs Scheme

Consider a scalar equation of the form in Eq. (3) with a linear flux function $f(u) = au$ in which $a = \text{const}$, also referred to as a wave propagation speed. To approximate the solution on the spatial domain which is a subinterval of $\mathbb{R}$, we discretise the interval to approximate the solution at discrete points by taking a uniform grid with mesh-size Δx. In addition we similarly discretise the time variable t using Δt. Denote u_i^n as an approximation of $u(x_i, t^n)$ at the point $x_i = i\Delta x$ which is the midpoint of the cell $[x_{i-1/2}, x_{i+1/2}]$, $t^n = n\Delta t$. Thus we need explicit conservation schemes to approximate the equation in (3) in the form:

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x}\left[f_{i-1/2} - f_{i+1/2}\right], \quad i \in \mathbb{Z}, \quad n \geq 0 \tag{26}$$

where $f_{i+1/2}$ is the intercell numerical flux and u_i^0 for which the continuous version is given in (4). For more details on discrete solution methods, the reader may consult [33, 50, 67]. A first-order scheme referred to as the upwind scheme can be defined as follows: for $a > 0$:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\big[au_i - au_{i-1}\big]$$

and for $a < 0$:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\big[au_{i+1} - au_i\big].$$

The above schemes can be generalised by defining the discrete flux function as

$$f_{i+1/2} = \frac{1}{2}\big(f(u_{i+1}) + f(u_i)\big) - \frac{1}{2}|a|\big(u_{i+1} - u_i\big).$$

This is the simplest case of the famous Godunov methods [33, 50, 67]. It is also referred to as the Rusanov Scheme.

In general,

$$f_{i+1/2} = F(u_{i-k+1}^n, \ldots, u_{i+k}^n) \tag{27}$$

where F is a continuous numerical flux. The Lax-Friedrichs Method takes the form:

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n).$$

One has to also consider stability issues when designing the method [33,50,67]. For example, the Lax-Friedrichs Method is stable for

$$\left|\frac{a\Delta t}{\Delta x}\right| \leq 1/2.$$

As a consequence the methods are versatile but due to the fact that waves of different families are averaged together at each computational cell, their resolution is very diffuse.

Alternatively, one discretises space leaving time continuous to obtain a system of ordinary differential equations. This is referred to as a semi-discretisation or a method of lines. A stability analysis can be carried out on the system of ordinary differential equations. For Cauchy-Problems consider a subinterval e.g. $[0, 1]$ and prescribe a boundary condition. Prescription of boundary conditions depends on the direction of transport (i.e. wind direction). For example, for $a > 0$, we need a condition at $x = 0$. On the other hand, periodical boundary conditions can be given as

$$u(0, t) = u(1, t), \quad \forall t \geq 0.$$

To construct finite difference or finite volume schemes of the form discussed above, we will discuss some properties that the numerical flux needs to satisfy. Let the numerical flux be defined as in Eq. (27) which is an approximation of $f(u)$,

a flux across the interface between neighbouring cells. The essential feature of the numerical scheme in Eq. (26) is their conservation form. This implies that the discrete flux over any spatial domain needs to depend on the discrete flux across the boundaries of the domain.

Definition 7 Now consider the hyperbolic conservation law in discretised form given in Eq. (26). The scheme is said to be consistent if

$$F(u, u, \ldots, u) = f(u), \quad \forall u.$$

This is a $(2k + 1)$-point scheme, for $k = 1$ it is a three-point scheme.

Definition 8 Consider Eq. (26) where $f_{i+1/2}$ is the intercell numerical flux and u_i^0 is given. The scheme (26) is also referred to as a conservative scheme and it is in conservative form.

Theorem 5 (Lax and Wendrof [47]) *Consider the conservative difference scheme (26), with consistent numerical flux. Let* $\Delta t \downarrow 0$ *with fixed grid-ratios* $\lambda_j = \dfrac{\Delta t}{\Delta x_j} \equiv const_j$, *and let* $u^{\Delta t} = \{u_i^n\}$ *denote the corresponding solution (parameterised with respect to the vanishing grid-size). Assume that* $u^{\Delta t}$ *"converges" strongly to a function u (in some sensible way), the limit u is a weak solution of Eq. (3).*

It can be pointed out that the Lax-Wendroff theorem is fundamental in designing the so called shock-capturing schemes. In these schemes, instead of tracking jump discontinuities (evolving smooth pieces of the approximate solution on both sides of the discontinuity), conservative schemes capture a discrete shock discontinuity.

To study stability of the scheme, use the (discrete) L^p-norms for the sequences $u^n = (u_j^n)$. Other notions which are used in understanding the convergence behavior of discrete sequences of solutions of conservation laws are monotonicity/TVD. These will be considered in the scalar case:

Definition 9
- A scheme is monotone if given two sequences $v^0 = (v_j^0)$ and $w^0 = (w_j^0)$, $v^0 \geq w^0$ then $v^1 \geq w^1$, where $v \geq w$ means for all j, $v_j \geq w_j$ and $v_j^1 = (v_j^1)$
- A scheme is Total Variation Diminishing (TVD) if $\forall v^0 = (v_j^0)$,

$$\mathrm{TV}(v^n) \leq \mathrm{TV}(v^0), \tag{28}$$

where

$$\mathrm{TV}(v) = \sum_{j \in \mathbb{Z}} |v_{j+1} - v_j|.$$

In general, monotonicity and thus the TVD property is ensured if

$$\min_k\{u_k^n\} \le u_j^{n+1} \le \max_k\{u_k^n\}.$$

Hence it transforms a monotone sequence, say non-decreasing one, into a monotone (non-decreasing) sequence. Therefore, oscillations can not occur. Numerical solutions of conservation laws that satisfy the TVD property (28), also possess the following properties [67]:

- Convergence: the piecewise-constant numerical solution, $u^{\Delta x}(x,t^n) = \sum_i u_i^n \mathscr{X}_i(x)$, where $\mathscr{X}_i$ is a characteristic function defined on the cell $[x_{i+1/2}, x_{i-1/2}]$, converges strongly to a limit function, $u(x,t^n)$, as the spatial grid is refined. This with equicontinuity in time (in the L^1-norm) and the Lax-Wendroff theorem, give a weak solution, $u(x,t)$, of the conservation law (3).
- Spurious solutions can not occur due to the TVD condition, also the BV condition in Sect. 2.
- Accuracy: one can derive schemes of accuracy higher than first-order. Monotone schemes are at most first-order accurate [36]. TVD schemes are instead not restricted to first-order at least in one-dimension. This means we replace L^1-contractive solutions with (the weaker) condition of bounded variation solutions.

Thus to apply the schemes, the discrete equations must be written in conservation form which is able to capture the correct speeds of discontinuities. To obtain $f_{i+1/2}$, one needs an extrapolation of the solution in the cell to the boundary of the cell i.e. need $u_{i+1/2}$. The Godunov first-order upwind method uses piecewise constant data to extrapolate the solution to cell edges. A higher-order method, the Modified Upwind Scheme for Conservation Law (MUSCL), modifies the piecewise constant data, replaces the Godunov approach by some monotone first-order centred scheme, avoiding explicitly solving the Riemann Problem as follows: Consider piecewise constant data $\{u_i^n\}$, replace constant states u_i^n, understood as integral averages in cells $I_i = [x_{i-1/2}, x_{i+1/2}]$, by piecewise linear functions $u_i(x)$:

$$u_i(x) = u_i^n + \frac{(x-x_i)}{\Delta x}\Delta_i, \qquad x \in [0, \Delta x]$$

where Δ_i is a suitably chosen slope of $u_i(x)$ in cell I_i. For a discussion of slope choices, we refer to [33, 50], in which slope limiters based on TVD constraints are employed to remove possible oscillations. Thus one chooses $\bar{\Delta}_i = \phi\Delta_i$ where ϕ is a slope limiter function, for example, minmod, van Leer [69], or Superbee [66]. The centre of x_i in local coordinates is $x = \frac{1}{2}\Delta x$ and $u_i(x_i) = u_i^n$. Thus the boundary extrapolated values are:

$$u_i^L = u_i(0) = u_i^n - \frac{1}{2}\bar{\Delta}_i;$$

$$u_i^R = u_i(\Delta x) = u_i^n + \frac{1}{2}\bar{\Delta}_i.$$

The values u_i^L and u_i^R are the new arguments of the numerical flux function, $f_{i+1/2}(u_{i+1}^L, u_i^R)$. Thus the problem reduces to a flux estimate at each interface. For ideas on different flux approximations, we refer to [45, 68].

