Conflict between Belief Functions: A New Measure Based on Their Non-conflicting Parts

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Abstract. When combining belief functions by conjunctive rules of combination, conflicts often appear, which are assigned to empty set by the non-normalised conjunctive rule or normalised by Dempster's rule of combination in Dempster-Shafer theory. Combination of conflicting belief functions and interpretation of their conflicts is often questionable in real applications; hence a series of alternative combination rules were suggested and a series of papers on conflicting belief functions have been published and conflicts of belief functions started to be investigated.

This theoretical contribution introduces a new definition of conflict between two belief functions on a general finite frame of discernment. Its idea is based on Hájek-Valdés algebraic analysis of belief functions, on our previous study of conflicts of belief functions, where internal conflicts of belief functions are distinguished from a conflict between belief functions, and on the decomposition of a belief function into its conflicting and non-conflicting parts. Basic properties of this newly defined conflict are presented, analyzed and briefly compared with our previous approaches to conflict as well as with Liu's degree of conflict.

Keywords: belief functions, Dempster-Shafer theory, uncertainty, Dempster's semigroup, internal conflict, conflict between belief functions, non-conflicting part of belief function, conflicting part of belief function.

1 Introduction

Complications of highly conflicting belief function combination, see e.g., [2, 6, 27], have motivated a theoretical investigation of conflicts between belief functions (BFs) [8, 16, 21–25]. The problematic issue of an essence of conflict between belief functions (BFs), originally defined as $m_{\bigcirc\!\!\!\!\cap}(\emptyset)$ by the non-normalised version of Dempster's rule $\bigcirc\!\!\!\!\cap$, was first mentioned by Almond [1] in 1995, and discussed further by Liu [22] in 2006. Almond's counter-example has been overcome by Liu's progressive approach. Unfortunately, the substance of the problem has not thus been solved as positive conflict value still may be detected in a pair of mutually non-conflicting BFs.

Further steps ahead were presented in our previous study [8]. New ideas concerning interpretation, definition, and measurement of conflicts of BFs were introduced there. Three new approaches to interpretation and computation of

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conflicts were presented: combinational conflict, plausibility conflict, and comparative conflict. Unfortunately, none of those captures the nature of conflict sufficiently enough yet; thus these approaches need further elaboration. Nevertheless, the very important distinction between conflict between BFs and internal conflicts in individual BFs is pointed out there; and the necessity to distinguish between a conflict and difference among BFs is emphasized.

When the mathematical properties of the three approaches to BF conflicts were analyzed, there appeared a possibility of expressing a BF Bel as Dempster's sum of non-conflicting BF Bel_0 with the same plausibility decisional support as the original BF Bel has and of indecisive BF Bel_S which does not prefer any of the elements in the corresponding frame of discernment — see [9]. A new measure of conflict between BFs is based on that approach.

2 Preliminaries

We assume classic definitions of basic notions from theory of *belief functions* (BFs) [26] on a finite frame of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$, see also [5–7]. We say that BF Bel is *non-conflicting* when conjunctive combination of Bel with itself does not produce any conflicting belief masses (when $(Bel \textcircled{\cap} Bel)(\emptyset) = 0$, i.e., $Bel \textcircled{\cap} Bel = Bel \oplus Bel$), i.e. whenever $Pl(\omega_i) = 1$ for some $\omega_i \in \Omega_n$. Otherwise, BF is *conflicting*, i.e., it contains some internal conflict [8].

Let us recall *normalised plausibility of singletons*[1] of Bel: Pl_P is the Bayesian BF (i.e., probability distribution on Ω_n in fact) $Pl\_P(Bel)$ (or simply Pl_P if corresponding Bel is obvious) such, that $Pl\_P(\omega_i) = \frac{Pl(\{\omega_i\})}{\sum_{\omega \in \Omega} Pl(\{\omega\})}$, where Pl is plausibility corresponding to Bel [3, 7] and alternative Smets' *pignistic probability* $BetP(\omega_i) = \sum_{\omega_i \in X} \frac{m(X)}{|X|}$. An *indecisive BF* (or non-discriminative BF) is a BF which does not prefer any $\omega_i \in \Omega_n$, i.e., a BF which gives no decisional support for any $\omega_i \in \Omega_n$ (it either gives no support as the vacuous BF (VBF), gives the same support to all elements as symmetric BFs give, or $Pl\_P(Bel) = U_n$ ($Pl\_P(\omega) = \frac{1}{n}$ for any $\omega \in \Omega_n$). $S_{Pl} = \{Bel \,|\, Pl\_P(Bel) = U_n\}$.

