Chapter 4

Lyapunov Inequalities

You know that I write slowly. This is chiefly because I am never satisfied until I have said as much as possible in a few words, and writing briefly takes far more time than writing at length.

Gauss (1777–1855).

In 1906 Lyapunov [105] proved an inequality giving the distance between two consecutive zeros of solutions of second order differential equations. It is proved that, if the differential equation

$$y^{''}(t) + p(t)y(t) = 0, \tag{4.0.1}$$

has a nontrivial solution $y(t)$ with $y(a) = y(b) = 0$ $(a < b)$ and $y(t) \neq 0$ for $t \in (a, b)$, then

$$\int_a^b p(t)dt > \frac{4}{b-a}, \tag{4.0.2}$$

where p is a positive real valued function defined on $[a, b]$. If the difference equation

$$\Delta^2 y(n) + p(n)y(n+1) = 0, \tag{4.0.3}$$

has a nontrivial solution $y(n)$ satisfying $y(0) = y(N) = 0$, where $p(n)$ is a positive sequence, then the Lyapunov inequality is given by

$$\sum_{k=0}^{N-1} p(n) \geq \begin{cases} \frac{2}{m+1}, & \text{if } N = 2m+2, \\ \frac{2m+1}{m(m+1)}, & \text{if } N = 2m+1. \end{cases}$$

The chapter is organized as follows. In Sect. 4.1 we present some Lyapunov type inequalities for second order linear dynamic equations and in Sect. 4.2

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we present results for half-linear dynamic equations. Section 4.3 considers dynamic equations with damping terms and in Sect. 4.4 we consider Hamiltonian systems on time scales.

Throughout this chapter (usually without mentioning) the integrals in the statements of the theorems are assumed to exist.

4.1 Second Order Linear Equations

In this section, we establish some Lyapunov type inequalities for Sturm–Liouville linear dynamic equations on time scales and then establish some sufficient conditions for disconjugacy of solutions. The results in this section are adapted from [48, 90, 123, 125, 128]. First, we consider the Sturm–Liouville dynamic equation

$$y^{\Delta\Delta}(t) + p(t)y^{\sigma}(t) = 0, \tag{4.1.1}$$

together with the quadratic functional

$$\mathcal{F}(y) = \int_a^b \left\{ (y^{\Delta}(t))^2 - p(y^{\sigma})^2(t) \right\} \Delta t,$$

where $p(t)$ is a positive rd-continuous function defined on $\mathbb{T}$.

By a solution of (4.1.1), we mean a continuous function $y : [a, \sigma^2(b)]_{\mathbb{T}} \to \mathbb{R}$, which is twice differentiable on $[a, b]_{\mathbb{T}}$ with y^{Δ^2} rd-continuous. It is known that (4.1.1) admits a unique solution when $y(a)$ and $y^{\Delta}(a)$ are prescribed. We say y has a generalized zero at some $c \in [a, \sigma(b)]_{\mathbb{T}}$ provided that $y(c)y^{\sigma}(c) \leq 0$ holds, and (4.1.1) is called disconjugate on $[a, b]_{\mathbb{T}}$ if there is no nontrivial solution of (4.1.1) with at least two generalized zeros in $[a, b]_{\mathbb{T}}$. Finally, (4.1.1) is said to be disfocal on $[a, \sigma^2(b)]_{\mathbb{T}}$ provided there is no nontrivial solution y of (4.1.1) with a generalized zero in $[a, \sigma^2(b)]_{\mathbb{T}}$ followed by a generalized zero of y^{Δ} in $[a, \sigma(b)]_{\mathbb{T}}$.

Lemma 4.1.1 *If x solves (4.1.1) and if $\mathcal{F}(y)$ is defined, then*

$$\mathcal{F}(y) - \mathcal{F}(x) = \mathcal{F}(y - x) + 2(y - x)(b)x^{\Delta}(b) - 2(y - x)x^{\Delta}(a).$$