The characteristic variables play an essential role: upwind schemes are generally derived for a scalar equation hence for systems, the quantities that are being convected are the characteristic variables. Therefore, to prevent spurious oscillations, characteristic variables must also be numerically transported.

3.2 The Relaxation Schemes

Below a scheme, which is simple to implement since it does not actively consider the direction of the flow, is presented [2, 5, 39]. The scheme is developed with the spirit of central schemes like the Rusanov or Lax-Friedrichs but based on kinetic considerations of the flow [1, 55]. The conservation law in Eq. (3) is re-written as a balance law in which the right-hand side is a stiff relaxation term as follows:

$$u_t + v_x = 0; \tag{29}$$

$$v_t + a^2 u_x = -\frac{1}{\varepsilon}(v - f(u)); \tag{30}$$

where a is a value yet to be determined. Equation (29)–(30) can be rewritten in the form:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ a^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = \frac{1}{\varepsilon} g(u).$$

The eigenvalues of this Jacobian matrix are $\pm a$. The characteristic variables take the form: $\begin{pmatrix} v + au \\ v - au \end{pmatrix} =: W$ from which we obtain a semi-linear system coupled by the source term that takes the form:

$$\frac{\partial W}{\partial t} + \Lambda \frac{\partial W}{\partial x} = \frac{1}{\varepsilon} G.$$

3.2.1 Spatial Discretisation

Let $\Delta x = x_{i+1/2} - x_{i-1/2}$, $\Delta t = t_{n+1} - t_n$, and $\omega_{i+1/2}^n := \omega(x_{i+1/2}, t_n)$ with the cell average

$$\omega_i^n = \frac{1}{\Delta x} \int_{x-1/2}^{x+1/2} \omega(x, t_n)\, dx.$$

Then the semi-discrete relaxation system takes the form:

$$\frac{du_i}{dt} + D_x v_i = 0; \tag{31}$$

$$\frac{dv_i}{dt} + a^2 D_x u_i = -\frac{1}{\varepsilon}(v_i - f(u_i)); \tag{32}$$

where D_x is the discrete equivalent of ∂_x. Now the flux at cell boundaries can be approximated as follows: consider interval $I_i = [x_{i-1/2}, x_{i+1/2}]$, denote an approximating polynomial on cell I_i by $p_i(x,t)$ then

$$\tilde{u}(x,t) = \sum_i p_i(x,t;u)\mathscr{X}_i(x);$$

where $\mathscr{X}$ is a characteristic function defined on cell I_i. Denote the values of u at cell boundary point between cell I_i and I_{i+1}, $x_{i+1/2}$, as:

$$u^R(x_{i+1/2};u) = p_{i+1}(x_{i+1/2};u);$$
$$u^L(x_{i+1/2};u) = p_i(x_{i+1/2};u).$$

To apply the MUSCL approach, characteristic variables are used in the reconstruction

$$(v-au)_{i+1/2} = (v-au)^R_{i+1/2} = p_{i+1}(x_{i+1/2}; v-au);$$
$$(v+au)_{i+1/2} = (v+au)^L_{i+1/2} = p_i(x_{i+1/2}; v+au);$$

giving:

$$u_{i+1/2} = \frac{1}{2a}\Big(p_i(x_{i+1/2}; v+au) - p_{i+1}(x_{i+1/2}; v-au)\Big);$$
$$v_{i+1/2} = \frac{1}{2}\Big(p_i(x_{i+1/2}; v+au) + p_{i+1}(x_{i+1/2}; v-au)\Big).$$

The first-order schemes are derived by choosing the polynomial: $p_i(x,u) = u_i$ giving $(v+au)_{i+1/2} = (v+au)_i$ and $(v-au)_{i+1/2} = (v-au)_{i+1}$ and

$$u_{i+1/2} = \frac{u_i + u_{i+1}}{2} - \frac{v_{i+1} - v_i}{2a};$$
$$v_{i+1/2} = \frac{v_i + v_{i+1}}{2} - a\frac{u_{i+1} - u_i}{2}.$$

A second-order polynomial with slope limiters can also be used, giving:

$$u_{i+1/2} = \frac{u_i + u_{i+1}}{2} - \frac{v_{i+1} - v_i}{2a} + \frac{\sigma_i^+ + \sigma_{i+1}^-}{4a};$$
$$v_{i+1/2} = \frac{v_i + v_{i+1}}{2} - a\frac{u_{i+1} - u_i}{2} + \frac{\sigma_i^+ - \sigma_{i+1}^-}{4}.$$

Using Sweby's notation: slopes of $v \pm au$ can be defined as:

$$\sigma^{\pm} = (v_{i+1} \pm au_{i+1} - v_i \mp au_i)\phi(\theta_i^{\pm});$$
$$\theta_i^{\pm} = \frac{v_i \pm au_i - v_{i-1} \mp au_{i-1}}{v_{i+1} \pm au_{i+1} - v_i \mp au_i}.$$

Below, some popular slope limiters are presented: Minmod Slope Limiter:

$$\phi(\theta) = \max(0, \min(1, \theta))$$

and van Leer slope limiter:

$$\phi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}.$$

It should be observed that if $\sigma_i^{\pm} = 0$ or $\phi = 0$, we obtain the first-order discretisation.

To complete the description of the relaxation systems initial conditions are approximated as: $u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) = f(u_0(x))$. It should also be pointed out that the choice of a, the free parameter, must be made such that the problem is well-defined. For this the regularisation properties of the relaxation framework are exploited in the choice of a as $\varepsilon \to 0$. The Chapman-Enskog asymptotic analysis [39] gives the sub-characteristic condition:

$$-a \leq f'(u) \leq a; \quad \forall u. \tag{33}$$

The limit $\varepsilon \to 0$ also gives the discretization of the original conservation law given in (3). In practice, for the cases where $\varepsilon \to 0$, the constraint $v = f(u)$ is used in Eqs. (29)–(30) (in which case the first-order scheme becomes the Rusanov scheme). This scheme is also referred to as the relaxed scheme. Otherwise one chooses a small value of ε, for example, $\varepsilon = 10^{-6}$, and solves (29)–(30) numerically. This scheme is referred to as the relaxing scheme.

3.2.2 Time Discretization

In general the time discretisation schemes can be derived using TVD Runge-Kutta schemes [34]. Consider Eq. (31). In this case, it can be observed that the strategy is as follows:

- Treat space discretizations separately using a MUSCL-type formulation.
- Treat time by an ordinary differential equation (ODE) solver (method of lines), for example, Implicit-Explicit (IMEX) schemes since the source term (RHS) is stiff.
- Choice of a^2 is wide as long as the sub-characteristic condition (33) is satisfied: a_i's can be chosen as global characteristic speeds, local speeds, depending on the stability and accuracy needs of implementation.
- Source terms are incorporated accordingly.

The semi-discrete formulation above is a system of ODEs which can be integrated using Implicit-Explicit (IMEX) Runge-Kutta approaches as follows: consider the semi-discrete equation (31) in a general form as follows [56–58]:

$$\frac{d\mathscr{Y}}{dt} = \mathscr{F}(\mathscr{Y}) - \frac{1}{\varepsilon}\mathscr{G}(\mathscr{Y}),$$

In general the following strategy is applied in solving the system:

1. Treat non-stiff stage, $\mathscr{F}$, with an explicit Runge-Kutta scheme.
2. Treat the stiff stage, $\mathscr{G}$, with a diagonally implicit Runge-Kutta (DIRK) scheme.
3. Scheme must be asymptotic-preserving.
4. The limiting Scheme i.e. as $\varepsilon \to 0$ must be Strong Stability Preserving (SSP) [34]

$$\|\mathscr{Y}^{n+1}\| \leq \|\mathscr{Y}^{n}\|.$$

Here $\mathscr{Y}^n$ is the approximate solution at $t = n\Delta t$. Examples of these schemes can be represented using the so called Butcher tables [58]:

$$\begin{array}{c|cccccc} \tilde{c}_1 & \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} & \cdots & \tilde{a}_{1s} \\ \tilde{c}_2 & \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} & \cdots & \tilde{a}_{2s} \\ \tilde{c}_3 & \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} & \cdots & \tilde{a}_{3s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{c}_s & \tilde{a}_{s1} & \tilde{a}_{s2} & \tilde{a}_{s3} & \cdots & \tilde{a}_{ss} \\ \hline & b_1 & b_2 & b_3 & \cdots & b_s \end{array} \qquad \begin{array}{c|cccccc} c_1 & a_{11} & a_{12} & a_{13} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & a_{23} & \cdots & a_{2s} \\ c_3 & a_{31} & a_{32} & a_{33} & \cdots & a_{3s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_s & a_{s1} & a_{s2} & a_{s3} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & b_3 & \cdots & b_s \end{array}$$

