Some Ideas on Constitutive Modeling of Debris Materials

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Abstract. Debris flows are dangerous natural hazards in countries with mountainous terrains. Debris materials show complex behavior depending on material composition and loading condition. Constitutive model plays an important role in better understanding the triggering mechanisms and reliable prediction of runout and deposition behavior. This paper reviews some constitutive models for debris materials in particular the Bingham fluid model. The hypoplastic constitutive model with critical state for granular materials is briefly recapitulated. Some ideas are presented to integrate the Bingham model into the hypoplastic constitutive model to account for both slow and fast flow of debris materials. The structure of this combined model and some specific formulations are discussed.

Keywords: debris materials, constitutive equation, hypoplasticity, Bingham fluid, yielding.

1 Introduction

Debris flows are fast moving, liquefied landslides of mixed and unconsolidated water and debris similar to flowing concrete. The complex behavior of debris materials is often described by nonnewtonian and Bingham fluid model. Debris flows can carry material ranging in size from clay to boulders, and may contain a large amount of woody debris such as logs and tree stumps. Both solid and fluid phases have major influence on the motion, distinguishing debris flows from other gravity driven flows such as rock avalanches and sediment-laden water floods [1].

Much research effort has been devoted to modelling the mechanical behaviour of debris materials. In early stage, a viscoplastic model of debris flow was proposed by Johnson [2]. Debris flow was simplified as a single phase continuum with Bingham or Coulomb yield strength. The rheological parameters, such as yield strength, cohesion and Bingham viscosity coefficient, were assumed by field investigation. However, this model cannot describe some important kinetic characteristics of debris flow, e.g. the interaction between particles and water. Takahashi [3] proposed a dilatant fluid model based on Bagnold's research. This model is considered equivalent to the two-phase mixture model in which the

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dynamic fluid effects are negligibly small. Some assumptions are applied in the model derivation leading to large discrepancy between the model predictions and field results. The Savage-Hutter avalanche model is a depth averaged model with the following simplifying assumptions: (i) density preserving, (ii) shallowness of the avalanche piles and small topographic curvatures, (iii) Coulomb-type sliding with bed friction angle δ and (iv) Mohr-Coulomb behaviour in the interior with the friction angle $\varphi \geq \delta$ and an ad-hoc assumption reducing the number of Mohr's circles in three dimensional stress states to one [4]. A constitutive model capable of describing the salient features of geophysical flows across three-dimensional terrain is still lacking because conceptual and computational problems have thwarted efforts to combine the influences of Coulomb friction, pore fluid stresses, bed topography, and flow inertia in a satisfactory manner. Iverson and Denlinger [5] suggested a Coulomb mixture model applicable to diverse geophysical flows, from dry granular avalanches to liquefied slurry floods. Use of a single model to describe the material behavior in different flow regimes helps clarify the physical basis of similarities and differences among the events. This model is a generalization of a previous mixture model of Iverson.

A perusal of the relevant models for debris flows in the past shows that most works focused on the simulation of the flow state rather than the mechanical behavior of debris materials. The hypoplastic model is a relatively new constitutive theory which has been developed to mathematically describe the non-linear and irreversible behavior of geomaterials. As compared to elastoplasticity, it does not a priori distinguish between elastic and plastic deformations. The model can be easily implemented into numerical algorithms. The hypoplastic model is well suitable to describe the complex behavior of debris materials before liquefication. Because debris flows characteristically originate as solid-like sediment masses, transform at least partly to fluid-like flows, and then transform back to solid-like deposits, reasonable models must simulate an evolution of material behavior without invoking preternatural changes in material properties [1]. In this paper we make use of hypoplasticity combined with rheological model to simulate this evolution of debris material.

