Spectral Theory for Neutron Transport

Mustapha Mokhtar-Kharroubi

In memory of Seiji Ukaï

1 Introduction

These notes resume a lecture given in the Cimpa School "Evolutionary equations with applications in natural sciences" held in South Africa (Muizenberg, July 22–August 2, 2013). However, the oral style of the lecture has been changed and the bibliography augmented. This version benefited also from helpful remarks and suggestions of a referee whom I would like to thank. The notes deal with various functional analytic tools and results around spectral analysis of neutron transport-like operators. A first section gives a detailed introduction (mostly without proofs) to fundamental concepts and results on spectral theory of (non-selfadjoint) operators in Banach spaces; in particular, we provide an introduction to spectral analysis of semigroups in Banach spaces and its consequences on their time asymptotic behaviour as time goes to infinity. A special attention is paid to positive semigroups in ordered spaces (i.e. semigroups leaving invariant the cone of positive elements) because of their fundamental interest in neutron transport theory. We focus on the analysis of essential spectra and isolated eigenvalues with finite multiplicities. A second section deals with spectral analysis of weighted shift (or collisionless transport) semigroups. A third section is devoted to spectral analysis of perturbed semigroups in Banach spaces, in particular to stability of essential type for perturbed semigroups. A last section deals with a thorough analysis of compactness problems for general models of neutron transport; the results are very different depending on whether we work in L^p spaces ($1 < p < \infty$) or in (the physical) L^1 space; this issue is the very core of spectral analysis of neutron transport operators and allow the abstract theory to cover them.

M. Mokhtar-Kharroubi (✉)
Département de Mathématiques, CNRS UMR 6623, Université de Franche-Comté,
16 Route de Gray, 25030 Besançon, France
e-mail: mmokhtar@univ-fcomte.fr

J. Banasiak, M. Mokhtar-Kharroubi (eds.), *Evolutionary Equations with Applications in Natural Sciences*, Lecture Notes in Mathematics 2126,
DOI 10.1007/978-3-319-11322-7_7

Transport theory provides a *statistical* description of large populations of "particles" moving in a host medium (see e.g. [17]) and is of interest in various fields such as radiative transfer theory, nuclear reactor theory, gas dynamics, plasma physics, structured population models in mathematical biology etc. Among the most classical kinetic equations, we mention the one governing the transport of neutrons through the uranium fuel elements of a nuclear reactor. The aim of this lecture is to present various functional analytic tools and results motivated by this class of equations. In a nuclear reactor, the proportion of neutrons with respect to the atoms of the host medium, is infinitesimal (about 10^{-11}), so the possible collisions between neutrons are negligible in comparison with the collisions of neutrons with the atoms of the host material. Thus (in absence of feedback temperature) neutron transport equations as well as radiative transfer equations for photons are *genuinely linear*. The population of particles is described by a density function $f(t, x, v)$ of particles at time $t > 0$, at position x and with velocity v. In particular

$$\int \int f(t, x, v)dxdv$$

is the *expected* number of particles at time $t > 0$. One sees immediately that L^1 spaces are natural settings in transport theory! Various models are used in nuclear reactor theory:

(1) *Inelastic model for neutron transport*

$$\frac{\partial f}{\partial t} + v.\frac{\partial f}{\partial x} + \sigma(x, v)f(t, x, v) = \int_V k(x, v, v')f(t, x, v')dv'$$

where $(x, v) \in \Omega \times V$, $\Omega \subset \mathbb{R}^3$, $V = \left\{v \in \mathbb{R}^3;\ c_0 \leq |v| \leq c_1\right\}$ $(0 \leq c_0 < c_1 < \infty)$ and dv is Lebesgue measure, with initial condition $f(0, x, v) = f_0(x, v)$ and boundary condition

$$f(t, x, v)_{|\Gamma_-} = 0$$

where

$$\Gamma_- := \{(x, v) \in \partial\Omega \times V;\ v.n(x) < 0\}$$

and $n(x)$ is the unit exterior normal at $x \in \partial\Omega$. The collision frequency $\sigma(., .)$ and the scattering kernel $k(., ., .)$ are nonnegative.

(2) *Multiple scattering:* This physical model differs from the previous "reactor model" by the fact that $\Omega = \mathbb{R}^3$ (no boundary condition) but $\sigma(x, v)$ and $k(x, v, v')$ are compactly supported in space.

(3) *The presence of delayed neutrons*

Besides the prompt neutrons (appearing instantaneously in a fission process), some neutrons may appear after a time delay as a decay product of radioactive fission fragments and induce a suitable source term in the usual equation

$$\frac{\partial f}{\partial t} + v.\frac{\partial f}{\partial x} + \sigma(x,v)f(t,x,v) = \int_{\mathbb{R}^3} k(x,v,v')f(t,x,v')dv' + \sum_{i=1}^{m} \lambda_i g_i$$

which is thus coupled to m differential equations

$$\frac{dg_i}{dt} = -\lambda_i g_i + \int_{\mathbb{R}^3} k_i(x,v,v')f(t,x,v')dv' \quad (1 \le i \le m)$$

where $\lambda_i > 0\,(1 \le i \le m)$ are the radioactive decay constants; see [48, Chapter 4] and references therein.

(4) *Multigroup models* (motivated by numerical calculations)

$$\frac{\partial f_i}{\partial t} + v.\frac{\partial f_i}{\partial x} + \sigma_i(x,v)f_i(t,x,v) = \sum_{j=1}^{m} \int_{V_j} k_{i,j}(x,v,v')f_j(t,x,v')\mu_j(dv'),$$

$(1 \le i \le m)$ where the spheres

$$V_j := \left\{v \in \mathbb{R}^3,\ |v| = c_j\right\},\ \ 1 \le j \le m,\ \ (c_j > 0)$$

are endowed with surface Lebesgue measures μ_j and $f_i(t,x,v)$ is the density of neutrons (at time $t > 0$ located at $x \in \Omega$) with velocity $v \in V_i$.

(5) *Partly inelastic models*

$$\frac{\partial f}{\partial t} + v.\frac{\partial f}{\partial x} + \sigma(x,v)f(t,x,v) = K_e f + K_i f$$

in $L^p(\Omega \times V)$ where, e.g. $V = \left\{v \in \mathbb{R}^3;\ c_0 \le |v| \le c_1\right\}$. The inelastic scattering operator is just

$$K_i f = \int_V k(x,v,v')f(x,v')dv'$$

while the elastic scattering operator is given by

$$K_e f = \int_{S^2} k(x,\rho,\omega,\omega')f(x,\rho\omega')dS(\omega')$$

where $v = \rho\omega$. The presence of an elastic scattering operator acting only on the *angles* $\omega \in S^2$ of velocities changes strongly the spectral structure of neutron transport operators [35, 68].

(6) *Diffusive models*

$$\frac{\partial f}{\partial t} - \triangle_x f + \sigma(x,v) f = \int_0^{+\infty} k(x,v,v') f(t,x,v') dv'$$

(motivated also by numerical calculations) where the transport operator $\frac{\partial f}{\partial t} + v.\frac{\partial f}{\partial x}$ is replaced by the parabolic operator $\frac{\partial f}{\partial t} - \triangle_x f$ where $\triangle_x$ denotes the Laplacian in space variable $x \in \Omega$ with Dirichlet boundary condition, (here $v > 0$ denotes a " kinetic energy" instead of a velocity); see e.g. [14, p. 133]. In the same spirit, we mention that diffusion (i.e. heat) equations with Dirichlet boundary condition turn out to be *asymptotic approximations* (as $\varepsilon \to 0$) of usual neutron transport equations appropriately rescaled by means of a small parameter ε (typically the mean free path); see e.g. [6] and references therein. We find in [49] an approach of the diffusion approximation of neutron transport (on the torus) via spectral theory.

In this lecture, we ignore the presence of delayed neutrons but deal with an abstract velocity measure $\mu(dv)$ (with support V) covering a priori different models, e.g. Lebesgue measure on $\mathbb{R}^n$ or on spheres or even combinations of the two.

In absence of scattering event (i.e. $k(x,v,v') = 0$) the density of neutral particles (e.g. neutrons) is governed by

$$\frac{\partial f}{\partial t} + v.\frac{\partial f}{\partial x} + \sigma(x,v) f(t,x,v) = 0$$

with initial condition f_0 and is solved explicitly by the method of characteristics

$$f(t,x,v) = e^{-\int_0^t \sigma(x-\tau v,v)d\tau} f_0(x-tv,v) 1_{\{t \leq s(x,v)\}}$$

where

$$s(x,v) = \inf\{s > 0;\ x - sv \notin \Omega\}$$

is the first exit time function. This defines a positive C_0-semigroup $(U(t))_{t\geqslant 0}$ on $L^p(\Omega \times \mathbb{R}^3; dx \otimes d\mu)$

$$U(t) : g \to e^{-\int_0^t \sigma(x-\tau v,v)d\tau} g(x-tv,v) 1_{\{t \leq s(x,v)\}}$$

called the *advection semigroup*. Its generator T is given (at least for smooth domains Ω) by

$$Tg = -v.\frac{\partial g}{\partial x} - \sigma(x,v) g(x,v),\ \ g \in D(T)$$

$$D(T) = \left\{ g \in L^p(\Omega \times \mathbb{R}^3);\ v.\frac{\partial g}{\partial x} \in L^p,\ g_{|\Gamma_-} = 0 \right\};$$

(see e.g. [12, 13] for a trace theory in neutron transport theory). Then the treatment of the full equation follows naturally by perturbation theory. For instance, if the scattering operator

$$K : g \to \int_{\mathbb{R}^3} k(x, v, v')g(x, v')\mu(dv')$$

is bounded on $L^p(\Omega \times \mathbb{R}^3)$ then, by standard perturbation theory,

$$A := T + K \quad (D(A) = D(T))$$

generates a positive C_0-semigroup $(V(t))_{t\geqslant 0}$ which solves the full neutron transport equation.

There are two basic eigenvalue problems in nuclear reactor theory:

(1) *Criticality eigenvalue problem*

This problem consists in looking for (γ, g) where $\gamma > 0$ and g is a nontrivial nonnegative solution to

$$0 = -v.\frac{\partial g}{\partial x} - \sigma(x, v)g(x, v) + \int_V k_s(x, v, v')g(x, v')\mu(dv') + \frac{1}{\gamma}\int_V k_f(x, v, v')g(x, v')\mu(dv'), \quad g_{|\Gamma_-} = 0;$$

here $k_s(x, v, v')$ and $k_f(x, v, v')$ are the scattering kernel and the fission kernel, see e.g. [41, 66].

(2) *The "time eigenelements"*

This problem consists in looking for (λ, g) with nontrivial g such that

$$-v.\frac{\partial g}{\partial x} - \sigma(x, v)g(x, v) + \int_V k(x, v, v')g(x, v')\mu(dv') = \lambda g(x, v), \quad g_{|\Gamma_-} = 0$$

and in relating them to time asymptotic behaviour ($t \to +\infty$) of the semigroup $(V(t))_{t\geqslant 0}$.

In this lecture, we focus on the second class of problems. There exists a considerable literature on the subject; we refer to [48] and references therein for the state of the art up to 1997. In these lecture, we present mostly new developments on this topic.

We note that this conventional neutron transport theory deals with the expected (or mean) behaviour of neutrons. In order to describe the fluctuations from the mean value of neutron populations, probabilistic formulations of neutron chain fissions were proposed very early, in particular in [7]. This leads to nonlinear problems governing *divergent* neutron chain fissions. Such problems are strongly related to spectral theory of usual (linear) neutron transport operators, see [30, 46, 57, 64].

We end this introduction by some historical notes. The beginning of spectral theory of neutron transport dates back to the beautiful and seminal paper by J. Lehner and M. Wing [36] devoted to a simplified model (constant cross sections) in slab geometry. The time asymptotic behaviour of neutron transport semigroups in bounded geometries is well-understood for a long time in the case when the velocities are *bounded away from zero*; this is a classical result by K. Jorgens:

Theorem 1 ([31]) *Let Ω be bounded and convex, let*

$$V = \left\{v \in \mathbb{R}^3;\ c_0 \leq |v| \leq c_1 < \infty\right\}$$

and let the scattering kernel $k(.,.,.)$ be bounded. If $c_0 > 0$ then $V(t)$ is compact on $L^2(\Omega \times V)$ for t large enough. In particular, for any $\alpha \in \mathbb{R}$

$$\sigma(A) \cap \{Re\lambda \geqslant \alpha\}$$

consists at most of finitely many eigenvalues with finite algebraic multiplicities $\{\lambda_1, \ldots .\lambda_m\}$ with spectral projections $\{P_1, \ldots .P_m\}$ and there exists $\beta < \alpha$ such that

$$V(t) = \sum_{j=1}^{m} e^{\lambda_j t} e^{tD_j} P_j + O(e^{\beta t})$$

where $D_j := (T - \lambda_j)P_j$.

The picture gets more complicated when arbitrarily small velocities must be taken into account. In this case, the (essential) spectrum of the generator T (of the advection semigroup $\{U(t); t \geqslant 0\}$) on $L^2(\Omega \times V)$ consists of a half-plane

$$\{\lambda \in \mathbb{C};\ Re\lambda \leq -\lambda^*\}$$

where "typically" $\lambda^* = \inf \sigma(x, v)$, see S. Albertoni and B. Montagnini [2]. Moreover, important compactness results were obtained very early, (see e.g. Demeru-Montagnini [16], Borysiewicz-Mika [8] and S. Ukai [74]) implying, for most physical scattering kernels, that the scattering operator K is T-compact on $L^2(\Omega \times V)$ i.e.

$$K : D(T) \to L^2(\Omega \times V)$$

is compact where $D(T)$ is endowed with the graph norm. It follows that the spectrum of $A = T + K$ consists of a left half-plane $\{\lambda \in \mathbb{C};\ Re\lambda \leq -\lambda^*\}$ and *at most* of isolated eigenvalues with finite algebraic multiplicities located in the right half-plane

$$\{\lambda \in \mathbb{C};\ Re\lambda > -\lambda^*\}.$$

(Note that it may happen that this set of isolated eigenvalues is empty for small bodies [2].) Then the time asymptotic behaviour of the solution is traditionally dealt with by means of inverse Laplace transform (Dunford calculus)

$$V(t)f = \lim_{\gamma \to +\infty} \frac{1}{2i\pi} \int_{\rho - i\gamma}^{\rho + i\gamma} e^{\lambda t}(\lambda - A)^{-1} f d\lambda$$

(with ρ large enough). If for some $\varepsilon > 0$

$$\sigma(T + K) \cap \{\lambda; Re\lambda > -\lambda^* + \varepsilon\} = \{\lambda_1, \dots .\lambda_m\}$$

(with spectral projections $\{P_1, \dots .P_m\}$) is finite and non-empty then, by shifting the path of integration and picking up the residues, we get an *asymptotic expansion*

$$V(t)f = \sum_{j=1}^{m} e^{\lambda_j t} e^{tD_j} P_j f + O_f(e^{\beta t}) \quad (\beta < -\lambda^* + \varepsilon);$$

for *smooth* initial data f; see, e.g. M. Borysiewicz and J. Mika [8] (see also M. Mokhtar-Kharroubi [45]). The drawbackof the approach is that we need very regular initial data (say $f \in D(A^2)$) to estimate the *transcient part* of the solution. To remedy this situation, a more relevant approach, initiated by I. Vidav [76], consists in studying the spectrum of the semigroup $(V(t))_{t \geqslant 0}$ itself instead of the spectrum of its generator because of the lack (in general) of a spectral mapping theorem relating spectra of semigroups and spectra of their generators. The perturbed semigroup $(V(t))_{t \geqslant 0}$ is expanded into a Dyson–Phillips series

$$V(t) = \sum_{n=0}^{\infty} U_n(t)$$

where $U_0(t) = U(t)$ is the advection semigroup and

$$U_{n+1}(t) = \int_0^t U(t-s) K U_n(s) ds \quad (n \geqslant 0).$$

Theorem 2 ([76]) *If some remainder term* $R_n(t) := \sum_{j=n}^{\infty} U_j(t)$ *is compact for large* t *then* $\sigma(V(t)) \cap \left\{\mu; |\mu| > e^{-\lambda^* t}\right\}$ *consists at most of isolated eigenvalues with finite multiplicities. In particular,* $\forall \varepsilon > 0$,

$$\sigma(T + K) \cap \{\lambda; Re\lambda \geqslant -\lambda^* + \varepsilon\} = \{\lambda_1, \dots .\lambda_m\}$$

is finite and

$$V(t) = \sum_{j=1}^{m} e^{\lambda_j t} e^{tD_j} P_j + O(e^{\beta t})$$

in operator norm where $\beta < -\lambda^* + \varepsilon$.

Vidav's result had relevant applications to realistic models of kinetic theory much later; see Y. Shizuta [71], G. Greiner [25], J. Voigt [77, 79], P. Takak [72], M. Mokhtar-Kharroubi [42, 43] and L. Weiss [82]. The role of positivity in peripheral spectral theory of neutron transport was emphasized by I. Vidav [75], T. Hiraoka-S. Ukaï [29], Angelescu-Protopopescu [4] and more recently, in others directions, e.g. by G. Greiner [26], J. Voigt [78] and M. Mokhtar-Kharroubi [43–45, 47].

2 Fundamentals of Spectral Theory

This section is a crash course (mostly without proofs) on the fundamental concepts and results on spectral theory of closed linear operators on complex Banach spaces with a special emphasis on generators of strongly continuous semigroups. Because of their importance in transport theory, the basic spectral properties of positive operators (i.e. leaving invariant the positive cone of a Banach lattice) are also given. Finally, we show the role of peripheral spectral theory of positive semigroups in their time asymptotic behaviour as $t \to +\infty$. Apart from Subsection 2.10, the material of this section is widely covered by the general references [15, 20–22, 32, 61, 73] and will be used in the sequel without explicit mention. Subsection 2.10 presents a class of positive semigroups whose real spectra can be described completely; this class covers weighted shift (i.e. advection) semigroups we deal with in Sect. 3.

2.1 Basic Definitions and Results

We start with some basic definitions and results. Let X be a complex Banach space and let

$$T : D(T) \subset X \to X$$

be a linear operator defined on a subspace $D(T)$. We say that T is a closed operator if its graph

$$\{(x, Tx);\ x \in D(T)\}$$

is closed in $X \times X$. We define the resolvent set of T by

$$\rho(T) := \{\lambda \in \mathbb{C};\ \lambda - T : D(T) \to X \ \text{ is bijective}\},$$

the spectrum of T by

$$\sigma(T) := \{\lambda \in \mathbb{C};\ \lambda \notin \rho(T)\}$$

and the resolvent operator by

$$(\lambda - T)^{-1} : X \to X \ \ (\lambda \in \rho(T)).$$

In particular, if there exists $x \in D(T) - \{0\}$ and $\lambda \in \mathbb{C}$ such that $Tx = \lambda x$ then $\lambda \in \sigma(T)$. In this case, λ is an eigenvalue of T and

$$\ker(T) := \{x \in D(T);\ (T - \lambda)x = 0\}$$

is the corresponding eigenspace. In contrast to finite dimensional spaces, in general, $\sigma(T)$ is *not* reduced to eigenvalues! For instance, one can show that the spectrum of the multiplication operator on $C([0,1])$ (endowed with the sup norm)

$$T : f \in C([0,1]) \to Tf \in C([0,1])$$

where $Tf(x) = xf(x)$ is equal to $[0,1]$ and that T has no eigenvalue. For *unbounded* operators, the spectrum may be empty or equal to $\mathbb{C}$! For example, let $X = C([0,1]\,;\ \mathbb{C})$ endowed with the sup-norm and

$$Tf = \frac{df}{dx}, \quad D(T) = C^1([0,1]).$$

Then $\forall \lambda \in \mathbb{C}$, $x \in [0,1] \to e^{\lambda x} \in \mathbb{C}$ is an eigenfunction of T so $\sigma(T) = \mathbb{C}$. If we replace $(T, D(T))$ by

$$\hat{T} f = \frac{df}{dx}, \quad D(\hat{T}) = \left\{f \in C^1([0,1]);\ f(0) = 0\right\}$$

then $\forall \lambda \in \mathbb{C}$ and $\forall g \in X$, the equation

$$\lambda f - \frac{df}{dx} = g,\ \ f(0) = 0$$

is uniquely solvable; thus $\rho(\hat{T}) = \mathbb{C}$ and $\sigma(\hat{T}) = \varnothing$.

It is useful to decompose the spectrum of T as follows: The point spectrum

$$\sigma_p(T) = \{\lambda \in \mathbb{C};\ \lambda - T : D(T) \to X \ \text{ is not injective}\}.$$

The approximate point spectrum

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C};\ \lambda - T : D(T) \to X \ \text{ not injective or } (\lambda - T)X \text{ not closed}\};$$

this terminology is motivated by the fact that $\lambda \in \sigma_{ap}(T)$ if and only if there exists a sequence $(x_n)_n \subset D(T)$ such that

$$\|x_n\| = 1,\ \|Tx_n - \lambda x_n\| \to 0.$$

The residual spectrum

$$\sigma_{res}(T) = \{\lambda \in \mathbb{C};\, (\lambda - T)X \ \text{ is not dense}\}.$$

We note that

$$\sigma(T) = \sigma_{res}(T) \cup \sigma_{ap}(T)$$

is a non-disjoint union. Among the first results, we note:

- $(\lambda - T)^{-1} : X \to X$ is a bounded operator for $\lambda \in \rho(T)$, i.e. $(\lambda - T)^{-1} \in \mathcal{L}(X)$, (by the closed graph theorem).
- $\rho(T)$ is an open subset of $\mathbb{C}$ (so $\sigma(T)$ is closed) and

  $$\lambda \in \rho(T) \to (\lambda - T)^{-1} \in \mathcal{L}(X)$$

  is holomorphic.

