On the Stochastic Least Action Principle for the Navier-Stokes Equation

Ana Bella Cruzeiro and Remi Lassalle

Abstract In this paper we extend the class of stochastic processes allowed to represent solutions of the Navier-Stokes equation on the two dimensional torus to certain non-Markovian processes which we call admissible. More precisely, using the variations of Ref. [3], we provide a criterion for the associated mean velocity field to solve this equation. Due to the fluctuations of the shift a new term of pressure appears which is of purely stochastic origin. We provide an alternative formulation of this least action principle by means of transformations of measure. Within this approach the action is a function of the law of the processes, while the variations are induced by some translations on the space of the divergence free vector fields. Due to the renormalization in the definition of the cylindrical Brownian motion, our action is only related to the relative entropy by an inequality. However we show that, if we cut the high frequency modes, this new approach provides a least action principle for the Navier-Stokes equation based on the relative entropy.

Keywords Navier-Stokes · Entropy

1 Introduction

Let (W_t) be a suitably renormalized Brownian motion on the space of vector fields on the two dimensional torus $\mathbb{T}^2$ with a well chosen Sobolev regularity. In the case where (u_t) is a deterministic vector field, it was shown that equations of the form

$$dg_t = (\circ dW_t + u_t dt)(g_t);\ g_t = e \tag{1.1}$$

A.B. Cruzeiro (✉)
GFMUL and Department Matemática Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisbon, Portugal
e-mail: abcruz@math.ist.utl.pt

R. Lassalle
GFMUL Instituto para a Investigação Interdisciplinar da Universidade de Lisboa, Av. Prof. Gama Pinto, 2, 1649-003 Lisbon, Portugal
e-mail: lassalle@telecom-paristech.fr

D. Crisan et al. (eds.), *Stochastic Analysis and Applications 2014*,
Springer Proceedings in Mathematics & Statistics 100,
DOI 10.1007/978-3-319-11292-3_6

could model the Navier-Stokes flows (see for instance the review article [2] and references within). More precisely it was shown that (u_t) solves the Navier-Stokes equation if and only if a certain associated action is stationary. Subsequently, models of the form

$$dg_t = (\circ dW_t + \dot{v}_t(\omega)dt)(g_t);\; g_t = e \tag{1.2}$$

where considered in Ref. [1], together with a notion of generalized stochastic flows with fixed marginals. In these latter models, the shift $\dot{v}_t(\omega)$ is allowed to be random: the drift changes from one realization of the noise to another which seems to fit accurately with the microscopic models of the Navier-Stokes equation one encounters in physics. In particular such processes are not necessarily Markovian.

In the case of (1.2) there is no reason why we should hope $\dot{v}(\omega)$ to solve the Navier-Stokes equation for any ω *a.s.*, and we should focus on the mean velocity field

$$u : (t,x) \in [0,1] \times \mathbb{T}^2 \to u(t,x) = E_\eta[\dot{v}_t(x)] \in T_x\mathbb{T}^2$$

where η is the underlying probability on the canonical path space, and where $T_x\mathbb{T}^2$ is the tangent space at x.

We extend here the criterion of Ref. [2] from equations with the form (1.1) to equations of type (1.2) for a wide class of stochastic drifts. Namely we focus on drifts v associated with a probability η with finite entropy with respect to the law μ of the renormalized Brownian motion on the corresponding path space. We exhibit a class of such drifts (they will be called admissible) whose mean velocity field solves the Navier-Stokes equation if and only if the associated action, which will be noted $\mathcal{S}(\eta|\mu)$, is critical. We then prove that this notion naturally extends the variational principle of Ref. [2]. One of the aspects of this model is to allow that the fluctuations of the drift itself may contribute to the pressure. Then we provide an alternative formulation to the least action principle by means of transformation of measure. However in this case, due to the renormalization involved in the definition of the cylindrical Brownian motion, our action for a process with law η is only related to the corresponding relative entropy

$$\mathcal{H}(\eta|\mu) := E_\eta\left[\ln \frac{d\eta}{d\mu}\right]$$

by an inequality. Nevertheless, by introducing a cut-off, the action $\mathcal{S}(\nu|\eta)$ becomes proportional to the relative entropy, and by cutting the high modes, we provide a least action principle to the Navier-Stokes equation by means of the relative entropy.

The structure of this paper is the following. In Sect. 2 we introduce the general framework as well as the main notations of the paper. In Sect. 3 we provide a characterization of solutions of the Navier-Stokes equation as critical flows of the action. In Sect. 4 this criterion is proved to extend those of Refs. [2, 3]. In Sect. 5 we introduce a cut-off in order to transform variations of the action in variations of the entropy. (Sect. 6).

2 Preliminaries and Notations

2.1 A Basis of Vector Fields on the Two Dimensional Torus

Let $M := \mathbb{T}^2$ be the set of pairs (θ_1, θ_2) of real numbers modulo 2π, and denote $m_{\mathbb{T}} = \frac{\lambda^L \otimes \lambda^L}{4\pi^2}$ where λ^L is the Lebesgue measure on $[0, 2\pi]$. Integration will often be noted dx instead of $m_{\mathbb{T}}(dx)$. A basis of the tangent space T_xM at $x = (\theta_1, \theta_2) \in M$ is given by $(\partial_i|_x) := (\frac{\partial}{\partial\theta_i}|_{x=(\theta_1,\theta_2)})$. We define a scalar product $\langle ., . \rangle_{T_xM}$ on each T_xM by setting $\langle \partial_i|_x, \partial_j|_x \rangle_{T_xM} = \delta^{i,j}$ where $\delta^{i,j} = 1$ if $i = j$ and 0 if $i \neq j$. When there is no ambiguity, we will sometimes note $X.Y$ instead of $\langle X, Y \rangle_{T_xM}$ for $X, Y \in T_xM$. If $\mathcal{X}(M)$ consists of sections of TM, $\mathcal{X}(M) = \{X : M \to T(M)\}$, and considering its L^2 equivalence class, we set

$$\mathcal{G} = \left\{ X \in \mathcal{X}(M) |\ \operatorname{div}(X) = 0 \text{ and } \int_M |X(x)|^2_{T_xM} dx < \infty \right\}$$

which is a separable Hilbert space with the product

$$\langle X, Y \rangle_{\mathcal{G}} := \int_M \langle X(x), Y(x) \rangle_{T_xM} dx$$