2 Integration of Bingham Fluid Model and Hypoplastic Model

2.1 Bingham Fluid Model [6]

In the early rheological descriptions, debris flows were treated as Bingham fluid with the following concepts: 1) the shear stress exhibits a linear dependence on the shear rate; 2) the shear stress is independent of the normal stress; 3) debris material behaves like a one-phase homogeneous material; 4) when the shear stress is below a threshold, Bingham materials behave like rigid or elastic bodies. These concepts can be described by the following equation,

$$\begin{cases} \dot{\gamma} = 0, \tau \leq \tau_c \\ \dot{\gamma} > 0, \tau = \tau_c + \mu\dot{\gamma}, \end{cases} \tag{1}$$

where $\dot{\gamma}$ is shear rate, μ is viscosity, τ and τ_c denote shear stress and yield stress, respectively. The three-dimensional representation of yielding of Bingham fluid model shown above has been summarized by Ancey (2007).

During debris flow the material is subjected to large shear deformation. For developing and evaluating constitutive models the planar simple shear motion is particularly relevant. In the case of simple shear, the stress tensor in a Cartesian frame can be written as follows (see Fig.1),

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix}, \tag{2}$$

The three stress invariants can be readily obtained, $I_1 = 3\sigma$, $I_2 = \tau^2$ and $I_3 = 0$.

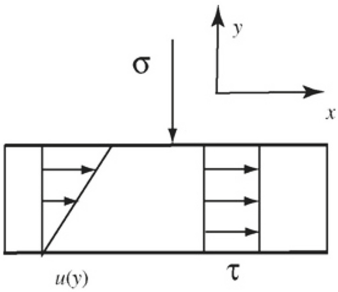


Fig. 1. Simple shear experiment [6]

According to the experimental observations, if the shear stress exceeds a critical value, the material yields. The yielding condition can be described by a scalar function of the stress-tensor invariant $f(I_1, I_2, I_3)$, frequently as a function of I_2. So the yield surface can be expressed as

$$f(I_1, I_2, I_3) = 0, \tag{3}$$

where f is called the plastic rule.

As shown in equation (1), the yield condition of Bingham fluid model is $\tau = \tau_c$. So the yield function of Bingham fluid model can be expressed as

$$f(I_2) = \sqrt{I_2} - \tau_c. \tag{4}$$

It is easy to show that the above function is the von Mises criterion in which the yield stress is a constant. Ancey (2007) summarized the derivation of the constitutive equation in tensorial form for a Bingham fluid. For debris flows it is important to know what happens beyond yielding. To this end, we assume that the following two conditions remain valid in the regime beyond yielding.

(1) Coaxiality principle: the principal directions of the extra-stress and strain-rate tensors coincide.

(2) Associate flow rule: the strain-rate tensor is directly proportional to the surplus of stress, i.e. the distance between the point representing the stress state and the yield surface, $\sqrt{I_2} - \tau_c$.

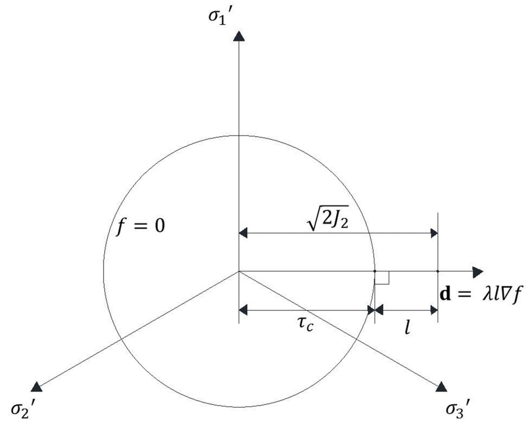


Fig. 2. Yield surface of Bingham fluid model on π-plane

As shown in Fig.2, the shear stress on π-plane is $\tau_\pi = \sqrt{2J_2}$. So the yield surface of Bingham fluid model can be described as

$$f(\boldsymbol{\sigma}) = \sqrt{2J_2} - \tau_c = 0. \tag{5}$$

Here, J_2 is the second deviatoric stress invariant , τ_c is the yield stress.