More precisely, if $\mu \in \rho(T)$ then $\lambda \in \rho(T)$ if $|\lambda - \mu| < \left\|(\mu - T)^{-1}\right\|^{-1}$ and then

$$(\lambda - T)^{-1} = \sum_{0}^{+\infty} (\mu - \lambda)^n \left[(\mu - T)^{-1}\right]^{n+1}.$$

It follows that $|\lambda - \mu| \geqslant \left\|(\mu - T)^{-1}\right\|^{-1}$ for any $\lambda \in \sigma(T)$ and then

$$dist(\mu, \sigma(T)) \geqslant \left\|(\mu - T)^{-1}\right\|^{-1}.$$

In particular $\left\|(\mu - T)^{-1}\right\| \to \infty$ as $dist(\mu, \sigma(T)) \to 0$.

Bounded operators $T \in \mathcal{L}(X)$ enjoy specific properties:

- $\sigma(T)$ is bounded and non-empty.
- The spectral radius of $T \in \mathcal{L}(X)$, defined by

  $$r_\sigma(T) := \sup\{|\lambda|;\ \lambda \in \sigma(T)\},$$

  is equal to $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \inf_n \|T^n\|^{\frac{1}{n}}$.

- In particular $r_\sigma(T) \leq \|T\|$ and $(\lambda - T)^{-1}$ is given by a Laurent's series

$$(\lambda - T)^{-1} = \sum_1^\infty \lambda^{-n} T^{n-1} \quad (|\lambda| > r_\sigma(T))$$

  with $T^m = \frac{1}{2i\pi} \int_C \lambda^m (\lambda - T)^{-1} d\lambda$ where C is any circle (positively oriented) centered at the origin with radius $> r_\sigma(T)$.

If $T : D(T) \subset X \to X$ is densely defined linear operator, we can define its dual operator

$$T' : D(T') \subset X' \to X'$$

by

$$\langle Tx, y' \rangle_{X,X'} = \langle x, T'y' \rangle_{X,X'}$$

with domain

$$D(T') = \left\{ y' \in X'; \exists c \geqslant 0, \ \left| \langle Tx, y' \rangle \right| \leq c \, \|x\| \ \ \forall x \in D(T) \right\}.$$

We note that T' is closed but not necessarily densely defined. But if X is reflexive then T' is densely defined, $(T')' = T$, $\sigma(T') = \sigma(T)$ and $(\lambda - T')^{-1} = \left((\lambda - T)^{-1}\right)'$. In particular if $T \in \mathcal{L}(X)$ then $r_\sigma(T') = r_\sigma(T)$.

We end this section with a *spectral mapping theorem* for bounded operators. Let $T \in \mathcal{L}(X)$ and let $\Omega \ni \lambda \to f(\lambda) \in \mathbb{C}$ be holomorphic on some open neighborhood Ω of $\sigma(T)$. Then there exists an open set ω such that $\sigma(T) \subset \omega \subset \overline{\omega} \subset \Omega$ and $\partial\omega$ consists of finitely many simple closed curves that do not intersect. One defines a Dunford integral

$$f(T) = \frac{1}{2i\pi} \int_{\partial\omega} f(\lambda)(\lambda - T)^{-1} d\lambda \in \mathcal{L}(X)$$

where $\partial\omega$ is properly oriented (the definition does not depend on the choice of ω). In particular if $f(\lambda)$ is a polynomial then $f(T)$ coincides with the usual meaning of $f(T)$. Then we have a spectral mapping theorem

$$\sigma(f(T)) = f(\sigma(T)).$$

2.2 Spectral Decomposition and Riesz Projection

Let X be a complex Banach space such that

$$X = X_1 \oplus X_2$$

(direct sum) where X_i $(i = 1, 2)$ are *closed* subspaces. Let $P : x \in X \to Px$ be the (continuous) projection on X_1 along X_2. Let

$$T : D(T) \subset X \to X$$

be a closed linear operator such that $P(D(T)) \subset D(T)$ and X_i $(i = 1, 2)$ are invariant under T. The parts T_i $(i = 1, 2)$ of T on X_i $(i = 1, 2)$ are defined by

$$D(T_i) = D(T) \cap X_i, \ \ T_i x = Tx \ \ (x \in D(T_i)).$$

We say that T is *reduced* by X_i $(i = 1, 2)$. Then

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$$

(not necessarily a disjoint union),

$$\sigma_p(T) = \sigma_p(T_1) \cup \sigma_p(T_2) \text{ and } \sigma_{ap}(T) = \sigma_{ap}(T_1) \cup \sigma_{ap}(T_2).$$

Similar results hold for any finite direct sum: $X = X_1 \oplus \ldots \oplus X_n$; see e.g. [73, Theorem 5.4, p. 289].

Let now $T : D(T) \subset X \to X$ be a closed linear operator such that $\sigma(T)$ is a *disjoint* union of two non-empty closed subsets σ_1 and σ_2 and let σ_1 be compact. Then there exists Γ, a finite number of rectifiable simple closed curves properly oriented enclosing an open set O which contains σ_1 and such that σ_2 is included in the exterior of O. Then

$$P := \int_{\Gamma} (\lambda - T)^{-1} d\lambda; \ \ P^2 = P$$

and $X = X_1 \oplus X_2$ ($X_1 = PX$ and $X_2 = (I - P)X = KerP$) reduces T (i.e. X_i are T invariant), $\sigma(T_i) = \sigma_i$ where $T_i := T_{|X_i}$ and T_1 is bounded. P is the *spectral projection* associated with σ_1. If σ_1 consists of finitely many points $(\lambda_1, \ldots, \lambda_n)$ then

$$P = P_1 + \ldots + P_n, \quad P_j P_k = \delta_{jk} P_j$$

$$P_j := \int_{\Gamma_j} (\lambda - T)^{-1} d\lambda$$

(where Γ_j is e.g. a small circle enclosing λ_j). P_j is the spectral projection associated with λ_j. We study now the structure of the resolvent around an isolated singularity. Let $\mu \in \sigma(T)$ be an *isolated* point of $\sigma(T)$. There exists a Laurent's series around μ

$$(\lambda - T)^{-1} = \sum_{n=-\infty}^{+\infty} (\lambda - \mu)^n U_n$$

where

$$U_n = \frac{1}{2i\pi} \int_C \frac{(\lambda - T)^{-1}}{(\lambda - \mu)^{n+1}} d\lambda \quad (n \in \mathbb{Z})$$

where C is a small circle positively oriented centered at μ. In particular, the residues

$$U_{-1} = \frac{1}{2i\pi} \int_C (\lambda - T)^{-1} d\lambda$$

is the spectral projection P. In addition

$$U_{-(n+1)} = (-1)^n (\mu - T)^n P \quad (n \geqslant 0).$$

We have

$$U_{-(n+1)} U_{-(m+1)} = U_{-(n+m+1)}$$

so μ is a *pole* of the resolvent (i.e. there exists $k > 0$ such that $U_{-k} \neq 0$ and $U_{-n} = 0 \ \forall n > k$) if and only if there exists $k > 0$ such that $U_{-k} \neq 0$ and $U_{-(k+1)} = 0$. Then k is the *order* of the pole. In this case, μ is an eigenvalue of T and $PX = Ker(\mu - T)^k$. The *algebraic multiplicity* $m_a \leq +\infty$ of μ is the dimension of PX. Conversely, if $m_a < +\infty$, i.e. P is of finite rank, then $(\mu - T)^{m_a} P = 0$ and then μ is a pole of the resolvent of order $\leq m_a$. Actually, the order k of the pole is the smallest $j \in \mathbb{N}$ such that $(\mu - T)^j P = 0$. The subspace $Ker(\mu - T)^k$ contains the *generalized* eigenvectors; it coincides with the eigenspace if and only if $PX = Ker(\mu - T)$, i.e. $k = 1$ (simple pole); μ is said to be a semi-simple eigenvalue. We say that μ is *algebraically simple* if $m_a = 1$.

2.3 Application to Riesz-Schauder Theory

As a first illustration of the interest of Riesz projections, we show why the non zero eigenvalues of compact operators have finite algebraic multiplicities. Let $T : X \to X$ be a *compact* operator (i.e. maps bounded sets into relatively compact ones). Then $\sigma(T)/\{0\}$ consists at most of isolated eigenvalues. Let $\alpha \in \sigma(T)$ with $\alpha \neq 0$. Define T_λ (in the neighborhood of α) by

$$(\lambda - T)^{-1} = \lambda^{-1} + T_\lambda.$$

Then $(\lambda - T)(\lambda^{-1} + T_\lambda) = I$ implies that $T_\lambda = T(\lambda^{-1}T_\lambda + \lambda^{-2}I)$ is compact. So (C being a small circle around α positively oriented) the spectral projection

$$U_{-1} = \frac{1}{2i\pi}\int_C (\lambda - T)^{-1}d\lambda = \frac{1}{2i\pi}\int_C \lambda^{-1}d\lambda + \frac{1}{2i\pi}\int_C T_\lambda d\lambda$$
$$= \frac{1}{2i\pi}\int_C T_\lambda d\lambda$$

is compact too. Since U_{-1} has a closed range then the open mapping theorem and Riesz theorem imply that U_{-1} has finite-dimensional range. Hence α has a finite algebraic multiplicity.

This result extends to *power compact* operators. Indeed, let $T \in \mathcal{L}(X)$ and $n \in \mathbb{N}$ ($n \geqslant 2$) such that T^n is compact. The spectral mapping theorem

$$\sigma(T^n) = (\sigma(T))^n$$

implies that $\sigma(T)/\{0\}$ consists at most of isolated points. Let $\alpha \in \sigma(T)$ with $\alpha \neq 0$. Then, for λ close to α, $(\lambda^n - T^n) = (\lambda^{n-1}I + \lambda^{n-2}T + \ldots + T^{n-1})(\lambda - T)$ implies

$$\begin{aligned}(\lambda - T)^{-1} &= (\lambda^n - T^n)^{-1}(\lambda^{n-1}I + \lambda^{n-2}T + \ldots + T^{n-1}) \\ &= [\lambda^{-n} + C_\lambda]\,(\lambda^{n-1}I + \lambda^{n-2}T + \ldots + T^{n-1}) \\ &= \lambda^{-n}(\lambda^{n-1}I + \lambda^{n-2}T + \ldots + T^{n-1}) \\ &\quad + C_\lambda(\lambda^{n-1}I + \lambda^{n-2}T + \ldots + T^{n-1})\end{aligned}$$

(where C_λ is compact) so the spectral projection

$$U_{-1} = \frac{1}{2i\pi}\int_C (\lambda - T)^{-1}d\lambda = \frac{1}{2i\pi}\int_C C_\lambda(\lambda^{n-1}I + \lambda^{n-2}T + \ldots + T^{n-1})d\lambda$$

is compact and we argue as previously.

2.4 Spectral Mapping Theorem for a Resolvent

Let $T : D(T) \subset X \to X$ be closed linear operator and $\lambda_0 \in \rho(T)$. The spectral links between T and its resolvent $(\lambda_0 - T)^{-1}$ are completely described by:

- $\sigma\left[(\lambda_0 - T)^{-1}\right]\backslash\{0\} = (\lambda_0 - \sigma(T))^{-1}$ (so $r_\sigma\left[(\lambda_0 - T)^{-1}\right] = [dist(\lambda_0, \sigma(T))]^{-1}$)
- $\sigma_p\left[(\lambda_0 - T)^{-1}\right]\backslash\{0\} = (\lambda_0 - \sigma_p(T))^{-1}$
- $\sigma_{ap}\left[(\lambda_0 - T)^{-1}\right]\backslash\{0\} = (\lambda_0 - \sigma_{ap}(T))^{-1}$
- $\sigma_{res}\left[(\lambda_0 - T)^{-1}\right]\backslash\{0\} = (\lambda_0 - \sigma_{res}(T))^{-1}$

- μ is an isolated point of $\sigma(T)$ if and only if $(\lambda_0 - \mu)^{-1}$ is an isolated point of $\sigma\left[(\lambda_0 - T)^{-1}\right]$. In this case, the residues and the orders of the pole of $(\lambda - T)^{-1}$ at μ and of $\left[\lambda - (\lambda_0 - T)^{-1}\right]^{-1}$ at $(\lambda_0 - \mu)^{-1}$ coincide.

See [20, Chapter IV]. These properties are of interest e.g. when we deal with Riesz-Schauder theory of operators with compact resolvent.

2.5 Fredholm Operators

A closed operator $T : D(T) \subset X \to X$ is said to be a Fredholm operator if $\dim Ker(T) < \infty$ and the range $R(T)$ of T is closed with finite codimension (i.e. $\dim \frac{X}{R(T)} < \infty$). Let $T : D(T) \subset X \to X$ be closed linear operator; its Fredholm domain is defined by

$$\rho_F(T) := \{\lambda \in \mathbb{C};\ \lambda - T : D(T) \to X \text{ is Fredholm}\}.$$

Then $\rho_F(T)$ is open and $\rho(T) \subset \rho_F(T)$. If λ_0 is an isolated eigenvalue of T with finite algebraic multiplicity then $\lambda_0 \in \rho_F(T)$, (see [32, Chapter IV]).

We recall that $T \in \mathcal{L}(X)$ is Fredholm if and only if there exists $S \in \mathcal{L}(X)$ such that $I - ST$ and $I - TS$ are finite rank operators (see [21] p. 190). The essential spectrum of T is defined by

$$\sigma_{ess}(T) := \mathbb{C} \backslash \rho_F(T).$$

Let $\mathcal{K}(X) \subset \mathcal{L}(X)$ be the closed ideal of compact operators. The Calkin algebra

$$\mathcal{C}(X) := \frac{\mathcal{L}(X)}{\mathcal{K}(X)}$$

is endowed with the quotient norm (for $\hat{T} := T + \mathcal{K}(X)$)

$$\left\| \hat{T} \right\|_{\mathcal{C}(X)} = \inf_{K \in \mathcal{K}(X)} \|T + K\| = dist(T, \mathcal{K}(X)).$$

Then

$$\rho_F(T) = \rho(\hat{T}) \text{ and } \sigma_{ess}(T) = \sigma(\hat{T}).$$

The essential norm of $T \in \mathcal{L}(X)$ is defined by

$$\|T\|_{ess} := \left\| \hat{T} \right\|_{\mathcal{C}(X)}.$$

In particular, $\|T\|_{ess} \leq \|T\|$ and the essential norm $\|.\|_{ess}$ is submultiplicative, i.e.

$$\|T_1T_2\|_{ess} \leq \|T_1\|_{ess}\,\|T_2\|_{ess} \quad (T_i \in \mathcal{L}(X),\ i = 1, 2).$$

The *essential* radius of $T \in \mathcal{L}(X)$ is defined by

$$r_{ess}(T) := r_\sigma(\hat{T}).$$

Then

$$r_{ess}(T) = \sup\left\{|\lambda|\,;\ \lambda \in \sigma(\hat{T})\right\} = \sup\{|\lambda|\,;\ \lambda \in \sigma_{ess}(T)\}.$$

In addition

$$r_{ess}(T) = \lim_{n\to\infty}\left\|\left(\hat{T}\right)^n\right\|_{\mathcal{C}(X)}^{\frac{1}{n}} = \lim_{n\to\infty}\left\|\hat{T^n}\right\|_{\mathcal{C}(X)}^{\frac{1}{n}} = \lim_{n\to\infty}\|T^n\|_{ess}^{\frac{1}{n}}.$$

The unbounded component of $\rho_F(T)$ consists of resolvent set and at most of isolated eigenvalues with finite algebraic multiplicities, (see [21, p. 204]). Then the essential radius of $T \in \mathcal{L}(X)$ is also given by

$$\inf\{r > 0;\ \lambda \in \sigma(T),\ |\lambda| > r \Rightarrow \lambda \in \sigma_{discr}(T)\ \}$$

where $\sigma_{discr}(T)$ refers to the isolated eigenvalues of T with finite algebraic multiplicities. Note that for any $\varepsilon > 0$, $\sigma(T) \cap \{|\lambda| \geqslant r_{ess}(T) + \varepsilon\}$ consists at most of finitely many eigenvalues with finite algebraic multiplicities. We point out that there exist several non equivalent definitions of essential spectrum for bounded operators but the corresponding essential radius is the same for all them, see [19, Corollary 4.11, p. 44].

2.6 Semigroups and Generators

Let X be a complex Banach space. By a C_0-semigroup on X we mean a family $(S(t))_{t\geqslant 0}$ of bounded linear operators on X indexed by $t \geqslant 0$ such that $S(0) = I,\ \ S(t)S(s) = S(t+s)$ and such that the *strong* continuity condition holds:

$$[0, +\infty[\ \ni t \to S(t)x \in X$$

is continuous for all $x \in X$. By the uniform boundedness theorem, $(S(t))_{t\geqslant 0}$ is locally bounded in $\mathcal{L}(X)$. The infinitesimal generator of $(S(t))_{t\geqslant 0}$ is the unbounded linear operator defined by

$$T : x \in D(T) \subset X \to \lim_{t\to 0}\frac{S(t)x - x}{t} \in X$$

with domain

$$D(T) = \left\{ x;\ \lim_{t\to 0} \frac{S(t)x - x}{t} \text{ exists in } X \right\}.$$

Then T is closed and densely defined. In addition, $D(T)$ is invariant under $S(t)$ and $S(t)Tx = TS(t)x\ \ \forall x \in D(T)$. Finally, $\forall x \in D(T)$,

$$f : t \geqslant 0 \to S(t)x \in X \ \text{ is } C^1$$

and

$$f'(t) = Tf(t),\ f(0) = x;$$

see e.g. [15].

If

$$p : \mathbb{R}_+ \to [-\infty, +\infty[$$

is subadditive (i.e. $p(t+s) \leq p(t) + p(s)$) and locally bounded from above then

$$\lim_{t\to+\infty} \frac{p(t)}{t} = \inf_{t>0} \frac{p(t)}{t}.$$

see e.g. [15]. Since

$$t \geqq 0 \to p(t) := \ln(\|S(t)\|) \in [-\infty, +\infty[$$

is subadditive and locally bounded from above then

$$\omega := \inf_{t>0} \frac{\ln(\|S(t)\|)}{t} = \lim_{t\to+\infty} \frac{\ln(\|S(t)\|)}{t} \in [-\infty, +\infty[\,.$$

In particular $(S(t))_{t\geqslant 0}$ is exponentially bounded, i.e.

$$\forall \alpha > \omega\ \exists M_\alpha \geqslant 1;\ \|S(t)\| \leq M_\alpha e^{\alpha t}\ \forall t \geqq 0;$$

ω is called the *type* or growth bound of $(S(t))_{t\geqslant 0}$. In addition, for any $t > 0$

$$\begin{aligned} r_\sigma(S(t)) &= \lim_{n\to+\infty} \|S(t)^n\|^{\frac{1}{n}} = \lim_{n\to+\infty} \|S(nt)\|^{\frac{1}{n}} \\ &= \lim_{n\to+\infty} \exp \frac{1}{n} \ln \|S(nt)\| = \lim_{n\to+\infty} \exp t \frac{1}{nt} \ln \|S(nt)\| = e^{\omega t}. \end{aligned}$$

We recall that $\{Re\lambda > \omega\} \subset \rho(T)$ and

$$(\lambda - T)^{-1} = \int_0^{+\infty} e^{-\lambda t} S(t)dt \quad (Re\lambda > \omega)$$

where the integral converges in operator norm. Thus $\sigma(T) \subset \{Re\lambda \leq \omega\}$ and the *spectral bound* of T

$$s(T) := \sup\{Re\lambda;\ \lambda \in \sigma(T)\} \leq \omega.$$

We end this section with the famous Hille–Yosida–Phillips–Miyadera–Feller theorem (commonly called Hille–Yosida theorem) which provides a general framework for a huge amount of linear evolution equations of mathematical physics and probability theory [22].

Theorem 3 *Let $T : D(T) \subset X \to X$ be a closed densely defined linear operator. Then T is the generator of a C_0-semigroup $(S(t))_{t\geqslant 0}$ satisfying the estimate $\|S(t)\| \leq Me^{\alpha t}$ $\forall t \geqq 0$ if and only if $\sigma(T) \subset \{Re\lambda \leq \alpha\}$ and*

$$\left\| \left[(\lambda - T)^{-1}\right]^n \right\| \leq \frac{M}{(Re\lambda - \alpha)^n} \quad (Re\lambda > \alpha)\ \forall n \in \mathbb{N}.$$

We note that if X is a reflexive complex Banach space and if $(S(t))_{t\geqslant 0}$ is a C_0-semigroup with generator T then the dual semigroup $(S'(t))_{t\geqslant 0}$ is *strongly continuous* and its generator is given by T'. In particular $(S(t))_{t\geqslant 0}$ and $(S'(t))_{t\geqslant 0}$ have the same type while T and T' have the same spectral bound.

2.7 Partial Spectral Mapping Theorems for Semigroups

In general, there exist *partial* spectral links between a C_0-semigroup and its generator, see [20, Chapter IV].

Theorem 4 *Let X be a complex Banach space and $(S(t))_{t\geqslant 0}$ be a C_0-semigroup on X with generator T.Then:*

(i) $e^{t\sigma_{ap}(T)} \subset \sigma_{ap}(S(t))\setminus\{0\}$.
(ii) $e^{t\sigma_p(T)} = \sigma_p(S(t))\setminus\{0\}$.
(iii) $e^{t\sigma_{res}(T)} = \sigma_{res}(S(t))\setminus\{0\}$.
(iv) $m_g(\lambda, T) \leq m_g(e^{\lambda t}, S(t))$
(v) $m_a(\lambda, T) \leq m_a(e^{\lambda t}, S(t))$
(vi) $k(\lambda, T) \leq k(e^{\lambda t}, S(t))$.

Here m_g (resp. m_a, resp. k) refers to geometric multiplicity (resp. algebraic multiplicity, resp. multiplicity of a pole). We note that the possible *failure* of the spectral mapping theorem stems from the approximate point spectrum. The link between the eigenvalues of $(S(t))_{t\geqslant 0}$ and those of its generator T is clarified further by:

Theorem 5 *Let X be a complex Banach space and $(S(t))_{t\geqslant 0}$ be a C_0-semigroup on X with generator T. Then:*

(i) $Ker(\mu - T) = \cap_{t\geqslant 0} Ker(e^{\mu t} - S(t))$.
(ii) $Ker(e^{\mu t} - S(t)) = \overline{lin}_{n\in\mathbb{Z}} Ker(\mu + \frac{2i\pi n}{t} - T)\ \forall t > 0$.