An Hilbertian basis of $\mathcal{G}$ is given by a subset $(e_\alpha)_{\alpha=1}^{\infty}$, whose definition is the following. Let $k : \alpha \in \mathbb{N}/\{0\} \to k(\alpha) := (k_1(\alpha), k_2(\alpha)) \in (\mathbb{Z} \times \mathbb{Z})/\{(0, 0)\}$ be a bijection such that $|k(\alpha)| := \sqrt{k_1^2(\alpha) + k_2(\alpha)^2} \uparrow \infty$; we set

$$e_\alpha(x) := \sum_j a^{\alpha,j}(x) \partial_j|_x$$

where

$$a^{\alpha,i}(x) := \begin{cases} 1 & \text{if } (\alpha, i) \in (1, 1) \cup (2, 2) \\ 0 & \text{if } (\alpha, i) \in (2, 1) \cup (1, 2) \\ \sqrt{2}\frac{k_2(m)}{|k(m)|} \cos(k(m).x) & \text{if } (\alpha, i) = (2m + 2, 1), m \geq 1 \\ -\sqrt{2}\frac{k_1(m)}{|k(m)|} \cos(k(m).x) & \text{if } (\alpha, i) = (2m + 2, 2), m \geq 1 \\ \sqrt{2}\frac{k_2(m)}{|k(m)|} \sin(k(m).x) & \text{if } (\alpha, i) = (2m + 1, 1), m \geq 1 \\ -\sqrt{2}\frac{k_1(m)}{|k(m)|} \sin(k(m).x) & \text{if } (\alpha, i) = (2m + 1, 2), m \geq 1 \end{cases}$$

and where, for $k = (k_1, k_2) \in \mathbb{Z} \times \mathbb{Z}$ and $x = (\theta_1, \theta_2) \in M$, $k.x := k_1\theta_1 + k_2\theta_2$. Any $X \in \mathcal{G}$ can be written

$$X(x) = \sum_j X^j(x)\partial_j|_x$$

where $X_j(x) = \sum_\alpha \langle X, e_\alpha\rangle_{\mathcal{G}} a^{\alpha,j}(x)$. Let $Y(x) := \sum_j Y_j(x)\partial_j|_x$ be another vector field: it is straightforward to check that we also have

$$\langle X, Y\rangle_{\mathcal{G}} = \int_M \sum_j X^j(x)Y^j(x)dx$$

We recall the following formulae

$$\operatorname{div}(X) := \sum_j \partial_j X^j,$$

$$\Delta X := \sum_i (\sum_j \partial^2_{j,j} X_i)\partial_i|_x$$

and

$$(X.\nabla)Y := \sum_j (\sum_i X_i(\partial_i Y_j))\partial_j|_x$$

2.2 *The Group of the Volume Preserving Homeomorphisms*

Let G be the group of the homeomorphisms of M which leaves $m_{\mathbb{T}}$ invariant

$$G := \{\phi : M \to M, \textit{homeomorphisms}, \phi_\star m_{\mathbb{T}} = m_{\mathbb{T}}\}$$

We note e the identity on G and $\phi.\psi$ the group operation of $\phi, \psi \in G$ (given by the composition of the two maps). We recall [6] that the subset of G consisting of maps which are, together with their inverses, in the Sobolev class H^r, for $r > 2$ is a Hilbert manifold and a topological group. It is not, strictly speaking, a Lie algebra since left translation is not smooth. T_eG is given by the set of the vector fields $X : x \in M \to X_x \in T_xM$ such that $\operatorname{div}(X) = 0$. Let $X \in T_eG$, and let

$$c : t \in \mathbb{R} \to c_t \in G; c_0 = e$$

be a smooth curve on G to which X is tangent. We recall that, by setting $\widehat{c} : (t, x) \in \mathbb{R} \times M \to c_t(x) \in M$, the value of X at $x \in M$ is given by

$$X(x) = \partial_t \widehat{c}(t, x)|_{t=0} \in T_xM$$

Furthermore X can be uniquely extended to a right invariant vector field $\widehat{X}$ on G by setting

$$\widehat{X} : \phi \in G \to \widehat{X}_\phi \in T_\phi G$$

where $\widehat{X}_\phi$ is given by

$$\widehat{X}_\phi : x \in M \to \widehat{X}_\phi(x) := X(\phi(x)) \in T_{\phi(x)} M$$

i.e. $\widehat{X}_\phi$ is tangent to the curve $c^\phi : t \in \mathbb{R} \to c_t.\phi \in G$. In particular for any smooth f on M and $x \in M$ denote f^x the map $\phi \in G \to f^x(\phi) := f(\phi(x)) \in \mathbb{R}$. Then f^x is smooth on G and we have

$$(\widehat{X}f^x)(\phi) := \widehat{X}_\phi f^x = \partial_t f(c_t.\phi(x))|_{t=0} = \partial_t f(\widehat{c}(t, \phi(x)))|_{t=0} = X(\phi(x))f := (Xf)(\phi(x))$$

In the sequel we will simply write X instead of $\widehat{X}$ since it will be clear from the context whether we consider X as an element of the tangent space, or as a right-invariant vector field on G. In order to kill the noise in the higher modes and to control the integrability of the derivatives, we introduce the following Sobolev spaces $(\mathcal{G}_\lambda)_{\lambda>1}$ and the associated abstract Wiener spaces $(W, H_\lambda, \mu_\lambda)$.

2.3 Sobolev Vector Fields

To any positive real number $\lambda > 1$ we associate a sequence $(\lambda_\alpha)_{\alpha\in\mathbb{N}}$ defined by

$$\lambda_\alpha = \frac{|k([\frac{\alpha-1}{2}])|^{2\lambda}}{K(\lambda)}$$

where [.] is the floor function and where $K(\lambda)$ is chosen so that

$$\sum_\alpha \frac{a^{\alpha,i}(x)}{\sqrt{\lambda_\alpha}} \frac{a^{\alpha,j}(x)}{\sqrt{\lambda_\alpha}} = \delta^{i,j}$$

Such a $K(\lambda)$ exists from standard results on Riemann series since $\lambda > 1$, and we have $K(\lambda) \uparrow \infty$ as $\lambda \downarrow 1$. For $\lambda > 1$, let S_λ be the positive, definite, trace class operator defined by

$$S_\lambda x := \sum_i \frac{1}{\lambda_i} \langle x, e_i \rangle_{\mathcal{G}} e_i$$

and let

$$\mathcal{G}_\lambda := \sqrt{S_\lambda}(\mathcal{G})$$

which is an Hilbert space for the scalar product $\langle ., . \rangle_{\mathcal{G}_\lambda}$ characterized by

$$\left\langle \sqrt{S_\lambda} x, \sqrt{S_\lambda} y \right\rangle_{\mathcal{G}_\lambda} = \langle x, y \rangle_{\mathcal{G}}.$$

A natural Hilbertian basis of $\mathcal{G}_\lambda$ is given by $(H_\alpha^\lambda)_{\alpha=1}^\infty$ where

$$H_\alpha^\lambda := \frac{e_\alpha}{\sqrt{\lambda_\alpha}} \tag{2.3}$$

We set

$$A_{\alpha,j}^\lambda(x) = \frac{a_{\alpha,j}}{\sqrt{\lambda_\alpha}}$$

so that

$$\sum_\alpha A_{\alpha,i}^\lambda(x) A_{\alpha,j}^\lambda(x) = \delta^{i,j} \tag{2.4}$$

and

$$H_\alpha^\lambda(x) = \sum_j A_{\alpha,j}^\lambda(x) \partial_j|_x$$

Since $\sqrt{S}_\lambda$ is Hilbert-Schmidt, it is well known that $|.|_{\mathcal{G}}$ is a measurable semi-norm on the Hilbert space $\mathcal{G}_\lambda$ (see [9]). In particular $(\mathcal{G}_\lambda, \mathcal{G})$ is an abstract Wiener space [9, 12], which allows to regard the cylindrical Brownian motion below as a Brownian sheet (note that we could have defined a Wiener measure directly on the Wiener space $(\mathcal{G}_\lambda, \mathcal{G})$, but we won't use this in the sequel since we are interested in the path space).