For practical purposes, the IMEX method is implemented as follows:

- For $l = 1, \ldots, s$,

  1. Evaluate $\mathbf{K}_l^*$ as:

$$\mathbf{K}_l^* = \mathscr{Y}^n + \Delta t \sum_{m=1}^{l-1} \tilde{a}_{lm}\mathscr{F}(\mathbf{K}_m) - \frac{\Delta t}{\varepsilon}\sum_{m=1}^{l-1} a_{lm}\mathscr{G}(\mathbf{K}_m).$$

2. Solve for $\mathbf{K}_l$:

$$\mathbf{K}_l = \mathbf{K}_l^* - \frac{\Delta t}{\varepsilon} a_{ll} \mathscr{G}(\mathbf{K}_l).$$

- Update $\mathscr{Y}^{n+1}$ as:

$$\mathscr{Y}^{n+1} = \mathscr{Y}^n + \Delta t \sum_{l=1}^{s} \tilde{b}_l \mathbf{F}(\mathbf{K}_l) - \frac{\Delta t}{\varepsilon} \sum_{l=1}^{s} b_l \mathscr{G}(\mathbf{K}_l).$$

Below, we present some examples of the Butcher tables for the second-order and third-order schemes:

Second-Order Scheme

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \begin{array}{c|cc} -1 & -1 & 0 \\ 2 & 1 & 1 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Third-Order Scheme

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \gamma & \gamma & 0 & 0 \\ 1-\gamma & \gamma-1 & 2-2\gamma & 0 \\ \hline & 0 & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \gamma & 0 & \gamma & 0 \\ 1-\gamma & 0 & 1-2\gamma & \gamma \\ \hline & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

where $\gamma = \frac{3+\sqrt{3}}{6}$. The advantage of this approach is that numerically neither linear algebraic nor nonlinear source terms can arise. As $\varepsilon \longrightarrow 0$ the time integration procedure tends to an SSP time integration scheme of the limit equation (3). The only restriction is the usual CFL condition

$$\text{CFL} = \max\left(\frac{\Delta t}{\Delta x}, \lambda \frac{\Delta t}{\Delta x}\right) \leq 1.$$

In the following section, we present some numerical examples for which the relaxation scheme have been applied. We also wish to point out that the results in Figs. 1 and 2 were also produced by the relaxation schemes of different accuracy. In the case below, we extend the application of the scheme to two dimensional Euler equations.

3.2.3 Some Numerical Results

In this section, examples of results produced using the above defined schemes are presented. The one-dimensional examples have been presented in Sect. 2 above. A dimension-by-dimension extension can be undertaken to solve problems in higher space dimensions [4, 5, 7, 61]. It must be mentioned here that one advantage of the finite volume formalism is the ability to deal with unstructured meshes in multi-dimensional problems while still applying the one-dimensional numerical schemes. Here, two examples of two dimensional inviscid Euler problems are presented:

$$\partial_t \rho + \partial_x m + \partial_y n = 0; \tag{35a}$$

$$\partial_t m + \partial_x(\rho u^2 + p) + \partial_y(\rho uv) = 0; \tag{35b}$$

$$\partial_t n + \partial_x(\rho uv) + \partial_y(\rho v^2 + p) = 0; \tag{35c}$$

$$\partial_t E + \partial_x(u(E + p)) + \partial_y(v(E + p)) = 0; \tag{35d}$$

$$p = (\gamma - 1)(E - \frac{\rho}{2}(u^2 + v^2)). \tag{35e}$$

Double Mach Reflection Problem [72]

Consider flow in the domain $\Omega = [0, 4] \times [0, 1]$, the bottom wall is a reflecting wall in interval: $[1/6, 4]$. A Mach 10 shock is introduced at $x = 1/6$, $y = 0$, 60^0 angle with x-axis. Further, exact post-shock condition at bottom boundary $[0, 1/6]$ are imposed. The rest of the boundary is a reflective boundary. On the top boundary exact motion of a Mach 10 shock is used. Results are displayed at $t = 0.2$ in Fig. 8.

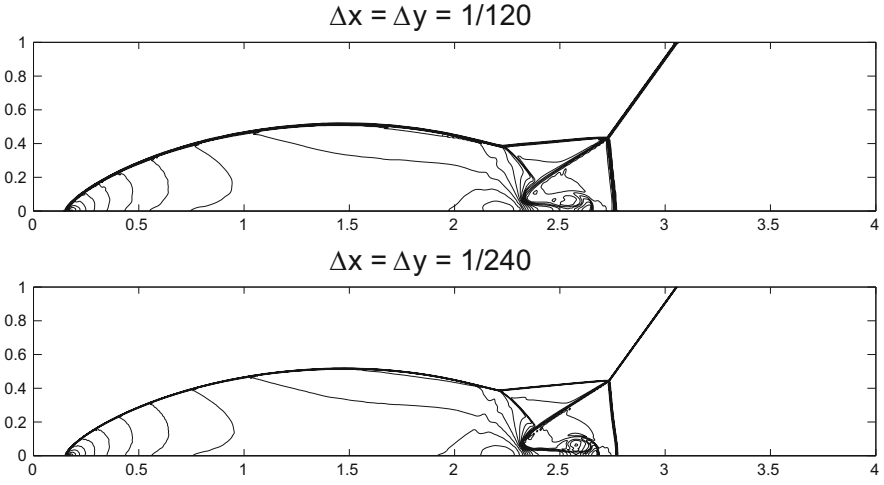

Fig. 8 Solution of the double Mach reflection problem obtained using the fifth-order relaxation scheme [5]

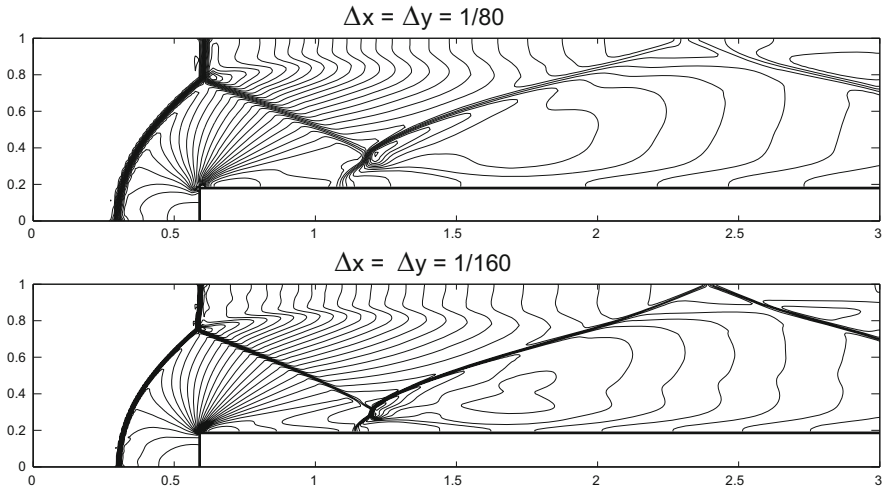

Fig. 9 Solution of forward-facing step problem obtained using the fifth-order relaxation scheme [5]

Forward-Facing Step Problem [72]

The next experiment involves a right-going Mach 3 uniform flow entering a wide tunnel 1 unit by 3 units long. The step is 0.2 units high located 0.6 units from left hand end of tunnel. As initial conditions a uniform right-going Mach 3 flow is considered. The boundary conditions are defined by a reflecting boundary condition along walls, inflow and outflow boundary conditions are applied at entrance and exit of tunnel. Results at $t = 4.0$ are given in Fig. 9.

3.2.4 Other Applications of the Relaxation Schemes

The relaxation approach has been applied in numerical methods for equations that avail themselves to being expressed in a kinetic formulation. In this formulation the $u(x,t)$ can be represented as an average of a 'microscopic' density function, $f(x,v,t)$. The formulation is based on classical kinetic models such as the Boltzmann equation. The useful tool is the velocity averaging which yields equations of moments of the kinetic model. These moments are solutions of the classical fluid flow equations like the incompressible Navier-Stokes equations. The most popular numerical approach for this is the Lattice Boltzmann Model (LBM). The relaxation approach can also be derived from the Boltzmann model by way of asymptotic analysis based on the Hilbert expansion in the small Knudsen and Mach number. Without going into details, this approach has been applied in simulating incompressible Navier-Stokes equations [7, 11, 40, 41], turbulent flow [6], the incompressible flow models with radiative transfer [10].