We can represent BFs by enumeration of their m-values, i.e., by $(2^n$–$1)$-tuples or by $(2^n$–$2)$-tuples as $m(\Omega_n) = 1 - \sum_{X \subsetneq \Omega_n} m(X)$; thus we have pairs $(a, b) = (m(\{\omega_1\}), m(\{\omega_2\}))$ for BFs on Ω_2.

Hájek-Valdés algebraic structure $\mathbf{D}_0$ of these pairs (called *d-pairs*) with Dempster's rule $\oplus$ (called *Dempster's semigroup*) and its analysis [19, 20, 28] were further studied and generalised by the author, e.g., in [5, 10]. There are distinguished d-pairs $0 = (0, 0)$ (i.e., *vacuous BF*) and $0' = (\frac{1}{2}, \frac{1}{2})$, $-(a, b) = (b, a)$, homomorphisms[2,3] $h:\ h(a, b) = (a, b) \oplus 0' = (\frac{1-b}{2-a-b}, \frac{1-a}{2-a-b})$ and $f:\ f(a, b) = (a, b) \oplus -(a, b) = (\frac{a+b-a^2-b^2-ab}{1-a^2-b^2}, \frac{a+b-a^2-b^2-ab}{1-a^2-b^2})$. We will use the following subsets of d-pairs: $S = \{(a, a)\}$, $S_1 = \{(a, 0)\}$, $S_2 = \{(0, b)\}$, and $G = \{(a, 1-a)\}$.

[1] Plausibility of singletons is called *contour function* by Shafer [26], thus $Pl\_P(Bel)$ is a normalisation of contour function in fact (thus $\sum_{\omega \in \Omega} Pl\_P(\omega) = 1$).

[2] Note that $h(a, b)$ is an abbreviation for $h((a, b))$, similarly for $f(a, b)$.

[3] $0'$ and h are generalised by $U_n = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}, 0, 0, ..., 0)$ and $h(Bel) = Bel \oplus U_n$ on Ω_n.

We can express BFs on Ω_2 (d-pairs) by a 2-dimensional triangle, see Fig. 1. Complexity of the structure grows exponentially with cardinality of the frame of discernment, e.g., we have a 6-dimensional structure on Ω_3, see [9, 10].

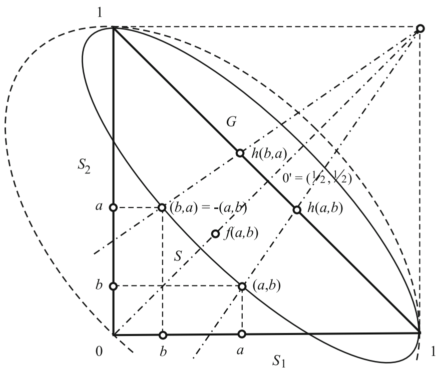

Fig. 1. $\mathbf{D}_0$. Homomorphism h is in this representation a projection of $\mathbf{D}_0$ to group $\mathbf{G}$ on $G = \{(a, 1-a)\}$ along the straight lines running through the point $(1, 1)$. All of the d-pairs lying on the same ellipse (running through points $(0, 1)$ and $(1, 0)$) are mapped by homomorphism f to the same d-pair in semigroup on $S = \{(s, s)\}$.

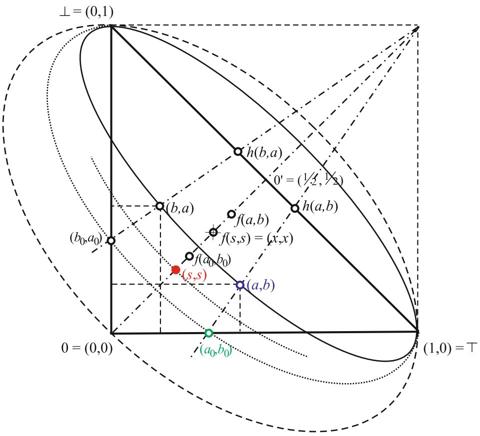

Fig. 2. Non-conflicting part (a_0, b_0) and conflicting part (s, s) of a BF (a, b) on a 2-element frame of discernment Ω_2

3 Conflicts of Belief Functions

Internal conflicts $IntC(m)$ which are included in particular individual BFs are distinguished from *conflict between BFs* $C(m_1, m_2)$ in [8]; the entire sum of conflicting masses is called *total conflict* $TotC(m_1, m_2)$; and three approaches to conflicts were introduced: combinational, plausibility and comparative.