2.4 Associated Wiener Spaces

The space

$$H_\lambda := \left\{ h : [0, 1] \to \mathcal{G}_\lambda : h := \int_0^{\cdot} \dot{h}_s ds, \int_0^1 |\dot{h}_s|_{\mathcal{G}_\lambda}^2 ds < \infty \right\}$$

is an Hilbert space whose product will be noted $\langle ., . \rangle_\lambda$. On the other hand the space

$$W := C_0 ([0, 1], \mathcal{G})$$

is a separable Banach space for the uniform convergence norm. We denote by i_λ the injection of H_λ in W. Since for $\lambda > 1$ $|.|_{\mathcal{G}}$ is a measurable semi-norm on $\mathcal{G}_\lambda$, it is a classical result on Wiener spaces that $(i_\lambda, W, H_\lambda)$ is also an abstract Wiener space. If μ_λ is the standard Wiener measure on W for the A.W.S. $(W, H_\lambda, i_\lambda)$, we recall that under this probability the coordinate process $t \to W_t(\omega) = \omega(t) \in \mathcal{G}$ is an abstract Brownian motion with respect to its own filtration $(\mathcal{F}_t)$ (see for instance [10, 12]). From the Itô Nisio theorem, we have $\mu_\lambda-$a.s.

$$W_t = \sum_\alpha W_t^\alpha H_\alpha^\lambda$$

with $W_t^\alpha := \widehat{\delta} H_\alpha(W_t)$, and where $\left\{\widehat{\delta}(X), X \in \mathcal{G}_\lambda\right\}$ is the isonormal Gaussian process on $\mathcal{G}_\lambda$. We recall that under μ_λ, $\left\{\widehat{\delta}(X)(W_s), X \in \mathcal{G}_\lambda, s \in [0,1]\right\}$ is a Gaussian process with covariance

$$E_{\mu_\lambda}[\widehat{\delta}(X)(W_s)\widehat{\delta}(Y)(W_t)] = (s \wedge t)\langle X, Y\rangle_{\mathcal{G}_\lambda}$$

so that $(W_\cdot^\alpha)$ is a family of real valued independent Brownian motions under μ_λ. Under μ_λ, the coordinate process $t \to W_t$ is called the cylindrical Brownian motion. The difference with respect to the case where the state space is finite dimensional is that it is a renormalized sum of independent Brownian motions, the renormalization appearing in (2.3). For a measure $\eta \ll \mu_\lambda$ and a $u \in L_a^0(\eta, H_\lambda)$, the stochastic integral $\delta^W u := \int_0^1 \dot{u}_s dW_s$ is well defined as an abstract stochastic integral [10, 12]. Let η be a probability which is absolutely continuous with respect to μ_λ. Then there is a unique $v \in L_a^0(\eta, H_\lambda)$ such that $\eta - a.s.$

$$\frac{d\eta}{d\mu_\lambda} := \exp\left(\delta^W v - \frac{|v|_\lambda^2}{2}\right) \tag{2.5}$$

Moreover $W^\eta := I_W - v$ is a $(\mathcal{F}_t)$-Brownian motion on $(W, \mathcal{F}, \eta)$. We call v the **velocity field** associated to η. The famous formula of [7] (which in fact holds in a more general framework: [10, 12]) reads

$$2\mathcal{H}(\eta|\mu_\lambda) = E_\eta\left[\int_0^1 |\dot{v}_t|_{\mathcal{G}_\lambda}^2 dt\right] \tag{2.6}$$

where

$$\mathcal{H}(\eta|\mu_\lambda) := E_\eta\left[\ln \frac{d\eta}{d\mu_\lambda}\right]$$

is the relative entropy of η with respect to μ_λ. Note that since $\mathcal{G}_\lambda \subset \mathcal{G} \subset T_e G$ it makes sense to consider $(Xf)(\phi)$ for $\phi \in G$, for f smooth on G and for $X \in \mathcal{G}_\lambda$ or $X \in \mathcal{G}$.

3 Navier-Stokes Flows with Stochastic Drifts

Henceforth and until the end of Sect. 5 we assume that the renormalization sequence is fixed for a $\lambda \geq 2$, and we drop the indices λ of the notations except for $\mathcal{G}_\lambda$.

3.1 Constraints on the Kinematics: Regular and Admissible Flows

Definition 1 A probability η which is absolutely continuous with respect to μ with finite entropy ($\mathcal{H}(\eta|\mu) < \infty$) is called a **regular flow** if $u \in C^1([0,1] \times M)$ and dt- $a.s.$ $\partial_t u \in \mathcal{G}$, where $u(t,x) := E_\eta[\dot{v}_t(x)]$, and where $v := \int_0^{\cdot} \dot{v}_s ds$ is the **velocity field** of η (see (2.5)). We call u the **mean velocity field** of η. Moreover we say that a regular flow is **admissible** if there is a $C^1([0,1] \times M)$ mapping $p^\star : [0,1] \times M \to \mathbb{R}$ such that

$$Cov(\dot{v}_t(x)) = p^\star(t,x) I_d$$

i.e. for $i, j \in \mathbb{N} \cap [1, d]$

$$E_\eta\left[\left(\dot{v}_t^i(x) - u_t^i(x)\right)\left(\dot{v}_t^j(x) - u_t^j(x)\right)\right] = p^\star(x,t)\delta^{i,j} \tag{3.7}$$

where $(\dot{v}_t^j(x))$ denotes the jth (random) component of $(\dot{v}_t^j)$ at x i.e. it is such that $\dot{v}_t(x) = \sum_j \dot{v}_t^j(x)\partial_j|_x$, and where $u_t^j(x) := E_\eta[\dot{v}_t^j(x)]$.