Proof. *Under the above assumptions we find*

$$\begin{aligned} &\mathcal{F}(y) - \mathcal{F}(x) - \mathcal{F}(y - x) \\ = &\int_a^b \left\{ (y^{\Delta})^2 - p(y^{\sigma})^2 - (x^{\Delta})^2 + p(x^{\sigma})^2 \right. \\ &\left. - (y^{\Delta} - x^{\Delta})^2 + p(y^{\sigma} - x^{\sigma}) \right\} (t)\Delta t \end{aligned}$$

$$
\begin{aligned}
&= \int_a^b \{(y^\Delta)^2 - p(y^\sigma)^2 - (x^\Delta)^2 + p(x^\sigma)^2 - (y^\Delta)^2 + 2y^\Delta x^\Delta - (x^\Delta)^2 \\
&\quad + p(y^\sigma)^2 - 2py^\sigma x^\sigma + p(x^\sigma)^2\}(t)\Delta t \\
&= 2\int_a^b \{y^\Delta x^\Delta - py^\sigma x^\sigma + p(x^\sigma)^2 - (x^\Delta)^2\}(t)\Delta t \\
&= 2\int_a^b \left\{y^\Delta x^\Delta + y^\sigma x^{\Delta^2} - x^\sigma x^{\Delta^2} - (x^\Delta)^2\right\}(t)\Delta t \\
&= 2\int_a^b \{yx^\Delta - xx^\Delta\}^\Delta \Delta(t) = 2\int_a^b \{(y-x)x^\Delta\}^\Delta \Delta t \\
&= 2(y(b)-x(b))x^\Delta(b) - 2(y(a)-x(a))x^\Delta(a),
\end{aligned}
$$

where we have used the product rule. ■

Lemma 4.1.2 *If $\mathcal{F}(y)$ is defined, then for any $r, s \in \mathbb{T}$ with $a \le r < s \le b$*

$$\int_r^s \left(y^\Delta(t)\right)^2 \Delta t \ge \frac{(y(s)-y(r))^2}{s-r}.$$

Proof. Let

$$x(t) = \frac{y(s)-y(r)}{s-r}t + \frac{sy(r)-ry(s)}{s-r}.$$

Then x solves the Sturm–Liouville equation (4.1.1) with $p = 0$ and therefore we may apply Lemma 4.1.1 to $\mathcal{F}_0$ defined by

$$\mathcal{F}_0(x) = \int_r^s (x^\Delta(t))^2 \Delta t,$$

to find

$$
\begin{aligned}
\mathcal{F}_0(y) &= \mathcal{F}_0(x) + \mathcal{F}_0(y-x) + (y-x)(s)x^\Delta(s) - (y-x)(r)x^\Delta(r) \\
&= \mathcal{F}_0(x) + \mathcal{F}_0(y-x) \\
&\ge \mathcal{F}_0(x) = \int_r^s \left\{\frac{y(s)-y(r)}{s-r}\right\}^2 \Delta t = \frac{(y(s)-y(r))^2}{s-r},
\end{aligned}
$$

and this completes the proof. ■

The following lemma will be used later (see [51]).

Lemma 4.1.3 *Equation (4.1.1) is disconjugate on* $[a,b]_{\mathbb{T}}$ *if and only if*

$$\mathcal{F}(y) = \int_a^b \left\{ (y^{\Delta}(t))^2 - p(y^{\sigma})^2(t) \right\} \Delta t > 0,$$

for all nontrivial solutions y *with* $y(a) = y(b) = 0$.

The following theorem gives the Lyapunov type inequality for the second order dynamic equation (4.1.1).

Theorem 4.1.1 *If* $y(t)$ *is a nontrivial solution of (4.1.1) with* $y(a) = y(b) = 0$ $(a < b)$, *then*

$$\int_a^b p(t)\Delta t > \frac{b-a}{f(d)}, \tag{4.1.2}$$

where $f(d) = \max\{f(t) : t \in [a,b]\}$ *and* $f(t) = (t-a)(b-t)$.

Proof. From Lemma 4.1.1, since y is a nontrivial solution of (4.1.1) with $y(a) = y(b) = 0$, we have that

$$\mathcal{F}(y) = \int_0^b \left\{ (y^{\Delta}(t))^2 - p(y^{\sigma})^2(t) \right\} \Delta t = 0.$$