The surplus of stress is denoted by l. Translated into mathematical terms, the two principles mentioned above lead to: $\mathbf{d} = \lambda l \nabla f$, with λ a proportionality coefficient and $\lambda^{-1} = \mu$, μ is viscosity, $\mathbf{d}$ is the strain rate tensor, ∇f denotes gradient of the yield surface. When $f > 0$, we obtain

$$\mathbf{s} = \mu \left(\frac{\sqrt{2J_2}}{\sqrt{2J_2} - \tau_c} \right) \mathbf{d} \tag{6}$$

in which $\mathbf{s} = \boldsymbol{\sigma} - tr(\boldsymbol{\sigma})\mathbf{1}/3$ is the stress deviator. Throughout this paper, we use bold letters to denote tensors and matrices. Equation (6) is the constitutive equation in tensorial form for Bingham fluid.

2.2 The Hypoplastic Constitutive Model [7]

In the expression of the Bingham model, debris materials behave like rigid bodies before yielding. This assumption impedes reliable predictions of yielding of debris materials. We try to use hypoplastic theory to depict the stress-strain relationship of debris materials in this stage. As compared to elastoplasticity, the distinctive feature of hypoplasticity is the continuously non-linear dependence of the material response on the direction of strain rate. Consider the following specific version of hypoplastic constitutive equation (Wu 1992),

$$\mathring{\boldsymbol{\sigma}} = c_1(tr\boldsymbol{\sigma})\dot{\epsilon} + c_2\frac{tr(\boldsymbol{\sigma}\dot{\boldsymbol{\epsilon}})}{tr\boldsymbol{\sigma}} + (c_3\frac{\boldsymbol{\sigma}^2}{tr\boldsymbol{\sigma}} + c_4\frac{\boldsymbol{\sigma}^{*2}}{tr\boldsymbol{\sigma}}) \parallel \dot{\boldsymbol{\epsilon}} \parallel . \tag{7}$$

In the extended model, the variation of the void ratio had been taken into account by introducing the density function, I_e. So the extended model can be expressed as

$$\mathring{\boldsymbol{\sigma}} = c_1(tr\boldsymbol{\sigma})\dot{\epsilon} + c_2\frac{tr(\boldsymbol{\sigma}\dot{\boldsymbol{\epsilon}})}{tr\boldsymbol{\sigma}} + (c_3\frac{\boldsymbol{\sigma}^2}{tr\boldsymbol{\sigma}} + c_4\frac{\boldsymbol{\sigma}^{*2}}{tr\boldsymbol{\sigma}}) \parallel \dot{\boldsymbol{\epsilon}} \parallel I_e, \tag{8}$$

where

$$I_e = (a-1)D_c + 1, D_c = \frac{e_{crt} - e}{e_{crt} - e_{min}}. \tag{9}$$

In the above equations, a is a material constant, D_c is the modified relative density, e_{min} and e_{crt} are the minimum and the critical void ratio, respectively.

In the framework of hypoplasticity, when stress rate tensor vanishes, strain rate does not vanish and the volume remains unchanged, a material element is said to be at failure. The flow rule and equation of failure surface derived from Equation (8) can be expressed as

$$F(\boldsymbol{\sigma}, e) = \frac{\{\dot{\boldsymbol{\epsilon}}\}}{\parallel \dot{\boldsymbol{\epsilon}} \parallel} = [\mathbf{L}]^{-1}\{\mathbf{N}\}I_e \tag{10}$$

and

$$f(\boldsymbol{\sigma}, e) = \{\mathbf{N}\}^T([\mathbf{L}]^T)^{-1}[\mathbf{L}]^{-1}\{\mathbf{N}\}{I_e}^2 - 1 = 0 \ , \tag{11}$$

respectively. $\mathbf{L}$ and $\mathbf{N}$ is the linear and nonlinear stiffness matrix, which depend on the specific constitutive equation, e.g. Equation (7).

The yield surface and direction of strain rate on π-plane are shown in Fig.3, which are obtained for dense sand.

2.3 Structure of a New Model for Debris Materials

We proceed to develop a new model by combining hypoplastic model with Bingham model. To this end, hypoplasticity will be used to describe debris materials before yielding, and the Bingham model will depict the rapid flow. It means that the new model is composed of a hypoplastic part and a Bingham fluid part in series. The structure of the new model is shown in Fig.4.