Theorem 6 ([24] Proposition 1.10 or [20] p. 283) *Let X be a complex Banach space and $(S(t))_{t\geqslant 0}$ be a C_0-semigroup on X with generator T and let $t > 0$ be fixed. Let $e^{\mu t}$ be a pole of $S(t)$ of order k and let Q be the corresponding residue. Then*

(i) For every $n \in \mathbb{Z}$, $\mu + \frac{2i\pi n}{t}$ is (at most) a pole of $(\lambda - T)^{-1}$ of order at most k and residue P_n.
(ii) $QX = \overline{lin}_{n\in\mathbb{Z}} P_n X$.

Corollary 1 *Let X be a complex Banach space and $(S(t))_{t\geqslant 0}$ be a C_0-semigroup on X with generator T and let $t > 0$ be fixed. Let $\alpha \neq 0$ be an isolated eigenvalue of $S(t)$ with finite algebraic multiplicity and with residue Q. Then $Q = \sum_{j=1}^{n} P_j$ where the P_j are the residues of $(\lambda - T)^{-1}$ at $\{\lambda_1, \ldots, \lambda_n\}$, the (finite and nonempty) set of eigenvalues of T such that $e^{\lambda_i t} = \alpha$.*

2.8 Essentially Compact Semigroups

The fact that $\|.\|_{ess}$ is submultiplicative implies that

$$t \geqq 0 \to p_{ess}(t) := \ln(\|S(t)\|_{ess}) \in [-\infty, +\infty[$$

is subadditive. It is also locally bounded from above so

$$\omega_{ess} := \inf_{t>0} \frac{\ln(\|S(t)\|_{ess})}{t} = \lim_{t\to+\infty} \frac{\ln(\|S(t)\|_{ess})}{t} \in [-\infty, \omega]\,.$$

In particular $\forall \alpha > \omega_{ess}\ \exists M_\alpha \geqslant 1$ such that

$$\|S(t)\|_{ess} \leq M_\alpha e^{\alpha t}\ \forall t \geqq 0;$$

ω_{ess} is called the *essential type* (or essential growth bound) of $(S(t))_{t\geqslant 0}$. For any $t > 0$

$$\begin{aligned} r_{ess}(S(t)) &= \lim_{n\to+\infty} \|S(t)^n\|_{ess}^{\frac{1}{n}} = \lim_{n\to+\infty} \|S(nt)\|_{ess}^{\frac{1}{n}} \\ &= \lim_{n\to+\infty} \exp\frac{1}{n}\ln\left(\|S(nt)\|_{ess}\right) \\ &= \lim_{n\to+\infty} \exp t\frac{1}{nt}\ln\left(\|S(nt)\|_{ess}\right) = e^{\omega_{ess}t}. \end{aligned}$$

A C_0-semigroup $(S(t))_{t\geqslant 0}$ on a complex Banach space X is said to be *essentially compact* if its essential type is less than its type (i.e. $\omega_{ess} < \omega$). Such semigroups have a nice finite-dimensional asymptotic structure.

Theorem 7 *Let X be a complex Banach space and $(S(t))_{t\geqslant 0}$ be an essentially compact C_0-semigroup on X with generator T. Then:*

(i) $\sigma(T) \cap \{Re\lambda > \omega_{ess}\}$ consists of a nonempty set of isolated eigenvalues with finite algebraic multiplicities.
(ii) For any ω' such that $\omega_{ess} < \omega' < \omega$, $\sigma(T) \cap \{Re\lambda \geqslant \omega'\}$ consists of a finite set (depending on ω') $\{\lambda_1, \ldots, \lambda_m\}$ of eigenvalues of T.
(iii) Let P_j be the residues of $(\lambda - T)^{-1}$ at λ_j and let $P := \sum_{j=1}^m P_j$. Then the projector P reduces $(S(t))_{t\geqslant 0}$ and

$$S(t) = \sum_{j=1}^{m} e^{\lambda_j t} e^{tD_j} P_j + O(e^{(\omega'-\varepsilon)t})$$

(for some $\varepsilon > 0$) where $D_j := (T - \lambda_j)P_j$ are nilpotent bounded operators ($D_j^{k_j} = 0$ where k_j is the order of the pole λ_j).

Proof Let ω' be such that $\omega_{ess} < \omega' < \omega$. Let $t > 0$ be fixed. Then $e^{\omega_{ess}t} < e^{\omega' t} < e^{\omega t}$ and

$$\sigma(S(t)) \cap \left\{\mu;\ |\mu| \geqslant e^{\omega' t}\right\}$$

consists of a finite (and nonempty) set of eigenvalues with finite algebraic multiplicities $\{\mu_1, \ldots, \mu_n\}$ while

$$\sigma(S(t)) \cap \left\{\mu;\ |\mu| < e^{\omega' t}\right\} \subset \left\{\mu;\ |\mu| < e^{(\omega'-\varepsilon)t}\right\}$$

for some $\varepsilon > 0$. For each j $(1 \leq j \leq n)$ let $\left\{\lambda_j^1, \ldots, \lambda_j^{l_j}\right\}$ be the (finite and nonempty) set of eigenvalues λ of T such that $e^{\lambda t} = \mu_j$. Then the residue of the pole μ_j of the resolvent of $S(t)$ is given by

$$Q_j = \sum_{k=1}^{l_j} P_j^k$$

where P_j^k is the residue of the λ_j^k of the resolvent of T. Let $Q = \sum_{j=1}^{n} Q_j$ be the spectral projection corresponding to the eigenvalues $\{\mu_1, \ldots, \mu_n\}$ of $S(t)$ in $\left\{\mu;\ |\mu| \geqslant e^{\omega' t}\right\}$. One sees that $Q = \sum_{j=1}^{n} \sum_{k=1}^{l_j} P_j^k$ is nothing but the spectral projection corresponding to the eigenvalues of T in $\{Re\lambda \geqslant \omega'\}$. We decompose $S(t)$ as $S(t)Q + S(t)(I - Q)$. We know that $\sigma(S(t)_{|ImQ}) = \{\mu_1, \ldots, \mu_n\}$ while $\sigma(S(t)_{|KerQ}) \subset \left\{\mu;\ |\mu| < e^{(\omega'-\varepsilon)t}\right\}$ so the type of $S(t)_{|KerQ}$ is $\leq \omega' - \varepsilon$. Finally, $S(t)_{|ImQ}$ is generated by the bounded operator

$$T(\sum_{j=1}^{m} P_j) = \sum_{j=1}^{m} TP_j = \sum_{j=1}^{m} \left[\lambda_j P_j + (T - \lambda_j)P_j\right] = \sum_{j=1}^{m} \left[\lambda_j P_j + D_j\right]$$

so $S(t)_{|ImQ} = \sum_{j=1}^{m} e^{\lambda_j t} e^{tD_j} P_j$.

2.9 Peripheral Spectral Theory and Applications

In ordered Banach spaces, *positive* semigroups (i.e. leaving invariant the positive cone) enjoy nice spectral properties. For the sake of simplicity, we restrict ourselves to Lebesgue spaces

$$X = L^p(\Omega, \mathcal{A}, \mu) \quad (1 \leq p \leq +\infty)$$

where $(\Omega, \mathcal{A}, \mu)$ is a measure space (i.e. Ω is a set, $\mathcal{A}$ is a σ-algebra of subsets of Ω and μ is a σ-finite measure on $\mathcal{A}$) although most of the results hold in general Banach lattices. For short, we will write $L^p(\mu)$ (or just L^p) instead of $L^p(\Omega, \mathcal{A}, \mu)$. Let $L_+^p(\mu)$ be the cone of nonnegative a.e. functions. Then

$$L^p(\mu) = L_+^p(\mu) - L_+^p(\mu).$$

More precisely

$$f = f_+ - f_-, \quad \forall f \in L^p(\mu)$$

where

$$f_+ = \sup\{f, 0\}, \ f_- = \sup\{-f, 0\}.$$

In particular

$$|f| = f_+ + f_-, \quad \|f\| = \||f|\|$$

where $|f|(x) := |f(x)|$. An operator $G \in L(X)$ is said to be *positive* if $Gf \in L_+^p(\mu)\ \forall f \in L_+^p(\mu)$. We write $G \geqslant 0$. In this case

$$|Gf| = |Gf_+ - Gf_-| \leq Gf_+ + Gf_- = G(|f|)$$

and consequently

$$\|G\| = \sup_{\|f\|\leq 1,\ f\in L_+^p} \|Gf\|.$$

It follows that if $0 \leq G_1 \leq G_2$ with $G_i \in \mathcal{L}(L^p)(i = 1, 2)$ then $\|G_1\| \leq \|G_2\|$. This last property applied to the iterates shows that $r_\sigma(G_1) \leq r_\sigma(G_2)$. It is easy to see that $G \in \mathcal{L}(L^p)$ is positive if and only if its dual operator $G' \in \mathcal{L}(L^{p'})$ is positive.

A C_0-semigroup $(S(t))_{t\geqslant 0}$ on X is said to be positive if $\forall t > 0$, $S(t)$ is a positive operator. A C_0-semigroup $(S(t))_{t\geqslant 0}$ with type ω and generator T is positive if and only if the resolvent $(\lambda - T)^{-1}$ is positive for $\lambda > \omega$; this follows from

$$(\lambda - T)^{-1} f = \int_0^{+\infty} e^{-\lambda t} S(t) f dt \quad (\lambda > \omega)$$

and the exponential formula

$$S(t)f = \lim_{n\to+\infty} (I - \frac{t}{n} T)^{-n} f.$$

A fundamental result for positive C_0-semigroups $(S(t))_{t\geqslant 0}$ on *Lebesgue* spaces $L^p(\mu)$ is that *the type of* $(S(t))_{t\geqslant 0}$ *coincides with the spectral bound* $s(T)$ *of its generator* T (see e.g. *[20]*). Another fundamental spectral property of positive operators $G \in \mathcal{L}(X)$ (in general Banach lattices) is that the spectral radius belongs to the spectrum

$$r_\sigma(G) \in \sigma(G).$$

Let us show an analogous property for a generator T of a positive semigroup $(S(t))_{t\geqslant 0}$:

$$s(T) > -\infty \Rightarrow s(T) \in \sigma(T).$$

Indeed, note first that

$$\left|(\lambda - T)^{-1} f\right| \leq \int_0^{+\infty} e^{-Re\lambda t} S(t) |f| dt \quad (\forall Re\lambda > s(T))$$

so $\left\|(\lambda - T)^{-1}\right\| \leq \left\|(Re\lambda - T)^{-1}\right\| \quad (\forall Re\lambda > s(T))$. By assumption there exists a sequence $(\beta_n)_n \subset \sigma(T)$ such that $Re\beta_n \to s(T)$. We build a sequence $(\lambda_n)_n$

with $Re\lambda_n > s(T)$ (so $(\lambda_n)_n \subset \rho(T)$), $Im\lambda_n = Im\beta_n$ and $Re\lambda_n \to s(T)$. Then $|\lambda_n - \beta_n| \to 0$ and $\left\|(\lambda_n - T)^{-1}\right\| \to +\infty$. Thus $\left\|(Re\lambda_n - T)^{-1}\right\| \to +\infty$ and consequently $s(T) \in \sigma(T)$.

Let $G \in \mathcal{L}(L^p)$ be positive. We say that G is *irreducible* if $\forall f \in L^p_+(\mu),\ f \neq 0$ and $\forall g \in L^{p'}_+(\mu),\ g \neq 0$ there exists $n \in \mathbb{N}$ (depending a priori on f and g) such that

$$\langle G^n f, g\rangle_{L^p, L^{p'}} > 0.$$

For $p < +\infty$, this is equivalent to saying that there is no closed subspace $L^p(\Omega', \mu)$ (with $\mu(\Omega') > 0$ and $\mu(\Omega/\Omega') > 0$) invariant by G. For instance, if $Gf > 0$ a.e. $\forall f \in L^p_+(\Omega),\ f \neq 0$ (we say that G is *positivity-improving*) then G is irreducible. A positive C_0-semigroup $(S(t))_{t\geqslant 0}$ is said to be irreducible if $\forall f \in L^p_+(\mu),\ f \neq 0$ and $\forall g \in L^{p'}_+(\mu),\ g \neq 0$ there exists $t > 0$ (depending a priori on f and g) such that

$$\langle S(t) f, g\rangle_{L^p, L^{p'}} > 0.$$

For $p < +\infty$, this is equivalent to saying that there is no closed subspace $L^p(\Omega', \mu)$ (with $\mu(\Omega') > 0$ and $\mu(\Omega/\Omega') > 0$) invariant by *all* $S(t)$. A positive C_0-semigroup $(S(t))_{t\geqslant 0}$ with generator T is irreducible if and only if $(\lambda - T)^{-1}$ is positivity-improving for some $\lambda > s(T)$. This follows easily from

$$\langle (\lambda - T)^{-1} f, g\rangle = \int_0^{+\infty} e^{-\lambda t} \langle S(t) f, g\rangle dt.$$

We recall a useful result combining compactness and irreducibility:

Theorem 8 ([63]) *If $G \in \mathcal{L}(X)$ is compact and irreducible then $r_\sigma(G) > 0$.*

The fact that $r_\sigma(G^n) = r_\sigma(G)^n$ implies easily:

Corollary 2 *If some power of $G \in L(X)$ is compact and positivity-improving then $r_\sigma(G) > 0$.*

The following result can be found in [61, Chapter CIII].

Theorem 9 *Let $(S(t))_{t\geqslant 0}$ be a positive C_0-semigroup on $L^p(\mu)$ with generator T. If $s(T)$ is a pole of the $(\lambda - T)^{-1}$ then the boundary spectrum*

$$\sigma_b(T) := \sigma(T) \cap (s(T) + i\mathbb{R})$$

consists of poles of the resolvent and is cyclic in the sense that there exists $\alpha \geqslant 0$ such that

$$\sigma_b(T) := s(T) + i\alpha\mathbb{Z}.$$

Corollary 3 *Let $(S(t))_{t\geqslant 0}$ be a positive C_0-semigroup on $L^p(\Omega, \mathcal{A}, \mu)$ with generator T. We assume that $(S(t))_{t\geqslant 0}$ is essentially compact (i.e. $\omega_{ess} < \omega$). Then*

$$\sigma_b(T) = \{s(T)\}$$

i.e. $s(T)$ is the leading eigenvalue and is strictly dominant (i.e. $\exists\varepsilon > 0$; $Re\lambda \leq s(T) - \varepsilon \ \forall\lambda \in \sigma(T),\ \lambda \neq s(T)$).

Proof According to the theorem above, $\sigma_b(T)$ is either unbounded or reduces to $\{s(T)\}$. The fact that $\omega_{ess} < \omega$ implies that $\sigma_b(T)$ is finite. ■

By combining essential compactness and positivity arguments we get a fundamental functional analytic result:

Theorem 10 ([61] Prop 3.5, p. 310) *Let $(S(t))_{t\geqslant 0}$ be an irreducible C_0-semigroup on $L^p(\Omega, \mathcal{A}, \mu)$ with generator T. We assume that $(S(t))_{t\geqslant 0}$ is essentially compact (i.e. $\omega_{ess} < \omega$). Then $s(T)$ is the leading eigenvalue, is strictly dominant and is algebraically simple. In particular there exists $\varepsilon > 0$ such that*

$$S(t)f = e^{s(T)t}\left(\int f(x)v(x)\mu(dx)\right)u + O(e^{(s(T)-\varepsilon)t})$$

where u is the (strictly positive almost everywhere) eigenfunction of T associated to $s(T)$ and v is the (strictly positive almost everywhere) eigenfunction of T' associated to $s(T') = s(T)$ with the normalization $\int u(x)v(x)\mu(dx) = 1$.

2.10 Semigroups with Dense Local Quasinilpotence Subspace

This subsection deals with a class of positive semigroups whose real spectra can be described completely. This class is well-suited to weighted shift semigroups we consider in the next section. We resume here some abstract results from [56]. For the sake of simplicity, we restrict ourselves to complex Lebesgue spaces $X = L^p(\mu)$ $(1 \leq p \leq \infty)$. Let $(S(t))_{t\geq 0}$ be a positive semigroup on $L^p(\mu)$. We define its local quasinilpotence subset by

$$Y = \left\{ f \in L^p(\mu);\ \lim_{t\to+\infty} \|S(t)\,|f|\|^{\frac{1}{t}} = 0 \right\}$$

where $|f|$ is the absolute value of $f \in L^p(\mu)$.

Lemma 1 *Y is a subspace of $L^p(\mu)$ invariant under $(S(t))_{t\geqslant 0}$.*

Proof

(i) Linearity: Clearly $\lambda f \in Y$ if $f \in Y$. Let $\varepsilon > 0$, $f, g \in Y$ be given. There exists $\overline{t} > 0$ depending on them such that

$$\|S(t)\,|f|\| \leq \varepsilon^t \text{ and } \|S(t)\,|g|\| \leq \varepsilon^t \quad \forall t \geqslant \overline{t}.$$

So $\|S(t)\,|f+g|\| \leq \|S(t)\,(|f|+|g|)\| \leq 2\varepsilon^t \quad \forall t \geqslant \overline{t}$ and

$$\|S(t)\,|f+g|\|^{\frac{1}{t}} \leq 2^{\frac{1}{t}}\varepsilon \leq 2\varepsilon \quad \forall t \geqslant \max(\overline{t}, 1).$$

(ii) Invariance: Let $\tau > 0$, $f \in Y$.

$$\|S(t)\,|S(\tau)f|\|^{\frac{1}{t}} \leq \|S(t)\,(S(\tau)\,|f|)\|^{\frac{1}{t}}$$
$$= \|S(t+\tau)\,(|f|)\|^{\frac{1}{t}} = \left(\|S(t+\tau)\,(|f|)\|^{\frac{1}{t+\tau}}\right)^{\frac{t+\tau}{t}} \to 0$$

as $t \to +\infty$; i.e. $S(\tau)f \in Y$.

Theorem 11 *Let $(S(t))_{t\geqslant 0}$ be a positive semigroup on $L^p(\mu)$ with type ω. If its local quasinilpotence subspace is dense in $L^p(\mu)$ then $[0, e^{\omega t}] \subset \sigma_{ap}(S(t))$.*

Proof Let $t > 0$ be fixed. Let $0 < \mu < e^{\omega t}$ and $y \in Y$. The equation

$$\mu x - S(t)x = y; \quad (y \in Y,\ \|y\| = 1)$$

can be solved by

$$x = \frac{1}{\mu}\sum_{k=0}^{\infty}\frac{1}{\mu^k}S(t)^k y = \frac{1}{\mu}\sum_{k=0}^{\infty}\frac{1}{\mu^k}S(kt)y$$

provided that this series converges. This is the case since

$$\left\|\frac{1}{\mu^k}S(kt)y\right\|^{\frac{1}{k}} = \frac{1}{\mu}\left(\|S(kt)y\|^{\frac{1}{kt}}\right)^t \to 0 \text{ as } k \to +\infty.$$

In particular $x \geqslant 0$ for $y \geqslant 0$ and

$$\|x\| \geqslant \frac{1}{\mu^{k+1}}\left\|S(t)^k y\right\| \quad \forall k \in \mathbb{N}.$$

There exists $z_k \in L^p_+(\mu)$ such that $\|z_k\| = 1$ and

$$\left\|S(t)^k z_k\right\| \geqslant \frac{1}{2}\left\|S(t)^k\right\|.$$

By the denseness of Y, $\exists\ y_k \in Y$ such that $\|y_k\| = 1$

$$\left\|S(t)^k y_k\right\| \geqslant \frac{1}{3}\left\|S(t)^k\right\|.$$

We may assume that $y_k \geqslant 0$ since $\left\|S(t)^k\,|y_k|\right\| \geqslant \left\|S(t)^k y_k\right\|$ and $|y_k| \in Y$. The solution $\hat{x}_k$ of

$$\mu\hat{x}_k - S(t)\hat{x}_k = y_k$$

satisfies

$$\|\hat{x}_k\| \geqslant \frac{1}{\mu^{k+1}}\left\|S(t)^k y_k\right\| \geqslant \frac{1}{3}\frac{1}{\mu^{k+1}}\left\|S(t)^k\right\|.$$

So

$$\lim\inf_{k\to+\infty}\|\hat{x}_k\|^{\frac{1}{k}} \geqslant \frac{1}{\mu}\lim_{k\to+\infty}\left\|S(t)^k\right\|^{\frac{1}{k}} = \frac{e^{\omega t}}{\mu} > 1$$

and then $\lim_{k\to+\infty}\|\hat{x}_k\| = \infty$. Finally $x_k := \frac{\hat{x}_k}{\|\hat{x}_k\|}$ is such that

$$\|x_k\| = 1 \text{ and } \|\mu x_k - S(t)x_k\| \to 0$$

i.e. $\mu \in \sigma_{ap}(S(t))$. The closedness of $\sigma_{ap}(S(t))$ ends the proof. ■

Lemma 2 *Let $(S(t))_{t\geqslant 0}$ be a positive semigroup on $L^p(\mu)$ with generator T. Let Y be the local quasinilpotence subspace of $(S(t))_{t\geqslant 0}$. Then, for any $\lambda > \omega$,*

$$\lim_{k\to\infty}\left\|(\lambda - T)^{-k}y\right\|^{\frac{1}{k}} = 0\ \ \forall y \in Y.$$

Proof For any $y \in Y$ and any $\varepsilon > 0$ there exists $t_{y,\varepsilon} > 0$ such that

$$\|S(t)y\| \leq \varepsilon^t\ \ \forall t \geqslant t_{y,\varepsilon}$$

i.e. (write $\varepsilon = e^{-A}$)

$$\|S(t)y\| \leq e^{-At}\ \ \forall t \geqslant t_{y,\varepsilon}$$

so $\exists M_{y,A} \geqslant 0$ such that

$$\|S(t)y\| \leq M_{y,A}e^{-At}\ \ \forall t \geqslant 0.$$

Hence

$$\begin{aligned}\left\|(\lambda - T)^{-k}y\right\| &= \left\|\int_0^{+\infty} dt_1 \ldots \int_0^{+\infty} dt_k e^{-\lambda(t_1+\ldots+t_k)} S(t_1 + \ldots + t_k)y\right\| \\ &\leq \int_0^{+\infty} dt_1 \ldots \int_0^{+\infty} dt_k e^{-\lambda(t_1+\ldots+t_k)} \left\|S(t_1 + \ldots + t_k)y\right\| \\ &\leq M_{y,A} \int_0^{+\infty} dt_1 \ldots \int_0^{+\infty} dt_k e^{-\lambda(t_1+\ldots+t_k)} e^{-A(t_1+\ldots+t_k)} \\ &= \frac{M_{y,A}}{(\lambda + A)^k}\end{aligned}$$

and

$$\lim \sup_{k\to+\infty} \left\|(\lambda - T)^{-k}y\right\|^{\frac{1}{k}} \leq \frac{1}{\lambda + A}$$

which ends the proof since $A > 0$ is arbitrary. ■

Theorem 12 *Let $(S(t))_{t\geqslant0}$ be a positive semigroup on $L^p(\mu)$ with generator T. Let $s(T)$ be the spectral bound of T. If the local quasinilpotence subspace of $(S(t))_{t\geqslant0}$ is dense in $L^p(\mu)$ then*

$$(-\infty, s(T)] \subset \sigma_{ap}(T).$$

Proof Let $\lambda < s(T) < \mu$ be fixed. Consider

$$\frac{1}{\mu - \lambda}x - (\mu - T)^{-1}x = y \in Y.$$

Arguing as for the semigroup, we show the existence of $(x_k)_k$ with $\|x_k\| = 1$ and

$$\left\|\frac{1}{\mu - \lambda}x_k - (\mu - T)^{-1}x_k\right\| \to 0$$

i.e. $\frac{1}{\mu-\lambda} \in \sigma_{ap}((\mu - T)^{-1})$ or equivalently $\lambda \in \sigma_{ap}(T)$. The closedness of $\sigma_{ap}(T)$ ends the proof. ■

Corollary 4 *Let $(S(t))_{t\geqslant0}$ be a positive semigroup on $L^p(\mu)$ with type ω and generator T. We assume that the local quasinilpotence subspace of $(S(t))_{t\geqslant0}$ is dense in $L^p(\mu)$.*

(i) If $\sigma(T)$ is invariant under translations along the imaginary axis then

$$\sigma(T) = \{\lambda \in \mathbb{C};\ Re\lambda \leq \omega\}.$$

(ii) If $\sigma(S(t))$ is invariant under rotations then

$$\sigma(S(t)) = \left\{\mu \in \mathbb{C};\ |\mu| \leq e^{\omega t}\right\}.$$

3 Spectral Analysis of Advection Semigroups

Neutron transport theory is mainly a perturbation theory (by scattering operators) of suitable weighted shift semigroups called advection semigroups

$$U(t) : g \to e^{-\int_0^t \sigma(x-\tau v,v)d\tau} g(x-tv,v)1_{\{t\leq s(x,v)\}}$$

where

$$s(x,v) = \inf\{s > 0;\ x - sv \notin \Omega\}$$

is the (first) exit time function from the spatial domain Ω. We describe here the spectra of such semigroups. This section resumes essentially [56]; (an alternative approach is given in [78]).