4 Applications in Networked Flows

Consider applications which have an inherent network and transport structure: traffic flow, gas or water transportation networks, telecommunication, blood flow or production systems (supply chain networks). The network is considered as a directed graph with arcs (edges/links) and vertices (nodes). Flow or Transport phenomena along each arc are sufficiently described by a one-dimensional hyperbolic equation, for example, a conservation law or a balance law (conservation law with source terms). To define flow or transport at the vertex, physical coupling conditions at each vertex are described by an algebraic condition. The mathematical problem can be considered as a coupled system of hyperbolic balance or conservation laws. In this section we discuss or develop an appropriate model for the arcs then define coupling conditions at nodes. To close the section, we will also investigate optimal control problems such as boundary stabilisation.

4.1 Modelling and Simulation of Networked Flow

In network flow problems, the mathematical model of the underlying physical process is formulated by an interconnected hierarchy of dynamics on the arcs/edges using mathematical modelling principles. These dynamics are coupled and interact with other processes by transmission conditions at the vertices/nodes of the graph/network. The transmission conditions might obey other independent dynamic laws defined based on physical, chemical or engineering considerations, the reader may refer to Banda et al. [8, 9] and references therein.

Arising mathematical issues deal with modelling and the interaction of different dynamics on arcs and the influence of the transmission conditions on the local and global dynamics of the network. This in particular involves the interaction between different dynamic and/or discrete models. Other problems arise due to the underlying complex physics and governing principles inside the vertices which are usually not known exactly, but need to be formulated in a concise mathematical formulation for further treatment.

From a practical point of view, network problems are usually large-scale and contain additional complexity due to their geometry. Hence, efficient numerical methods and new techniques are required to compute solutions for given accuracy and within a reasonable time. In most cases real-time solutions would be the desired goal.

To address such problems, one considers modelling of the dynamics on the arcs and development or application of efficient and accurate numerical methods for resolving these dynamics. Already developed models are available and can be applied. Where necessary a careful derivation of simplified dynamics which yields a trade-off between accuracy and computability can be considered. The second step is the modelling of transmission or coupling conditions informed

by the underlying physical processes and mathematical analysis of the derived conditions. In particular, the compatibility of the coupling with both mathematical and engineering needs are of interest. For example, in the case of gas dynamics in pipelines, results are already available. The details of mathematical proofs of existence of solutions to a coupled system of dynamics on arcs and vertices are presented in [8, 9]. Here we give a summary of the results that are known from literature:

1. Gas Networks: conditions of conservation of mass and equal pressure at the nodes are well-posed.
2. Compressor condition and conservation of mass through compressor are well-posed.
3. Assumption of gas with minor losses at the junction was not taken into consideration (i.e. pressure drop factor—depends on geometry, flow and density). In addition results on three-dimensional flow, which is more realistic have not been considered.

4.1.1 Flow in a Single Pipe

To present flow in a network of pipes, it is necessary to consider flow in a single pipe. Thereafter extensions of single pipe flow to coupled pipes in a network will be discussed. The link between the two are algebraic relations which are defined using physical considerations.

The flow model is a non-linear partial differential equation (PDE) presented in Eq. (7). Some simplifications assumed on the Euler equations are as follows. For pipelines the cross-section of the pipes is very small compared to the length of the pipeline segments (pipes). Such being the case, the flow is assumed to be one-dimensional. Hence, pressure and velocity variations across the cross-sectional area are assumed negligible. The temperature of the gas is assumed to be constant. This is the case since most of the pipes in the real-world are buried underground and the soil is assumed to be a large heat sink. Therefore, the flow is assumed to be in thermodynamic equilibrium. Only two forces acting on the gas are considered significant: internal friction of the pipe and inclination of the pipe due to topography.

With these assumptions the isothermal Euler equations in one space dimension augmented with non-linear source terms are obtained [8, 9]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0; \tag{36a}$$

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{\rho} + p(\rho)\right) = s_1(\rho, u) + s_2(x, \rho). \tag{36b}$$

As described in Sect. 1, the first equation defines the conservation of mass. The second equation is based on Newton's laws of motion and describes the conservation of momentum (mass flux of the gas) $q = \rho u$ where $\rho(x, t)$ is the mass density of the

gas, $u(x,t)$ is the gas velocity and $p(\rho)$ is a pressure law. The pressure law satisfies the following properties:

(**P**) $p \in C^2(\mathbb{R}^+;\mathbb{R}^+)$ with $p(0) = 0$ and $p'(\rho) > 0$ and $p'' \geq 0$ for all $\rho \in \mathbb{R}^+$.

With the above assumptions, $p(\rho)$ is often chosen as

$$p(\rho) = \frac{Z\mathscr{R}T}{M_g}\rho = a^2\rho, \tag{37}$$

where Z is the natural gas compressibility factor, $\mathscr{R}$ the universal gas constant, T the absolute gas temperature, and M_g the gas molecular weight. Since temperature has been assumed constant, one considers the constant a to be the speed of sound in the gas. Also note that a depends on the type of gas as well as the temperature.

For the sake of simplicity, further assumptions are made [28, 54, 65]: pipe wall expansion or contraction under pressure loads is negligible; hence, pipes have constant cross-sectional area. The diameter, D, of the pipe is constant.

The source term modelling the influence of friction is modelled by assuming steady state friction for all pipes [54, 73]. The friction factor f_g is calculated using Chen's equation [17]:

$$\frac{1}{\sqrt{f_g}} := -2\log\Big(\frac{\varepsilon/D}{3.7065} - \frac{5.0452}{N_{Re}}\log\Big(\frac{1}{2.8257}\Big(\frac{\varepsilon}{D}\Big)^{1.1098} + \frac{5.8506}{0.8981 N_{Re}}\Big)\Big) \tag{38}$$

where N_{Re} is the Reynolds number $N_{Re} = \rho u D/\mu$, μ the gas dynamic viscosity and ϵ the pipeline roughness, which are again assumed to be the same for all pipes. With the above assumptions, the friction term takes the form

$$s_1(\rho,u) = -\frac{f_g}{2D}\rho u|u|.$$

The pipe inclination takes the form

$$s_2(x,\rho) = -g\rho\sin\alpha(x)$$

where g is acceleration due to gravity and $\alpha(x)$ is the slope of the pipe.

In addition, two additional assumptions need to be made:

A1. There are no vacuum states, i.e. $\rho > 0$.
A2. All flow states are subsonic, i.e. $\frac{q}{\rho} < a$.

These assumptions are backed by physical considerations. It is reasonable to assume that atmospheric pressure is the lower bound for the pressure in the pipes. Lower pressure can occur due to waves travelling through the pipe. Waves which would create vacuum states are untenable since they would cause the pipe to explode or implode. Generally pipelines are operated at high pressure (40–60 bar) and the gas velocity is very low (< 10 m/s). As the speed of sound in natural gas is around

370 m/s, the second assumption also makes sense. In pipeline networks, all states are a reasonable distance from the sonic states so that travelling waves cannot create sonic or supersonic states.

Briefly, some mathematical properties of the above system will be discussed [50]. These mathematical properties are used to discuss the well-posedness of the flow problem in a single pipe. With the above assumptions Eq. (36) is also strictly hyperbolic. To solve Riemann problems, characteristic fields based on Lax-curves are employed [26, 50]. The discussion of the Riemann problems has been given in Sect. 2.

4.1.2 Flow in Networks or Coupling Pipes and Nodes

To complete the discussion of flows in networks it is necessary to discuss coupling of different pipes at a node. Coupling of pipes is believed to be the major part of gas networks [65]. Approaches in the engineering community are exploited.

Definition 10 A network is a finite directed graph $(\mathscr{J}, \mathscr{V})$. The arcs, $I_j \in \mathscr{J}$, are connected together by vertices or nodes, $v \in \mathscr{V}$.

We need to study a Cauchy problem on the whole network. This depends on the solution at the vertices.