Before yielding, the Bingham fluid part behaves like a rigid body, the deformation is only due to the hypoplastic part. After yielding, the stress in the friction element exceeds the yield value and this allows deformation in all elements.

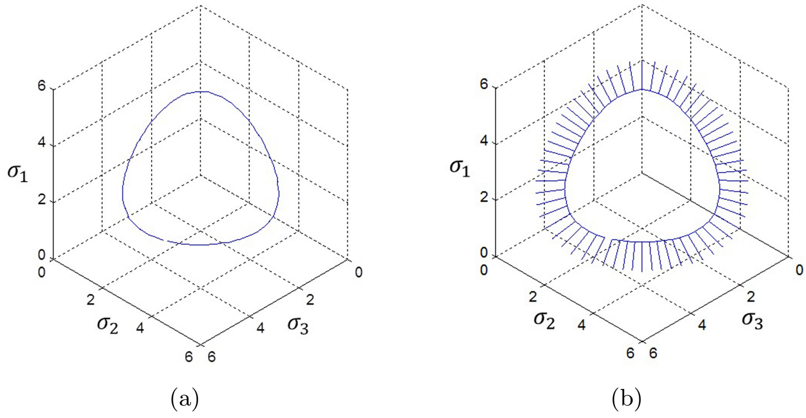


Fig. 3. Failure surface and flow rule of hypoplastic model: (a) failure surface; (b) flow rule

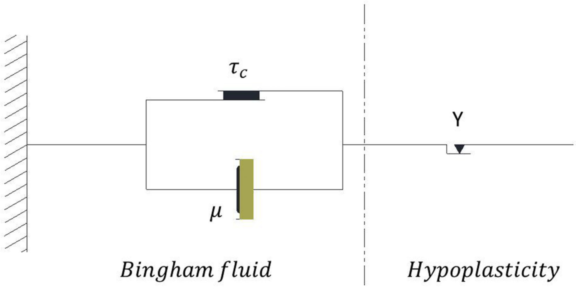


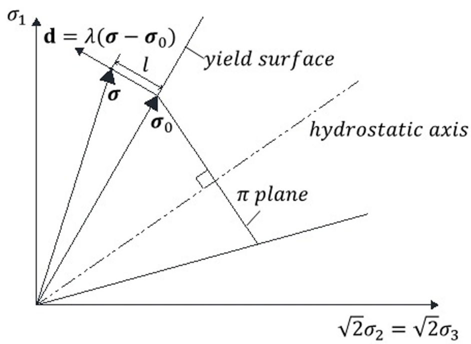
Fig. 4. Schematic diagram of the new model

As mentioned before, the yield stress τ_c is a constant. Equation (5) and (6) are derived under this condition. In hypoplasticity, however, the yield stress is usually a function of normal stress. So, in the new model, the yield condition should be determined by the yield function of the hypoplastic part, and Equation (6) should be modified based on the new yield condition by combining these two models.

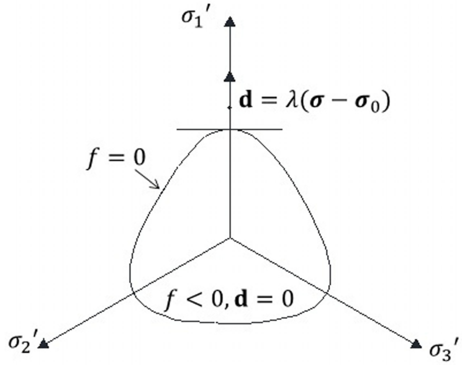
As shown in Fig.5, material yield at the stress state $\boldsymbol{\sigma}_0$ and then reach a new stress state $\boldsymbol{\sigma}$. We assume that the former mentioned coaxiality principle still holds. The strain rate after yielding has the same direction with the vector $(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0)$ and is proportional to the module of this vector, $\| \boldsymbol{\sigma} - \boldsymbol{\sigma}_0 \|$, rather than the distance between the point $\boldsymbol{\sigma}$ and the yield surface since hypoplastic model has non-associate flow rule. Translated into mathematical terms, the strain rate tensor can be expressed as

$$\mathbf{d} = \lambda(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0), \tag{12}$$

where λ is a proportionality coefficient.