Unfortunately, there are not yet any precise formulas, but only bounding inequalities for *combinational conflicts*: $\frac{1}{2}TotC(m, m)) \leq IntC(m) \leq TotC(m, m)$, $TotC(m_1, m_2) - (IntC(m_1) + IntC(m_2)) \leq C(m_1, m_2) \leq TotC(m_1, m_2)$.

Internal plausibility conflict of BF Bel is defined as *Pl-IntC*$(Bel) = 1 - max_{\omega \in \Omega} Pl(\{\omega\})$, where Pl is the plausibility equivalent to Bel.

Plausibility conflict between BFs Bel_1 and Bel_2 is defined by the formula *Pl-C*$(Bel_1, Bel_2) \ = min(\sum_{\omega \in \Omega_{PlC}(Bel_1, Bel_2)} \frac{1}{2} |Pl\_P(Bel_1)(\omega) - Pl\_P(Bel_2)(\omega)|$, $(m_1 \circledcirc m_2)(\emptyset)$), where *conflicting set* $\Omega_{PlC}(Bel_1, Bel_2)$ is the set of elements $\omega \in \Omega$ with conflicting Pl_P values [8]. For an analysis and improvement of *Pl-C* and analogously defined *pignistic conflict Bet-C* see [11, 12, 15].

The idea of comparative conflictness / non-conflictness is a specification of basic belief masses (m-values) to smaller focal elements, which fit focal elements of the other BF as much as possible. *The comparative conflict between BFs Bel_1 and Bel_2* is defined as the smallest difference of such more specified basic belief assignments derived from the input m_1 and m_2.

The above three approaches were compared with Liu's degree of conflict cf in [8]; cf is defined as $cf(m_i, m_j) = (m_{\bigcirc\!\!\!\!\cap}(\emptyset), difBetP_{m_i}^{m_j})$ in [22], $difBetP_{m_i}^{m_j}$ is defined as $difBetP_{m_i}^{m_j} = max_{A \subseteq \Omega}(|BetP_{m_i}(A) - BetP_{m_j}(A)|)$.

Analysing these three approaches to conflicts [8], especially plausibility conflict *Pl-C*, the most elaborated of the approaches, a possibility of decomposition of a belief function into its conflicting and non-conflicting parts was observed.

We can use the important property of Dempster's combination, which is respecting the homomorphisms h and f, i.e., respecting the h-lines and f-ellipses, when two BFs are combined on a two-element frame of discernment [5, 19, 20, 28], see Fig 2. Using this property and two technical lemmata from [9] we obtain:

Theorem 1. *Any BF (a, b) on a 2-element frame of discernment Ω_2 is Dempster's sum of its unique* non-conflicting part $(a_0, b_0) \in S_1 \cup S_2$ *and of its unique* conflicting part $(s, s) \in S$, *which does not prefer any element of Ω_2, i.e., $(a, b) = (a_0, b_0) \oplus (s, s)$. It holds true that $s = \frac{b(1-a)}{1-2a+b-ab+a^2} = \frac{b(1-b)}{1-a+ab-b^2}$ and $(a, b) = (\frac{a-b}{1-b}, 0) \oplus (s, s)$ for $a \geq b$; and similarly that $s = \frac{a(1-b)}{1+a-2b-ab+b^2} = \frac{a(1-a)}{1-b+ab-a^2}$ and $(a, b) = (0, \frac{b-a}{1-a}) \oplus (s, s)$ for $a \leq b$.*

An algebraic analysis of Dempster's semigroup on Ω_3 is currently in development. We have only a simple description of the set of indecisive BFs, the most basic algebraic substructures on Dempster's semigroup on Ω_3 now [10]. Thus we do not have an analogy of Theorem 1 for BFs defined on general finite frames, and existence of their unique conflicting part is still an open problem.

On the other hand, we have already proven homomorphic properties of h : $h(Bel) = Bel \oplus U_n$ and also existence of a unique non-conflicting part Bel_0 for any BF Bel on any finite frame of discernment Ω_n [9].