3.1 On Advection Semigroups

Let $\Omega \subset \mathbb{R}^n$ be an open subset and let μ be a positive Borel measure on $\mathbb{R}^n$ with support V. Let

$$\sigma : \Omega \times V \to \mathbb{R}_+$$

be measurable and such that

$$\lim_{t\to 0}\int_0^t \sigma(x-\tau v,v)d\tau = 0 \text{ a.e.}$$

Let

$$s(x,v) = \inf\{s > 0;\ x - sv \notin \Omega\}$$

be the so-called exit time function. Then

$$S(t) : g \to e^{-\int_0^t \sigma(x-\tau v,v)d\tau} g(x-tv,v)1_{\{t\leq s(x,v)\}}$$

defines a positive semigroup on $L^p(\Omega \times V; dx \otimes \mu(dv))$ (for any $1 \leq p \leq +\infty$), strongly continuous when $p < +\infty$, see e.g. [78]. The dual streaming semigroups in $L^{p'}(\Omega \times V)$ are given by

$$S'(t) : f \to e^{-\int_0^t \sigma(x+\tau v,v)d\tau} f(x+tv,v) 1_{\{t \le s(x,-v)\}}.$$

3.2 Invariance Property of Transport Operators

Let $\mu\{0\} = 0$ and

$$\alpha : (x,v) \in \Omega \times V \to \frac{x.v}{|v|^2}.$$

For any $\eta > 0$

$$M_\eta : f \in L^p(\Omega \times V) \to e^{-i\eta\alpha(x,v)} f \in L^p(\Omega \times V)$$

is an isometric isomorphism.

Theorem 13 ([78]) $M_\eta^{-1} S(t) M_\eta = e^{i\eta t} S(t)$. *In particular* $\sigma(S(t))$ *is invariant by rotations.*

Proof We have

$$\begin{aligned} S(t)M_\eta f &= e^{-\int_0^t \sigma(x-\tau v,v)d\tau} M_\eta f(x-tv,v) 1_{\{t \le s(x,v)\}} \\ &= e^{-\int_0^t \sigma(x-\tau v,v)d\tau} e^{-i\eta\frac{(x-tv).v}{|v|^2}} f(x-tv,v) 1_{\{t \le s(x,v)\}} \end{aligned}$$

so

$$M_\eta^{-1} S(t) M_\eta f = e^{i\eta\frac{x.v}{|v|^2}} e^{-i\eta\frac{(x-tv).v}{|v|^2}} S(t) f = e^{i\eta t} S(t) f$$

so $M_\eta^{-1} S(t) M_\eta = e^{i\eta t} S(t)$. Hence $\sigma(M_\eta^{-1} S(t) M_\eta) = \sigma(e^{i\eta t} S(t))$. On the other hand, by similarity,

$$\sigma(M_\eta^{-1} S(t) M_\eta) = \sigma(S(t))$$

and

$$\sigma(e^{i\eta t} S(t)) = e^{i\eta t} \sigma(S(t))$$

so we are done. ■

As previously we have:

Theorem 14 *Let T be the generator of a streaming semigroup $(S(t))_{t\geqslant 0}$. Then $M_\eta^{-1}TM_\eta = T + i\eta I$. In particular $\sigma(T)$ is invariant by translation along the imaginary axis.*

Proof Let $f \in D(T)$. Then

$$
\begin{aligned}
\frac{S(t)M_\eta f - M_\eta f}{t} &= M_\eta \frac{M_\eta^{-1}S(t)M_\eta f - f}{t} \\
&= M_\eta \frac{e^{i\eta t}S(t)f - f}{t} \\
&= M_\eta \frac{e^{i\eta t}S(t)f - e^{i\eta t}f}{t} + M_\eta \frac{e^{i\eta t}f - f}{t} \\
&= e^{i\eta t}M_\eta \frac{S(t)f - f}{t} + \frac{e^{i\eta t} - 1}{t}M_\eta f \\
&\to M_\eta Tf + i\eta M_\eta f
\end{aligned}
$$

so $M_\eta f \in D(T)$ and $TM_\eta f = M_\eta Tf + i\eta M_\eta f$ or $M_\eta^{-1}TM_\eta = T + i\eta I$. By similarity, $\sigma(T) = \sigma(M_\eta^{-1}TM_\eta) = \sigma(T) + i\eta \forall \eta \in \mathbb{R}$. ■

3.3 Decomposition of the Phase Space

We consider the partition of the phase space $\Omega \times V$ according to

$$E_1 = \{(x,v) \in \Omega \times V;\ s(x,-v) < +\infty\},$$

$$E_2 = \{(x,v) \in \Omega \times V;\ s(x,-v) = +\infty,\ s(x,v) < +\infty\},$$

$$E_3 = \{(x,v) \in \Omega \times V;\ s(x,-v) = +\infty,\ s(x,v) = +\infty\}.$$

This induces a direct sum

$$L^p(\Omega \times V; dx \otimes \mu(dv)) = L^p(E_1) \oplus L^p(E_2) \oplus L^p(E_3)$$

where, we identify $L^p(E_i)$ to the closed subspace of functions $f \in L^p(\Omega \times V)$ vanishing almost everywhere on $\Omega \times V \setminus E_i$. If some set E_i has zero measure then we drop out $L^p(E_i)$ from the direct sum above.

Theorem 15 *The subspaces $L^p(E_i)$ $(i = 1,2,3)$ are invariant under $(S(t))_{t\geqslant 0.}$. For each $i = 1,2,3$, we denote by $(S_i(t))_{t\geqslant 0}$ the part of $(S(t))_{t\geqslant 0.}$ on $L^p(E_i)$ and* *by T_i its generator. Then*

$$\sigma(S(t)) = \sigma(S_1(t)) \cup \sigma(S_2(t)) \cup \sigma(S_3(t))$$

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_2) \cup \sigma(T_3).$$

We have also similar results where $\sigma(.)$ is replaced by $\sigma_p(.)$ or $\sigma_{ap}(.)$. In addition, if $\sigma(.,.)$ is bounded then $(S_3(t))_{t\geqslant 0}$ extends to a positive group.

Proof We check that the direct sum $L^p(\Omega \times V) = L^p(E_1) \oplus L^p(E_2) \oplus L^p(E_3)$ reduces $(S(t))_{t\geqslant 0.}$ We restrict ourselves to $L^p(E_1)$. Let $f \in L^p(E_1)$, i.e. f vanishes almost everywhere on $E_2 \cup E_3$. We have to show that $S(t)f \in L^p(E_1)$ i.e. $S(t)f$ vanishes almost everywhere on $E_2 \cup E_3$. Since

$$S(t)f(x,v) = e^{-\int_0^t \sigma(x-\tau v,v)d\tau} f(x-tv,v)1_{\{t\leq s(x,v)\}}$$

is zero for $t > s(x,v)$, we assume from the start that $t \leq s(x,v)$. One notes that $(x,v) \in E_2 \cup E_3 \Leftrightarrow s(x,-v) = +\infty$ and

$$s(x-tv,-v) = t + s(x,-v)$$

so that $(x-tv,-v) \in E_2 \cup E_3$ and $f(x-tv,v) = 0$. Since the projection P_i on $L^p(E_i)$ along $L^p(\Omega \times V \setminus E_i)$ commutes with $(S(t))_{t\geqslant 0.}$ then the direct sum above reduces also the generator T. Finally, on E_3 (if $\sigma(.,.)$ is bounded) $(S_3(t))_{t\geqslant 0}$ extends to a positive group where

$$S_3(t)^{-1}g = e^{\int_0^t \sigma(x+\tau v,v)d\tau} f(x+tv,v) \ \ (t > 0).$$

3.4 Spectra of the First Reduced Advection Semigroup

Lemma 3 *Let $t > 0$ be fixed. For any $f \in L^p(E_1)$*

$$\|S_1(t)f\|^p = \int_{\{t<s(y,-v)\}\cap\{s(y,-v)<\infty\}} e^{-p\int_0^t \sigma(y+\tau v,v)d\tau} |f(y,v)|^p \, dx\mu(dv).$$

Proof We have to compute the norm of $S_1(t)f$ on the set

$$\{t \leq s(x,v)\} \cap \{s(x,-v) < +\infty\},$$

so $\|S_1(t)f\|^p$ is equal to

$$\int_{\{t\leq s(x,v)\}\cap\{s(x,-v)<+\infty\}} e^{-p\int_0^t \sigma(x-\tau v,v)d\tau} |f(x-tv,v)|^p \, dx\mu(dv).$$

Since $s(x - tv, -v) = t + s(x, -v)$ is finite if and only if $s(x, -v)$ is finite then the change of variable

$$y := x - tv \in \Omega$$

gives $s(y, -v) > t$ and

$$\|S_1(t) f\|^p = \int_{\{t<s(y,-v)\}\cap\{s(y,-v)<\infty\}} e^{-p\int_0^t \sigma(y+\tau v,v)d\tau} |f(y, v)|^p \, dy\mu(dv).$$

The type of $(S_1(t))_{t\geqslant 0}$ is equal to $-\lambda_1^*$ where

$$\lambda_1^* = \lim_{t\to+\infty} \inf_{\{t<s(y,-v)\}\cap\{s(y,-v)<\infty\}} \frac{1}{t}\int_0^t \sigma(y + \tau v, v)d\tau.$$

because

$$\|S_1(t)\| = \sup_{\{t<s(y,-v)\}\cap\{s(y,-v)<\infty\}} e^{-\int_0^t \sigma(y+\tau v,v)d\tau}$$
$$= e^{-\inf_{\{t<s(y,-v)\}\cap\{s(y,-v)<\infty\}} \int_0^t \sigma(y+\tau v,v)d\tau}$$

so

$$\frac{\ln\|S_1(t)\|}{t} = -\inf_{\{t<s(y,-v)\}\cap\{s(y,-v)<\infty\}} \frac{1}{t}\int_0^t \sigma(y + \tau v, v)d\tau$$

and

$$\omega_1 = -\lim_{t\to+\infty} \inf_{\{t<s(y,-v)\}\cap\{s(y,-v)<\infty\}} \frac{1}{t}\int_0^t \sigma(y + \tau v, v)d\tau.$$

We have

$$\sigma(S_1(t)) = \left\{\mu \in \mathbb{C}; |\mu| \le e^{-\lambda_1^* t}\right\}, \ \sigma(T_i) = \{\lambda \in \mathbb{C}; Re\lambda \le -\lambda_1^*\}.$$

Indeed, it suffices to show that the local quasinilpotence subspace of $(S_1(t))_{t\geqslant 0}$ is *dense* in $L^p(E_1)$. Let

$$O_m := \{x, v) \in \Omega \times V;\ s(x, -v) \le m\}.$$

We note that $\cup_m L^p(O_m)$ is dense in $L^p(E_1)$ because of

$$\cup_m O_m = \{x, v) \in \Omega \times V;\ s(x, -v) < +\infty\}.$$

Finally

$$\|S_1(t)f\|^p = \int_{\{t<s(y,-v)\}\cap\{s(y,-v)<\infty\}} e^{-p\int_0^t \sigma(y+\tau v,v)d\tau} |f(y,v)|^p \, dy\mu(dv)$$

shows that, for $f \in L^p(O_m)$, $\|S_1(t)f\| = 0$ for $t > m$ so $\cup_m L^p(O_m)$ is included in the local quasinilpotence subspace of $(S_1(t))_{t\geqslant 0}$. ■

3.5 Spectra of the Second Reduced Advection Semigroup

We deal now with $(S_2(t))_{t\geqslant 0}$ on $L^p(E_2)$ where

$$E_2 = \{(x,v) \in \Omega \times V;\ s(x,-v) = +\infty,\ s(x,v) < +\infty\}.$$

We consider first the case $1 < p < +\infty$. Indeed, by duality $\sigma(S_2(t)) = \sigma(S_2'(t))$ where

$$S_2'(t)f = e^{-\int_0^t \sigma(x+\tau v,v)d\tau} f(x+tv,v).$$

Thus $\|S_2(t)f\|^{p'}$ is equal to

$$\int_{\{s(x,-v)=\infty,\ s(x,v)<\infty\}} e^{-p'\int_0^t \sigma(y+\tau v,v)d\tau} |f(x+tv,v)|^p \, dx\mu(dv)$$
$$= \int_{\{s(y,-v)=\infty,\ t\leq s(y,v)<\infty\}} e^{-p'\int_0^t \sigma(y+\tau v,v)d\tau} |f(y,v)|^p \, dy\mu(dv).$$

Introducing the sets

$$O_m' := \{x,v) \in E_2;\ s(y,v) \leq m\}$$

one sees that $\cup_m L^{p'}(O_m')$ is dense in $L^{p'}(E_2)$ because of

$$\cup_m O_m = E_2.$$

Since in $L^{p'}(O_m')$, $\|S_2(t)f\| = 0$ for $t > m$ then the local quasinilpotence subspace of $(S_2'(t))_{t\geqslant 0}$ is dense. This ends the proof because $\sigma(S_2(t)) = \sigma(S_2'(t))$ and $\sigma(T_2) = \sigma(T_2')$. ■

3.6 Spectra of the Third Reduced Advection (Semi)group

Theorem 16 *Let $\mathcal{S} := \sigma(T_3) \cap \mathbb{R}$ be the real spectrum of T_3. Then*

$$\sigma(T_3) = \mathcal{S} + i\mathbb{R} \text{ and } \sigma(S_3(t)) = e^{t\sigma(T_3)}.$$

Moreover, $sup\mathcal{S} = -\lambda_3^*$ *and* $\inf \mathcal{S} = -\lambda_3^{**}$ *where*

$$\lambda_3^* = \lim_{t\to+\infty} \inf_{\{ s(y,-v)=\infty,\ s(y,v)=\infty\}} \frac{1}{t}\int_0^t \sigma(y+\tau v, v)d\tau$$

$$\lambda_3^* = \lim_{t\to+\infty} \sup_{\{ s(y,-v)=\infty,\ s(y,v)=\infty\}} \frac{1}{t}\int_0^t \sigma(y+\tau v, v)d\tau.$$

Proof The fact that $\sigma(T_3)$ is invariant by translation along the imaginary axis and that $e^{t\sigma(T_3)}$ is invariant under the rotations is a general feature of streaming semigroups in arbitrary geometry. The spectral mapping property for the real spectrum is due to the fact that $(S_3(t))_{t\in\mathbb{R}}$ is a positive C_0-group (see [27]). The type $-\lambda_3^*$ of $(S_3(t))_{t\geqslant 0}$ is obtained as for $(S_1(t))_{t\geqslant 0}$ or $(S_2(t))_{t\geqslant 0}$. Finally, λ_3^* is the spectral bound of the generator of $(S_3(-t))_{t\geqslant 0}$, (i.e. $-T_3$) and is obtained similarly. ■

Theorem 17 *If* $\sigma : \Omega \times V \to \mathbb{R}_+$ *is space-homogeneous then* $\mathcal{S} := \sigma(T_3) \cap \mathbb{R}$ *is nothing but the essential range of* $-\sigma(.)$.

See the details in [56]; in particular, $S := \sigma(T_3) \cap \mathbb{R}$ need *not* be connected. The description of $\sigma(T_3) \cap \mathbb{R}$ for general collision frequency $\sigma : \Omega \times V \to \mathbb{R}_+$ seems to be open. When $\Omega = \mathbb{R}^n$, the situation is well understood for bounded and compactly supported (in space) collision frequencies; see [28].

3.7 Reminders on Sun-Dual Theory

To study $\sigma(S_2(t))$ in L^1 spaces, we need to recall some material. Let X be a complex Banach space and let $(S(t))_{t\geqslant 0}$ be a C_0-semigroup on X with generator T. Let $(S'(t))_{t\geqslant 0}$ be the dual semigroup on the dual space X'. If X is not reflexive then a priori $(S'(t))_{t\geqslant 0}$ is *not* strongly continuous. Let

$$L^{\odot} := \left\{x' \in X';\ \left\|S'(t)x' - x'\right\| \to 0 \text{ as } t \to 0\right\}$$

the subspace of strong continuity of $(S'(t))_{t\geqslant 0}$. Then

- $L^{\odot}$ is a closed subspace of X' invariant under $(S'(t))_{t\geqslant 0}$.
- $L^{\odot} = \overline{D(T')}$ (the closure in X').

We denote by $(S^{\odot}(t))_{t\geqslant 0}$ the restriction of $(S'(t))_{t\geqslant 0}$ to $L^{\odot}$ (*sun-dual* C_0-semigroup). Its generator is given by

$$D(T^{\odot}) = \left\{x' \in D(T'),\ T'x' \in L^{\odot}\right\} \text{ and } T^{\odot}x' = T'x'$$

and we have $\sigma(T) = \sigma(T') = \sigma(T^{\odot})$ and $\sigma(S(t)) = \sigma(S'(t)) = \sigma(S^{\odot}(t))$; see e.g. [20, Chapter IV].

3.8 Sun-Dual Theory for Advection Semigroups

We consider $(S_2(t))_{t\geqslant 0}$ in $L^1(E_2)$ where

$$E_2 = \{(x, v) \in \Omega \times V;\ s(x, -v) = +\infty,\ s(x, v) < +\infty\}$$

and assume that

$$\sigma : \Omega \times V \to \mathbb{R}_+ \text{ is bounded.}$$

Since

$$\sigma(S_2(t)) = \sigma(S_2'(t)) = \sigma(S_2^{\odot}(t)),$$

it suffices to identify $\sigma(S_2^{\odot}(t))$. Because of the boundedness of σ,

$$L^{\odot} = \left\{ f \in L^{\infty}(E_2),\ \sup_{(x,v)} |f(x + tv, v) - f(x, v)| \to 0 \text{ as } t \to 0 \right\}.$$

Actually, we are going to work in the smaller closed subspace

$$L_0^{\odot} := \left\{ f \in L^{\odot},\ \sup_{\{(y,v),\ s(y,v)\geqslant r\}} |f(y, v)| \to 0 \text{ as } r \to \infty \right\}.$$

Lemma 4 $(S'(t))_{t\geqslant 0}$ *leaves invariant* $L_0^{\odot}$.