3.2 Constraints on the Dynamics: Critical Flows

Definition 2 Let η be a regular flow whose velocity field is denoted by v^η (see (2.5)). For any $k \in C^1([0,1]; \mathcal{G})$ we set

$$L_k \mathcal{S}(\eta|\mu) := E_\eta\left[\int_0^1 \left(\int_M < \dot{v}_t^\eta(x), \partial_t k + (\dot{v}_t^\eta.\nabla)k + \frac{\Delta k}{2} >_{T_x M} dx\right) dt\right]$$

The probability η is said to be **critical** if and only if for any $k \in C_0^1([0,1], \mathcal{G})$

$$L_k \mathcal{S}(\eta|\mu) = 0$$

where

$$C_0^1([0,1], \mathcal{G}) := \left\{k \in C^1([0,1]; \mathcal{G}) : k(0,.) = k(1,.) = 0\right\}$$

The dynamic of the mean velocity field of a critical flow is given by the following theorem

Theorem 1 *Let η be a regular flow with a velocity field v and a mean velocity field $u \in \mathcal{G}_\lambda$. Then η is critical (Definition 2) if and only if there is a function $\widehat{p}(t,x)$ such that*

$$\partial_t u + E_\eta[(\dot{v}_t(x).\nabla)\dot{v}_t(x)] = \frac{\Delta u}{2} - \nabla\widehat{p}(t,x) \tag{3.8}$$

In other words, let

$$\beta(t,x) := E_\eta[((\dot{v}_t(x) - u_t(x)).\nabla)(\dot{v}_t(x) - u_t(x))] \tag{3.9}$$

Then u solves, in the weak L^2 sense, the following equation :

$$\partial_t u + (u_t.\nabla)u = \frac{\Delta u}{2} - \nabla\widehat{p} - \beta \tag{3.10}$$

Proof For any $k \in {C_0}^1([0,1];\mathcal{G})$ we have $k(0,.) = k(1,.) = 0$, so that an integration by parts yields

$$L_k\mathcal{S}(\eta|\mu) = -\int_M\int_0^1 \left(\partial_t u + E_\eta[(\dot{v}_t.\nabla)\dot{v}_t] - \frac{\Delta u}{2}\right)(t,x).k(t,x)dxdt \tag{3.11}$$

from which we obtain (3.8). Since

$$\begin{aligned}\beta(t,x) &:= E_\eta\left[[\dot{v}_t(x) - u_t(x)].\nabla)[\dot{v}_t(x) - u_t(x)]\right]\\ &= E_\eta\left[(\dot{v}_t(x).\nabla)\dot{v}_t(x)\right] + (u_t(x).\nabla)u_t(x) - E_\eta\left[(\dot{v}_t(x).\nabla)u_t(x)\right] - E_\eta\left[(u(x).\nabla)\dot{v}_t(x)\right]\\ &= E_\eta\left[(\dot{v}_t(x).\nabla)\dot{v}_t(x)\right] + (u_t(x).\nabla)u_t(x) - (E_\eta\left[\dot{v}_t(x)\right].\nabla)u_t(x) - (u(x).\nabla)E_\eta\left[\dot{v}_t(x)\right]\\ &= E_\eta\left[(\dot{v}_t(x).\nabla)\dot{v}_t(x)\right] - (u_t(x).\nabla)u_t(x)\end{aligned}$$

we obtain (3.10) from (3.8). □

3.3 Navier-Stokes Flows

Definition 3 A regular flow η (see Definition 1) is a Navier-Stokes flow if its mean velocity field u solves the Navier-Stokes equation, i.e. if and only if there is a function $p : [0,1] \times M \to \mathbb{R}$ which is such that u solves, in the weak L^2 sense, the Navier-Stokes equation

$$\partial_t u + u.\nabla u = \frac{\Delta u}{2} - \nabla p$$

we have:

Corollary 1 *An admissible flow is a Navier-Stokes flow if and only if it is critical.*

Proof Let η be an admissible flow. We recall that by definition there exists a mapping $p^\star$ such that

$$Cov(\dot{v}_t(x)) = p^\star(x,t)I_d \tag{3.12}$$

where $v := \int_0^\cdot \dot{v}_s ds$ is the velocity field of η (see (2.5)). We also recall that

$$u(t,x) := E_\eta[\dot{v}_t(x)]$$

The idea is to apply Theorem 1 and to set

$$p := p^\star + \widehat{p}$$

We have

$$\beta^i(t,x) = \sum_j \partial_j Cov(\dot{v}_t(x))^{i,j}$$

Indeed (repeated indices are summed over) we have

$$\begin{aligned}
\beta^i(t,x) &= E_\eta\left[\left(\dot{v}_t^j(x) - u_t^j(x)\right)\partial_j\left(\dot{v}_t^i(x) - u_t^i(x)\right)\right]\\
&= \partial_j E_\eta\left[\left(\dot{v}_t^i(x) - u_t^i(x)\right)\left(\dot{v}_t^j(x) - u_t^j(x)\right)\right] - E_\eta\left[\left(\dot{v}_t^i(x) - u_t^i(x)\right)\partial_j\left(\dot{v}_t^j(x) - u_t^j(x)\right)\right]\\
&= \partial_j E_\eta\left[\left(\dot{v}_t^i(x) - u_t^i(x)\right)\left(\dot{v}_t^j(x) - u_t^j(x)\right)\right] - E_\eta\left[\left(\dot{v}_t^i(x) - u_t^i(x)\right)\operatorname{div}(\dot{v}_t(x) - u_t(x))\right]\\
&= \partial_j E_\eta\left[\left(\dot{v}_t^i(x) - u_t^i(x)\right)\left(\dot{v}_t^j(x) - u_t^j(x)\right)\right]\\
&= \partial_j Cov(\dot{v}_t(x))^{i,j}
\end{aligned}$$

Assumption (3.12) then yields $\beta^i(t,x) = \partial_i p^\star$ i.e.

$$\beta = \nabla p^\star \tag{3.13}$$

□

Remark 1 Note that by this proof, for critical flows, $p^\star$ appears as a part of the pressure which is originated from the stochastic model. Specifically it expresses the fluctuations of the drift itself. Indeed by (3.13) and (3.9) for an admissible flow η we have

$$\nabla p^\star(t,x) = E_\eta[(\dot{v}_t(x).\nabla)\dot{v}_t(x)] - (u_t(x).\nabla)u_t(x) \tag{3.14}$$

where $p^\star$ is the function associated to the admissible flow η by formula (3.7).

4 Interpretation of Critical Flows by Means of the Stochastic Exponential

In this section we prove that the quantities $L_k\mathcal{S}(\eta|\mu)$ defined in Definition 2 can still be interpreted in terms of certain variations along deterministic paths which extend those of Ref. [3].

4.1 The Stochastic Exponential

Let $C_G = C_e([0,1], G)$ be the space of continuous paths starting from e and with values in G. The coordinate function $(t,\gamma) \in [0,1] \times C_G \to \gamma_t(\omega)$ generates a filtration $\left(\mathcal{F}_t^G\right)$ and we denote $\mathcal{F}^G := \mathcal{F}_1^G$.

Proposition 1 *The equation*

$$dX_t = \circ dB_t;\ X_0 = e \tag{4.15}$$

has a continuous strong solution on the space $\left(W, \mathcal{F}_\cdot^W, \mu\right)$ *with the canonical Brownian* $t \to W_t \in \mathcal{G}$. *We note* g *this solution. By this we mean that for* μ*−a.s.* $g \in C_G$ *and, for any smooth* f *on* G,

$$f(g_t) = f(e) + \sum_\alpha \int_0^t (H_\alpha f)(g_t) \circ dW_t$$

where $\circ$ *denotes the Stratonovich integral.*

Proof See [11]. □

Girsanov theorem on (W, H, μ) implies the following:

Proposition 2 *Let* η *be a probability which is absolutely continuous with respect to* μ *whose velocity field is noted* v, *and set* $\widetilde{W} := I_W - v$. *Then* $(g, \widetilde{W})$ *is a solution of*

$$dX_t = (\circ dB_t + \dot{v}_t dt); X_0 = e \tag{4.16}$$

on $(W, \mathcal{F}_\cdot, \eta)$.