Theorem 2. *For any BF Bel defined on a general finite Ω_n there exists a unique consonant BF Bel_0 such that,*

$$h(Bel_0 \oplus Bel_S) = h(Bel)$$

for any BF Bel_S for which $Bel_S \oplus U_n = U_n$ (Especially also $h(Bel_0) = h(Bel)$).

Algorithm 1. *(Computing the non-conflicting part of a BF).* Take all element(s) with maximal contour value (plausibility of singletons); they create the least focal element of Bel_0; assign to it the m-value equal to the difference between the max and max but one (different) contour values. A cycle: among the remaining elements (if any remains) take again all the element(s) with maximal contour value and add them to the previous focal element, thus you obtain a new focal element of Bel_0 (m-value: the corresponding difference between different contour values again). Repeat the cycle until Ω_n is obtained with m-value equal to min contour value. For a positive minimal contour value include Ω_n among focal elements of Bel_0. For a non-consistent BF Bel ($Pl(\{\omega_i\}) < 1$ for any $\omega_i \in \Omega_n$) we need final normalisation of Bel_0.

More formally (a construction of the set of focal elements SFE and basic belief assignment m defined on SFE):

FE := $\emptyset$; SFE := $\emptyset$; Ω := Ω_n
Max_Pl := $Pl(\{\omega\})$, where $\omega \in \Omega_n$ s. t. $Pl(\{\omega\}) \geq Pl(\{\omega'\})$ for any $\omega' \in \Omega_n$
Min_Pl := $Pl(\{\omega\})$ where $\omega \in \Omega_n$ s. t. $Pl(\{\omega\}) \leq Pl(\{\omega'\})$ for any $\omega' \in \Omega_n$
Max1 := $\{\omega \in \Omega \mid Pl(\{\omega\}) = \text{Max\_Pl}\}$
$\Omega := \Omega \setminus$ Max1
Max2 := $\{\omega \in \Omega \mid Pl(\{\omega\}) \geq Pl(\{\omega'\})$ for any $\omega' \in \Omega\}$
while Max2 $\neq \emptyset$ **do**
  FE := FE $\cup$ Max1; SPE := SPE $\cup$ {FE}
  m(FE) := $Pl(\{\omega_1\}) - Pl(\{\omega_2\})$, where $\omega_1 \in$ Max1, $\omega_2 \in$ Max2
  Max1 := Max2; $\Omega := \Omega \setminus$ Max1
  Max2 := $\{\omega \in \Omega \mid Pl(\{\omega\}) \geq Pl(\{\omega'\})$ for any $\omega' \in \Omega\}$
end while
if Min_Pl > 0 **then**
  SFE := SPE $\cup$ Ω_n (as FE $\cup$ Max1 = Ω_n now)
  $m(\Omega_n)$:= Min_Pl
end if
if Max_Pl < 1 **then** normalisation of m (because $\sum_{X \subseteq \Omega_n} m(X) = \text{Max\_Pl}$)
end if

4 Conflict between Belief Functions Based on Their Non-conflicting Parts

4.1 Motivation and Definition of a New Measure of Conflict

One of the main problems of the previous definitions of conflict between BFs is the fact that the defined conflict usually includes some part (or even entire in the original Shafer's definition) of internal conflicts which are included inside the BFs in question. The other frequent problem is that the definitions of conflict between BFs are incorrectly related to distance or difference between the BFs.

In the following Theorem 1 we have unique decomposition of any belief function Bel on Ω_2 into its non-conflicting and conflicting parts $Bel = Bel_0 \oplus Bel_S$. There is no conflict in Bel_0 and entire internal conflict of Bel is included in Bel_S (as we suppose Bel_S to be non-conflicting with any BF for all Bel_S such that $h(Bel_S) = 0'$). Unfortunately, we do not have such a decomposition for BFs on a general finite frame of discernment Ω_n (this topic is still under investigation). Nevertheless, according to Theorem 2, we have a unique non-conflicting part Bel_0 for any BF Bel on Ω_n, such that Bel_0 does not include any part of internal conflict of the original BF Bel.