Proof

$$\left|\left(S'(t)f\right)(y, v)\right| \leq |f(y + tv, v)|$$

and $s(y+tv, v) = s(y, v)+t \to \infty$ if and only if $s(y, v) \to \infty$ so that $S'(t)f \in L_0^{\odot}$ if $f \in L_0^{\odot}$. ■

Let $\left(S_0^{\odot}(t)\right)_{t\geqslant 0}$ be the restriction of $\left(S_2^{\odot}(t)\right)_{t\geqslant 0}$ to $L_0^{\odot}$ and let $T_0^{\odot}$ be its generator. Then $\sigma_{ap}(S_0^{\odot}(t)) \subset \sigma_{ap}(S_2^{\odot}(t))$ and $\sigma_{ap}(T_0^{\odot}) \subset \sigma_{ap}(T_2^{\odot})$. In particular, $\sigma_{ap}(S_0^{\odot}(t)) \subset \sigma(S_2(t))$ and $\sigma_{ap}(T_0^{\odot}) \subset \sigma(T_2)$. Let

$$L_{00}^{\odot} := \left\{ f \in L^{\odot},\ \exists r > 0,\ f(y, v) = 0 \text{ for } s(y, v) \geqslant r \right\}.$$

Theorem 18 *$L_{00}^{\odot}$ is dense in $L_0^{\odot}$.*

Corollary 5 *The local quasinilpotence subspace of $\left(S_0^{\odot}(t)\right)_{t\geqslant 0}$ is dense in $L_0^{\odot}$.*

Proof of Corollary 5: The local quasinilpotence subspace of $\left(S_0^{\odot}(t)\right)_{t\geqslant 0}$ contains $L_{00}^{\odot}$. ■

Before proving Theorem 18, we need:

Lemma 5 *$L^{\odot}$ is an algebra.*

Proof For each $m \in \mathbb{N}$, let $\gamma_m : [0,+\infty[\to [0,1]$ be smooth (say C^1) and such that

$$\gamma_m(s) = \begin{cases} 1 \; if \; s \leq m \\ 0 \; if \; s \geqslant 2m. \end{cases}$$

Lemma 6 *$\forall m \in \mathbb{N},\ (x,v) \to \gamma_m(s(x,v))$ belongs to $L_{00}^{\odot}$.*

Proof We have just to show that $(x,v) \to \gamma_m(s(x,v))$ belongs to $L^{\odot}$. Since γ_m is Lipschitz then

$$\begin{aligned} |\gamma_m(s(x+tv,v)) - \gamma_m(s(x,v))| &= |\gamma_m(s(x,v)+t) - \gamma_m(s(x,v))| \\ &\leq Ct \;\; \forall (x,v). \end{aligned}$$

■

Proof of Theorem 18: Let $f \in L_0^{\odot}$ then $\forall m \in \mathbb{N}, (x,v) \to \gamma_m(s(x,v)) f(x,v)$ belongs to $L_{00}^{\odot}$.

$$\begin{aligned} |\gamma_m(s(x,v)) f(x,v) - f(x,v)| &= |(1-\gamma_m(s(x,v))) f(x,v)| \\ &\leq \sup_{s(x,v)\geqslant 2m} |f(x,v)| \to 0 \text{ as } m \to \infty \end{aligned}$$

since $f \in L_0^{\odot}$. ■

By the general theory,

$$\sigma(T_0^{\odot}) = \left\{Re\lambda \leq \omega_0^{\odot}\right\},\ \sigma(S_0^{\odot}(t)) = \left\{\mu;\ |\mu| \leq e^{\omega_0^{\odot} t}\right\}$$

where $\omega_0^{\odot}$ is the type of $\left(S_0^{\odot}(t)\right)_{t\geqslant 0}$. We have to *identify* $\omega_0^{\odot}$! The fact that $\left\|S_0^{\odot}(t)\right\| \leq \left\|S_2'(t)\right\| = \|S_2(t)\|$ implies

$$\omega_0^{\odot} \leq \omega_2 = \text{ type of } (S_2(t))_{t\geqslant 0}\,.$$

On the other hand,

$$g_m : (x,v) \to \gamma_m(s(x,v))$$

belongs to $L_0^{\odot}$ and $\|g_m\| \leq 1$ so that

$$\left\|S_0^{\odot}(t)\right\| \geqslant \left\|S_0^{\odot}(t)g_m\right\| = \sup_{(y,v)} \left(e^{-\int_0^t \sigma(y+\tau v,v)d\tau}\gamma_m(s(y+tv,v))\right)$$
$$= \sup_{(y,v)} \left(e^{-\int_0^t \sigma(y+\tau v,v)d\tau}\gamma_m(s(y,v)+t)\right) \quad \forall m \in \mathbb{N}.$$

But $\gamma_m(s(y,v)+t) = 1$ if $s(y,v) \leq m-t$ so

$$\left\|S_0^{\odot}(t)\right\| \geqslant \sup_{\{s(y,v)\leq m-t\}} e^{-\int_0^t \sigma(y+\tau v,v)d\tau} \quad \forall m \in \mathbb{N}.$$

Finally

$$\left\|S_0^{\odot}(t)\right\| \geqslant \sup_{\{s(y,v)<+\infty\}} e^{-\int_0^t \sigma(y+\tau v,v)d\tau} = \|S_2(t)\|$$

whence $\omega_2 \leq \omega_0^{\odot}$ and we are done. ■

A similar theory can be built for general vector fields. Indeed, consider $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$ a Lipschitz vector field and denote by $\Phi(x,t)$ the unique global solution to

$$\frac{d}{dt}X(t) = \mathcal{F}(X(t)), \quad t \in \mathbb{R}$$
$$X(0) = x.$$

Let $\Omega \subset \mathbb{R}^n$ be an open set and let

$$s_{\pm}(x) := \inf\{s > 0;\ \Phi(x,\pm s) \notin \Omega\}$$

be the exit times from Ω (with the convention that $\inf\emptyset = +\infty$). We define a weighted shift semigroup

$$U(t) : f \to U(t)f$$

where

$$U(t)f = e^{-\int_0^t \nu(\Phi(x,-s))ds} f(\Phi(x,-t))\chi_{\{t<s_-(x)\}}(x).$$

We introduce the sets

$$\Omega_1 = \{x \in \Omega;\ s_+(x) < \infty\},\ \Omega_2 = \{x \in \Omega;\ s_+(x) = \infty,\ s_-(x) < \infty\}$$

and

$$\Omega_3 = \{x \in \Omega;\ s_+(x) = \infty,\ s_-(x) = \infty\}.$$

Then $L^p(\Omega_i)$ $(i = 1, 2, 3)$ are invariant under $(U(t))_{t\geqslant 0}$ and we can extend the previous spectral theory of advection semigroups, see [39].

4 Spectra of Perturbed Operators

This section deals with functional analytic results on stability of essential spectra for perturbed generators or perturbed semigroups on Banach spaces. Let X be a complex Banach space and $D \subset \mathbb{C}$ be open and connected. A compact operator valued meromorphic mapping

$$A : D \to \mathcal{C}(X)$$

($\mathcal{C}(X) \subset \mathcal{L}(X)$ is the closed subspace of compact operators) is called *essentially meromorphic* on D if A is holomorphic on D except at a discrete set of points $z_k \in D$ where A has poles with Laurent expansions

$$A(z) = \sum_{n=-m_k}^{\infty} (z - z_k)^m A_n(z_k) \quad (0 < m_k < \infty)$$

where $A_n(z_k)(n = -1, -2, \ldots, -m_k)$ are finite rank operators. We recall now a fundamental analytic Fredholm alternative:

Theorem 19 (*[65] Corollary II*) *Let X be a complex Banach space and $D \subset \mathbb{C}$ be open and connected. Let*

$$A : D \to \mathcal{C}(X)$$

be essentially meromorphic. Then

(i) Either $\lambda = 1$ is an eigenvalue of all $A(z)$
(ii) or $[I - A(z)]^{-1}$ exists except for a discrete set of points and $[I - A(z)]^{-1}$ is essentially meromorphic on D.

Let $T : D(T) \subset X \to X$ be a closed operator. We define its "essential resolvent set" as

$$\rho_e(T) = \rho(T) \cup \sigma_{discr}(T)$$

where $\sigma_{discr}(T)$ refers to the isolated eigenvalues of T with finite algebraic multiplicities. This set is open. We note that if $T \in \mathcal{L}(X)$ then the unbounded component of $\rho_e(T)$ coincides with the unbounded component of the Fredholm domain $\rho_F(T)$ (see [21, p. 204]). We give first a result from [77] and some of its consequences.

Theorem 20 *Let $T : D(T) \subset X \to X$ be a closed operator and let Ω be a connected component of $\rho_e(T)$. Let $B : D(T) \to X$ be T-bounded such that there exists $n \in \mathbb{N}$ and*

$$\left[B(\lambda - T)^{-1}\right]^n \text{ is compact } (\lambda \in \Omega \cap \rho(T)).$$

We assume that there exists some $\lambda \in \Omega \cap \rho(T)$ such that $I - \left[B(\lambda - T)^{-1}\right]^n$ is invertible (i.e. 1 is not an eigenvalue of $\left[B(\lambda - T)^{-1}\right]^n$). Then $\Omega \subset \rho_e(T + B)$ and

$$\left[B(\lambda - T - B)^{-1}\right]^n \text{ is compact } (\lambda \in \Omega \cap \rho(T + B)).$$

Proof We note that $(\lambda - T)^{-1}$ is essentially meromorphic on Ω. Then $B_\lambda := B(\lambda - T)^{-1}$ and $B_\lambda^n = \left[B(\lambda - T)^{-1}\right]^n$ are also essentially meromorphic on Ω. Since B_λ^n is operator compact valued then, by the analytic Fredholm alternative $(I - B_\lambda^n)^{-1}$ is also essentially meromorphic on Ω. On the other hand

$$I - B_\lambda^n = (I - B_\lambda)(I + B_\lambda + \ldots + B_\lambda^{n-1})$$

shows that

$$(I - B_\lambda)^{-1} = (I - B_\lambda^n)^{-1}(I + B_\lambda + \ldots + B_\lambda^{n-1})$$

is also essentially meromorphic on Ω and then so is

$$(\lambda - T - B)^{-1} = (\lambda - T)^{-1}(I - B_\lambda)^{-1}$$

i.e. $\Omega \subset \rho_e(T + B)$. Finally

$$\left[B(\lambda - T - B)^{-1}\right]^n = \left[B_\lambda(I - B_\lambda)^{-1}\right]^n = B_\lambda^n(I - B_\lambda)^{-n}$$

is also compact on $\Omega \cap \rho(T + B)$. ■

We give now a more precise version of the theorem above under an additional assumption.

Corollary 6 *Let $T : D(T) \subset X \to X$ be a closed operator and let Ω be a connected component of $\rho_e(T)$. Let $B : D(T) \to X$ be T-bounded such that there exists $n \in \mathbb{N}$ and*

$$\left[B(\lambda - T)^{-1}\right]^n \text{ is compact } (\lambda \in \Omega \cap \rho(T)).$$

We assume that there exists a sequence $\left(\lambda_j\right)_j \subset \Omega \cap \rho(T)$ *such that*

$$\left\| B(\lambda_j - T)^{-1} \right\| \to 0 \; (j \to +\infty).$$

Then $\left(\lambda_j\right)_j \subset \rho(T+B)$ *for* j *large enough and* $\left\| B(\lambda_j - T - B)^{-1} \right\| \to 0$ *as* $j \to +\infty$. *Furthermore* Ω *is a component of* $\rho_e(T+B)$.

Proof We note that for j large enough

$$\left\| B(\lambda_j - T - B)^{-1} \right\| = \left\| B_{\lambda_j}(I - B_{\lambda_j})^{-1} \right\| \leq \sum_{k=1}^{\infty} \left\| B_{\lambda_j} \right\|^k \to 0 \;\; (j \to +\infty)$$

Let Ω' be the component of $\rho_e(T+B)$ which contains Ω. We know that

$$\left[B(\lambda - T - B)^{-1} \right]^n$$

is compact ($\lambda \in \Omega \cap \rho(T+B)$). By analyticity, this extends to $\Omega' \cap \rho(T+B)$. By considering T as $(T+B) - B$ and reversing the arguments in the previous theorem one gets $\Omega' \subset \rho_e(T)$ and consequently $\Omega = \Omega'$. ■

Corollary 7 *Let* $T, B \in \mathcal{L}(X)$. *We assume that* $\left[B(\lambda - T)^{-1} \right]^n$ *is compact on the unbounded component of* $\rho(T)$, *i.e.*

$$\left[B(\lambda - T)^{-1} \right]^n \;\; \textit{is compact} \; (\lambda \in \Omega \cap \rho(T))$$

where Ω *is the unbounded connected component of* $\rho_e(T)$. *The unbounded components of* $\rho_e(T)$ *and* $\rho_e(T+B)$ *coincide and then*

$$r_e(T) = r_e(T+B).$$

Proof Since $\left\| (\lambda - T)^{-1} \right\| \to 0$ as $|\lambda| \to \infty$ we apply the corollary above. ■

Corollary 8 *Let* $T, B \in \mathcal{L}(X)$.

(i) If B *is compact then* $r_e(T) = r_e(T+B)$.
(ii) If $X = L^1(\mu)$ *and* B *is weakly compact then* $r_e(T) = r_e(T+B)$.

Proof The case (i) is clear with $n = 1$. In the case (ii) $B(\lambda - T)^{-1}$ is also weakly compact on $L^1(\mu)$ and consequently its square is compact. ■

4.1 Strong Integral of Operator Valued Mappings

Let (Ω, μ) be a *finite* measure space and let X, Y be two Banach spaces. Let

$$G : \omega \in \Omega \to G(\omega) \in \mathcal{L}(X, Y)$$

be bounded and *strongly* measurable in the sense that for each $x \in X$

$$\omega \in \Omega \to G(\omega)x \in Y$$

is (Bochner) measurable. We define the *strong integral* of G on Ω as the bounded operator

$$\int_{\Omega} G : x \in X \to \int_{\Omega} G(\omega)x\mu(d\omega) \in Y.$$

We note that strongly continuous mappings appear everywhere in semigroup theory!

Theorem 21 ([33, 80, 82]) *Under the conditions above, assume in addition that* $\forall \omega \in \Omega, G(\omega) \in \mathcal{C}(X, Y)$ *(i.e.* $G(\omega)$ *is a compact operator). Then*

$$\int_{\Omega} G \in \mathcal{C}(X, Y).$$

In the statement above, we can replace "compact" by "weakly compact" [69]. Direct proofs in Lebesgue spaces relying on Kolmogorov's compactness criterion and the Dunford-Pettis criterion of weak compactness are given in [50].

4.2 Spectra of Perturbed Generators

Theorem 22 *Let* X *be a complex Banach space. Let* $(U(t))_{t \geqslant 0}$ *be a* C_0*-semigroup with generator* T *and let* $K \in \mathcal{L}(X)$*. We denote by* $(V(t))_{t \geqslant 0}$ *the* C_0*-semigroup generated by* $T + K$*. We assume that* K *is* T*-power compact i.e. there exists* $n \in \mathbb{N}$ *such that (for* λ *in some right half-plane included in* $\rho(T + K)$*)*

$$\left[K(\lambda - T)^{-1}\right]^n \text{ is compact.}$$

Then the components of $\rho_e(T)$ *and* $\rho_e(T + K)$ *containing a right half-plane coincide. In particular*

$$\sigma(T + K) \cap \{Re\lambda > s(T)\}$$

consists at most of isolated eigenvalues with finite algebraic multiplicities.

The proof follows from Corollary 6 since $\left\|(\lambda - T)^{-1}\right\| \to 0$ as $Re\lambda$ goes to $+\infty$. This theorem is a refinement of a first version due to I. Vidav [75].

4.3 Dyson–Phillips Expansions

The perturbed semigroup $(V(t))_{t\geqslant 0}$ is related to the unperturbed one $(U(t))_{t\geqslant 0}$ by an integral equation (Duhamel equation)

$$V(t) = U(t) + \int_0^t U(t)KV(t-s)ds.$$

The integrals are interpreted in a strong sense i.e.

$$V(t)x = U(t)x + \int_0^t U(t)KV(t-s)xds \ \ (x \in X).$$

The Duhamel equation is solved by standard iterations

$$V_{j+1}(t)x = U(t)x + \int_0^t U(t)KV_j(t-s)xds \ (j \geqslant 0) \ \ V_0 = 0$$

and

$$\forall C > 0, \ \sup_{t\in[0,C]} \left\| V_j(t) - V(t) \right\|_{\mathcal{L}(X)} \to 0 \ \text{ as } j \to +\infty.$$

Finally, $(V(t))_{t\geqslant 0}$ is given by a Dyson–Phillips series

$$V(t) = \sum_0^{+\infty} U_j(t)$$

where

$$U_{j+1}(t) = \int_0^t U_0(t)KU_j(t-s)ds \ (j \geqslant 0) \ \ \ U_0(t) = U(t).$$

By remainder terms of Dyson–Phillips expansions we mean

$$R_m(t) := \sum_{j=m}^{+\infty} U_j(t) \ \ (m \geqslant 0).$$

For any strongly continuous mappings $f, g : \mathbb{R}_+ \to \mathcal{L}(X)$, we define their convolution by (the strong integral)

$$f * g = h(t) := \int_0^t f(s)g(t-s)ds$$

and note $[f]^n$ the n-fold convolution of f with the convention $[f]^1 = f$. Then we can express $U_j(.)$ for $j \geqslant 1$ as

$$U_j(.) = [UK]^j * U = U * [KU]^j \quad (j \geqslant 1).$$

Theorem 23 ([48] Chapter 2) *Let $n \in \mathbb{N}_*$ be given. Then $U_n(t)$ is compact for all $t \geqslant 0$ if and only if $R_n(t) := \sum_{j=n}^{+\infty} U_j(t)$ is compact for all $t \geqslant 0$. If $X = L^1(\mu)$ then we can replace "compact" by "weakly compact".*

Proof If $U_n(t)$ is compact for all $t \geqslant 0$ then

$$U_{n+1}(t) = \int_0^t U(s)KU_n(t-s)ds$$

is compact for all $t \geqslant 0$ as a strong integral of "compact operator" valued mappings. By induction $U_j(t)$ is compact (for all $t \geqslant 0$) for all $j \geqslant n$ and then $R_n(t)$ is compact for all $t \geqslant 0$ since the series converges in operator norm. Conversely, let $R_n(t)$ be compact for all $t \geqslant 0$. Then

$$\begin{aligned} R_{n+1}(t) = \sum_{j=n+1}^{+\infty} U_j(t) &= \sum_{j=n+1}^{+\infty} \left([UK]^j * U\right) \\ &= [UK] * \sum_{j=n+1}^{+\infty} \left([UK]^{j-1} * U\right) \\ &= [UK] * \sum_{j=n}^{+\infty} \left([UK]^j * U\right) \\ &= [UK] * R_n(t) = \int_0^t U(s)KR_n(t-s)ds \end{aligned}$$

shows that $R_{n+1}(t)$ is also compact for all $t \geqslant 0$ as a strong integral of "compact operator" valued mappings. Finally $U_n(t) = R_n(t) - R_{n+1}(t)$ is also compact for all $t \geqslant 0$. ■

The following result is given in [81] for unbounded perturbations. We give here a slightly different (and simpler) presentation of the proof thanks to the boundedness of K.

Theorem 24 *Let X be a complex Banach space. Let $(U(t))_{t\geqslant 0}$ be a C_0-semigroup with generator T and let $K \in \mathcal{L}(X)$. We denote by $(V(t))_{t\geqslant 0}$ the C_0-semigroup generated by $T + K$. If some remainder term*

$$R_n(t) := \sum_{j=n}^{+\infty} U_j(t)$$

is compact for t large enough then $\omega_e(V) \leq \omega_e(U)$ where $\omega_e(V)$ (resp. $\omega_e(U)$) is the essential type of $(V(t))_{t\geqslant 0}$ (resp. $(U(t))_{t\geqslant 0}$). If $X = L^1(\mu)$ then we can replace "compact" by "weakly compact".

Proof Let $\beta > \omega_e(U)$ then there exists a *finite range projection* P_β commuting with $(U(t))_{t\geqslant 0}$ such that for any $\beta' > \beta$

$$\left\| U(t)(I - P_\beta) \right\| \leq M_{\beta'} e^{\beta' t} \quad (t \geqslant 0).$$

(with $P_\beta = 0$ if $\beta > \omega(U)$ the type of $(U(t))_{t\geqslant 0}$). On the other hand, by stability of essential radius by compact (or weakly compact if $X = L^1(\mu)$) perturbation

$$r_e(V(t)) = r_e(\sum_{j=0}^{n-1} U_j(t)) = r_e(\sum_{j=0}^{n-1} [UK]^j * U)$$

for t large enough. We note that

$$\begin{aligned} [UK]^j &= \left[U(I - P_\beta + P_\beta)K(I - P_\beta + P_\beta)\right]^j \\ &= \left[U(I - P_\beta)K(I - P_\beta)\right]^j + C_j(t) \end{aligned}$$

where $C_j(t)$ is a sum of convolutions where each convolution involves at least one term of the form $U(I - P_\beta)KP_\beta$, $UP_\beta KP_\beta$ or $UP_\beta K(I - P_\beta)$. Such terms are compact (of finite rank) because of P_β so that the convolutions are compact for all time as strong integrals of "compact operator" valued mappings. Thus $C_j(t)$ is compact for all $t \geqslant 0$. Once again, the stability of essential radius by compact perturbation gives for t large enough

$$\begin{aligned} r_e(V(t)) &= r_e((I - P_\beta)U(t)(I - P_\beta) + \sum_{j=1}^{n-1} \left[U(I - P_\beta)K(I - P_\beta)\right]^j * U) \\ &\leq \left\| (I - P_\beta)U(t)(I - P_\beta) + \sum_{j=1}^{n-1} \left[U(I - P_\beta)K(I - P_\beta)\right]^j * U \right\| \\ &\leq \left\| (I - P_\beta)U(t)(I - P_\beta) \right\| + \sum_{j=1}^{n-1} \left\| \left[U(I - P_\beta)K(I - P_\beta)\right]^j * U \right\|. \end{aligned}$$

Observe that

$$\left[U(I - P_\beta)K(I - P_\beta)\right] * U = \left[\tilde{U} K\right] * \tilde{U}$$

where $\left(\tilde{U}(t)\right)_{t\geqslant 0}$ is the semigroup $\left(U(t)(I-P_\beta)\right)_{t\geqslant 0}$. More generally

$$\left[U(I-P_\beta)K(I-P_\beta)\right]^j * U = \left[\tilde{U}K\right]^j * \tilde{U}.$$

By using the estimate

$$\left\|\tilde{U}(t)\right\| \leq M_{\beta'} e^{\beta' t} \ \ (t \geqslant 0),$$

an elementary calculation shows that $\left\|\left[\tilde{U}K\right]^j * \tilde{U}\right\| \leq c_j t^j e^{\beta' t}$ so

$$r_e(V(t)) \leq p_n(t) e^{\beta' t}$$

where $p_n(t)$ is a polynomial of degree n. To end the proof, let $\beta'' > \beta'$. Then there exists a constant $M_{\beta''}$ such that

$$r_e(V(t)) \leq M_{\beta''} e^{\beta'' t}$$

for t large enough. Let $\omega_e(V)$ be the essential type of $(V(t))_{t\geqslant 0}$. The fact that

$$r_e(V(t)) = e^{\omega_e(V)t}$$

implies that $\omega_e(V) \leq \beta''$. Hence $\omega_e(V) \leq \omega_e(U)$ since $\beta' > \omega_e(U)$ and $\beta'' > \beta'$ are chosen arbitrarily. ■

Remark 1 A classical weaker estimate $\omega_e(V) \leq \omega(U)$ (where $\omega(U)$ is the type of $(U(t))_{t\geqslant 0}$) is due to I. Vidav [76]. The estimate $\omega_e(V) \leq \omega_e(U)$ is also derived in [70, 82] by using the properties of measure of noncompactness of strong integrals.