Also, since y is nontrivial, we see that

$$M := \max\{y^2(t) : t \in [a,b] \cap \mathbb{T}\}, \tag{4.1.3}$$

is defined and positive. Now let $c \in [a,b]$ be such that $y^2(c) = M$. Applying the above and Lemma 4.1.2, twice (once with $r = a$ and $s = c$ and a second time with $r = c$ and $s = b$), we find

$$\begin{aligned} M\int_0^b p(t)\Delta t &\geq \int_0^b \left\{ p(y^{\sigma})^2(t) \right\} \Delta t \\ &= \int_0^b (y^{\Delta}(t))^2 \Delta t = \int_0^b (y^{\Delta}(t))^2 \Delta t + \int_0^b (y^{\Delta}(t))^2 \Delta t \\ &\geq \frac{(y(c)-y(a))^2}{c-a} + \frac{(y(b)-y(c))^2}{b-c} \\ &= y^2(c)\left\{ \frac{1}{c-a} + \frac{1}{b-c} \right\} = M\frac{b-a}{f(c)} \geq M\frac{b-a}{f(d)}, \end{aligned}$$

where the last inequality holds since $f(d) = \max\{f(t) : t \in [a,b] \cap \mathbb{T}\}$. Hence, dividing by $M > 0$ yields the desired inequality. The proof is complete. ■

Example 4.1.1 *We use the notation from the proof of Theorem 4.1.1.*

(i). If $\mathbb{T}=\mathbb{R}$ *, then*

$$\min\left\{\left|\frac{a+b}{2}-s\right| : s\in[a,b]\right\}=0, \text{ and so that } d=\frac{a+b}{2}.$$

Hence $f(d)=((b-a)^2/4)$ *and the Lyapunov inequality from Theorem 4.1.1 reads*

$$\int_0^b p(t)dt \geq \frac{4}{b-a}.$$

(ii). If $\mathbb{T}=\mathbb{Z}$*, then we consider two cases. First, if* $a+b$ *is even, then*

$$\min\left\{\left|\frac{a+b}{2}-s\right| : s\in[a,b]\cap\mathbb{Z}\right\}=0, \text{ and so that } d=\frac{a+b}{2}.$$

Hence $f(d)=((b-a)^2/4)$ *and the Lyapunov inequality reads*

$$\sum_{t=a}^{b-1} p(t) \geq \frac{4}{b-a}.$$

If $a+b$ *is odd, then*

$$\min\left\{\left|\frac{a+b}{2}-s\right| : s\in[a,b]\cap\mathbb{Z}\right\}=\frac{1}{2}, \text{ and so that } d=\frac{a+b-1}{2}.$$

Then, we have $f(d)=((b-a)^2-1/4)$ *and the Lyapunov inequality reads*

$$\sum_{t=a}^{b-1} p(t) \geq \frac{4}{b-a}\left\{\frac{1}{1-(1/(b-a)^2)}\right\}.$$

As an application of Theorem 4.1.1, we now prove a sufficient condition for the disconjugacy of (4.1.1).

Theorem 4.1.2 *If* p *satisfies*

$$\int_a^b p(t)\Delta(t) < \frac{b-a}{f(d)}, \tag{4.1.4}$$

then (4.1.1) is disconjugate on $[a,b]_{\mathbb{T}}$.

Proof. Suppose that (4.1.4) holds. For the sake of contradiction we assume that (4.1.1) is not disconjugate. But then, by Lemma 4.1.3, there exists a nontrivial y with $y(a) = y(b) = 0$ such that $\mathcal{F}(y) \leq 0$. Using this y, we now define M by (4.1.3) and we find

$$M\int_a^b p(t)\Delta t \geq \int_a^b p(t)(y^\sigma(t))^2\Delta t \geq \int_a^b (y^\Delta(t))^2\Delta t \geq \frac{M(b-a)}{f(d)},$$

where the last inequality follows as in the proof of Theorem 4.1.1. Hence, after dividing by $M > 0$, we arrive at

$$\int_a^b p(t)\Delta t \geq \frac{b-a}{f(d)},$$

which contradicts (4.1.4) and hence completes the proof. ■

Remark 4.1.1 *Note that in both condition (4.1.2) and (4.1.4) we could replace $(b-a)/f(d)$ by $4/(b-a)$, and Theorems 4.1.1 and 4.1.2 would remain true. This is because for $a \leq c \leq b$, we have*

$$\frac{1}{c-a} + \frac{1}{b-c} = \frac{(a+b-2c)^2}{(b-a)(c-a)(b-c)} + \frac{4}{b-a} \geq \frac{4}{b-a}.$$

In the following, we apply Opial type inequalities on time scales to prove some Lyapunov type inequalities for the second-order dynamic equation (4.1.1).