Thus $(m'_0 \circledcirc m''_0)(\emptyset) = TotC(Bel'_0, Bel''_0) = C(Bel'_0, Bel''_0)$ holds true for any couple of BFs Bel', Bel'' on Ω_n, their con-conflicting parts Bel'_0, Bel''_0 and the related bbas m'_0, m''_0. Using these facts, we can define the conflict between BFs

Bel' and Bel'' as the conflict between their non-conflicting parts Bel'_0 and Bel''_0 (as conflicting parts Bel'_S, Bel''_S are mutually non-conflicting and both of them are non-conflicting with both non-conflicting parts Bel'_0 and Bel''_0):

Definition 1. *Let Bel', Bel'' be two belief functions on n-element frame of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$. Let Bel'_0 and Bel''_0 be their non-conflicting parts and m'_0, m''_0 the related basic belief assignments (bbas). We define* conflict between BFs Bel' and Bel'' *as*

$$Conf(Bel', Bel'') = m_{Bel'_0 \circledcirc Bel''_0}(\emptyset) = (m'_0 \circledcirc m''_0)(\emptyset).$$

Example 1. Let us suppose Ω_3, now; and four BFs m', m'', m''', and m'''' given as follows:

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$	Ω_3
$m'(X)$:	0.375	0.100	0.225	0.10			0.20
$m''(X)$:	0.250	0.175	0.175	0.20	0.05	0.05	0.10
$m'''(X)$:	0.350	0.250		0.25	0.05		0.10
$m''''(X)$:	0.100	0.200		0.40	0.00	0.00	0.30

$Pl\_P' = (0.45, 0.20, 0.35)$, $Pl\_P'' = (0.40, 0.35, 0.25)$, $Pl\_P''' = (0.50, 0.40, 0.10)$, $Pl\_P'''' = (0.40, 0.45, 0.15)$, $m'_0 = (\frac{10}{45}, 0, 0, 0, \frac{15}{45}, 0; \frac{20}{45})$, $m''_0 = (0.125, 0, 0, 0.25, 0, 0; 0.625)$, $m'''_0 = (0.20, 0, 0, 0.60, 0, 0; 0.20)$, $m''''_0 = (0, \frac{5}{45}, 0, \frac{25}{45}, 0, 0; \frac{15}{45})$. Thus,
$Conf(Bel', Bel'') = 0 = Conf(Bel', Bel''') = Conf(Bel'', Bel''')$; and
$Conf(Bel', Bel'''') = \frac{10}{45} \cdot \frac{5}{45} + \frac{5}{45} \cdot \frac{15}{45} = \frac{5}{81}$; $Conf(Bel'', Bel'''') = \frac{5}{45} \cdot \frac{5}{40} = \frac{1}{72}$;
$Conf(Bel''', Bel'''') = \frac{10}{50} \cdot \frac{5}{45} = \frac{1}{45}$.

Let us also present some highly conflicting examples. For simplicity we consider Ω_3 again; for examples on larger frames of discernment see [14].

Example 2. Let us suppose Ω_3, again; and two pairs of highly conflicting BFs m', m'', and m''', m'''' now:

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$	Ω_3
$m'(X)$:	0.9		0.1				
$m''(X)$:		0.9	0.1				
$m'''(X)$:			1.0				
$m''''(X)$:	0.3	0.1		0.6			

$Pl\_P' = (0.9, 0.0, 0.1)$, $Pl\_P'' = (0.0, 0.9, 0.1)$, $Pl\_P''' = (0.0, 0.0, 1.0)$, $Pl\_P'''' = (\frac{9}{16}, \frac{7}{16}, 0.0)$, $m'_0 = (\frac{8}{9}, 0, 0, 0, \frac{1}{9}, 0; 0)$, $m''_0 = (0, \frac{8}{9}, 0, 0, 0, \frac{1}{9}, ; 0)$, $m'''_0 = (0, 0, 1.0, 0, 0, 0; 0)$, $m''''_0 = (\frac{2}{9}, 0, 0, \frac{7}{9}, 0, 0; 0)$.

Thus, $Conf(Bel', Bel'') = \frac{8}{9} \cdot \frac{8}{9} + \frac{8}{9} \cdot \frac{1}{9} + \frac{8}{9} \cdot \frac{1}{9} = \frac{80}{81} = 0.98765432$.
$Conf(Bel', Bel''') = \frac{8}{9} = Conf(Bel'', Bel''')$; and $Conf(Bel''', Bel'''') = \frac{2}{9} + \frac{7}{9} = 1.0$.
(Of course there is $Conf(Bel', Bel'''') = 0$ and small conflict $Conf(Bel'', Bel'''')$.)

