# A Sphere Decoding Algorithm for Underdetermined OFDM/SDMA Uplink System with an Effective Radius Selection

K.V. Shahnaz and C.K. Ali

**Abstract.** Multiuser Detection (MUD) Techniques for orthogonal frequency division multiplexing/space division multiple access (OFDM/SDMA) system remain a challenging area especially when the number of transmitters exceed receivers. Maximum Likelihood (ML) detection is the optimal one, but infeasible due to the high complexity when large number of antennas are used together with high order modulation scheme. Sphere Decoding (SD) algorithm with less complexity but performance near to ML has been explored widely for determined and overdetermined MIMO channels. Very few papers that efficiently deal with an underdetermined OFDM/SDMA channel have been published so far. In this paper a simple pseudo-antenna augmentation scheme has been employed to utilize SD in a rank-deficient case. An effective radius selection method is also included.

## 1 Introduction

Combination of orthogonal frequency division multiplexing (OFDM) with smart antenna designs have emerged in recent years [1]. They are applied with the main objective of combating the effects of multipath fading on the desired signals, there by increasing both the performance and capacity of wireless systems. An application of smart antennas is space division multiple access (SDMA). Here the users are identified with the help of their spatial signature. Channel distortion due to multipath propagation is easily mitigated with OFDM, while bandwidth efficiency can be increased with use of SDMA. The performance improvement of OFDM/SDMA has

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become an attractive research topic for emerging future wireless systems. Channel estimation and multiuser detection are two major tasks to be handled effectively. In this paper, focus is on multiuser detection (MUD), assuming perfect channel knowledge at the receiver. To solve the problem of multiuser detection in OFDM/SDMA systems, various classical solutions are available, having varying complexity, like zero forcing (ZF), minimum mean square error (MMSE), successive interference cancellation (SIC), parallel interference cancellation (PIC) [2].

OFDM/SDMA systems confront with three channels. Underloaded (number of BS antennas ($N_r$) larger than transmitters ($N_t$)), fullyloaded ($N_r$ same as $N_t$) and overloaded ($N_r$ less than $N_t$) which is also called a rank-deficient or underdetermined channel. In rank-deficient systems, the $N_r \times N_t$ dimensional OFDM/SDMA channel matrix becomes singular and, hence, noninvertible, thus making the degree of freedom of the detector insufficiently high for detecting the signals of all the transmitters. Above detectors perform well for the first two cases but exhibit very poor performance in the third case which we know is the practical scenario.

We use an efficient Sphere Decoding (SD) algorithm to tackle this problem. SD algorith was introduced by Finke and Phost [3] in 1985. Since then it has been widely used in various communication applications. It searches a lattice point in a hypershere centred at a given vector, there by reducing the search space and hence the tedious computations required for ML detection. There are a few papers in literature that treat underdetermined MIMO channels. But all require some detailed procedures. The pseudo-antenna augmentation scheme in [5] gives a simple and powerful approach to treat rank-deficient MIMO channels and has been extended to an OFDM/SDMA system. The linearly reducing radius selection based on noise variance is also simple and effective.

## 2 System Model

The multiuser OFDM/SDMA system considered supports $N_t$ mobile stations (MSs) simultaneously transmitting in the Up Link (UL) to the Base Station (BS). Each of the users is equipped with a single transmit antenna, whereas the BS employs an array of $N_r$ antennas. It is assumed that a time division multiple access protocol organizes the division of the available time domain resources into OFDM/SDMA time slots. Instead of one, $N_t$ MSs are assigned to each slot that is allowed to simultaneously transmit their streams of OFDM-modulated symbols to the BS [1].