Theorem 4.1.3 *Assume that y is a nontrivial solution of the second-order dynamic equation (4.1.1) with $y(a) = y^{\Delta\sigma}(b) = 0$. Then, we have*

$$K_P(\sigma(b), a) = \left(2\int_a^{\sigma(b)} |P(t)|^2 \left[\sigma(t) - a\right]\Delta t\right)^{1/2} \geq 1, \tag{4.1.5}$$

where

$$P(t) := \int_t^{\sigma(b)} p(s)\Delta s, \quad \text{for} \quad t \in [a, \sigma(b)]_{\mathbb{T}}. \tag{4.1.6}$$

Proof. Now

$$\begin{aligned}\int_a^{\sigma(b)} y^\sigma(t)y^{\Delta^2}(t)\Delta t &= y^\sigma(b)y^{\Delta\sigma}(b) - y(a)y^\Delta(a) - \int_a^{\sigma(b)} \left[y^\Delta(t)\right]^2\Delta t \\ &= -\int_a^{\sigma(b)} \left[y^\Delta(t)\right]^2\Delta t, \end{aligned} \tag{4.1.7}$$

and using (4.1.6), we get that

$$\int_a^{\sigma(b)} p(t)\big[y^\sigma(t)\big]^2\Delta t = -\int_a^{\sigma(b)} P^\Delta(t)\big[y^\sigma(t)\big]^2\Delta t$$
$$= P(a)\big[y(a)\big]^2 + \int_a^{\sigma(b)} P(t)\Big[\big[y(t)\big]^2\Big]^\Delta\Delta t$$
$$= \int_a^{\sigma(b)} P(t)\Big[\big[y(t)\big]^2\Big]^\Delta\Delta t$$
$$= \int_a^{\sigma(b)} P(t)\big(\big[y(t)+y^\sigma(t)\big]y^\Delta(t)\big)\Delta t$$
$$\leq \int_a^{\sigma(b)} \big|P(t)\big|\big(\big[y(t)+y^\sigma(t)\big]y^\Delta(t)\big)\Delta t. \tag{4.1.8}$$

Multiplying (4.1.1) by y^σ and integrating from a to $\sigma(b)$ and using Theorem 3.1.7, (4.1.7) and (4.1.8), we get

$$\int_a^{\sigma(b)} \big(y^\Delta(t)\big)^2\,\Delta t \leq \int_a^{\sigma(b)} \big|P(t)\big|\big(\big[y(t)+y^\sigma(t)\big]y^\Delta(t)\big)\Delta t$$
$$\leq K_P(\sigma(b),a)\int_a^{\sigma(b)} \big[y^\Delta(t)\big]^2, \tag{4.1.9}$$

Clearly, (4.1.5) follows from (4.1.9) by dividing by

$$\int_a^{\sigma(b)} \big[y^\Delta(t)\big]^2\Delta t,$$

on both sides. The proof is complete. ■

Remark 4.1.2 *The conclusion of Theorem 4.1.3 also holds for the second order dynamic inequality*

$$y^{\Delta^2}(t)+p(t)y^\sigma(t)\geq 0, \qquad \text{for} \quad t\in[a,b]_{\mathbb{T}}, \tag{4.1.10}$$

with $y(a)=0$ *and* $y(b)y^{\Delta\sigma}(b)\leq 0$.

Similar reasoning by considering Theorem 3.1.8 instead of Theorem 3.1.7 yields the following result.

Theorem 4.1.4 *Assume that* x *is a nontrivial solution of (4.1.1) with* $x^\Delta(a)=x^{\sigma^2}(b)=0$. *Then, we have*

$$L_P(\sigma^2(b),a)=\left(2\int_a^{\sigma^2(b)}(P(t))^2\left[\sigma^2(b)-t\right]\Delta t\right)^{1/2}\geq 1,$$

where

$$P(t):=\int_a^t p(s)\Delta s, \quad \text{for} \quad t\in[a,\sigma(b)]_{\mathbb{T}}.$$

Remark 4.1.3 *The conclusion of Theorem 4.1.4 also holds for* (4.1.10) *with* $x(a)x^{\Delta}(a) \geq 0$ *and* $x^{\sigma^2}(b) = 0$.

In the following, we establish a disconjugacy result for solutions of (4.1.1).