(a)



(b)

Fig. 5. Schematic diagram of strain rate on (a) Rendulic plane and (b) π-plane beyond yielding

Substitution Equation (10) into Equation (12) yields the expression of stress tensor beyond yielding,

$$\boldsymbol{\sigma} = \mu F(\boldsymbol{\sigma}_0, e) \parallel \dot{\boldsymbol{\epsilon}} \parallel + \boldsymbol{\sigma}_0, \tag{13}$$

in which μ is viscosity, $\boldsymbol{\sigma}_0$ is yield stress tensor and $\dot{\boldsymbol{\epsilon}}$ substitutes $\mathbf{d}$ to denote strain rate tensor.

The derivative of Equation (13) with respect to time is

$$\dot{\boldsymbol{\sigma}} = \mu F(\boldsymbol{\sigma}_0, e) \parallel \ddot{\boldsymbol{\epsilon}} \parallel . \tag{14}$$

This is the constitutive equation in tensorial form for Bingham fluid based on the yield condition of the extended hypoplastic model, and can be used to integrate with this model. So the new model for debris materials can be expressed as

$$\overset{\circ}{\boldsymbol{\sigma}} =(1-k)[c_1(tr\boldsymbol{\sigma})\dot{\epsilon} + c_2\frac{tr(\boldsymbol{\sigma}\dot{\boldsymbol{\epsilon}})}{tr\boldsymbol{\sigma}} + (c_3\frac{\boldsymbol{\sigma}^2}{tr\boldsymbol{\sigma}} + c_4\frac{\boldsymbol{\sigma}^{*2}}{tr\boldsymbol{\sigma}}) \parallel \dot{\boldsymbol{\epsilon}} \parallel I_e] + k\mu F(\boldsymbol{\sigma}_0, e) \parallel \ddot{\boldsymbol{\epsilon}} \parallel, \tag{15}$$

where k is the phase changing coefficient and can be expressed as

$$k = [\mid 0.5 - 5R_e \mid + (0.5 - 5R_e)]^2. \tag{16}$$

R_e is the ratio of mean effective stresses

$$R_e = \frac{tr\boldsymbol{\sigma}}{tr\boldsymbol{\sigma}_i}, \tag{17}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_i$ are the instantaneous effective stress tensor and initial effective stress tensor, respectively. It is assumed that, when R_e is less than 0.1, the effect of the hypoplastic part becomes negligible and the Bingham part dominates the motion progressively.

From the Bingham fluid part, it can be predicted that, when $\parallel \ddot{\boldsymbol{\epsilon}} \parallel = 0$ or the angle between vector $F(\boldsymbol{\sigma}_0, e)$ and $\overset{\circ}{\boldsymbol{\sigma}}$ is greater than $90°$, the debris material will cease to flow.

3 Conclusions

Based on the yield criterion and flow rule of hypoplastic model, a new model for debris materials is presented. After yielding, the value of stress rate is assumed proportional to the magnitude of strain acceleration with a proportional coefficient μ, which represents viscosity, and the direction of strain rate is determined by the flow rule of the integrated hypoplastic model. Equation (15) shows the form of the new model, in which the specific version of the hypoplastic part can be determined by matching the experiment results from element tests, e.g. annular shear tests and channel flow tests. Some experimental data of annular shear tests by Savage (1984) and Daniel (1985) can be used for the calibration of the new model. One problem of Equation (15) is that the phase change is controlled artificially. The phase changing coefficient varies from 0 to 1 to trigger the Bingham part beyond materials yield. As a matter of fact, flow of debris materials is subjected to the combined influence of material properties, boundary condition, the path of loading and so on.

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