Applying assumption **A2** and the notation in [8, 35], a set of edges is denoted by $\mathscr{J}$ and a set of nodes is denoted by $\mathscr{V}$. Both sets are taken to be non-empty. Each edge $j \in \mathscr{J}$ corresponds to a pipe parametrised by an interval $I_j := [x_j^a, x_j^b]$. Each node $v \in \mathscr{V}$ corresponds to a single intersection of pipes. For each node $v \in \mathscr{V}$, the set of all indices of pipes $j \in \mathscr{J}$ ingoing and outgoing to the node can be separated into two sets δ_v^- and δ_v^+, respectively. The set of all pipes intersecting at a node $v \in \mathscr{V}$ can be denoted as $\delta_v = \delta_v^- \cup \delta_v^+$. In addition, the degree of a vertex $v \in \mathscr{V}$ is the number of pipes connected to the node.

Further, the nodes $\mathscr{V}$ can also be classified according to their physical use. Any node of degree one, i.e. $|\delta_v^- \cup \delta_v^+| = 1$, is either an inflow ($\delta_v^- = \emptyset$) or an outflow ($\delta_v^+ = \emptyset$) boundary node for the network. These nodes can be interpreted as suppliers or consumers of gas and the sets of such nodes can be denoted as $\mathscr{V}_I$ or $\mathscr{V}_O$, respectively. Some nodes of degree two are controllable nodes (for example, compressor stations or valves). This subset of nodes is denoted by $\mathscr{V}_C \subset \mathscr{V}$. The rest of the nodes, $\mathscr{V}_P = \mathscr{V} \setminus (\mathscr{V}_I \cup \mathscr{V}_O \cup \mathscr{V}_C)$, can be considered the standard pipe-to-pipe intersections.

In addition to the above assumptions, the following can also be imposed:

A3. All pipes have the same diameter D. The cross-sectional area is given by $A = \frac{D^2}{4}\pi$. Similar to a single pipe, the walls do not expand or contract due to pressure load.

A4. The friction factor, f_g, is the same for all pipes.

In general, the value at a node v depends only on the flow in the ingoing and outgoing pipes and where needed some (possibly) time-dependent controls. Thus

for the network, on each edge $l \in \mathscr{J}$, assume that the dynamics is governed by the isothermal Euler equation (36) for all $x \in [x_l^a, x_l^b]$ and $t \in [0, T]$ supplemented with initial data U_l^0. In addition, on each vertex $v \in \mathscr{V}$ systems of the type (36) are coupled by suitable coupling conditions:

$$\Psi(\rho^{(l_1)}, q^{(l_1)}, \ldots, \rho^{(l_n)}, q^{(l_n)}) = \Pi(t), \qquad \delta_v = \{l_1, \ldots, l_n\}.$$

Hence the network model for gas flow in pipelines consisting of m pipes takes the form

$$\frac{\partial \rho^{(l)}}{\partial t} + \frac{\partial q^{(l)}}{\partial x} = 0; \tag{39a}$$

$$\frac{\partial q^{(l)}}{\partial t} + \frac{\partial}{\partial x}\left(\frac{(q^{(l)})^2}{\rho} + a^2 \rho^{(l)}\right) = s(x, \rho^{(l)}, q^{(l)}), \qquad l \in \{1, \ldots, n\}; \tag{39b}$$

$$\Psi(\rho^{(l_1)}, q^{(l_1)}, \ldots, \rho^{(l_m)}, q^{(l_n)}) = \Pi(t). \tag{39c}$$

Now let us consider different junctions and the resulting coupling conditions.

Inflow and Outflow Nodes

The inflow and outflow nodes $v \in \mathscr{V}_I \cup \mathscr{V}_O$ can be identified with boundary conditions for the Eq. (36), modelling time-dependent inflow pressure or the outgoing mass flux of the gas. This formulation is well known and widely discussed in the literature [50].

Standard Junctions

On junctions $v \in \mathscr{V}_P$, the amount of gas is conserved at pipe-to-pipe intersections:

$$\sum_{l \in \delta_v^+} q^{(l)} = \sum_{\bar{l} \in \delta_v^-} q^{(\bar{l})}$$

In addition, pressure is assumed equal for all pipes at the node [25, 28, 70], i.e.

$$p(\rho^{(l)}) = p(\rho^{(\bar{l})}), \qquad \forall\, l, \bar{l} \in \delta_v^- \cup \delta_v^+.$$

An introduction of such coupling conditions was undertaken in [9] and analysed in [8]. The conservation of mass at the nodes is unanimously agreed upon in the literature. The pressure conditions, however provide room for debate. The engineering community apply pressure tables but it is still not clear how these can

be represented in mathematical terms. Different conditions can be applied [19]. Here we assume there is 'good mixing' at each vertex so that the condition of equal pressure can be imposed. Hence at the vertex the pipes are coupled using the following conditions:

$$\Psi(\rho^{(l_1)}, q^{(l_1)}, \ldots, \rho^{(l_n)}, q^{(l_n)}) = \begin{pmatrix} \sum_{l \in \delta_v^+} q^{(l)} - \sum_{\bar{l} \in \delta_v^-} q^{(\bar{l})} \\ p(\rho^{(l_1)}) - p(\rho^{(l_2)}) \\ \cdots \\ p(\rho^{(l_1)}) - p(\rho^{(l_n)}) \end{pmatrix} = 0 \tag{40}$$

for $\delta_v^- \cup \delta_v^+ = \{l_1, \ldots, l_n\}$.

To solve the problem at the vertex, ideas from solutions of the standard Riemann problem are adopted [15, 26, 50] also presented in Sect. 2. Half-Riemann problems are solved instead [37]. In addition to that a few remarks on the process are in order. Given a node with n pipes coupled through coupling conditions (40), in each of the n pipes assume a constant subsonic state $(\bar{\rho}^{(l)}, \bar{q}^{(l)})$. The states $(\bar{\rho}^{(1)}, \bar{q}^{(1)}), \ldots, (\bar{\rho}^{(n)}, \bar{q}^{(n)}))$ need not satisfy the coupling conditions. Hence, similar to the Riemann problem for a simple domain, we construct the intermediate states that develop at the node as in the state u_m in Sect. 2.4. Unfortunately, solutions to this problem are only realisable if the waves generated travel with negative speed in incoming pipes and with positive speed in outgoing pipes, which means the coupling conditions do not provide feasible solutions to all choices of states $(\bar{\rho}^{(1)}, \bar{q}^{(1)}), \ldots, (\bar{\rho}^{(n)}, \bar{q}^{(n)}))$.

A solution of a Riemann problem is a vector $(u^{(1)}(x,t), \ldots, u^{(n)}(x,t))$ of functions $u^{(l)} : I_l \times (0, \infty) \to \Omega$ satisfying:

- for every $l \in \{1, \ldots, n\}$, $u^{(l)}$ is a restriction to $I_l \times (0, \infty)$ of the solution to the classical Riemann Problem:

$$\begin{aligned} &(u^{(l)})_t + f(u^{(l)})_x = 0, \quad t > 0, x \in \mathbb{R}; \\ &u^{(l)}(x, 0) = \bar{u}^{(l)}, \qquad x < 0; \\ &u^{(l)}(x, 0) = v^{(l)}(0), \quad x > 0 \end{aligned}$$

  Coupling or transmission conditions in Eq. (40) guarantee that we obtain a well-posed model. Of course, we have to prove that we indeed yield a well-posed mathematical model for the junction. See [8, 9, 12] for details. In addition, the derivation of $v^{(l)}(0)$ is a result of the process of solving Riemann Problems at the junction [8, 9].
- for every $l \in \{1, \ldots, n\}$

$$\lim_{t \to 0^+} u^{(l)}(\cdot, t) = \bar{u}^{(l)}$$

  with respect to the L^1_{loc} topology.

The coupling conditions as presented above are well-posed. This result was discussed in [8, 37]. Below a numerical example will be presented. In addition, this work has been extended further to prove well-posedness for multi-phase flows. The special case of the drift-flux model has been investigated in [12].

In summary, the following is the general result: consider the Cauchy Problem and a Riemann solver. Under some strict technical assumptions on the initial conditions as the case might be, there exists a unique weak solution at vertex v: $(u^{(1)}(x,t), u^{(2)}(x,t), \ldots, u^{(n)}(x,t))$ such that

1. for every $l \in \{1, \ldots, n\}$, $u^{(l)}(x,0) = u_0^{(l)}(x)$ for a.e. $x \in I_l$;
2. for a.e. $t > 0$ a solution for the Riemann solver exists.