Proof We have

$$\widetilde{W}_s = \sum_\alpha \widehat{\delta}(H_\alpha)(W_s)H_\alpha - \sum_\alpha \langle v, H_\alpha\rangle_\lambda H_\alpha = \sum_\alpha \widehat{\delta}(H_\alpha)(\widetilde{W}_s)H_\alpha$$

Since $\widetilde{W}_\star\eta = \mu$, $\widetilde{W}_\cdot^\alpha := \widehat{\delta}(H_\alpha)(\widetilde{W}_\cdot)$ are independent Brownian motions on (W, H, η), by Itô's formula we have, $\eta - a.s.$,

$$f(g_t) = f(e) + \int_0^1 \sum_\alpha (H_\alpha f)(g_t) \circ d\widetilde{W}_t^\alpha + \sum_\alpha \int_0^1 (H_\alpha f)(g_s)\langle \dot{v}_s, H_\alpha \rangle_{\mathcal{G}_\lambda} ds$$

i.e.

$$f(g_t) = f(e) + \int_0^1 (H_\alpha f)(g_s) \circ d\widetilde{W}_t^\alpha + \int_0^1 (\dot{v}_t(\omega)f)(g_s)ds \qquad \square$$

Proposition 3 *Let η be a probability absolutely continuous with respect to μ, $v := \int_0^\cdot \dot{v}_s ds$ the associated velocity field, $\widetilde{W} = I_W - v$ and $\widetilde{W}_\cdot^\alpha = \widehat{\delta}(H_\alpha)(\widetilde{W}_\cdot)$. For any smooth function f on $[0,1] \times M$ we have η−a.s.*

$$f(t, g_t(x)) = f(0,x) + \int_0^t \left(\frac{\Delta}{2} f + (\dot{v}_\sigma.\nabla)f + \partial_\sigma f)(\sigma, g_\sigma(x)\right) d\sigma + \int_0^t \sum_\alpha (H_\alpha f)(\sigma, g_\sigma(x)) d\widetilde{W}_\sigma^\alpha \tag{4.17}$$

and η−a.s.

$$\lim_{\delta \to 0} E_\eta \left[\frac{f(t+\delta, g_{t+\delta}(x)) - f(t, g_t(x))}{\delta} \middle| \mathcal{F}_t \right] = \left(\partial_t f + (\dot{v}_t(\omega).\nabla)f + \frac{\Delta f}{2} \right)(t, g_t(x)) \tag{4.18}$$

Proof Let $x \in M, f \in C^\infty(M)$. The main part of the proof will be to prove that

$$\sum_\alpha (H_\alpha^2 f^x)(\phi) = (\Delta f)(\phi(x)) \tag{4.19}$$

To see this recall that $f^x : \phi \in G \to f(\phi(x)) \in \mathbb{R}$. We have

$$(H_\alpha f^x)(\phi) := H_\alpha(\phi) f^x = H_\alpha(\phi(x)) f = (H_\alpha f)(\phi(x)) = (H_\alpha f)^x(\phi) \tag{4.20}$$

so that by iterating (4.20) we obtain

$$\sum_\alpha (H_\alpha^2 f^x)(\phi) = \sum_\alpha (H_\alpha^2 f)(\phi(x)) \tag{4.21}$$

On the other hand

$$\sum_\alpha (H_\alpha^2 f)(\phi(x)) = (\Delta f)(\phi(x)) \tag{4.22}$$

Indeed by using the fact that for any α the vector field H^α is divergence free together with (2.4) we obtain

$$\sum_\alpha H_\alpha^2 f = \sum_{\alpha,j} A^{\alpha,j} \partial_j (H_\alpha f)$$

$$
\begin{aligned}
&= \sum_{\alpha,i,j} A^{\alpha,j} A^{\alpha,i} (\partial_j \partial_i f) + A^{\alpha,j} (\partial_j A^{\alpha,i})(\partial_i f) \\
&= \sum_i (\partial^2_{i,i} f) + \sum_{\alpha,i,j} A^{\alpha,j} (\partial_j A^{\alpha,i})(\partial_i f) \\
&= \Delta f + \sum_{\alpha,i,j} A^{\alpha,j} (\partial_j A^{\alpha,i})(\partial_i f) \\
&= \Delta f + \sum_{\alpha,i,j} \partial_j (A^{\alpha,j} A^{\alpha,i})(\partial_i f) - \sum_{\alpha,i,j} (\partial_j A^{\alpha,j}) A^{\alpha,i} (\partial_i f) \\
&= \Delta f + \sum_{i,j} \partial_j (\sum_{\alpha} A^{\alpha,j} A^{\alpha,i})(\partial_i f) - \sum_{\alpha,i} (\mathrm{div}(H^\alpha) A^{\alpha,i} (\partial_i f) \\
&= \Delta f
\end{aligned}
$$

Finally by putting together (4.21) and (4.22) we get (4.19) which yields

$$
\begin{aligned}
f(t, g_t(x)) &= f^x(t, g_t) \\
&= f^x(s, g_s) + \int_s^t (H_\alpha f^x)(g_\sigma) \circ d\widetilde{W}^\alpha_\sigma + \int_s^t (\partial_\sigma f^x + \dot{v}_\sigma f^x)(g_\sigma) d\sigma \\
&= f(s, g_s(x)) + \int_s^t \left(\frac{\Delta}{2} f + (\dot{v}_\sigma . \nabla) f + \partial_\sigma f \right) (\sigma, g_\sigma(x)) d\sigma \\
&\quad + \int_s^t \sum_\alpha (H_\alpha f_\sigma)(g_\sigma(x)) d\widetilde{W}^\alpha_\sigma
\end{aligned}
$$

On the other hand by the Girsanov theorem, $(\widetilde{W}_t)$ is a $(\mathcal{F}_t)$-Brownian motion on (W, η) so that (4.18) follows from (4.17). □

4.2 Perturbations of the Energy Along Deterministic Paths

For $k \in C^0([0, 1], \mathcal{G}_\lambda)$, $k := \int_0^{\cdot} \dot{k}_s ds$, we define $e(k)$ to be the solution of the ordinary differential equation on G

$$d(e_t(k)) = (\dot{k}_t dt)(e_t(k)); \; e_0 = e$$

i.e. for any smooth $F : G \to \mathbb{R}$,

$$F(e_t(k)) = F(e) + \int_0^t (\dot{k}_s F)(e_s(k)) ds. \tag{4.23}$$

Note that $e_{\cdot}(0_H) = e$ i.e. the exponential of the function which is constant and equal to 0_H is constant and equal to e. We denote by $(e_t^i(k))$ the ith component of $(e_t(k))$ in the canonical chart.