Theorem 4.1.5 *Assume that* y *is a nontrivial solution of (4.1.1) with* $y(a) = y^{\sigma^2}(b) = 0$, *and let* $P \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$ *be a function satisfying* $P^{\Delta} \equiv p$ *on* $[a,b]_{\mathbb{T}}$. *Then, we have*

$$\min_{c\in[a,\sigma^2(b)]_{\mathbb{T}}} \big\{ \max \big\{ K_P(\sigma^2(b), c), L_P(c,a) \big\} \big\} \geq 1. \qquad (4.1.11)$$

Proof. Similar reasoning as in the proof of Theorem 4.1.3 yields the desired inequality (4.1.11) by applying Corollary 3.1.2 instead of Theorem 3.1.7. ■

Corollary 4.1.1 *Assume that* y *is a nontrivial solution of (4.1.1) with* $y(a) = 0$, *and let* $P \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$ *be a function as in Theorem 4.1.5. If*

$$\min_{c\in[a,\sigma^2(b)]_{\mathbb{T}}} \big\{ \max \big\{ K_P(\sigma^2(b), c), L_P(c,a) \big\} \big\} < 1,$$

then $y^{\sigma^2}(b) \neq 0$.

Next we consider the second order dynamic equation on $[a,b]$

$$\big[r(t)y^{\Delta}(t)\big]^{\Delta} + q(t)y^{\sigma}(t) = 0, \qquad t \in [a,b], \qquad (4.1.12)$$

on an arbitrary time scale $\mathbb{T}$, where r is a positive rd-continuous function and q is rd-continuous function and

$$\int_{\alpha}^{\beta} 1/r(t)\Delta t < \infty, \ \text{ and } \int_{\alpha}^{\beta} |q(t)|\,\Delta t < \infty. \qquad (4.1.13)$$

We obtain lower bounds for the spacing $\beta-\alpha$ where y is a solution of (4.1.12) satisfying some conditions at α and β.

By a solution of (4.1.12) on an interval $\mathbb{T}$, we mean a nontrivial real-valued function $y \in C_{rd}(\mathbb{T})$, which has the property that $r(t)y^{\Delta}(t) \in C^1_{rd}(\mathbb{T})$ and satisfies Eq. (4.1.12) on $\mathbb{T}$. We say that (4.1.12) is right disfocal (left disfocal) on $[\alpha,\beta]_{\mathbb{T}}$ if the solutions of (4.1.12) such that $y^{\Delta}(\alpha) = 0$ ($y^{\Delta}(\beta) = 0$) have no generalized zeros in $[\alpha,\beta]_{\mathbb{T}}$.

Theorem 4.1.6 *Suppose* y *is a nontrivial solution of (4.1.12). If* $y(\alpha) = y^{\Delta}(\beta) = 0$, *then*

$$\left[\sqrt{2}\left(\int_{\alpha}^{\beta} \frac{Q^2(t)}{r(t)}\left(\int_{\alpha}^{t} \frac{\Delta u}{r(u)}\right)\Delta t\right)^{\frac{1}{2}} + \sup_{\alpha\leq t\leq\beta}\left|\mu(t)\frac{Q(t)}{r(t)}\right|\right] \geq 1, \qquad (4.1.14)$$

where $Q(t)=\int_t^\beta q(s)ds$. *If instead* $y^\Delta(\alpha)=y(\beta)=0$, *then*

$$\left[\sqrt{2}\left(\int_\alpha^\beta \frac{Q^2(t)}{r(t)}\left(\int_t^\beta \frac{\Delta u}{r(u)}\right)\Delta t\right)^{\frac{1}{2}}+\sup_{\alpha\le t\le\beta}\left|\mu(t)\frac{Q(t)}{r(t)}\right|\right]\ge 1, \tag{4.1.15}$$

where $Q(t)=\int_\alpha^t q(s)ds$.