Moreover, for up to a 2×2 system, the solution depends in a Lipschitz continuous way on the initial conditions. As pointed out above, the main assumption is the initial data needs to be sub-sonic and small TV-norm of initial data and

$$\det\Big(D_1\Psi(\bar{u}), \ldots, D_n\Psi(\bar{u})\Big) \neq 0 \quad \text{where} \quad D_j\Psi(\bar{u}) = \frac{\partial}{\partial u_j}\Psi(\bar{u}).$$

A solution may contain shock waves just as in the Cauchy problem.

4.1.3 Numerical Results

At this point an example taken from [8] is presented. A time-splitting approach can be applied:

$$\partial_t \begin{pmatrix} \rho^{(l)} \\ q^{(l)} \end{pmatrix} + \partial_x \begin{pmatrix} q^{(l)} \\ q^{(l)2}/\rho^{(l)} + P^{(l)}(\rho^{(l)}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

$$\partial_t \begin{pmatrix} \rho^{(l)} \\ q^{(l)} \end{pmatrix} = \begin{pmatrix} 0 \\ s(\rho^{(l)}, q^{(l)}) \end{pmatrix}.$$

1. A high resolution scheme is applied to the homogeneous system.
2. An exact solution can be computed for the ordinary differential equations.

A One-to-One Network Example

Consider an incoming wave on pipe $l = 1$, which cannot freely pass the intersection, due to a given low flux profile on the outgoing pipe $l = 2$. Friction in the pipes is taken to be $f_g = 10^{-3}$. The sound speeds are $a_l = 360\,\text{m/s}$ and the pipe diameter is $D = 0.1$ m. An inflow of $q_{in} = 70\ \text{kg m}^{-2}\,\text{s}^{-1}$ at $x = x_1^a$ and an outflow

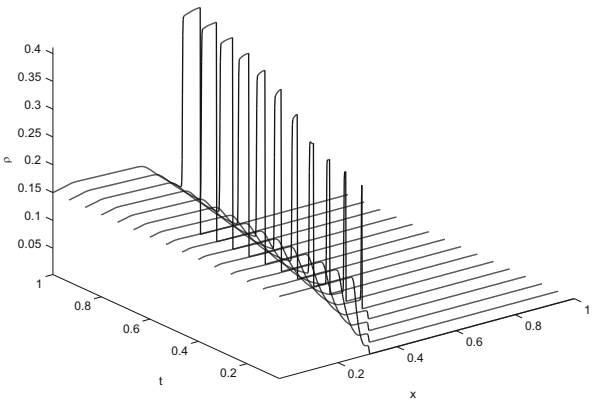

Fig. 10 Snapshots of the solution ρ_l to the problem of two connected pipes at different times t [8]

of $q_{out} = 1$ at $x = x_2^b$ is prescribed. Both pipes have the same initial condition $U_j^0 = [\frac{1}{360}, 1]$. In Fig. 10, snapshots of the density evolution are shown. Take note that the inflow profile moves on pipe 1 until it reaches the intersection. Since the maximal flow on the outgoing pipe is $q^* = 1$, a backwards moving shock wave on pipe 1 develops and increases the density near the intersection.

A Five-Pipe Network

In the following example, we consider a small network with five pipes, refer to Fig. 11. The initial conditions are defined as follows: on pipes 1,3,5 we set the density and flux as: $(4, 2)$, $(4, 4)$, $(4, 6)$.

The initial conditions on pipe 2,4 are a realisation of a pressure increase and decrease as follows:

$$U_2^0(x) = \begin{cases} (4,2), & x < \frac{1}{2}; \\ (4 + \frac{1}{2}\sin(\pi(2x-1)), 2), & x > \frac{1}{2}; \end{cases}$$

$$U_4^0(x) = \begin{cases} (4 + \frac{1}{2}\sin(4\pi(x-\frac{1}{4})), 2), & \frac{1}{2} < x < \frac{3}{4} \\ (4,2), & \text{else.} \end{cases}$$

The flow in the pipes 1, 3, 5 is displayed in Fig. 12. In the figure it can be seen that the pressure equality has been achieved. The evolution of the shocks due to the pressure variations is also visible.

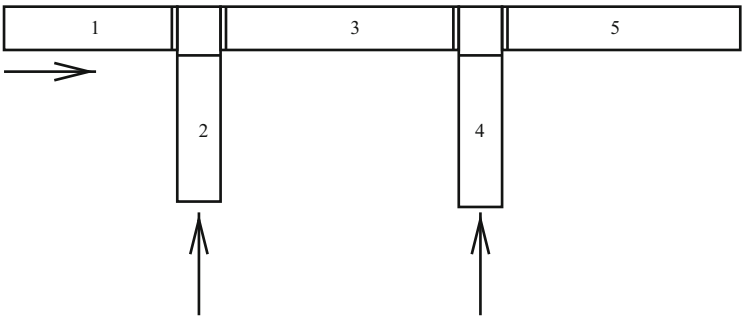

Fig. 11 Network setup

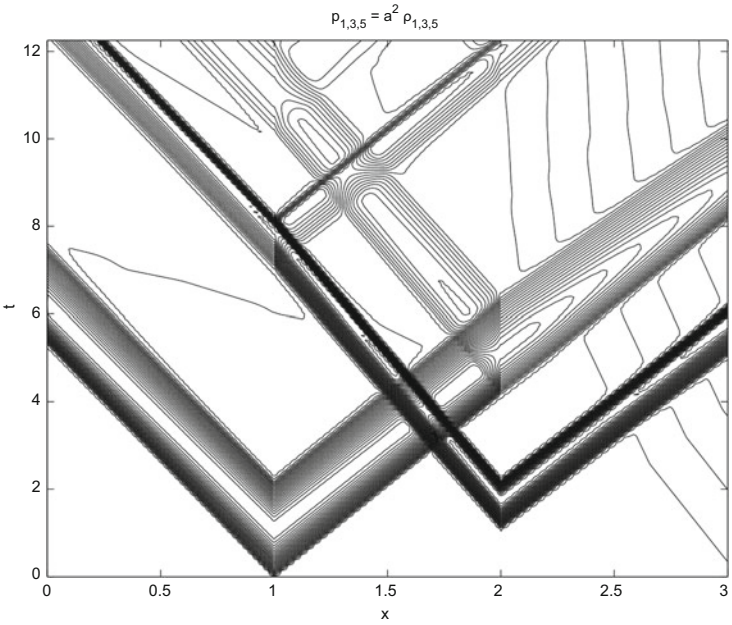

Fig. 12 Pressure distribution in time for pipes 1, 3, 5

4.2 Boundary Stabilisation of a Dynamic System of Hyperbolic Conservation Equations

Consider dynamic systems described by hyperbolic partial differential equations. The problem now is to find methods for designing certain control action at the boundary of a domain in order to attain a desired stable equilibrium status. This kind of need can be observed in daily activities such as the use of traffic lights on traffic flow networks or the installation of gates for hydraulic networks. For such

networks to guarantee navigation, a critical issue is to stabilise the water level in the reaches taking into consideration the variations of the water flow rate of the river. The stability of a boundary control system is crucial to attain a desired equilibrium status. Convergence of the solution towards the desired equilibrium is due to boundary damping only.

The significant tool that has been introduced in the study of boundary stabilisation problems are suitable Lyapunov functions. For stability of the solution to the partial differential equation, the exponential decay of such a Lyapunov function can be established in a variety of cases—see [22], for example. For a more detailed discussion, the reader may refer to [21] and the references therein. Our focus was the analysis of the numerical discretisation analogous to the continuous results. Therefore, the conditions under which, in a numerical scheme, an exponential decay of the discrete solution to the hyperbolic system can be observed were derived [3]. Mathematical proofs for explicit decay rates for general three-point finite volume schemes, using the discrete counterpart of the Lyapunov function, are presented in [3]. In general, for a numerical discretisation the exponential decay of discrete L^2-Lyapunov functions up to a given finite terminal time only has been proved.