4.2 An Analysis of Properties of the Measure of Conflict *Conf*

Let us present properties of *Conf* now, for proofs of the statements see [14].

Lemma 1. *Conflict Conf between belief functions is symmetric:* $Conf(Bel', Bel'') = Conf(Bel'', Bel')$.

Lemma 2. *Any BF* $Bel_{SPl} \in S_{Pl}$ *is non-conflicting with any other BF on* Ω_n: $Conf(Bel_{SPl}, Bel) = 0$ *for any* Bel *defined on* Ω_n *and any BF* $Bel_{SPl} \in S_{Pl} = \{Bel \,|\, Pl\_P(Bel) = U_n\}$.

Corollary 1. *(i) Any BF* $Bel_{S0} \in S_0 = \{(a, a, ..., a, 0, 0, ...0; 1-na) \,|\, 0 \leq a \leq \frac{1}{n}\}$ *is non-conflicting with any other BF Bel defined on* Ω_n*, i.e.,* $Conf((a, a, ..., a, 0, 0, ...0; 1-na), Bel) = 0$*. This specially holds true also for* 0 *and* U_n.
(ii) Any symmetric BF $Bel_S \in S = \{Bel \,|\, m(X) = m(Y) \text{ for } |X| = |Y|\}$ *is non-conflicting with any other BF Bel defined on* Ω_n*, i.e.,* $Conf(Bel_S, Bel) = 0$.

Theorem 3. *Let* Bel' *and* Bel'' *be general BFs defined on an* n*-element frame of discernment* Ω_n*, let* Bel'_0 *and* Bel''_0 *be their unique non-conflicting parts, and* $X' = \{\omega \in \Omega_n \,|\, Pl'(\{\omega\}) \geq Pl'(\{\omega'\}) \text{ for any } \omega' \in \Omega_n\}$, $X'' = \{\omega \in \Omega_n \,|\, Pl''(\{\omega\}) \geq Pl''(\{\omega'\}) \text{ for any } \omega' \in \Omega_n\}$*. The following statements are equivalent:*
(i) BFs Bel' *and* Bel'' *are mutually non-conflicting, i.e.* $Conf(Bel', Bel'') = 0$,
(ii) The least focal elements of Bel'_0 *and* Bel''_0 *have non-empty intersection,*
(iii) $X' \cap X'' \neq \emptyset$.

Corollary 2. *(i) For any BF Bel on* Ω_n *the following holds:* $Conf(Bel, Bel) = 0$.
(ii) For any couple of BF Bel' *and* Bel'' *defined on* Ω_n *such that* $Pl\_P' = Pl\_P''$ *the following holds true:* $Conf(Bel', Bel'') = 0$.
(iii) For any couple of BFs (a, b) *and* (c, d) *defined on* Ω_2 *such that* $BetP(a, b) = BetP(c, d)$ *the following holds true:* $Conf((a, b), (c, d)) = 0$.

Note that assertion (iii) holds true just for BFs defined on Ω_2. Thus, on general Ω_n, there exist mutually conflicting BFs with same pignistic probabilities, see Example 3.

Example 3. Let us suppose Ω_3, now; and two BFs m' and m'' given as follows:

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	Ω_3
$m'(X)$:	0.21	0.22	0.44	0.10	0.03
$m''(X)$:	0.01	0.02	0.44	0.50	0.03

$BetP' = (0.27, 0.28, 0.45) = BetP''$, $Pl\_P' = (\frac{34}{116}, \frac{35}{116}, \frac{47}{116})$, $Pl\_P'' = (\frac{54}{156}, \frac{55}{156}, \frac{47}{156})$, $m'_0 = (0, 0, \frac{12}{47}, 0, 0, \frac{1}{47}; \frac{34}{47})$, $m''_0 = (0, \frac{1}{55}, 0, \frac{7}{55}, 0, 0; \frac{47}{55})$, $Conf(Bel', Bel'') = \frac{8 \cdot 12}{47 \cdot 55} = \frac{96}{2585} = 0.037137$. Thus the conflict between BFs is small, but it is positive.