At the $k^{th}$ subcarrier of the $n^{th}$ OFDM symbol received by the $N_r$ - element BS antenna array we have the received complex signal vector $y[n,k]$, which is constituted by the superposition of the independently faded signals associated with the $N_t$ mobile users and contaminated by AWGN, expressed as:

$$y = Hs + n \tag{1}$$

where the $N_r \times 1$ dimensional vector $y$, the $N_t \times 1$ dimensional vector $s$ and the $N_r \times 1$ dimensional vector $n$ are the received, transmitted and noise signals, respectively. The indices $[n,k]$ have been omitted for notational convenience. Here,

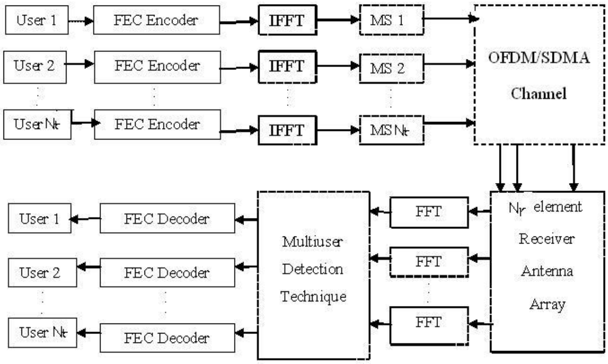


**Fig. 1** Schematic of the OFDM/SDMA Uplink System

$y = (y_1, y_2, y_3 ... y_{N_r})^T$, $s = (s^{(1)}, s^{(2)}, s^{(3)} ... s^{N_t)})^T$ and $n = (n_1, n_2, n_3 ... n_{N_r})^T$. $N_r \times N_t$ dimensional matrix $H$ contains the frequency domain channel transfer functions (FD-CHTF) of $N_t$ users.

## 3 Sphere Decoding Algorithm

Any multiuser detection technique for single carrier MIMO can be used for the latest proposed OFDM/SDMA also, as it is done on per-carrier basis. It is intuitive that the ML Detector gives the optimum result as it searches through all possibility of transmitted symbols. Exhaustive search by optimum ML detector requires $2^{mN_t}$ evaluations of the decision metric for finding the most likely transmitted $N_t$ user symbol vector $\widehat{s}$.

$$\widehat{s} = arg\left\{min_{s \in M^{N_t}} \|y - Hs\|^2\right\} \tag{2}$$

The set of $M^{N_t}$ number of trial vectors are elements of $M_c$ that denotes the set containing the $M = 2^m$ number of legitimate complex constellation points associated with the specific modulation scheme employed, while $m$ denotes the number of bits per symbol.

To reduce the complexity of ML, SD limits the searching space by radius $r$. Then the decision criterion of ML detection will be

$$\|y - Hs\|^2 < r^2 \tag{3}$$

By QR decomposition of channel matrix $H$, Eqn. 3 can be decomposed as follows.

$$\|y - Hs\|^2 = \|y - QRs\|^2 < r^2 \tag{4}$$

which again can be written as

$$\left\|Q^H y - Rs\right\|^2 = \|z - Rs\|^2 < r^2 \tag{5}$$

Q is $Nr \times Nr$ orthogonal matrix and R is $N_t \times N_t$ upper triangular matrix. $z$ is $Q^H y$ and a $N_t \times 1$ vector. By the property of upper triangular matrix R, the rule of ML detection can be mapped to a tree-searching problem. Metric for tree search is defined as Partial Euclidian Distance (PED) which is given by

$$PED : T_i\left(s^i\right) = \left\|b_{i+1} - r_{ii}s_j\right\|^2 + T_{i+1}\left(s^{i+1}\right) \quad (6)$$

where $b_{i+1} = z_i - \sum_{j=i+1}^{N_t} r_{ij}s_j$ and $T_{N_{t+1}}\left(s^{N_{t+1}}\right) = 0$

Tree searching is started at level $i = N_t$. After calculating PED, the next node to visit is determined by SD algorithm. Various SD algorithms can be classified into three categories according to their tree searching stategies which are Fincke and Phost (FP), Schnorr-Euchner (SE) and K-Best(KB) Algorithm. This paper consider SE-SD since it can guarantee fixed throughput without severe performance degradation [4].