4.2.1 The Continuous Results

The following results are (probably) the most generally available continuous stabilisation results [20, 22]. Consider the non-linear hyperbolic partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \qquad u(x,0) = u_0(x), \tag{41}$$

where $t \in [0, +\infty)$, $x \in [0, L]$, $u : [0, L] \times [0, +\infty) \to \mathbb{R}^k$ and $f : \mathbb{R}^k \to \mathbb{R}^k$ denotes a possibly non-linear smooth flux function. The gas dynamics model discussed above is a special case of this with $k = 2$. The flux function f is assumed to be strictly hyperbolic. If the solution, u, is smooth, solving Eq. (41) is equivalent to solving

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \; x \in [0, L], t \in [0, +\infty), u \in \mathbb{R}^k \tag{42}$$

with $A(u) = D_u f(u)$ a $k \times k$ real matrix and

$$u(x, 0) = u^0(x). \tag{43}$$

It must be emphasised that here solution in H^2 are considered, and not general discontinuous solutions as in the previous sections. Consider the case where $L = 1$ and hence $x \in [0, 1]$. In addition, $\|a\|_q$ denotes the q-norm of the vector $a \in \mathbb{R}^k$, $k > 1$, and $\|x\|$ denotes the absolute value of a real number $x \in \mathbb{R}$.

Feedback boundary conditions [21,22,51] for (42) can be prescribed as follows:

$$\begin{pmatrix} u_+(0,t) \\ u_-(1,t) \end{pmatrix} = G \begin{pmatrix} u_+(1,t) \\ u_-(0,t) \end{pmatrix}, \quad t \in [0,+\infty) \tag{44}$$

where $G : \mathbb{R}^k \to \mathbb{R}^k$ is a possible non-linear function. The variables u_+ and u_- will be defined at a later stage below. Denote by Λ_i for $i = 1,\ldots,k$ the eigenvalues of $A(0)$. As in [22], $A(0)$ may be assumed to be diagonal (otherwise an appropriate state transformation is applied). Due to the strict hyperbolicity the eigenvalues are distinct, i.e. $\Lambda_i \neq \Lambda_j$ for $i \neq j$. In addition, assume that $\Lambda_i \neq 0$ for all $i = 1,\ldots,k$. The eigenvalues are ordered such that $\Lambda_i > 0$ for $i = 1,\ldots,m$ and $\Lambda_i < 0$ for $i = m+1,\ldots,k$. For any $u \in \mathbb{R}^k$, define $u_+ \in \mathbb{R}^m$ and $u_- \in \mathbb{R}^{k-m}$ by requiring

$$u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}.$$

Hence, $u_\pm$ are the components of the vector $u \in \mathbb{R}^k$ corresponding to the positive and negative eigenvalues of the diagonal matrix $A(0)$. Thus u is now defined in the eigenvector basis of $A(0)$ corresponding to ordered eigenvalues. Similarly, for A (and G), define for $A_+ : \mathbb{R}^k \to \mathbb{R}^{m\times k}$, $G_+ : \mathbb{R}^k \to \mathbb{R}^m$ and $A_- : \mathbb{R}^k \to \mathbb{R}^{k-m\times k}$, $G_- : \mathbb{R}^k \to \mathbb{R}^{k-m}$ by

$$A(u) = \begin{pmatrix} A_+(u) \\ A_-(u) \end{pmatrix} \text{ and } G(u) = \begin{pmatrix} G_+(u) \\ G_-(u) \end{pmatrix}.$$

Again, assume that $A_\pm(u)$ and $G_\pm(u)$ are diagonal, for all u. Let $G_+(u) = G_+((u_+,u_-)^T)$. The partial derivatives with respect to u_+ and u_- will be denoted by $G'_{+,u_+}(u)$ and $G'_{+,u_-}(u)$, respectively and analogously for G_-. This notation is as in [22]. Then, the definition of stability of the equilibrium solution $u \equiv 0$ is given by:

Definition 11 (Definition 2.2[22]) The equilibrium solution $u \equiv 0$ of the non-linear system (42)–(43) is exponentially stable (in the H^2-norm) if there exists $\epsilon > 0$, $\nu > 0$, $C > 0$ such that, for every $u^0 \in H^2((0,1);\mathbb{R}^k)$ satisfying $\|u^0\|_{H^2((0,1);\mathbb{R}^k)} \leq \epsilon$ and the compatibility conditions

$$\begin{pmatrix} u^0_+(0) \\ u^0_-(1) \end{pmatrix} = G \begin{pmatrix} u^0_+(1) \\ u^0_-(0) \end{pmatrix};$$

$$A_+(u^0(0))u^0_x(0) = G'_{+,u_+}\begin{pmatrix} u^0_+(1) \\ u^0_-(0) \end{pmatrix} A_+(u^0(1))u^0_x(1) + G'_{+,u_-}\begin{pmatrix} u^0_+(1) \\ u^0_-(0) \end{pmatrix} A_-(u^0(0))u^0_x(0);$$

$$A_-(u^0(1))u^0_x(1) = G'_{-,u_+}\begin{pmatrix} u^0_+(1) \\ u^0_-(0) \end{pmatrix} A_+(u^0(1))u^0_x(1) + G'_{-,u_-}\begin{pmatrix} u^0_+(1) \\ u^0_-(0) \end{pmatrix} A_-(u^0(0))u^0_x(0);$$

the classical solution u to the Cauchy problem (42) and (43) with boundary conditions (44) is defined for all $t \in [0, +\infty)$ and satisfies

$$\|u(\cdot, t)\|_{H^2((0,1);\mathbb{R}^k)} \leq C \exp(-\nu t)\|u^0\|_{H^2((0,1);\mathbb{R}^p)}. \tag{45}$$

For any real $k \times k$ matrix A, $\rho_1(A) := \inf\{\|\Delta A \Delta^{-1}\|_2 : \Delta \in \mathscr{D}_{k,+}\}$, where $\mathscr{D}_{k,+}$ is the set of all real $k \times k$ diagonal matrices with strictly positive diagonal elements. Then, the following general result holds:

Theorem 6 (Theorem 2.3[22]) *Assume $A(0)$ is diagonal with distinct and non-zero eigenvalues and A is of class $C^2(\mathbb{R}^k; \mathbb{R}^k)$ in a neighbourhood of zero. Take $A(0) = diag(\Lambda_i)_{i=1}^k$ and $\Lambda_i > 0$ for $i = 1, \ldots, m$ and $\Lambda_i < 0$ for $i = m+1, \ldots, k$ and $\Lambda_i \neq \Lambda_j$ for $i \neq j$. Assume G is of class $C^2(\mathbb{R}^k; \mathbb{R}^k)$ in a neighbourhood of zero and $G(0) = 0$.*

Let $\rho_1(G'(0)) < 1$, then the equilibrium $u \equiv 0$ for (42) with boundary conditions given by (44) is exponentially stable.

In this context, in [3] an analysis of the Lyapunov function for a discrete case was developed. Up to now a discrete counter-part to Theorem 6 has not been proved. A weaker result concerning the L^2-stabilisation was developed in [3].

4.2.2 Stabilisation of a Discrete Problem

The finite volume numerical schemes for boundary L^2-stabilisation of one-dimensional non-linear hyperbolic systems were adopted. In order to facilitate the proofs, we will need additional assumptions discussed below.

The boundary conditions are considered as in (44) with G being a diagonal matrix. To simplify the discussion, let

$$A(u) = \text{diag}(\Lambda_i(u))_{i=1}^m, \ \Lambda_i(u) > 0, \ \Lambda_i(u) \neq \Lambda_j(u), i \neq j. \tag{46}$$

In the following $\mathbf{u}$ denotes all components of u, i.e., $\mathbf{u} := (u_j)_{j=1}^m$. Let $\delta > 0$ and denote by $M_\delta(0) := \{\mathbf{u} : |u_j| \leq \delta, j = 1, \ldots, m\}$. Assume δ is sufficiently small such that

$$M_\delta(0) \subset B_\epsilon(0).$$

Further, let Δx denote the cell width of a uniform spatial grid and N the number of cells in the discretisation of the domain $[0, 1]$ such that $\Delta x N = 1$ with cell centres at $x_i = (i + \frac{1}{2})\Delta x$, $i = 0, \ldots, N-1$. Further x_{-1} and x_N denote the cell centres of the cells outside the computational domain on the left-hand and right-hand side of the domain, respectively. The interfacial numerical fluxes are computed at cell boundaries $x_{i-1/2} = i\Delta x$ for $i = 0, \ldots, N$. The left and right boundary points are

at $x_{-1/2}$ and $x_{N-1/2}$, respectively. The temporal grid is chosen such that the CFL condition holds:

$$\lambda \frac{\Delta t}{\Delta x} \leq 1, \ \lambda := \max_{j=1,\ldots,m} \max_{\mathbf{u} \in M_\delta(0)} \Lambda_j(\mathbf{u}) \tag{47}$$

and $t^n = n\Delta t$ for $n = 0, 1, \ldots, K$ where by possibly further reducing Δt, it can be assumed that $K\Delta t = T$. The value of $u_j(x,t)$ at the cell centre x_i is approximated by $u^n_{i,j}$ for $i = 0, \ldots, N-1$ and for each component j at time t^n for $n = 0, 1, \ldots, K$. The left boundary is discretized using $u^n_{-1,j}$. The initial condition is discretized as

$$u^0_{i,j} := \frac{1}{\Delta x} \int_{x_i - 1/2}^{x_i + 1/2} u^0_j(x) dx$$

where $u^0_j(x), \ j = 1, \ldots, m$ denotes the components of solution vector $\mathbf{u}$.