Proposition 4 *If η is a probability of finite entropy with respect to μ, for any $k \in C_0^1([0,1], \mathcal{G}_\lambda)$ we have*

$$L_k \mathcal{S}(\eta|\mu) = \frac{d}{d\epsilon} E_\eta \left[\int_0^1 \left(\int_M \frac{|D^\eta e_t(\epsilon k).g_t(x)|^2_{T_{g_t(x)}M}}{2} dx \right) dt \right] |_{\epsilon=0} \tag{4.24}$$

where $L_k \mathcal{S}(\eta|\mu)$ has been defined in Definition 2 and where $D^\eta e_t(\epsilon k).g_t(x)$ is defined a.e. by

$$D^\eta e_t(\epsilon k).g_t(x) := \sum_i \lim_{\delta \to 0} E_\eta \left[\frac{e^i_{t+\delta}(\epsilon k).g_{t+\delta}(x) - e^i_t(\epsilon k).g_t(x)}{\delta} \middle| \mathcal{F}_t \right] \partial_i|_{g_{t(x)}} \tag{4.25}$$

Proof By (4.18) of Proposition 3 we first obtain

$$D^\eta e_t(\epsilon k).g_t(x) := \sum_i \left(\partial_t e_t^i(\epsilon k) + (\dot{v}_t(\omega).\nabla) e_t^i(\epsilon k) + \frac{\Delta e_t^i(\epsilon k)}{2} \right) (g_t(x)) \partial_i|_{g_t(x)} \tag{4.26}$$

On the other hand let $x \in M$ and denote by f a smooth function on M. Considering $F := f^x$ in (4.23) we have

$$f(e_t(\epsilon k)(x)) = f(x) + \epsilon \int_0^t (\dot{k}_s f)(e_s(\epsilon k)(x)) ds$$

Since $e_{\cdot}(0_H)(x) = e(x) = x$, we get :

$$\frac{d}{d\epsilon}|_{\epsilon=0} f(e_t(\epsilon k)(x)) = \int_0^t (\dot{k}_s f)(x) ds = (k_t f)(x)$$

so that

$$\frac{d}{d\epsilon}|_{\epsilon=0} e_t(\epsilon k)(x) = k_t(x) \tag{4.27}$$

By (4.26) and (4.27) we obtain

$$\frac{d}{d\epsilon} D^\eta e_t(\epsilon k).g_t(x)|_{\epsilon=0} = \left(\partial_t k_t + \dot{v}_t.\nabla k_t + \frac{\Delta k_t}{2} \right) (g_t(x)) \tag{4.28}$$

For convenience of notations we denote by A the right hand term of (4.24). By first differentiating the product, then by applying (4.26) at $\epsilon = 0$, then by applying (4.28), and finally by using that g_t preserves the measure we obtain

$$A = E_\eta\left[\int_0^1\left(\int_M \langle D^\eta g_t(x), \frac{d}{d\epsilon}D^\eta e_t(\epsilon k).g_t(x)|_{\epsilon=0}\rangle_{T_{g_t(x)}M}dx\right)dt\right]$$

$$= E_\eta\left[\int_0^1\left(\int_M \langle \dot{v}_t(g_t(x)), \frac{d}{d\epsilon}D^\eta e_t(\epsilon k).g_t(x)|_{\epsilon=0}\rangle_{T_{g_t(x)}M}dx\right)dt\right]$$

$$= E_\eta\left[\int_0^1\left(\int_M \langle \dot{v}_t(g_t(x)), \left(\partial_t k_t + \dot{v}_t.\nabla k_t + \frac{\Delta k_t}{2}\right)(g_t(x))\rangle_{T_{g_t(x)}M}dx\right)dt\right]$$

$$= E_\eta\left[\int_0^1\left(\int_M \langle \dot{v}_t(x), \partial_t k_t(x) + \dot{v}_t.\nabla k_t(x) + \frac{\Delta k_t}{2}(x)\rangle_{T_x M}dx\right)dt\right]$$

which proves (4.24). □

5 Variations of the Energy Along Translations

Let η be a probability which is absolutely continuous with respect to μ (as mentioned in the beginning of Sect. 3 we work with a fixed $\lambda \geq 2$) and with velocity field v^η. The stochastic action of η is defined by

$$\mathcal{S}(\eta|\mu) := E_\eta\left[\int_0^1 \frac{|\dot{v}_s^\eta|^2_{\mathcal{G}}}{2}ds\right] \tag{5.29}$$

The motivation for this definition is that, by taking $\epsilon = 0$ in (4.26) and using the fact that g_t preserves the measure, we also have

$$\mathcal{S}(\eta|\mu) = E_\eta\left[\int_0^1\left(\int_M \frac{|D^\eta g_s(x)|^2_{T_{g_s(x)}M}}{2}dx\right)ds\right]$$

with the notations of Proposition 4. By (2.6), $\mathcal{G}_\lambda \subset \mathcal{G}$ implies that whenever the entropy is finite we have

$$\mathcal{S}(\eta|\mu) < \infty$$

as well. More accurately, by a classical result on abstract Wiener spaces together with (2.6), there exists a $c > 0$ such that for any $\eta \ll \mu$

$$\mathcal{S}(\eta|\mu) \le c\mathcal{H}(\eta|\mu)$$

In this section we introduce another kind of variations for the functional $\mathcal{S}(\eta|\mu)$, namely we study its variations along translations, These variations are generally different from those introduced above; however, when restricted to admissible flows, they are the same. We also investigate similar variations for the relative entropy. Proposition 5 computes the values of the variations of these quantities along deterministic translations.

Proposition 5 *Let η be a probability absolutely continuous with respect to μ with velocity field v^η and mean velocity $u_s(x) := E_\eta[\dot{v}_s^\eta(x)]$. If $\mathcal{S}(\eta|\mu) < \infty$ we have,*

$$\frac{d}{d\epsilon}\mathcal{S}(\tau_{\epsilon h}\eta|\mu)|_{\epsilon=0} = \int_0^1 \langle u_s, \dot{h}_s\rangle_{\mathcal{G}} ds \tag{5.30}$$

and if $\mathcal{H}(\eta|\mu) < \infty$ we have

$$\frac{d}{d\epsilon}\mathcal{H}(\tau_{\epsilon h}\eta|\mu)|_{\epsilon=0} = \int_0^1 \langle u_s, \dot{h}_s\rangle_{\mathcal{G}_\lambda} ds \tag{5.31}$$

where $\tau_h\eta$ is the image measure of η by the mapping τ_h defined by

$$\tau_h : \omega \in W \to \omega + h \in W$$

Proof A straightforward application of the Cameron-Martin theorem shows that for any $h := \int_0^{\cdot} \dot{h}_s ds \in H$, the velocity field field $v^{\tau_h\eta}$ of $\tau_h\eta$ is given by

$$v^{\tau_h\eta} = \tau_h \circ v^\eta \circ \tau_{-h} = v^\eta \circ \tau_{-h} + h \tag{5.32}$$

Hence by (5.29) we have

$$\mathcal{S}(\tau_h\eta|\mu) = E_\eta\left[\int_0^1 \frac{|\dot{v}_s^\eta + \dot{h}_s|_{\mathcal{G}}^2}{2} ds\right]$$

which yields (5.30). Similarly (5.31) follows by (2.6) and (5.32). □

Let

$$C_0^n([0,1],\mathcal{G}_{\lambda+2}) := \left\{k \in C^n([0,1],\mathcal{G}_{\lambda+2}) : k(0,.) = k(1,.) = 0\right\} \tag{5.33}$$

and let Π be the Helmoltz projection on divergence free vector fields. We set

$$\mathcal{K}_0^\eta := \left\{ h := \int_0^{\cdot} \dot{h}_s(\omega)ds \middle| \exists k \in C_0^n([0,1], \mathcal{G}_{\lambda+2}), ds - a.s., \dot{h}_s = \partial_s k_s + \Pi((u_s.\nabla)k_s) + \frac{\Delta k_s}{2} \right\} \tag{5.34}$$

so that it makes sense to say that any $h \in \mathcal{K}_0^\eta$ is associated to a $k \in C_0^n([0,1], \mathcal{G}_{\lambda+2})$. For n sufficiently large we have $\mathcal{K}^\eta \subset H$.