The complexity of SD also depends on the radius $r$ of the hypersphere. If $r$ is too large, the hypersphere contains too many lattice points, if $r$ is too small, the hypersphere may contain no lattice point at all. There are no general guidelines for selecting $r$ and it depends on the particular application. Next section discusses a simple method to choose an appropriate $r$.

## *3.1 Radius Selection*

A deterministic method for selecting an initial hypershere radius has been proposed by Qiao in [8]. This method is designed for communication application.

The steps involved are:

1. Solve $z = Rs$.
2. Round the entries of $s$ to their nearest integer, $\widetilde{s} = \lceil s \rfloor$
3. Set $r = \|R\widetilde{s} - z\|_2$

This $\widetilde{s}$ is known as Babai estimate. The Sphere $\|Rs - z\|_2 = r$ contains atleast one lattice point, namely $\widetilde{s}$, the integer vector closest to the real least squares solution $s$. In communication application, this procedure is nothing but ZF equalization.
Let us examine the size of r.
Take $d = \widetilde{s} - s$,

then $R\widetilde{s} - z = R(s+d) - z = Rd$, as $z = Rs$

Since $d = \lceil s \rfloor - s$, $\|d\|_2 \leq \sqrt{N_t}/2$.

Thus, $r = \|Rd\|_2 \leq \sqrt{N_t}/2\,\|R\|_2 = \sqrt{N_t}/2\,\|H\|_2 = \|H\|_2$ as we take $N_t = 4$ for simulation.

In this paper, the channel considered is a multipath wi-fi indoor channel modelled by Rayleigh fading. Due to channel power constraint, $\|H\|_2$ is not large. Consequently the radius of the search sphere is not large. The SD algorithm using $r = \|R\widetilde{s} - z\|_2$ will find the integer least square solution in this sphere, possibly on its surface. Due to imperfection in channel estimation and rounding errors in calculation of $r$, the SD algorithm fails to find a lattice point in the computed hypersphere at low SNR. So a value greater than the computed $r$ which does not compromise the complexity and yet give a good performance has to be selected for the simulation.

### 3.2 Pseudo-antenna Augmentation Scheme

The sphere decoders designed for the case where $N_r \geq N_t$ fail when $N_r < N_t$ since the channel matrix $H$ does not have full column rank and therefore cannot be QR factorized. In paper [5], a modification has been done to the channel matrix estimated, as

$$\widetilde{H} = \begin{bmatrix} eI_{(N_t-N_r)} & 0_{(N_t-N_r)\times N_r} \\ \multicolumn{2}{c}{H} \end{bmatrix} \tag{7}$$

to make it a matrix with full column rank.

Here the bottom $Nr$ rows comprise the original channel matrix. $I$ is the identity matrix. $e$ is either a small real or complex number depending on the modulation scheme.

Zeros are augmented to first $N_t - N_r$ rows of final received vector

$$\widetilde{y}_{N_t\times 1} = \begin{bmatrix} 0_{(N_t-N_r)\times N_r} \\ y_{N_r\times 1} \end{bmatrix} \tag{8}$$

Then the psudo received vector may be defined as

$$\begin{bmatrix} es_1 \\ \vdots \\ es_{N_t-N_r} \\ \sum_{i=1}^{N_t} h_{1i}s_i + n_1 \\ \vdots \\ \sum_{i=1}^{N_t} h_{N_r,i}s_i + n_{N_r} \end{bmatrix} \tag{9}$$

and the noise vector as

$$\begin{bmatrix} -es_1 \\ \vdots \\ -es_{N_t-N_r} \\ 0_{(Nr\times 1)} \end{bmatrix} \tag{10}$$

$\widetilde{H}$ has full column rank by this augmentation and can be decomposed using standard QR factorizing algorithms.