We have seen that if some remainder term $R_n(t) := \sum_{j=n}^{+\infty} U_j(t)$ is compact (or weakly compact when $X = L^1(\mu)$) for large t then $\omega_e(V) \leq \omega_e(U)$. We show now that if some remainder term is compact (or weakly compact when $X = L^1(\mu)$) for *all* $t \geqslant 0$ then $\omega_e(V) = \omega_e(U)$. We need a preliminary result:

Lemma 7 ([48] Chapter 2) *Let X be a complex Banach space. Let $(U(t))_{t\geqslant 0}$ be a C_0-semigroup with generator T and let $K \in \mathcal{L}(X)$. We denote by $(V(t))_{t\geqslant 0}$ the C_0-semigroup generated by $T+K$. Let $V(t) = \sum_0^{+\infty} U_j(t)$ be the Dyson–Phillips expansion of $(V(t))_{t\geqslant 0}$. Let $U(t) = \sum_0^{+\infty} V_j(t)$ be the Dyson–Phillips expansion of $(U(t))_{t\geqslant 0}$ considered as a perturbation of $(V(t))_{t\geqslant 0}$ (i.e. $T = (T+K)+(-K)$). For any $j \in \mathbb{N}_*$, $U_j(t)$ is compact for all $t \geqslant 0$ if and only if $V_j(t)$ is compact for all $t \geqslant 0$. If $X = L^1(\mu)$ then we can replace "compact" by "weakly compact".*

By reversing the role of $(U(t))_{t\geqslant 0}$ and $(V(t))_{t\geqslant 0}$ we obtain:

Corollary 9 *Let X be a complex Banach space. Let $(U(t))_{t\geqslant 0}$ be a C_0-semigroup with generator T and let $K \in \mathcal{L}(X)$. We denote by $(V(t))_{t\geqslant 0}$ the C_0-semigroup* *generated by* $T + K$*. Let*

$$U_{j+1}(t) = \int_0^t U_0(t)KU_j(t-s)ds \ (j \geqslant 0) \quad U_0(t) = U(t)$$

be the terms of the Dyson–Phillips expansion of $(V(t))_{t\geqslant 0}$*. If for some* $n \in \mathbb{N}_*$, $U_n(t)$ *is compact (resp. weakly compact if* $X = L^1(\mu)$*) for all* $t \geqslant 0$ *then*

$$\omega_e(V) = \omega_e(U).$$

Remark 2 We note that the stability of essential type appears also in [70,81,82] but under stronger assumptions.

4.4 Short Digression on Resolvent Approach

The following "resolvent characterization" is due to S. Brendle [9].

Theorem 25 *Let* $n \in \mathbb{N}_*$*. Then* $U_n(t)$ *is compact for all* $t \geqslant 0$ *if and only if:*

(i) $t \geqslant 0 \to U_n(t) \in \mathcal{L}(X)$ *is continuous in operator norm*
and
(ii) $\left[(\alpha + i\beta - T)^{-1}K\right]^n (\alpha + i\beta - T)^{-1}$ *is compact for some* $\alpha > \omega(U)$ *and all* $\beta \in \mathbb{R}$.

This is useful in some applications (for example for kinetic equations involving boundary operators relating the incoming and outgoing fluxes) where the unperturbed semigroup $(U(t))_{t\geqslant 0}$ is not explicit while the resolvent $(\lambda - T)^{-1}$ is "tractable"! The cost of the approach is that we need a priori that

$$t \geqslant 0 \to U_n(t) \in \mathcal{L}(X)$$

is continuous in operator norm. The following result gives sufficient conditions of continuity in operator norm.

Theorem 26 ([9]) *Let* $n \in \mathbb{N}$*. If* X *is a Hilbert space and if*

$$\left\| \left[(\alpha + i\beta - T)^{-1}K\right]^n (\alpha + i\beta - T)^{-1} \right\| \to 0 \ \text{ as } |\beta| \to +\infty$$

then $t \geqslant 0 \to U_{n+2}(t) \in \mathcal{L}(X)$ *is continuous in operator norm.*

Note that the continuity of $t \geqslant 0 \to U_1(t) \in \mathcal{L}(X)$ (if we want to show the compactness of $V(t) - U(t)$) is out of reach of this theorem. We give now Sbihi's criterion of continuity in operator norm.

Theorem 27 ([67]) *Let X be a Hilbert space and let T be dissipative (i.e. $Re\,(Tx,x) \leq 0\ \forall x \in D(T)$). If*

$$\left\| K^{*}(\lambda - T)^{-1}K \right\| + \left\| K(\lambda - T)^{-1}K^{*} \right\| \to 0 \ \ as \ \ |Im\lambda| \to +\infty$$

then $t \geqslant 0 \to U_1(t) \in \mathcal{L}(X)$ is continuous in operator norm.

Useful applications of this result are given in [34, 38, 67].

5 Collisional Transport Theory

In this section, we show how the previous functional analytic tools apply to neutron transport theory. We start with an unperturbed (advection) semigroup in $L^p(\Omega \times V;\ dx \otimes \mu(dv))$

$$U(t) : g \to e^{-\int_0^t \sigma(x-\tau v,v)d\tau} g(x-tv,v)1_{\{t \leq s(x,v)\}}\ (t \geqslant 0)$$

(with generator T) where

$$s(x,v) = \inf\{s > 0;\ x - sv \notin \Omega\}$$

is the first exit time function from the spatial domain Ω. We regard the scattering operator

$$K : f \to \int_V k(x,v,v')f(x,v')\mu(dv')$$

as a bounded perturbation of T (we refer to [58] and references therein for generation results with unbounded scattering operators) and denote by $(V(t))_{t\geqslant 0}$ the perturbed neutron transport semigroup. We are faced with *two* main questions:

- When is $\left[(\lambda - T)^{-1}K\right]^n$ compact in $L^p(\Omega \times V)$ for some $n \in \mathbb{N}_*$?
- When is some remainder term $R_m(t)$ compact in $L^p(\Omega \times V)$?

We point out that the resolvent $(\lambda - T)^{-1}$ of T cannot be compact (e.g. in bounded geometries, $\sigma(T)$ is a half-plane when $0 \in V$!). The scattering operator

$$K : f \to \int_V k(x,v,v')f(x,v')\mu(dv')$$

is *local* with respect to the space-variable x so that K cannot be compact on $L^p(\Omega \times V)$. The good news is that compactness will emerge from subtle combinations of properties of T and those of K. For information, we recall some classical results: Under quite general assumptions on the scattering kernel $k(x,v,v')$, for bounded domains Ω and Lebesgue measure dv on $\mathbb{R}^n$ as velocity measure, the *second order* remainder term $R_2(t)$ is compact in $L^p(\Omega \times V)$ $(1 < p < \infty)$ or weakly compact in $L^1(\Omega \times V)$; see e.g. [25, 43, 45, 72, 79, 82].

We introduce now a useful class of scattering operators. Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 1)$ be an open subset and let μ be a positive Radon measure with support V. Let

$$X := L^p(\Omega \times V; dx \otimes \mu(dv))$$

with $1 \leq p < +\infty$. Let

$$k : (x, v, v') : \Omega \times V \times V \to k(x, v, v') \in \mathbb{R}_+$$

be measurable and such that

$$K : f \in L^p(\Omega \times V) \to \int_V k(x, v, v') f(x, v') dv' \in L^p(\Omega \times V)$$

is a bounded operator on $L^p(\Omega \times V)$. Since K is *local* in space variable, we may interpret it as a family of bounded operators on $L^p(V)$ indexed by the parameter $x \in \Omega$ i.e. a mapping

$$K : x \in \Omega \to K(x) \in \mathcal{L}(L^p(V)).$$

Then

$$\|K\|_{\mathcal{L}(L^p(\Omega \times V))} = \sup_{x \in \Omega} \|K(x)\|_{\mathcal{L}(L^p(V))}.$$

5.1 L^p Theory $(1 < p < \infty)$

In this section, we restrict ourselves to $1 < p < \infty$. A scattering operator K is called regular if

(i) $\{K(x);\ x \in \Omega\}$ is a set of *collectively compact* operators on $L^p(V)$, i.e. the set

$$\left\{K(x)\varphi;\ x \in \Omega,\ \|\varphi\|_{L^p(V)} \leq 1\right\}$$

is relatively compact in $L^p(V)$.

(ii) For any $\psi' \in L^{p'}(V)$, the set $\{K'(x)\psi';\ x \in \Omega\}$ is relatively compact in $L^{p'}(V)$ where p' is the conjugate exponent and $K'(x)$ is the dual operator of $K(x)$.

We note that the compactness of K with respect to velocities (at least in $L^2(V)$) is satisfied by most physical models [8].

Theorem 28 ([51]) *The class of regular scattering operators is the closure in the operator norm of $\mathcal{L}(L^p(\Omega \times V))$ of the class of scattering operators with kernels*

$$k(x, v, v') = \sum_{j \in J} \alpha_j(x) f_j(v) g_j(v')$$

where $\alpha_j \in L^\infty(\Omega)$, f_j, g_j continuous with compact supports and J finite.

The first part of the following lemma is given in ([48, Chapter 3, p. 32]) and the second part in [51].

Lemma 8 *Let μ be a finite Radon measure on $\mathbb{R}^n$.*

(i) If the hyperplanes (through the origin) have zero μ-measure then

$$\sup_{e \in S^{n-1}} \mu\{v;\ |v.e| \leq \varepsilon\} \to 0 \ \ as\ \varepsilon \to 0.$$

(ii) If the affine (i.e. translated) hyperplanes have zero μ-measure then

$$\sup_{e \in S^{n-1}} \mu \otimes \mu\left\{(v, v');\ \left|(v - v').e\right| \leq \varepsilon\right\} \to 0 \ \ as\ \varepsilon \to 0.$$

Theorem 29 ([51]) *We assume that Ω has finite Lebesgue measure and the scattering operator is regular in $L^p(\Omega \times V; dx \otimes \mu(dv))$ $(1 < p < \infty)$. If the hyperplanes have zero μ-measure then $(\lambda - T)^{-1} K$ and $K(\lambda - T)^{-1}$ are compact on $L^p(\Omega \times V; dx \otimes \mu(dv))$.*

Remark 3 The compactness of $K(\lambda - T)^{-1}$ can be expressed as an averaging lemma in open sets Ω with finite volume, i.e. if F is a bounded subset of $D(T)$ (for the graph norm) then

$$\{K\varphi;\ \varphi \in F\} \ \text{ is relatively compact in } L^p(\Omega \times V).$$

We note that if

$$\sup_{e \in S^{n-1}} \mu\{v; |v.e| \leq \varepsilon\} \leq c\varepsilon^\alpha$$

and if Ω is bounded and convex then the compactness can be measured in terms of fractional Sobolev regularity [23], (see also [1]).

Corollary 10 *We assume that Ω has finite Lebesgue measure, the scattering operator is regular in $L^p(\Omega \times V; dx \otimes \mu(dv))$ $(1 < p < \infty)$ and the hyperplanes have zero μ-measure. Then*

$$\sigma_{ess}(T + K) = \sigma_{ess}(T).$$

In particular $\sigma(T+K)\cap\{Re\lambda > s(T)\}$ consists at most of isolated eigenvalues with finite algebraic multiplicities where

$$s(T) = -\lim_{t\to+\infty}\inf_{(y,v)}\frac{1}{t}\int_0^t \sigma(y+\tau v, v)d\tau.$$

Theorem 30 ([51]) *We assume that Ω has finite Lebesgue measure and the scattering operator is regular in $L^p(\Omega\times V; dx\otimes\mu(dv))$ $(1 < p < \infty)$. If the hyperplanes have zero μ-measure then*

$$V(t) - U(t) \text{ is compact on } L^p(\Omega\times V) \quad (t \geqslant 0),$$

i.e. the first remainder $R_1(t) = \sum_{j=1}^{+\infty} U_j(t)$ is compact on $L^p(\Omega\times V)$.

Strategy of the proof:

- $R_1(t) = \sum_{j=m}^{+\infty} U_j(t)$ is compact for all $t \geqslant 0$ if and only if $U_1(t) = \int_0^t U(t-s)KU(s)ds$ is. So we deal with $U_1(t)$.
- $U_1(t)$ depends (linearly and) continuously on the scattering operator K. So, by approximation, we may assume that

$$k(x,v,v') = \sum_{j\in J}\alpha_j(x)f_j(v)g_j(v')$$

  where $\alpha_j \in L^\infty(\Omega)$, f_j, g_j continuous with compact supports and J is finite.
- By linearity, we may even choose

$$k(x,v,v') = \alpha(x)f(v)g(v')$$

  where $\alpha \in L^\infty(\Omega)$, f, g are continuous with compact supports. In this case, $U_1(t)$ operates on all $L^q(\Omega\times V)$ $(1 \le q \le +\infty)$. So, by an interpolation argument, we may restrict ourselves to the case $p = 2$.
- *Domination* arguments: In L^p spaces $(1 < p < \infty)$, if O_i $(i = 1, 2)$ are two positive operators such that

$$O_1 f \le O_2 f \quad \forall f \in L^p_+$$

  and if O_2 is compact then O_1 is also compact; see [3]. So we may assume that V is compact, $\alpha = f = g = 1$ and $\sigma = 0$.
- Because of $\sigma = 0$

$$U(t)\varphi = \varphi(x-tv, v)1_{\{t\le s(x,v)\}}$$

  where $s(x,v) = \inf\{s > 0;\ x - sv \notin \Omega\}$. If $\tilde{\varphi} \in L^2(\mathbb{R}^n\times V)$ is the trivial extension of φ by zero outside $\Omega\times V$ then, for nonnegative φ

$$U(t)\varphi(x,v) \le \tilde{\varphi}(x-tv, v) \ \ \forall (x,v)\in\Omega\times V.$$

so $U(t)\varphi \leq \mathcal{R}U_\infty(t)\mathcal{E}\varphi$ where $\mathcal{E} : L^2(\Omega \times V) \to L^2(\mathbb{R}^n \times V)$ is the trivial extension operator, $\mathcal{R} : L^2(\mathbb{R}^n \times V) \to L^2(\Omega \times V)$ is the restriction operator and

$$U_\infty(t) : \psi \in L^2(\mathbb{R}^n \times V) \to \psi(x - tv, v) \in L^2(\mathbb{R}^n \times V).$$

- Thus

$$\begin{aligned} U_1(t) &= \int_0^t \mathcal{R}U_\infty(t-s)\mathcal{E}K\mathcal{R}U_\infty(s)\mathcal{E}ds \\ &= \mathcal{R}\left(\int_0^t U_\infty(t-s)\mathcal{E}K\mathcal{R}U_\infty(s)ds\right)\mathcal{E} \\ &\leq \mathcal{R}\left(\int_0^t U_\infty(t-s)KU_\infty(s)ds\right)\mathcal{E}. \end{aligned}$$

and we are led to deal with the compactness of

$$\mathcal{R}\int_0^t U_\infty(t-s)KU_\infty(s)ds : L^2(\mathbb{R}^n \times V) \to L^2(\Omega \times V).$$

- Note that

$$\int_0^t U_\infty(t-s)KU_\infty(s)\psi ds = \int_0^t ds \int_V \psi(x - (t-s)v - sv', v')\mu(v').$$

For any $\psi(.,.) \in L^2(\mathbb{R}^n \times V)$, we denote by

$$\hat{\psi}(\zeta, v) = \lim_{M\to\infty} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{|\zeta|\leq M} \psi(x, v)e^{-i\zeta.x} d\zeta$$

its partial Fourier transform with respect to *space* variable where the limit holds in $L^2(\mathbb{R}^n \times V)$ norm. Then

$$\|\psi\|^2_{L^2(\mathbb{R}^n\times V)} = \int_V \int_{\mathbb{R}^n} \left|\hat{\psi}(\zeta, v)\right|^2 d\zeta\mu(dv)$$

and

$$\psi(x, v) = \lim_{M\to\infty} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{|\zeta|\leq M} \hat{\psi}(\zeta, v)e^{i\zeta.x} d\zeta$$

where the limit holds in $L^2(\mathbb{R}^n \times V)$ norm.

Hence

$$\begin{aligned}
&\int_0^t U_\infty(t-s)KU_\infty(s)\psi ds \\
&\quad = \int_0^t ds \int_V \psi(x-(t-s)v-sv',v')\mu(dv') \\
&\quad = \lim_{M\to\infty} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{|\zeta|\leq M} e^{ix.\zeta} \int_0^t ds \int_V \hat{\psi}(\zeta,v) e^{-i((t-s)v+sv').\zeta} \mu(dv') \\
&\quad = \lim_{M\to\infty} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{|\zeta|\leq M} \int_V e^{ix.\zeta} \hat{\psi}(\zeta,v) \left(\int_0^t dse^{-i((t-s)v+sv').\zeta} ds \right) \mu(dv')
\end{aligned}$$

where the limit holds in $L^2(\mathbb{R}^n \times V)$ norm. For each $M > 0$, let

$$O_M : L^2(\mathbb{R}^n \times V) \to L^2(\mathbb{R}^n \times V)$$

$$\psi \to \int_{|\zeta|\leq M} \int_V e^{ix.\zeta} \hat{\psi}(\zeta,v) \left(\int_0^t dse^{-i((t-s)v+sv').\zeta} ds \right) \mu(dv').$$

We observe that $\mathcal{R}O_M$ is a Hilbert Shmidt operator because Ω has finite volume and V is compact (keep in mind that $x \in \Omega$!). It suffices to show that

$$O_M\psi \to \int_0^t ds \int_V \psi(x-(t-s)v-sv',v')\mu(dv')$$

in $L^2(\mathbb{R}^n \times V)$ *uniformly* in $\|\psi\|_{L^2(\mathbb{R}^n\times V)} \leq 1$, i.e.

$$\int_{|\zeta|>M} e^{ix.\zeta} \int_V \hat{\psi}(\zeta,v') \left(\int_0^t e^{-i((t-s)v+sv').\zeta} ds \right) \mu(dv') \to 0$$

in $L^2(\mathbb{R}^n \times V)$ uniformly in $\|\psi\|_{L^2(\mathbb{R}^n\times V)} \leq 1$. By the Parseval identity, this amounts to

$$\int_V \mu(dv) \int_{|\zeta|>M} d\zeta \left| \int_V \hat{\psi}(\zeta,v') \left(\int_0^t e^{-i((t-s)v+sv').\zeta} ds \right) \mu(dv') \right|^2 \to 0$$

uniformly in $\|\psi\|_{L^2(\mathbb{R}^n\times V)} \leq 1$. Using Cauchy-Schwarz inequality, we majorize by

$$\begin{aligned}
&\sup_{|\zeta|>M} \int_{V\times V} \mu(dv')\mu(dv) \left| \int_0^t e^{-i((t-s)v+sv').\zeta} ds \right|^2 \int_{\mathbb{R}^n\times V} \left| \hat{\psi}(\zeta,v') \right|^2 \\
&\quad \leq \sup_{|\zeta|>M} \int_{V\times V} \mu(dv')\mu(dv) \left| \int_0^t e^{-i((t-s)v+sv').\zeta} ds \right|^2 \quad \forall\ \|\psi\| \leq 1.
\end{aligned}$$

Now

$$\int_{V\times V}\mu(dv')\mu(dv)\left|\int_0^t e^{-i((t-s)v+sv').\zeta}ds\right|^2$$
$$=\int_{V\times V}\mu(dv')\mu(dv)\left|\int_0^t e^{is(v-v').\zeta}ds\right|^2$$
$$=\int_{V\times V}\mu(dv')\mu(dv)\left|\int_0^t e^{is|\zeta|(v-v').e}ds\right|^2$$

where $\zeta=|\zeta|\,e,\ (e\in S^{n-1})$ and

$$\int_{V\times V}\mu(dv')\mu(dv)\left|\int_0^t e^{is|\zeta|(v-v').e}ds\right|^2$$
$$=\int_{\{|(v-v').e|\le\varepsilon\}}\mu(dv')\mu(dv)\left|\int_0^t e^{is|\zeta|(v-v').e}ds\right|^2$$
$$+\int_{\{|(v-v').e|>\varepsilon\}}\mu(dv')\mu(dv)\left|\int_0^t e^{is|\zeta|(v-v').e}ds\right|^2$$
$$\le t^2\int_{\{|(v-v').e|\le\varepsilon\}}\mu(dv')\mu(dv)$$
$$+\int_{\{|(v-v').e|>\varepsilon\}}\mu(dv')\mu(dv)\left|\int_0^t e^{is|\zeta|(v-v').e}ds\right|^2.$$

The first term can made arbitrarily small for ε small enough (assumption on the velocity measure μ) and, for ε fixed, the second term goes to zero as $|\zeta|\to\infty$ because of $\int_0^t e^{is|\zeta|(v-v').e}ds$ (Riemann-Lebesgue lemma). This ends the proof. ■

Corollary 11 *We assume that Ω has finite Lebesgue measure, the scattering operator is regular in $L^p(\Omega\times V;dx\otimes\mu(dv))$ $(1<p<\infty)$ and the hyperplanes have zero μ-measure. Then*

$$\sigma_{ess}(V(t))=\sigma_{ess}(U(t)).$$

In particular $\omega_e(V)=\omega_e(U)$ and $\sigma(V(t))\cap\left\{\beta;\ |\beta|>e^{s(T)t}\right\}$ consists at most of isolated eigenvalues with finite algebraic multiplicities.