Proposition 6 *Let η be a smooth flow whose mean velocity field is given by u. Then u solves the Navier-Stokes equation if and only if for any $h \in \mathcal{K}_0^\eta$*

$$\frac{d}{d\epsilon}\mathcal{S}(\tau_{\epsilon h}\eta|\mu)|_{\epsilon=0} = 0$$

Proof By Proposition 5, and by definition of Π, for any h (which is associated to k) we have

$$\frac{d}{d\epsilon}\mathcal{S}(\tau_{\epsilon h}\eta|\mu)|_{\epsilon=0} = \int_M \int_0^1 \left(\partial_s k + \Pi((u.\nabla)k) + \frac{\Delta k}{2}\right)(s,x).u(s,x)dxds \tag{5.35}$$

$$= \int_M \int_0^1 \left(\partial_s k + (u.\nabla)k + \frac{\Delta k}{2}\right)(s,x).u(s,x)dxds \tag{5.36}$$

and, since $k(0,.) = k(1,.) = 0$, the result directly follows from an integrating by parts. □

We now relate these variations to the ones of Sect. 4. Namely we prove that, for admissible flows, these variations of measure by quasi-invariant transformations yield exactly the same variations as the exponential variations of Sect. 4.

Proposition 7 *Let η be an admissible flow. Then, for any $h \in \mathcal{K}_0^\eta$ (see (5.34)) associated with a $k \in C_0^n([0,1], \mathcal{G}_{\lambda+2})$ (see (5.33)) we have*

$$\frac{d}{d\epsilon}\mathcal{S}(\tau_{\epsilon h}\eta|\mu)|_{\epsilon=0} = L_k\mathcal{S}(\eta|\mu)$$

Proof Let u be the mean velocity field of η. Since η is admissible we have, by (3.14)

$$\langle u_t, (u_t.\nabla)k_t\rangle_{\mathcal{G}} = -\langle (u_t.\nabla)u_t, k_t\rangle_{\mathcal{G}} = -E_\eta[\langle(\dot{v}_t.\nabla)\dot{v}_t, k_t\rangle_{\mathcal{G}}] = E_\eta[\langle \dot{v}_t, (\dot{v}_t.\nabla)k_t\rangle_{\mathcal{G}}]$$

Hence, using (5.36),

$$\frac{d}{d\epsilon}\mathcal{S}(\tau_{\epsilon h}\eta|\mu)|_{\epsilon=0} = E_\eta\left[\int_0^1\left(\int_M \langle \dot{v}_t^\eta(x), \partial_t k + (\dot{v}_t^\eta.\nabla)k + \frac{\Delta k}{2}\rangle_{T_xM}dx\right)dt\right]$$

which is exactly the definition of $L_k\mathcal{S}(\eta|\mu)$ (Definition 2). □

6 Generalized Flows with a Cut-off

In Sect. 5 we have seen that in the infinite dimensional case, the relative entropy was generally not proportional to the action $\mathcal{S}(\cdot|\mu)$. The reason is that the renormalization procedure gives a different weight to the different modes: hard modes have a weaker weight in the energy than in the relative entropy. However if instead of renormalizing we introduce a cutoff, and rescale the noise accordingly, $\mathcal{S}(\cdot|\mu)$ becomes proportional to the relative entropy $\mathcal{H}(\cdot|\mu)$. Within this framework, we investigate the existence of generalized flows with a given marginal.

6.1 General Framework for a Cut-off at Scale n

We recall that (e_α) denotes the Hilbertian basis of $\mathcal{G}$ of Sect. 2. By induction we define $(I_l)_{l=1}^\infty$ by $I_1 = 1$ and

$$I_{l+1} = \min\left(\{m \geq I_l : |k(m)| > |k(I_l)|\}\right)$$

For $N \in \mathbb{N}, N > 1$ we set

$$n := 2I_N$$

We define $\mathcal{G}^n = Vect(e_1, \ldots, e_n) \subset \mathcal{G}$ and recall that we work under the hypothesis

$$e_\alpha(x) = \sum_j a^{\alpha,j}(x)\partial_j|_x$$

The cut-off has been chosen so that $\exists S(N)$ such that

$$\sum_{\alpha=1}^n a^{\alpha,i}(x)a^{\alpha,j}(x) = \mathcal{S}(N)\delta^{i,j}$$

where $\mathcal{S}(N) \uparrow \infty$. We note

$$H_n := \left\{ h : [0,1] \to \mathcal{G}^n, h := \int_0^{\cdot} \dot{h}_s ds, \int_0^1 |\dot{h}_s|^2_{\mathcal{G}} ds < \infty \right\}$$

and $\langle ., . \rangle_{H_n}$ the associated scalar product. We set $W_n := C([0,1], \mathcal{G}^n)$ endowed with the norm of uniform convergence, and μ_n the Wiener measure on (W_n, H_n) with a parameter

$$\sigma(N) := \frac{2\nu}{\mathcal{S}(N)}$$

$t \to W_t$ is the coordinate process. Define g^n to be the solution of

$$dg^n_t := (\circ dW_t)(g^n_t);\ g^n_0 = e$$

on the Wiener space (W_n, H_n, μ_n) i.e., satisfying, for every smooth f,

$$f(g^n_t) = f(e) + \int_0^t \sum_{\alpha=1}^n (e_\alpha f)(g^n_s) \circ dW^\alpha_s$$

where $W^\alpha := \langle W_t, e_\alpha \rangle_{G_n}$. We are now working with the Wiener measure with parameter $\sigma(N)$. Still by the Girsanov theorem, for any $\eta \ll \mu_n$ there is a unique $v \in L^0(\eta, H_n)$ such that

$$\frac{d\eta}{d\mu} = \exp\left(\delta^W v - \frac{\sigma(N)|v|^2_{H_n}}{2} \right)$$

and $\widetilde{W} := I_W - \sigma(N)v$ is a Brownian motion with parameter $\sigma(N)$ under η. We call v the velocity field of η. Furthermore, Föllmer's formula (c.f. [8]) then reads

$$\mathcal{H}(\eta|\mu_n) = \sigma(N) E_\eta \left[\frac{|v|^2_{H_n}}{2} \right]$$

Hence $(g, \widetilde{W})$ is a solution to

$$dg^n_t := \circ(dW^\nu_t + \sigma(N)\dot{v}_t dt))(g^n_t);\ g^n_0 = e$$

on the probability space (W_n, η) for the filtration generated by the coordinate process $t \to W_t$, i.e., for every smooth f,

$$f(g_t^n) = f(e) + \int_0^t \sum_{\alpha=1}^n (e_\alpha f)(g_s) \circ d\widetilde{W}_s^\alpha + \sigma(N) \int_0^t \sum_{\alpha=1}^n (e_\alpha f)(g_s)\langle \dot{v}_s, e_\alpha \rangle ds$$

Within this framework, by an admissible flow we mean a probability η of finite entropy with respect to η satisfying the same conditions as in Definition 1 with μ_n (resp. $\mathcal{G}_n$) instead of μ (resp. of $\mathcal{G}$).