The effect of value taken by $e$ has been analyzed in [5]. If $e$ is small, lower bound on the radius with which correct symbol vector included is essentially independent of $e$. But if $e$ is large, radius needs to be large. With very small $e$, complexity of SD is independent of $e$ and same as that of usual SD algorithm which is roughly $O(N_t^3)$. So value of $e$ was selected as 0.01.

## 4 Complexity Comparison

Complexity of ZF, MMSE, ML and SD are given in terms of number of complex operations involved, in Table 1. For easiness of presentation we take $N_r = N_t$. $N$ is the number of sub carriers used in OFDM. The computational complexity of calculating $\|y - Hs\|^2$ is determined in [7] to be

$$C_o = 2N_t^2 + 2N_t - 1 \tag{11}$$

Since ML detection searches entire symbol alphabet, its complexity is $2^{mN_t}C_o$ for single ofdm carrier, where $2^m$ is the number of legitimate complex constellation points associated with the specific modulation scheme as mentioned. The paper [6] examine complexity of SD analytically. We know that the complexity is mainly due to number of lattice points in hypershere with selected radius, and dimensions $k = 1, \dots N_t$. Let the number of search points searched by the algorithm in the hypershere of a given radius considering all the influencing factors be $r_o$. Then the complexity per carrier will be proportional to $r_o C_o$. The worst case value of complexity is exponential, but we always select radius in such a way to restrict the complexity proportional to an integer multiple of of $N_t$. It is true especially when radius is calculated using Babai estimate. In such case, complexity is almost same as ZF. So let us assume that the complexity will not go beyond $2N_t C_o$. By using Gauss-Jordan Elimination algorithm to compute matrix inversions, complexity of ZF and MMSE per ofdm carrier is given in [7] and this has been used to compute the number of complex operations required.

**Table 1** Computational Complexity Comparison of ZF, MMSE, ML and SD

| OFDM/SDMA detector | Numer of Complex Operations |
|---|---|
| ZF | $N((14/3)N_t^3 + 5N_t^2 - (8/3)N_t)$ |
| MMSE | $N((14/3)N_t^3 + 5N_t^2 - (2/3)N_t + 1)$ |
| ML | $N(2^{mN_t} C_o)$ |
| SD | $N(2N_t C_o)$ |