Theorem 4. *Let* Bel' *and* Bel'' *be arbitrary BFs on a general finite frame of discernment* Ω_n *given by bbas* m' *and* m''*. For conflict Conf between* Bel' *and* Bel'' *it holds that*

$$Conf(Bel', Bel'') \leq (m' \circledcirc m'')(\emptyset).$$

Despite a simple idea and the simple definition of conflict $Conf$ between BFs, there are many variants of explicit formulas for computation of the conflict, due to different ordering of m-values of focal elements of the BFs. For illustration, we present only the simplest case of BFs on a 2-element frame of discernment:

$$Conf((a,b),(c,d)) \;=\; \frac{a-b}{1-b}\cdot\frac{d-c}{1-c} \;\text{ if }\; a>b \;\&\; c<d,$$

analogously for $a<b$ & $c>d$, $Conf((a,b),(c,d)) = 0$ otherwise.

In general, we have just to follow Definition 1: to compute non-conflicting parts Bel'_0 and Bel''_0 of both BFs in question (Algorithm 1) and simply apply $\circledcirc$.

Martin's Axioms of Conflict between Belief Functions. There are the following axioms of conflict between belief functions presented in [23]:

(A1) : $Conf(Bel', Bel'') \geq 0$,
(A2) : $Conf(Bel, Bel) = 0$,
(A3) : $Conf(Bel', Bel'') = Conf(Bel'', Bel')$,
(A4) : $Conf(Bel', Bel'') \leq 1$.

All of these axioms[4] are satisfied by the conflict $Conf$ according our Definition 1. Martin underlines, that he does not assume triangle inequality $Conf(Bel', Bel''') \leq Conf(Bel', Bel'') + Conf(Bel'', Bel''')$. Note, that our definition of the conflict is the case, where triangle inequality does not hold true, see Example 4.

In addition to these axioms, we should mention also important properties from Theorem 4 and Lemma 1 resp. its corollary on symmetric belief functions.

Example 4. Let $Bel' = (0.4, 0.1, 0.1, 0.2, 0, 0.1; 0.1)$, $Pl\_P' = (\frac{7}{15}, \frac{5}{15}, \frac{3}{15})$, $Bel'' = (0.3, 0.2, 0.1, 0.1, 0, 0.1; 0.2)$, $Pl\_P'' = (\frac{6}{16}, \frac{6}{16}, \frac{4}{16})$, $Bel''' = (0.1, 0.2, 0.3, 0.1, 0, 0.2; 0.1)$, $Pl\_P''' = (\frac{3}{15}, \frac{6}{15}, \frac{6}{15})$, $Bel'_0 = (\frac{2}{7}, 0, 0, \frac{2}{7}, 0, 0; \frac{3}{7})$, $Bel''_0 = (0, 0, 0, \frac{2}{6}, 0, 0; \frac{4}{6})$, $Bel'''_0 = (0, 0, 0, 0, 0, \frac{3}{6}; \frac{3}{6})$, $Conf(Bel', Bel''') = \frac{1}{7} \not\leq 0 = Conf(Bel', Bel'') + Conf(Bel'', Bel''')$.

5 Open Problems for Future Research

We have a simply defined conflict between two belief functions on a general finite frame of discernment. Nevertheless, to complete this study of conflicts of BFs we will have to define and analyze also internal conflicts of individual BFs.

Two main open issues remain: The first one is a question of precise interpretation of the conflicting part of a belief function and its relationship to the internal conflict of the BF on a 2-element frame of discernment. First results are presented in [13].

The second, more complex issue is a study of internal conflict of BFs on a general finite frame of discernment. This also includes a question whether a decomposition of a general BF exists to its non-conflicting and conflicting parts; consequently, a generalisation of Hájek-Valdés algebraic analysis of BFs to a general frame of discernment is concerned, namely a generalisation of the operation $-(a,b) = (b,a)$ and of homomorphism f.

As another open question remains a further elaboration of the theoretic principles of the presented results with those from [16] and [23].

[4] There is also (A5), unfortunately mistyped or incorrectly formulated in [23], see [14].

6 Conclusion

In this study, we introduced a new definition of conflict *Conf* between belief functions on a general finite frame of discernment. Its properties were compared with our previous approaches [8], and also with Liu's approach [22]. *Conf* is a simplification of plausibility conflict *Pl-C*, while keeping its nature. *Conf* also specifies the size of the conflict between belief functions in a way which is compatible with the combinational conflict. Thus, we can consider *Conf* as an improvement of both the combinational- and the plausibility-conflict approaches.

The presented theoretical results improve general understanding of conflict between belief functions and the entire nature of belief functions. Correct understanding of conflicts may, consequently, improve combination of conflicting belief functions in their practical applications.

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