6.2 Variations of the Action

We now define the action for the cutoff $n \in \mathbb{N}$ by

$$\mathcal{S}(\eta|\mu_n) := E_\eta\left[\int_0^1 \frac{|D_s^\eta g_s^n|_{\mathcal{G}}^2}{2} ds\right] = E_\eta\left[\int_0^1 \frac{|\sigma(N)\dot{v}_s|_{\mathcal{G}}^2}{2} ds\right] = \sigma(N)^2 E_\eta\left[\int_0^1 \frac{|\dot{v}_s|_{\mathcal{G}}^2}{2} ds\right]$$

Therefore

$$\mathcal{S}(\eta|\mu_n) = \sigma(N)\mathcal{H}(\eta|\mu_n) \tag{6.37}$$

Similarly to Proposition 7 we note

$$\mathcal{K}_0^\eta(n) := \left\{h \in H_n : \exists k \in C_0^1([0,1], \mathcal{G}^n), ds - a.s., \dot{h}_s = \partial_s k + \pi_n \Pi((\sigma(N)u_s.\nabla)k) + \nu\Delta k\right\}$$

where π_n is the orthogonal projection $\pi_n : \mathcal{G} \to \mathcal{G}_n$ and we say that a $h \in \mathcal{K}_0^\eta(n)$ is associated to a $k \in C_0^1([0,1], \mathcal{G}^n)$.

Proposition 8 *For any smooth flow* η

$$u^n(t,x) := \sigma(N)E_\eta[\dot{v}_t(x)]$$

solves the Navier-Stokes equation if and only if for any $h \in \mathcal{K}_0^\eta(n)$ *we have*

$$\frac{d}{d\epsilon}\mathcal{H}(\tau_{\epsilon h}\eta|\mu_n)]|_{\epsilon=0} = 0$$

for any h associated with a $k \in C_0^1([0,1], \mathcal{G}^n)$*. Moreover whenever* η *is an admissible flow, and* $h \in \mathcal{K}_0^\eta(n)$ *is associated to* $k \in C_0^1([0,1], \mathcal{G}^n)$ *we have*

$$\frac{d}{d\epsilon}\mathcal{H}(\tau_{\epsilon h}\eta|\mu_n)]|_{\epsilon=0} = \frac{d}{d\epsilon}E_\eta\left[\int_0^1 \left(\int_M \frac{|D^\eta e_t(\epsilon k).g_t^n(x)|_{T_{g_t(x)}M}^2}{2} dx\right) dt\right]|_{\epsilon=0}$$

where the notations are those of Sect. 4.

Proof The first part of the proof is the same as in Proposition 6. We now prove the second part of the claim which is similar to Proposition 7. As in the first subsection we have

$$\sum_{\alpha=1}^{n} e_\alpha^2 f = \mathcal{S}(N)\Delta f$$

Therefore by setting

$$A := \lim_{\epsilon\to 0}\left(\frac{E_\eta\left[\int_0^1\left(\int_M |D^\eta e_t(\epsilon k).g_t^n(x)|^2_{T_{g_t(x)}M}dx\right)dt\right] - E_\eta\left[\int_0^1\left(\int_M |D^\eta g_t^n(x)|^2_{T_{g_t(x)}M}dx\right)dt\right]}{2\epsilon}\right)$$

and using the fact g_t preserves the measure we get

$$A = E_\eta\left[\int_0^1 \langle \dot{v}_t, \partial_t k + \sigma(N)\dot{v}_t.\nabla k + \nu\Delta k\rangle_{\mathcal{G}} dt\right]$$

If η is assumed to be admissible, then similarly to the proof of Proposition 7 we obtain

$$A = \frac{d}{d\epsilon}\mathcal{H}(\tau_{\epsilon h}\eta|\mu_n)]|_{\epsilon=0}$$

□

Concerning existence of Lagrangian Navier-Stokes flows with a cut-off they have been shown to exist in Ref. [4] for deterministic L^2 drifts. Examples of random solutions of Navier-Stokes equations were constructed in Ref. [5] but we did not prove existence of the corresponding flows.

Acknowledgments We thank the anonymous referee for a careful reading of the manuscript and valuable remarks. We acknowledge support from the FCT project PTDC/MAT/120354/2010.

References

1. A. Antoniouk, A.B. Cruzeiro, M. Arnaudon, Generalized Navier-Stokes flows and applications to incompressible viscous fluids. Bull. Sci. Math. **138**(4), 565–584 (2014)
2. M. Arnaudon, A.B. Cruzeiro, Stochastic Lagrangian flows and the Navier-Stokes equation, preprint, http://gfm.cii.fc.ul.pt/people/abcruzeiro/abc-marnaudon-2012-11-08.pdf (2012)
3. A.B. Cruzeiro, F. Cipriano, Navier-Stokes equation and diffusions on the group of homeomorphisms of the torus. Commun. Math. Phys. **275**, 255–269 (2007)
4. A.B. Cruzeiro, F. Cipriano, Variational principle for diffusions on the diffeomorhism group with the H_2 metric, in *Mathematical analysis of random phenomena*, World. Sci. Publ., Hackensack (2007)
5. A.B. Cruzeiro, Zh Qian, Backward stochastic differential equations associated with the vorticity equations. J. Funct. Anal. **267**(3), 660–677 (2014)
6. D.J. Ebin, J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid. Ann. Math. **2**(92), 102–163 (1970)

7. H. Föllmer, *Time Reversal on Wiener Space*, Lecture Notes in Mathematics (Springer, Berlin, 1986)
8. H. Föllmer, *Random Fields and Diffusion Processes, Saint Flour XV-XVII* (Springer, New York, 1988)
9. H. Kuo, *Gaussian Measures in Banach Spaces*, Lecture Notes in Mathematics (Springer, New York, 1975), p. 463
10. R. Lassalle, Invertibility of adapted perturbations of the identity on abstract Wiener space. J. Funct. Anal. **262**(6), 2734–2776 (2012)
11. P. Malliavin, *Stochastic Analysis* (Springer, New York, 1997)
12. A.S. Üstünel, M. Zakai, *Transformation of Measure on Wiener Space* (Springer, New York, 1999)