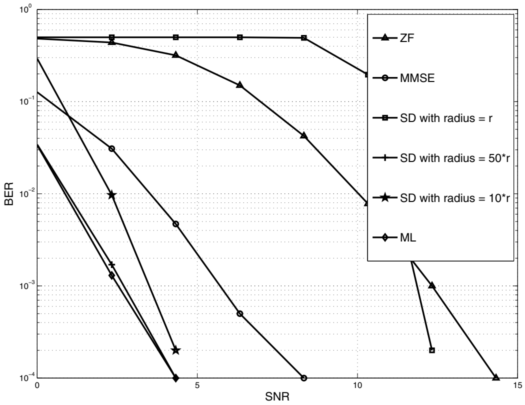


**Fig. 2** BER vs SNR performance comparison of SD aided OFDM/SDMA with ML, MMSE, and ZF when Nt = Nr = 4

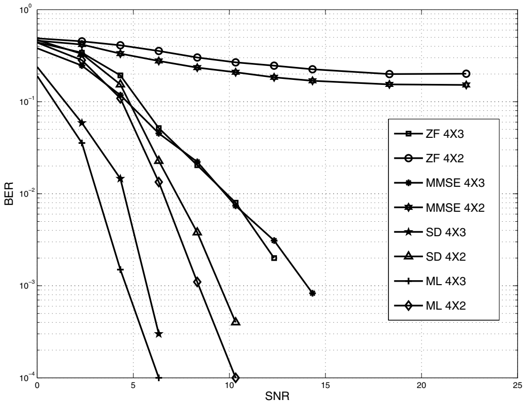


**Fig. 3** BER vs SNR performance comparison of SD aided OFDM/SDMA with ML, MMSE and ZF when Nt = 4, Nr = 3, 2

# 5 Simulation Results

A schematic of multiuser OFDM/SDMA uplink system has been shown in Fig. 1, based on which simulation was performed in MATLAB. IEEE 802.11n high speed WLAN standards are used for the simulation. Bandwidth is 20 MHz. Table 1 give the parameters used for the simulation. MATLAB inbuilt commands ('poly2trellis', and 'convenc' with [133 171] as generator polynomial) was used for half-rate error control coding. Input data was scrambled before convolutional encoding and interleaved after encoding. All these schemes improved the overall perfomance but with some increase in data overhead and complexity. Perfect Channel State Information (CSI) is assumed at the receiver and more focus is given to multiuser detection when channel is underdetermined.

Figure 2 shows the performance comparison of of SD with ML, ZF and MMSE. As explained in Subsect. 3.1, initial radius was calculated using Babai estimate. At low SNRs the performance of SD seems to be very much degraded compared to all other detectors as the algorithm doesnt find any lattice point inside this sphere due to some practical errors as discussed before. The complexity nears that of ZF at this radius. At a radius 50 times that of $r$, the performance of SD is almost same as ML. It is intuitive that complexity also nears that of ML. At a value less than this, say when radius is 10 times $r$, the performance was still closer to ML but at a lower complexity for sure. A bit error rate (BER) of $10^{-3}$ is achieved at SNR $3.5dB$.

Next, Fig. 3 shows the performance of SD when number of receivers are less than transmitters. The radius was chosen as 20 times $r$ as it gives a good performance at lower complexity than ML. Performance of ZF and MMSE keep on degrading severely as the channel become more underdetermined. But performance of SD was found closer to ML, which is really a notable advantage for implementing a practical system.

From the above two plots it is clear that performance of SD algorithm is poor at low SNRs. So selecting large radii at low SNRs and small radii at High SNRs will improve the overall performance of the algorithm. In Table 3, a comparison is shown between two simulations performed when radius is constant for all SNRs, and a linearly reducing radii. The simulation is done for $N_r = 3$ and $N_t = 4$. There was a slight improvement in the initial BER. The significant advantage is the reduced simulation time for linearly varying radii. Simulation time is the time taken to run the MATLAB simulation keeping all other parameters same for both cases. The values of radii vary slightly in each simulation as value of $R$ and $\widetilde{s}$ vary.

**Table 2** Simulation Parameters of OFDM/SDMA

| Parameters | Specifications |
|---|---|
| FFT size | $N = 64$ |
| cyclic Prefix length | $L = 16$ |
| modulation | BPSK |
| channel model | Multipath Rayleigh fading |
| number of multipaths selected | 10 |
| channel State Information | Perfect |
| number of BS Antennas | $N_r = 4, 3, 2$ |
| number of simultaneous users | $N_t = 4$ |

**Table 3** Performance Comparison of SD algorithm for constant and varying radii

| Radius | Simulation Time | BER at SNR 0dB |
|---|---|---|
| 18 (on an average) for all SNR | 211 seconds | 0.2375 |
| [82.7532 18.9945 17.1404 15.6659 12.2056 9.6533 7.2722 6.2088 5.3234 5.0695 3.2076 1.2024] | 188 seconds | 0.1803 |

## 6 Conclusion

At full-load, complexity of ZF and MMSE are far less than ML and give acceptable performance. But we can see that SD works far better than them at a complexity less than ML, and nears the performance of ML. Performance of MMSE and ZF are completely degraded when channel is underdetermined. A simple pseudo-antenna augmentation scheme incorporated in SD algorithm makes it an efficient detection scheme for an over-loaded OFDM/SDMA system where the conventional detectors fail severely. The linrarly varying radii selection adopted ensures better BER at an acceptable complexity.

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