Here $\omega_e(U)$ denotes the essential type of $(U(t))_{t\geqslant 0}$ etc. The assumption on the velocity measure μ is "optimal":

Theorem 31 ([51]) *Let μ be finite, Ω bounded and*

$$K:\varphi\in L^2(\Omega\times V)\to\int_V\varphi(x,v)\mu(dv).$$

Let there exist a hyperplane $H = \{v; v.e = c\}$ $(e \in S^{n-1},\ c \in \mathbb{R})$ *with positive* μ*-measure. Then there exists* $\bar{t} > 0$ *such that* $V(t) - U(t)$ *is not compact on* $L^2(\Omega \times V)$ *for* $0 < t \leq \bar{t}$.

and

Theorem 32 ([51]) *In the general setting above, if every ball centred at zero contains at least a section (by a hyperplane) with positive* μ*-measure then* $V(t) - U(t)$ *is not compact on* $L^2(\Omega \times V)$ *for all* $t > 0$.

The assumption that the scattering operator is regular is "nearly optimal":

Theorem 33 ([51]) *Let* μ *be an arbitrary positive measure. We assume that its support* V *is bounded. If* $V(t) - U(t)$ *is compact on* $L^p(\Omega \times V)$ *for all* $t > 0$ *then, for any open ball* $B \subset \Omega$*, the strong integral*

$$\int_B K(x)dx$$

is a compact operator on $L^p(V)$.

Corollary 12 *Besides the conditions of Theorem 33, we assume that*

$$x \in \Omega \to K(x) \in \mathcal{L}(L^p(V))$$

is measurable (not simply strongly measurable!) e.g. is piecewise continuous in operator norm. Then $K(x)$ *is a compact operator on* $L^p(V)$ *for almost all* $x \in \Omega$.

Proof

$$\frac{1}{|B|}\int_B K(x)dx \to K(x) \text{ in } \mathcal{L}(L^p(V)) \text{ as } |B| \to 0$$

at the Lebesgue points x of $K : \Omega \to \mathcal{L}(L^p(V))$. ■

5.2 L^1 Theory

As noted previously, L^1 space is the physical setting for neutron transport because

$$\int_\Omega \int_V f(x, v, t)dx\mu(dv)$$

is the expected number of particles. The L^1 mathematical results are very different from those in L^p theory $(p > 1)$ and the analysis is much more involved! Weak compactness is a fundamental tool for spectral theory of neutron transport in L^1 spaces. To this end, we recall first some useful results. Let (E, m) be a σ-finite measure space. A bounded subset $B \subset L^1(E,m)$ is relatively weakly compact if

$$\sup_{f \in B} \int_A |f|\,dm \to 0 \text{ as } m(A) \to 0$$

and (if $m(E) = \infty$) there exists measurable sets $E_n \subset E$, $m(E_n) < +\infty$, $E_n \subset E_{n+1}$, $\cup E_n = E$ such that

$$\sup_{f \in B} \int_{E_n^c} |f|\,dm \to 0 \text{ as } n \to \infty.$$

A bounded subset $B \subset L^1(E,m)$ is relatively weakly compact if and only if B is relatively sequentially weakly compact. A bounded operator G on $L^1(E,m)$ is said to be weakly compact if G sends a bounded set into a relatively weakly compact one. If $G_i : L^1(E,m) \to L^1(E,m)$ $(i = 1,2)$ are positive operators and $G_1 f \leq G_2 f \ \ \forall f \in L^1_+(E,m)$ then G_1 is weakly compact if G_2 is; this follows easily from the above criterion of weak compactness. If G_i $(i = 1,2)$ are two weakly compact operators on $L^1(E,m)$ then $G_1 G_2$ is a compact operator [18]. We are going to present (weak) compactness results for neutron transport operators and show their spectral consequences. We give here an overview of [52]. We treat first a model case where

$$\sigma(x,v) = 0, \ \ \mu \text{ is finite and } k(x,v,v') = 1.$$

We start with *negative* results:

Theorem 34 ([52]) *Let $\Omega \subset \mathbb{R}^n$ be an open set with finite Lebesgue measure and μ an arbitrary finite positive Borel measure on $\mathbb{R}^n$ with support V. Let $n \geqslant 2$ (or $n = 1$ and $0 \in V$). Let*

$$K : \varphi \in L^1(\Omega \times V) \to \int_V \varphi(x,v)\mu(dv) \in L^1(\Omega).$$

Then $K(\lambda - T)^{-1}$ is not weakly compact.

Proof We can assume without loss of generality that $0 \in \Omega$. Consider just the case $n \geqslant 2$. Let $(f_j)_j \subset C_c(\Omega \times V)$ a normalized sequence in $L^1(\Omega \times V)$ converging in the weak star topology of measures to the Dirac mass $\delta_{(0,\overline{v})}$. Then for any $\psi \in C_c(\Omega)$

$$\begin{aligned}
\langle K(\lambda - T)^{-1} f_j, \psi \rangle &= \int_\Omega \psi(x)dx \int_V \mu(dv) \int_0^{s(x,v)} e^{-\lambda t} f_j(x - tv, v)dt \\
&= \int_\Omega \int_V f_j(y,v) \left[\int_0^{s(y,-v)} e^{-\lambda t} \psi(y + tv)dt \right] dy\mu(dv) \\
&\to \int_0^{s(0,-\overline{v})} e^{-\lambda t} \psi(t\overline{v})dt
\end{aligned}$$

i.e. $K(\lambda - T)^{-1} f_j$ tends, in the weak star topology of measures, to a (non-trivial) Radon measure

$$m : \psi \in C_c(\Omega) \to \int_0^{s(0,-\overline{v})} e^{-\lambda t} \psi(t\overline{v}) dt$$

with support included in a segment. Hence $m \notin L^1(\Omega)$ and $K(\lambda - T)^{-1}$ is not weakly compact. ■

We note that this property was noted for the first time in the whole space in [23]. We have also:

Theorem 35 ([52]) *Let $n \geqslant 3$ and let $\Omega \subset \mathbb{R}^n$ be an open set with finite Lebesgue measure. Let μ be an arbitrary finite positive Borel measure on $\mathbb{R}^n$ with support V and*

$$K : \varphi \in L^1(\Omega \times V) \to \int_V \varphi(x, v)\mu(dv) \in L^1(\Omega).$$

Then

(i) $(\lambda - T - K)^{-1} - (\lambda - T)^{-1}$ *is not weakly compact.*
(ii) $V(t) - U(t)$ *is not weakly compact.*

We note that this theorem is false for $n = 1$ while the case $n = 2$ is open, see [52]. We recall now a necessary condition on μ.

Theorem 36 ([52]) *We assume that the velocity measure is invariant under the symmetry $v \to -v$. Let there exist $m \in \mathbb{N}$ such that*

$$\left[K(\lambda - T)^{-1}\right]^m$$

is compact on $L^1(\Omega \times V)$. Then the hyperplanes (through zero) have zero μ-measure.

We note that for any $m \in \mathbb{N}$

$$\left[K(\lambda - T)^{-1} K\right]^m \leq \mathcal{R} \left[K(\lambda - T_\infty)^{-1} K\right]^m \mathcal{E}$$

where

$$\mathcal{E} : L^1(\Omega \times V) \to L^1(\mathbb{R}^n \times V)$$

is the trivial extension operator,

$$\mathcal{R} : L^1(\mathbb{R}^n \times V) \to L^1(\Omega \times V)$$

is the restriction operator and T_∞ is the generator of

$$U_\infty(t) : \psi \in L^1(\mathbb{R}^n \times V) \to \psi(x - tv, v) \in L^1(\mathbb{R}^n \times V).$$

We start with a fundamental observation:

Lemma 9 *Let μ be an arbitrary finite positive measure on $\mathbb{R}^n$ with support V and*

$$K : \varphi \in L^1(\Omega \times V) \to \int_V \varphi(x, v)\mu(dv) \in L^1(\Omega).$$

For any $\lambda > 0$ there exists a finite positive measure β on $\mathbb{R}^n$ (depending on λ) such that

$$K(\lambda - T_\infty)^{-1} K\varphi = \beta * K\varphi.$$

Moreover,

$$\hat{\beta}(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{\mu(dv)}{\lambda + i\zeta.v}.$$

Proof

$$\begin{aligned} K(\lambda - T_\infty)^{-1} K\varphi &= \int_{\mathbb{R}^n} \mu(dv) \int_0^\infty e^{-\lambda t} (K\varphi)(x - tv) dt \\ &= \int_0^\infty e^{-\lambda t} dt \int_{\mathbb{R}^n} (K\varphi)(x - tv)\mu(dv) \\ &= \int_0^\infty e^{-\lambda t} dt \int_{\mathbb{R}^n} (K\varphi)(x - z)\mu_t(dz) \\ &= \int_0^\infty e^{-\lambda t} \left[\mu_t * K\varphi\right] dt \end{aligned}$$

where μ_t is the image of μ under the dilation $v \to tv$. So

$$K(\lambda - T_\infty)^{-1} K\varphi = \beta * K\varphi$$

where $\beta := \int_0^\infty e^{-\lambda t} \mu_t dt$ (strong integral). Finally $\hat{\beta}(\zeta)$ is given by

$$\begin{aligned} \int_0^\infty e^{-\lambda t} \hat{\mu_t}(\zeta) dt &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_0^\infty e^{-\lambda t} \left[\int_{\mathbb{R}^n} e^{-i\zeta.v} \mu_t(dv)\right] dt \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_0^\infty e^{-\lambda t} \left[\int_{\mathbb{R}^n} e^{-it\zeta.v} \mu(dv)\right] dt \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{\mu(dv)}{\lambda + i\zeta.v}. \end{aligned}$$

This ends the proof. ■

Remark 4 $\hat{\beta}(\zeta) = \int_{\mathbb{R}^n} \frac{\mu(dv)}{\lambda + i\zeta.v} \to 0$ for $|\zeta| \to \infty$ if and only if the hyperplanes (through zero) have zero μ-measure; see [52]. We are going to show that the compactness results rely on how fast $\hat{\beta}(\zeta)$ goes to zero as $|\zeta| \to \infty$.

Theorem 37 ([52]) *We assume that $\Omega \subset \mathbb{R}^n$ is an open set with finite Lebesgue measure. Let the velocity measure μ be finite and such that*

$$\int_{\mathbb{R}^n} \left|\hat{\beta}(\zeta)\right|^{2m} d\zeta < +\infty$$

for some $m \in \mathbb{N}$. Then $\left[K(\lambda - T)^{-1}K\right]^m$ is weakly compact in $L^1(\Omega \times V)$ and $\left[K(\lambda - T)^{-1}K\right]^{m+1}$ is compact in $L^1(\Omega \times V)$.

Proof We start from

$$\left[K(\lambda - T_\infty)^{-1}K\right]^m \varphi = \beta^{(m)} * K\varphi$$

where

$$\beta^{(m)} = \beta * \ldots . * \beta \ \ (m \text{ times}).$$

Since

$$\hat{\beta^{(m)}}(\zeta) = \left(\hat{\beta}(\zeta)\right)^m$$

then our assumption amounts to $\hat{\beta^{(m)}} \in L^2(\mathbb{R}^n)$. In particular $\beta^{(m)} \in L^2(\mathbb{R}^n)$ ($\beta^{(m)}$ is now a function!). It follows that

$$\left[K(\lambda - T_\infty)^{-1}K\right]^m \varphi = \beta^{(m)} * K\varphi \in L^2(\mathbb{R}^n)$$

and then $\left[K(\lambda - T_\infty)^{-1}K\right]^m$ maps continuously$L^1(\mathbb{R}^n \times V)$ into $L^2(\mathbb{R}^n)$. Hence

$$\mathcal{R}\left[K(\lambda - T_\infty)^{-1}K\right]^m : L^1(\mathbb{R}^n \times V) \to L^1(\Omega)$$

is weakly compact because the imbedding of $L^2(\Omega)$ into $L^1(\Omega)$ is weakly compact since Ω has finite Lebesgue measure (a bounded subset of $L^2(\Omega)$ is equi-integrable). Finally $\left[K(\lambda - T)^{-1}K\right]^m$ is also weakly compact by a domination argument. It follows that $\left[K(\lambda - T)^{-1}K\right]^{2m}$ is compact as a product of two weakly compact operators. Actually

$$\left[K(\lambda - T)^{-1}K\right]^{m+1} = K(\lambda - T)^{-1}K\left[K(\lambda - T)^{-1}K\right]^m$$

is compact because $K(\lambda - T)^{-1}$ is a *Dunford-Pettis* operator, see below. ■

We give now a geometrical condition on μ implying the compactness results.

Theorem 38 ([52]) *Let $\Omega \subset \mathbb{R}^n$ be an open set with finite Lebesgue measure. Let the velocity measure μ be finite and there exist $\alpha > 0$, $c > 0$ such that*

$$\sup_{e \in S^{n-1}} \mu\{v; |v.e| \leq \varepsilon\} \leq c\varepsilon^{\alpha}.$$

Then $\left[K(\lambda - T)^{-1}K\right]^{m+1}$ is compact in $L^1(\Omega \times V)$ for $m > \frac{n(\alpha+1)}{2\alpha}$.

Proof Note that $\hat{\beta}(\zeta)$ is a continuous function. According to the preceding theorem, we need just check the integrability of $\left|\hat{\beta}(\zeta)\right|^{2m}$ *at infinity*. Up to a factor $(2\pi)^{-\frac{n}{2}}$

$$\left|\hat{\beta}(\zeta)\right| = \left|\int_{\mathbb{R}^n} \frac{\mu(dv)}{\lambda + i\zeta.v}\right| \leq \int_{\mathbb{R}^n} \frac{\mu(dv)}{\sqrt{\lambda^2 + |\zeta|^2 |e.v|^2}}$$

where $e = \frac{\zeta}{|\zeta|}$. So, for any $\varepsilon > 0$,

$$\begin{aligned} \left|\hat{\beta}(\zeta)\right| &\leq \int_{\{|e.v|<\varepsilon\}} \frac{\mu(dv)}{\sqrt{\lambda^2 + |\zeta|^2 |e.v|^2}} + \int_{\{|e.v|\geqslant\varepsilon\}} \frac{\mu(dv)}{\sqrt{\lambda^2 + |\zeta|^2 |e.v|^2}} \\ &\leq \lambda^{-1}\mu\{|e.v| < \varepsilon\} + \frac{\|\mu\|}{|\zeta|\,\varepsilon} \leq \lambda^{-1}c\varepsilon^{\alpha} + \frac{\|\mu\|}{|\zeta|\,\varepsilon}. \end{aligned}$$

Optimizing with respect to ε yields

$$\left|\hat{\beta}(\zeta)\right|^{2m} \leq \frac{C}{|\zeta|^{\frac{2m\alpha}{\alpha+1}}}$$

for some positive constant C depending on λ. We are done if $\frac{2m\alpha}{\alpha+1} > n$ i.e. if $m > \frac{n(\alpha+1)}{2\alpha}$. ■

In the same spirit (but with more involved estimates) we can show:

Theorem 39 ([52]) *Let $\Omega \subset \mathbb{R}^n$ be an open set with finite Lebesgue measure. Let the velocity measure μ be finite and there exist $\alpha > 0$, $c > 0$ such that*

$$\sup_{e \in S^{n-1}} \mu \otimes \mu \left\{(v, v');\ \left|(v - v').e\right| \leq \varepsilon\right\} \leq c\varepsilon^{\alpha}.$$

Then $U_m(t)$ is weakly compact in $L^1(\Omega \times V)$ for all $t \geqslant 0$ and for $m \geqslant m_0$ where m_0 is the smallest odd integer greater than $\frac{n(\alpha+1)}{2\alpha} + 1$.

We point out that in Theorems 38 and 39, the condition on m *does not depend* on the constant c in the statement; this fact is fundamental if we want to pass from model cases to more general models. We show now how to treat by approximation more general velocity measures and scattering kernels. Indeed, in the approximation procedure, a general (a priori infinite) velocity measure μ is approximated, by truncation, by a sequence of finite measures μ_j such that

$$\sup_{e\in S^{n-1}} \mu_j \otimes \mu_j \left\{(v,v');\left|(v-v').e\right| \leq \varepsilon\right\} \leq c_j\varepsilon^{\alpha}$$

where α is *independent* of j.

A scattering operator K in $L^1(\Omega \times V)$ is said to be regular if $\{K(x);\ x \in \Omega\}$ is a set of *collectively weakly compact operators* on $L^1(V)$, i.e. the set

$$\left\{K(x)\varphi;\ x \in \Omega,\ \|\varphi\|_{L^1(V)} \leq 1\right\}$$

is relatively weakly compact in $L^1(V)$. This class of scattering kernels appears (with Lebesgue measure dv on $\mathbb{R}^n$) in P. Takak [72] and L. W. Weis [82]. See B. Lods [37] for the extension of P. Takak's construction to abstract velocity measures μ.

Theorem 40 ([37]) *Let K be a regular scattering operator in $L^1(\Omega \times V)$. Then there exists a sequence $(K_j)_j$ of scattering operators such that:*

(i) $0 \leq K_j \leq K$
(ii) $\left\|K - K_j\right\|_{\mathcal{L}(L^1(\Omega\times V))} \to 0$ *as* $j \to +\infty$.
(iii) For each K_j there exists $f_j \in L^1(V)$ such that

$$K_j\varphi \leq f_j(v)\int \varphi(x,v')\mu(dv')\ \ \forall\varphi \in L^1_+(\Omega \times V).$$

We are ready to show:

Theorem 41 ([52]) *Let $\Omega \subset \mathbb{R}^n$ be an open set with finite Lebesgue measure and let K be a regular scattering operator in $L^1(\Omega \times V)$. We assume that the velocity measure μ is such that: There exists $\alpha > 0$ such that for any $c_1 > 0$ there exists $c_2 > 0$ such that*

$$\sup_{e\in S^{n-1}} \mu\{v;|v| \leq c_1,\ |v.e| \leq \varepsilon\} \leq c_2\varepsilon^{\alpha}.$$

Then the components of $\rho_e(T)$ and $\rho_e(T+K)$ containing a right half-plane coincide. In particular

$$\sigma(T+K)\cap\{Re\lambda > s(T)\}$$

consists at most of isolated eigenvalues with finite algebraic multiplicities where

$$s(T) = -\lim_{t\to+\infty}\inf_{(y,v)}\frac{1}{t}\int_0^t \sigma(y+\tau v,v)d\tau.$$

Proof Let us show that $\left[K(\lambda - T)^{-1}K\right]^{m+1}$ is weakly compact in $L^1(\Omega \times V)$ for $m > \frac{n(\alpha+1)}{2\alpha}$. We fix $m > \frac{n(\alpha+1)}{2\alpha}$. It suffices to show the weak compactness of $\left[K_j(\lambda - T)^{-1}K_j\right]^{m+1}$ for all j (K_j from Theorem 40). By domination, we may replace K_j by $\hat{K}_j$ where

$$\widehat{K_j}\varphi = f_j(v)\int \varphi(x, v')\mu(dv').$$

By approximation again we may suppose that f_j is continuous with compact support. Let j be fixed and denote by V_j the support of f_j. We note that $\left[\widehat{K_j}(\lambda - T)^{-1}\widehat{K_j}\right]^m$ leaves invariant $L^1(\Omega \times V_j)$ and $\widehat{K_j}$ maps $L^1(\Omega \times V)$ into $L^1(\Omega \times V_j)$. By replacing $f_j(v)$ by its supremum, the model case (dealt with previously) in $L^1(\Omega \times V_j)$ insures that $\left[K_j(\lambda - T)^{-1}K_j\right]^m$ is weakly compact on $L^1(\Omega \times V_j)$ so

$$\left[K_j(\lambda - T)^{-1}K_j\right]^{m+1} = \left[K_j(\lambda - T)^{-1}K_j\right]^m \left[K_j(\lambda - T)^{-1}K_j\right]$$

is weakly compact on $L^1(\Omega \times V)$. Finally some power of $K(\lambda - T)^{-1}$ is compact on $L^1(\Omega \times V)$ and we conclude by the general theory. ■

Theorem 42 ([52]) *Let $\Omega \subset \mathbb{R}^n$ be an open set with finite Lebesgue measure and let K be a regular scattering operator in $L^1(\Omega \times V)$. We assume that the velocity measure μ is such that: There exists $\alpha > 0$ such that for any $c_1 > 0$ there exists $c_2 > 0$ such that*

$$\sup_{e \in S^{n-1}} \mu \otimes \mu \left\{(v, v'); |v|, \left|v'\right| \leq c_1 \, , \, \left|(v - v').e\right| \leq \varepsilon\right\} \leq c_2 \varepsilon^\alpha.$$

Then $(V(t))_{t\geqslant 0}$ and $(U(t))_{t\geqslant 0}$ have the same essential type. In particular

$$\sigma(V(t)) \cap \left\{\alpha \in \mathbb{C};\ |\alpha| > e^{s(T)t}\right\}$$

consists at most of isolated eigenvalues with finite algebraic multiplicities.

Proof Let us show that $R_{m+1}(t)$ is weakly compact in $L^1(\Omega \times V)$ for all $t \geqslant 0$ and for $m \geqslant m_0$ where m_0 is the smallest odd integer greater than $\frac{n(\alpha+1)}{2\alpha} + 1$. We fix $m \geqslant m_0$. It suffices to show that $R^j_{m+1}(t)$ is weakly compact in $L^1(\Omega \times V)$ for all j where $R^j_{m+1}(t)$ is the remainder term of order $m+1$ corresponding to a perturbation K_j in place of K (K_j from Theorem 40). By domination, we may replace K_j by $\widehat{K_j}$ where

$$\widehat{K_j}\varphi = f_j(v)\int \varphi(x, v')\mu(dv').$$

By approximation again we may suppose that f_j is continuous with compact support. Let j be fixed and denote by V_j the support of f_j. We note that $R_m^j(t)$ leaves invariant $L^1(\Omega \times V_j)$ and $\widehat{K_j}$ *maps* $L^1(\Omega \times V)$ *into* $L^1(\Omega \times V_j)$. Note also that

$$R_{m+1}^j = R_m^j * \left[\widehat{K_j} U\right] = \int_0^t R_m^j(t-s)\widehat{K_j}U(s)ds.$$

By dominating f_j by its supremum, the previous model case (and a domination argument) shows that $R_m^j(t-s)$ is weakly compact in $L^1(\Omega \times V_j)$. Thus $R_m^j(t-s)\widehat{K_j}U(s)$ is weakly compact in $L^1(\Omega \times V)$ and then so is R_{m+1}^j as a strong integral of "weakly compact operator" valued mapping. We conclude by the general theory. ■

Remark 5 The conditions on μ in Theorems 41 and 42 are satisfied e.g. for Lebesgue measure on $\mathbb{R}^n$ or on spheres (multigroup models).