Hence, the following discretisation of (42)–(44) for $n = 0, \ldots, K-1$ is introduced:

$$u^{n+1}_{i,j} = u^n_{i,j} - \frac{\Delta t}{\Delta x} \Lambda_j(\mathbf{u}^n_i) \left(u^n_{i,j} - u^n_{i-1,j} \right), \quad i = 0, \ldots, N-1; \tag{48a}$$

$$u^{n+1}_{-1,j} = \kappa_j u^{n+1}_{N-1,j}; \tag{48b}$$

$$u^0_{i,j} = \frac{1}{\Delta x} \int_{x_i - 1/2}^{x_i + 1/2} u^0_j(x) dx, \ i = 0, \ldots, N-1; \quad j = 1, \ldots, m; \tag{48c}$$

$$u^0_{-1,j} = \kappa_j u^0_{N-1,j}. \tag{48d}$$

Note that the last condition is the discrete compatibility condition for u^0.

The discrete Lyapunov function at time t^n with positive coefficients $\mu_j, \ j = 1, \ldots, m$ takes the form

$$L^n = \Delta x \sum_{i=0}^{N-1} \sum_{j=1}^{m} \left(u^n_{i,j} \right)^2 \exp(-\mu_j x_i). \tag{49}$$

The numerical result is as follows:

Theorem 7 *Let $T > 0$ and assume (46) holds. For any $\kappa_j, \ j = 1 \ldots, m$ such that*

$$0 < \kappa_j < \sqrt{\frac{D_j^{\min}}{D_j^{\max}}}, \tag{50}$$

where $\max_{\mathbf{u}\in M_\delta(0)} \frac{\Delta t}{\Delta x}\Lambda_j(\mathbf{u}) =: D_j^{\max} \leq 1$ *and* $\frac{\Delta t}{\Delta x}\Lambda_j(\mathbf{u}_i^n) \geq \min_{\mathbf{u}\in M_\delta(0)} \frac{\Delta t}{\Delta x}\Lambda_j(\mathbf{u}) =: D_j^{\min} > 0$ *the following holds:*

there exists $\mu_j > 0, j = 1, \ldots, m$ *and* $\delta > 0$ *such that for all initial data* $u_{i,j}^0$ *with*
$\|u_{i,j}^0\| \leq \delta$, $\|\frac{u_{i,j}^0 - u_{i-1,j}^0}{\Delta x}\| \leq \delta$, *and* $\|\frac{u_{0,j}^n - u_{-1,j}^n}{\Delta x}\| \leq \delta \exp\left(t^n \max_{\xi\in M_\delta(0)} \|\nabla_{\mathbf{u}}\Lambda_j(\xi)\|_\infty\right)$
for all $i = 0, \ldots, N-1$; $j = 1, \ldots, m$, *the numerical solution* $u_{i,j}^n$ *defined by (48) satisfies*

$$L^n \leq \exp(-\nu t^n) L^0,\ n = 0, 1, \ldots, K \tag{51}$$

for some $\nu > 0$. *Moreover,* $u_{i,j}^n$ *is exponentially stable in the discrete* L^2*-norm*

$$\Delta x \sum_{i=0}^{N-1}\sum_{j=1}^{m}\left(u_{i,j}^n\right)^2 \leq \tilde{C}\exp(-\nu t^n)\Delta x \sum_{i=0}^{N-1}\sum_{j=1}^{m}\left(u_{i,j}^0\right)^2, n = 0, 1, \ldots, K. \tag{52}$$

The by-product of this analysis is that, assuming (50), the explicit form of bounds and constants such as μ_j, δ, ν are derived and $\tilde{C} = \frac{C_1}{C_0} = \max_{j=1,\ldots,m} \exp(\mu_j x_{N-1})$. The grid size $\Delta x, \Delta t$ is not fixed and can be chosen arbitrarily provided that the CFL condition is satisfied.

Remark 1 One observes that the presented result is weaker than the corresponding continuous result obtained in [22, Theorem 2.3]. Therein, *no* assumption on the boundedness of T is required. The continuous Lyapunov function can be shown to be equivalent to the H^2-norm of u. The exponential decay of this Lyapunov function therefore yields the global existence of u.

A simple linear transport is considered [3]:

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0,\ x \in [0, 1], t \in [0, T] \tag{53}$$

and subject to the boundary and initial conditions

$$u_1(t, 0) = \kappa\, u_2(t, 0),\ u_2(t, 1) = \kappa\, u_1(t, 1),\ \ u_i(0, x) = u_i^0. \tag{54}$$

In this example, the analytical decay rates ν of the Lyapunov function is of interest. In the following numerical computations, a three point scheme given in Eq. (48a) above is used in each component as in Eq. (48). A time horizon fixed at $T = 12$ is taken and constant initial data $u_1^0 = -\frac{1}{2}$ and $u_2^0 = \frac{1}{2}$ are prescribed. The value of the Lyapunov function is computed by $L_n^{\text{exact}} := L^0\exp(-\nu t^n)$ with ν. These values are compared to those of the numerical Lyapunov function L^n. The

Table 1 The number of cells in the spatial domain [0, 1] is denoted by N

N	L^{inf}	L^2	μ	ν
100	4.32E−03	7.48E−04	5.66E−01	5.69E−01
200	2.18E−03	2.67E−04	5.70E−01	5.72E−01
400	1.09E−03	9.49E−05	5.73E−01	5.73E−01
800	5.48E−04	3.36E−05	5.74E−01	5.74E−01
1,600	2.74E−05	1.19E−05	5.75E−01	5.75E−01

L^{inf} denotes the norm $\|(L_n^{\text{exact}})_n - (L^n)_n\|_\infty$ and L^2 the norm $\|(L_n^{\text{exact}})_n - (L^n)_n\|_2$. The CFL constant is equal to one and $\kappa = \frac{3}{4}$ [3]

L^2- and L^∞-difference between both for different choices of the computational grid, boundary damping κ and values of the CFL constant are considered.

The parameter is fixed at $\kappa = \frac{3}{4}$ and the sharpest possible bound is set for the CFL constant, CFL $= 1$. In Table 1, the results for different grid sizes $\Delta x = \frac{1}{N}$ are presented. In the L^2 and L^{inf} the expected first-order convergence of the numerical discretisation is observed. Further, the values of μ and ν also converge towards the theoretical value in the case $\Delta x \to 0$ which are given by $\mu = \ln \kappa^{-2} = 5.75\text{E-}01$ and $\nu = \mu$.

In conclusion, we claim that the analysis of the stabilisation process using a three-point scheme gives very good indicators of the parameters to be used in the stabilisation process.

5 Summary

In this chapter, a review of the evolution equations also referred to as conservation laws have been presented. These are hyperbolic partial differential equations. In general, they model flow or transport processes and their applications are abundant. Furthermore, for the interesting cases, these equations are nonlinear and pose a lot of challenges in mathematical analysis. Examples of their peculiarities include the fact that admissible solutions include discontinuous functions. In such cases the concept of a solution is defined in a weak topology. To identify unique solutions physical insight is employed, especially the idea of entropy. For general approximations of solutions, numerical methods are required. All these ideas have been demonstrated in the chapter with a few examples to demonstrate the ideas.

Towards the end of the chapter extensions of applications of conservation laws to networked flows have been discussed. The idea of solving a Riemann problem at a network vertex has been discussed and some computational ideas and results also presented. In addition boundary stabilisation has been discussed. This is the case in which a particular equilibrium profile is desired. To achieve this profile, boundary conditions are manipulated.

This is not a complete presentation in developments involving conservation laws. A vast collection of literature exists including manuscripts which have been cited in the course of the discussion in the chapter. The reader is encouraged to consult these references to have more detailed and relevant ideas on the mathematical analysis, computational methods as well as applications in networked flows, design, shape optimisation, as well as boundary stabilisation just to mention a few.

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