5.3 Dunford-Pettis Operators in Transport Theory

A bounded operator $G \in \mathcal{L}(L^1(\nu))$ is called a Dunford-Pettis (or a completely continuous) operator if G maps a weakly compact subset into a (norm) compact subset. For instance, a weakly compact operator on $L^1(\nu)$ is Dunford-Pettis; this explains why the product of two weakly compact operators on $L^1(\nu)$ is compact. More generally, if G_2 is weakly compact on $L^1(\nu)$ and G_1 is Dunford-Pettis on $L^1(\nu)$ then G_1G_2 is compact on $L^1(\nu)$. We have seen that various relevant operators for neutron transport are *not* weakly compact in $L^1(\Omega \times V)$, for instance:

$$K(\lambda - T)^{-1},\ (\lambda - T - K)^{-1} - (\lambda - T)^{-1} \text{ and } V(t) - U(t).$$

We can show however that they are *all* Dunford-Pettis, (see [50] for more information). This explains why we claimed in the proof of Theorem 37 that

$$\left[K(\lambda - T)^{-1}K\right]^{m+1} = K(\lambda - T)^{-1}K\left[K(\lambda - T)^{-1}K\right]^m$$

is compact since $\left[K(\lambda - T)^{-1}K\right]^m$ is weakly compact and $K(\lambda - T)^{-1}$ is Dunford-Pettis. We restrict ourselves to:

Theorem 43 ([50]) *Let $\Omega \subset \mathbb{R}^n$ be an open set with finite Lebesgue measure and let K be a regular scattering operator in $L^1(\Omega \times V)$. If the affine hyperplanes have zero μ-measure then $V(t) - U(t)$ is a Dunford-Pettis operator on $L^1(\Omega \times V)$.*

Proof We note first that $V(t) - U(t) = \sum_1^\infty U_j(t)$ is Dunford-Pettis for all $t \geqslant 0$ if and only if $U_1(t)$ is [50]. By approximation, we may assume that their exists f

continuous with support in $\{|v| < \overline{c}\}$ such that

$$K\varphi \leq f(v) \int \varphi(x, v')\mu(dv') \ \ \forall \varphi \in L^1_+(\Omega \times V).$$

Let $E \subset L^1(\Omega \times V)$ be relatively weakly compact. In particular

$$\sup_{\varphi \in E} \int_{|v| \geqslant c} \mu(dv) \int_{\Omega} |\varphi(x, v)|\, dx \to 0 \ \text{ as } \ c \to +\infty.$$

We decompose $\varphi \in E$ as

$$\varphi = \varphi\chi_{\{|v|<c\}} + \varphi\chi_{\{|v|\geqslant c\}}$$

so

$$\left\| U_1(t)(\varphi\chi_{\{|v|\geqslant c\}}) \right\| \leq \|U_1(t)\| \left\| \varphi\chi_{\{|v|\geqslant c\}} \right\| \to 0 \ \text{ as } \ c \to +\infty$$

uniformly in $\varphi \in E$. On the other hand $\varphi\chi_{\{|v|<c\}}$ is zero for $|v| > c$ and (for $c > \overline{c}$) for any $\psi \in L^1(\Omega \times V)$, $K\psi$ is zero for $|v| > c$. So we may assume from the beginning that V is bounded and then K maps also $L^2(\Omega \times V)$ into itself. We decompose $\varphi \in E$ as

$$\varphi = \varphi^1_\alpha + \varphi^2_\alpha := \varphi\chi_{\{|\varphi|<\alpha\}} + \varphi\chi_{\{|\varphi|\geqslant\alpha\}}$$

and note that

$$\int |\varphi| \geqslant \int_{\{|\varphi|\geqslant\alpha\}} |\varphi| \geqslant \alpha \left(dx \otimes \mu \{|\varphi| \geqslant \alpha\}\right)$$

so

$$dx \otimes \mu \{|\varphi| \geqslant \alpha\} \to 0 \ \text{ as } \alpha \to +\infty.$$

The equi-integrability of E implies that

$$\left\|\varphi^2_\alpha\right\|_{L^1} \to 0 \ \text{ as } \alpha \to +\infty$$

uniformly in $\varphi \in E$ and finally $\left\|U_1(t)\varphi^2_\alpha\right\|_{L^1} \to 0$ as $c \to +\infty$ uniformly in $\varphi \in E$. Since, $\left\{\varphi^1_\alpha\right\}$ is bounded in L^1 and in L^∞ then the interpolation inequality

$$\left\|\varphi^1_\alpha\right\|_{L^2} \leq \left\|\varphi^1_\alpha\right\|^{\frac{1}{2}}_{L^1} \left\|\varphi^1_\alpha\right\|^{\frac{1}{2}}_{L^\infty}$$

shows that $\{\varphi_\alpha^1\}$ is also bounded in $L^2(\Omega \times V)$. We know (from the L^2 theory) that $U_1(t)$ is compact in $L^2(\Omega \times V)$ so $\{U_1(t)\varphi_\alpha^1\}$ is relatively compact in $L^2(\Omega \times V)$ and then relatively compact in $L^1(\Omega \times V)$ since $\Omega \times V$ has a finite measure. Thus $\{U_1(t)\varphi;\ \varphi \in E\}$ is as close to a relatively compact subset of $L^1(\Omega \times V)$ as we want and finally $\{U_1(t)\varphi;\ \varphi \in E\}$ is relatively compact. ■

6 Comments

6.1 *Measure Convolution Operators in Transport Theory*

The compactness results for neutron transport operators in L^p spaces with $1 < p < \infty$ (see Subsection 5.1) can also be derived from the analysis of just two particular measure convolution operators on $\mathbb{R}^n$, see [59].

6.2 *Unbounded Geometries*

The compactness results given in this lecture in spatial domains Ω with finite Lebesgue measure need no be true in general domains, e.g. the results are false in the whole space (i.e. $\Omega = \mathbb{R}^n$) and *space homogeneous* cross-sections. However, under suitable assumptions on the cross-sections, we can recover the compactness results above in unbounded geometries [60]. Actually, for general geometries and cross-sections, the relevant perturbation theory does not concern the essential spectra (and the essential types) but rather the *critical* spectra (and the critical types); we refer to [10, 53, 54, 59, 62, 67] for the abstract theory and how to use it in the context of neutron transport theory.

6.3 *Leading Eigenvalue of Neutron Transport*

The time asymptotic behaviour of neutron transport semigroup is meaningful if the latter has a spectral gap or equivalently if its generator has a leading eigenvalue. This topic relies on peripheral spectral analysis of neutron transport. The first relevant question concerns irreducibility criteria of neutron transport semigroups for which we refer e.g. to [26, 47, 78] and [48, Chapter 5]. The second relevant question concerns the effective existence of leading eigenvalues; besides the isotropic case dealt with by [29], we refer to ([48, Chapter 5]) for general tools. Variational characterizations of the leading eigenvalue in L^p spaces ($InfSup$ or $SupInf$ criteria) and lower bounds of this eigenvalue are given in [55]. The criticality eigenvalue problem is dealt with in [41, 66] and [48, Chapter 5]. Finally, we refer to [40] for variational characterizations of the criticality eigenvalue.

6.4 Partly Elastic Scattering Operators

Most of the literature on spectral theory of neutron transport is devoted either to one speed models or to inelastic models. Despite their apparent difference, these two models can be covered by a unique general formalism as we did in Sect. 5. However, more complex models which take into account of both elastic and inelastic scatterings appear e.g. in [35]:

$$\frac{\partial f}{\partial t} + v.\frac{\partial f}{\partial x} + \sigma(x, v) f(t, x, v) = K_e f + K_i f$$

where

$$K_i f = \int_{\mathbb{R}^3} k(x, v, v') f(x, v') dv' \quad \text{(inelastic operator)}$$

and

$$K_e f = \int_{S^2} k(x, \rho, \omega, \omega') f(x, \rho\omega') dS(\omega') \text{ (elastic operator)}$$

where $v = \rho\omega$. The peculiarity of the elastic scattering operator is that it is *not* compact "in velocities" in contrast to usual inelastic scattering operators. This explains the complexity of $\sigma(T + K_e)$ which consists of a half-plane and various *"curves"*[35]. We find in [68] various compactness results (due to K_i) and spectral results. In particular the semigroups generated by $T + K_e$ and by $T + K_e + K_i$ have the same essential type.

6.5 Generalized Boundary Conditions

We point out that "zero incoming flux" is the natural (i.e. physical) boundary condition for neutron transport. However, various boundary conditions (relating e.g. the incoming and outgoing fluxes) were also considered in kinetic theory [11] or in structured cell population models (see e.g. [5]). The literature on the subject is considerable and we do not try to comment on it. We just note that the corresponding advection semigroup is no longer explicit and, for this reason, spectral analysis of such kinetic models with non local boundary conditions is much more technical. We refer for instance to [38] and references therein.

References

1. V.I. Agoshkov, Spaces of functions with differential-difference characteristics and smoothness of solutions of the transport equations. Sov. Math. Dokl. **29**, 662–666 (1987)
2. S. Albertoni, B. Montagnini, On the spectrum of neutron transport equations in finite bodies. J. Math. Anal. Appl. **13**, 19–48 (1966)
3. D. Aliprantis, O. Burkinshaw, Positive compact operators on Banach lattices. Math. Z **174** 289–298 (1980)
4. N. Angelescu, V. Protopopescu, On a problem in linear transport theory. Rev. Roum. Phys. **22**, 1055–1061 (1977)
5. O. Arino, Some spectral properties for the asymptotic behavior of semigroups connected to population dynamics. SIAM Rev. **34**(4), 445–476 (1992)
6. C. Bardos, R. Santos, R. Sentis, Diffusion approximation and computation of the critical size. Trans. Am. Math. Soc. **284**(2), 617–649 (1984)
7. G.I. Bell, Stochastic theory of neutron transport. J. Nucl. Sci. Eng. **21**, 390–401 (1965)
8. M. Borysiewicz, J. Mika, Time behaviour of thermal neutrons in moderating media. J. Math. Anal. Appl. **26**, 461–478 (1969)
9. S. Brendle, On the asymptotic behaviour of perturbed strongly continuous semigroups. Math. Nachr. **226**, 35–47 (2001)
10. S. Brendle, R. Nagel, J. Poland, On the spectral mapping theorem for perturbed strongly continuous semigroups. Archiv Math. **74**, 365–378 (2000)
11. C. Cercignani, *The Boltzmann Equation and Its Applications* (Springer, New York, 1988)
12. M. Cessenat, Théorèmes de trace L^p pour des espaces de fonctions de la neutronique. C. R. Acad. Sci. Paris Ser. I **299**, 831–834 (1984)
13. M. Cessenat, Théorèmes de trace pour les espaces de fonctions de la neutronique. C. R. Acad. Sci. Paris Ser. I **300**, 89–92 (1985)
14. R. Dautray, J.L. Lions (eds.), *Analyse mathématique et calcul numérique pour les sciences et les techniques*. Tome 1 (Masson, Paris, 1985)
15. E.-B. Davies, *One-Parameter Semigroups* (Academic Press, London, 1980)
16. M.L. Demeru, B. Montagnini, Complete continuity of the free gas scattering operator in neutron thermalization theory. J. Math. Anal. Appl. **12**, 49–57 (1965)
17. J.J. Duderstadt, W.R. Martin, *Transport Theory* (John Wiley & Sons, Inc, New York, 1979)
18. N. Dunford, J.T. Schwartz, *Linear Operators*, Part I (John Wiley & Sons, Inc, Interscience New York, 1958)
19. D.E. Edmunds, W.D. Evans, *Spectral Theory and Differential Operators* (Clarendon Press, Oxford, 1989)
20. K.-J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations* (Springer, New York, 2000)
21. I. Gohberg, S. Goldberg, M.A. Kaashoek, *Classes of Linear Operators*, vol. 1 (Birkhauser, Basel, 1990)
22. J.A. Goldstein, *Semigroups of Linear Operators and Applications*. Oxford Mathematical Monographs (Oxford University Press, Oxford, 1985)
23. F. Golse, P.L. Lions, B. Perthame, R. Sentis, J. Funct. Anal. **76**, 110–125 (1988)
24. G. Greiner, Zur Perron-Frobenius Theorie stark stetiger Halbgruppen. Math. Z. **177**, 401–423 (1981)
25. G. Greiner, Spectral properties and asymptotic behavior of the linear transport equation. Math. Z. **185**, 167–177 (1984)
26. G. Greiner, An irreducibility criterion for the linear transport equation, in *Semesterbericht Funktionalanalysis* (University of Tubingen, 1984)
27. G. Greiner, A spectral decomposition of strongly continuous group of positive operators. Q. J. Math. Oxf. **35**, 37–47 (1984)
28. A. Huber, Spectral properties of the linear multiple scattering operator in L^1 Banach lattice. Int. Equ. Opt. Theory **6**, 357–371 (1983)

29. T. Hiraoka, S. Ukaï, Eigenvalue spectrum of the neutron transport operator for a fast multiplying system. J. Nucl. Sci. Technol. **9**(1), 36–46 (1972)
30. K. Jarmouni-Idrissi, M. Mokhtar-Kharroubi, A class of non-linear problems arising in the stochastic theory of neutron transport. Nonlinear Anal. **31**(3–4), 265–293 (1998)
31. K. Jorgens, Commun. Pure. Appl. Math. **11**, 219–242 (1958)
32. T. Kato, *Perturbation Theory of Linear Operators* (Springer, Berlin, 1984)
33. M. Kunze, G. Schluchtermann, Strongly generated Banach spaces and measures of non-compactness. Math. Nachr. **191**, 197–214 (1998)
34. K. Latrach, B. Lods, Spectral analysis of transport equations with bounce-back boundary conditions. Math. Methods Appl. Sci. **32**, 1325–1344 (2009)
35. E.W. Larsen, P.F. Zweifel, On the spectrum of the linear transport operator. J. Math. Phys. **15**, 1987–1997 (1974)
36. J. Lehner, M. Wing, On the spectrum of an unsymmetric operator arising in the transport theory of neutrons. Commun. Pure. Appl. Math. **8**, 217–234 (1955)
37. B. Lods, On linear kinetic equations involving unbounded cross-sections. Math. Methods Appl. Sci. **27**, 1049–1075 (2004)
38. B. Lods, M. Sbihi, Stability of the essential spectrum for 2D-transport models with Maxwell boundary conditions. Math. Methods Appl. Sci. **29**, 499–523 (2006)
39. B. Lods, M. Mokhtar-Kharroubi, M. Sbihi, Spectral properties of general advection operators and weighted translation semigroups. Commun. Pure. Appl. Anal. **8**(5), 1–24 (2009)
40. B. Lods, Variational characterizations of the effective multiplication factor of a nuclear reactor core. Kinet. Relat. Models **2**, 307–331 (2009)
41. J. Mika, Fundamental eigenvalues of the linear transport equation. J. Quant. Spectr. Radiat. Transf. **11**, 879–891 (1971)
42. M. Mokhtar-Kharroubi, La compacité dans la théorie du transport des neutrons. C. R. Acad. Sci. Paris Ser. I **303**, 617–619 (1986)
43. M. Mokhtar-Kharroubi, Les équations de la neutronique: Positivité, Compacité, Théorie spectrale, Comportement asymptotique en temps. Thèse d'Etat (French habilitation), Paris (1987)
44. M. Mokhtar-Kharroubi, Compactness properties for positive semigroups on Banach lattices and applications. Houston J. Math. **17**(1), 25–38 (1991)
45. M. Mokhtar-Kharroubi, Time asymptotic behaviour and compactness in transport theory. Eur. J. Mech. B Fluids **11**(1), 39–68 (1992)
46. M. Mokhtar-Kharroubi, On the stochastic nonlinear neutron transport equation. Proc. R. Soc. Edinb. **121**(A), 253–272 (1992)
47. M. Mokhtar-Kharroubi, Quelques applications de la positivité en théorie du transport. Ann. Fac. Toulouse **11**(1), 75–99 (1990)
48. M. Mokhtar-Kharroubi, *Mathematical Topics in Neutron Transport Theory, New Aspects*, vol. 46 (World Scientific, Singapore, 1997)
49. M. Mokhtar-Kharroubi, L. Thevenot, On the diffusion theory of neutron transport on the torus. Asymp. Anal. **30**(3–4), 273–300 (2002)
50. M. Mokhtar-Kharroubi, On the convex compactness property for the strong operator topology and related topics. Math. Methods Appl. Sci. **27**(6), 687–701 (2004)
51. M. Mokhtar-Kharroubi, Optimal spectral theory of the linear Boltzmann equation. J. Funct. Anal. **226**, 21–47 (2005)
52. M. Mokhtar-Kharroubi, On L1-spectral theory of neutron transport. Differ. Int. Equ. **18**(11), 1221–1242 (2005)
53. M. Mokhtar-Kharroubi, M. Sbihi, Critical spectrum and spectral mapping theorems in transport theory. Semigroup Forum **70**(3), 406–435 (2005)
54. M. Mokhtar-Kharroubi, M. Sbihi, Spectral mapping theorems for neutron transport, L^1-theory. Semigroup Forum **72**, 249–282 (2006)
55. M. Mokhtar-Kharroubi, On the leading eigenvalue of neutron transport models. J. Math. Anal. Appl. **315**, 263–275 (2006)

56. M. Mokhtar-Kharroubi, Spectral properties of a class of positive semigroups on Banach lattices and streaming operators. Positivity **10**(2), 231–249 (2006)
57. M. Mokhtar-Kharroubi, F. Salvarani. Convergence rates to equilibrium for neutron chain fissions. Acta. Appl. Math. **113**, 145–165 (2011)
58. M. Mokhtar-Kharroubi, New generation theorems in transport theory. Afr. Mat. **22**, 153–176 (2011)
59. M. Mokhtar-Kharroubi, On some measure convolution operators in neutron transport theory. Acta Appl. Math. (2014). doi:10.1007/s10440-014-9866-3.
60. M. Mokhtar-Kharroubi, Compactness properties of neutron transport operators in unbounded geometries (work in preparation)
61. R. Nagel (ed.), *One-Parameter Semigroups of Positive Operators*. Lecture Notes in Mathematics, vol. 1184 (Springer, Berlin, 1986)
62. R. Nagel, J. Poland, The critical spectrum of a strongly continuous semigroup. Adv. Math **152**(1), 120–133 (2000)
63. B. de Pagter, Irreducible compact operators. Math. Z. **192**, 149–153 (1986)
64. A. Pazy, P. Rabinowitz, A nonlinear integral equation with applications to neutron transport theory. Arch. Rat. Mech. Anal. **32**, 226–246 (1969)
65. M. Ribaric, I. Vidav, Analytic properties of the inverse $A(z)^{-1}$ of an analytic linear operator valued function $A(z)$. Arch. Ration. Mech. Anal. **32**, 298–310 (1969)
66. R. Sanchez, The criticality eigenvalue problem for the transport with general boundary conditions. Transp. Theory Stat. Phys. **35**, 159–185 (2006)
67. M. Sbihi, A resolvent approach to the stability of essential and critical spectra of perturbed C_0-semigroups on Hilbert spaces with applications to transport theory. J. Evol. Equ. **7**(1), 35–58 (2007)
68. M. Sbihi, Spectral theory of neutron transport semigroups with partly elastic collision operators. J. Math. Phys. **47**, 123502 (2006).
69. G. Schluchtermann, On weakly compact operators. Math. Ann. **292**, 263–266 (1992)
70. G. Schluchtermann, Perturbation of linear semigroups, in *Recent Progress in Operator Theory, (Regensburg 1995)*. Oper. Theory Adv. Appl., vol. 103 (Birkhauser, Basel, 1998), pp. 263–277
71. Y. Shizuta, On the classical solutions of the Boltzmann equation. Commun. Pure. Appl. Math. **36**, 705–754 (1983)
72. P. Takak, A spectral mapping theorem for the exponential function in linear transport theory. Transp. Theory Stat. Phys. **14**(5), 655–667 (1985)
73. A.E. Taylor, D.C. Lay, *Introduction to Functional Analysis* (Krieger Publishing Company, Melbourne, 1980)
74. S. Ukaï, Eigenvalues of the neutron transport operator for a homogeneous finite moderator. J. Math. Anal. Appl. **30**, 297–314 (1967)
75. I. Vidav, Existence and uniqueness of nonnegative eigenfunctions of the Boltzmann operator. J. Math. Anal. Appl. **22**, 144–155 (1968)
76. I. Vidav, Spectra of perturbed semigroups with application to transport theory. J. Math. Anal. Appl. **30**, 264–279 (1970)
77. J. Voigt, A perturbation theorem for the essential spectral radius of strongly continuous semigroups. Monatsh. Math. **90**, 153–161 (1980)
78. J. Voigt, Positivity in time dependent linear transport theory. Acta Appl. Math. **2**, 311–331 (1984)
79. J. Voigt, Spectral properties of the neutron transport equation. J. Math. Anal. Appl. **106**, 140–153 (1985)
80. J. Voigt, On the convex compactness property for the strong operator topology. Note di Math. **12**, 259–269 (1992)
81. J. Voigt, Stability of essential type of strongly continuous semigroups. Proc. Steklov Inst. Math. **3**, 383–389 (1995)
82. L.W. Weis, A generalization of the Vidav-Jorgens perturbation theorem for semigroups and its application to transport theory. J. Math. Anal. Appl. **129** 6–23 (1988)