Proof. We prove (4.1.14). Multiplying (4.1.12) by y^σ and integrating by parts, we have

$$\begin{aligned}\int_\alpha^\beta y^\sigma(t)\left(r(t)y^\Delta(t)\right)^\Delta \Delta t &= y(t)r(t)y^\Delta(t)\big|_\alpha^\beta-\int_\alpha^\beta r(t)(y^\Delta(t))^2\Delta t\\ &= -\int_\alpha^\beta q(t)\left(y^\sigma(t)\right)^2\Delta t.\end{aligned}$$

Using the assumptions that $y(\alpha)=y^\Delta(\beta)=0$ and $Q(t)=\int_t^\beta q(s)\Delta s$, we get that

$$\int_\alpha^\beta r(t)\left(y^\Delta(t)\right)^2\Delta t=\int_\alpha^\beta q(t)\left(y^\sigma(t)\right)^2\Delta t=-\int_\alpha^\beta Q^\Delta(t)\left(y^\sigma(t)\right)^2\Delta t.$$

Integrating by parts the right-hand side and using the fact that $y(\alpha)=0=Q(\beta)$, we see that

$$\begin{aligned}\int_\alpha^\beta r(t)\left(y^\Delta(t)\right)^2\Delta t &= \int_\alpha^\beta Q(t)\left(y(t)+y^\sigma(t)\right)y^\Delta(t)\Delta t\\ &\le \int_\alpha^\beta |Q(t)|\,|y(t)+y^\sigma(t)|\left|y^\Delta(t)\right|\Delta t.\end{aligned}$$

Applying the inequality (3.1.23) with $s=Q$, we have

$$\begin{aligned}\int_\alpha^\beta r(t)\left(y^\Delta(t)\right)^2\Delta t &\le \left[\sqrt{2}\left(\int_\alpha^\beta \frac{Q^2(t)}{r(t)}\left(\int_\alpha^t \frac{\Delta u}{r(u)}\right)\Delta t\right)^{\frac{1}{2}}+\sup_{\alpha\le t\le\beta}\left|\mu(t)\frac{Q(t)}{r(t)}\right|\right]\\ &\quad\times\int_\alpha^\beta r(t)\left|y^\Delta(t)\right|^2\Delta t.\end{aligned}$$

This implies that

$$\left[\sqrt{2}\left(\int_\alpha^\beta \frac{Q^2(t)}{r(t)}\left(\int_\alpha^t \frac{\Delta u}{r(u)}\right)\Delta t\right)^{\frac{1}{2}}+\sup_{\alpha\le t\le\beta}\left|\mu(t)\frac{Q(t)}{r(t)}\right|\right]\ge 1,$$

which is the desired inequality (4.1.14). The proof of (4.1.15) is similar to the proof of (4.1.14) by using integration by parts and Theorem 3.1.12 instead of Theorem 3.1.11. The proof is complete. ■

As a special case of Theorem 4.1.6, when $r(t)=1$, we have the following results for Eq. (4.1.1).

Corollary 4.1.2 *Suppose y is a nontrivial solution of (4.1.1). If $y(\alpha) = y^{\Delta}(\beta) = 0$, then*

$$\left[\sqrt{2}\left(\int_{\alpha}^{\beta} Q^2(t)\,(t-\alpha)\,\Delta t\right)^{\frac{1}{2}} + \sup_{\alpha\le t\le\beta}\left(\mu(t)\,|Q(t)|\right)\right] \ge 1,$$

where $Q(t) = \int_t^{\beta} q(s)ds$. If instead $y^{\Delta}(\alpha) = y(\beta) = 0$, then

$$\left[\sqrt{2}\left(\int_{\alpha}^{\beta} Q^2(t)\,(\beta-t)\,\Delta t\right)^{\frac{1}{2}} + \sup_{\alpha\le t\le\beta}\left(\mu(t)\,|Q(t)|\right)\right] \ge 1,$$

where $Q(t) = \int_{\alpha}^{t} q(s)ds$.

Remark 4.1.4 *Note that if $\mathbb{T} = \mathbb{R}$ then $\mu(t) = 0$ and Eq. (4.1.12) (when $r(t) = 1$) becomes*

$$y^{''}(t) + q(t)y(t) = 0. \tag{4.1.16}$$

In this case the result in Corollary 4.1.2 reduces to a result obtained by Brown and Hinton [57].

Corollary 4.1.3 ([57]). *Suppose y is a solution of Eq. (4.1.16). If $y(\alpha) = y^{'}(\beta) = 0$, then*

$$2\int_{\alpha}^{\beta} Q^2(s)(s-\alpha)ds > 1, \tag{4.1.17}$$

where $Q(t) = \int_t^{\beta} q(s)ds$. If instead $y^{'}(\alpha) = y(\beta) = 0$, then

$$2\int_{\alpha}^{\beta} Q^2(s)(\beta-s)ds > 1, \tag{4.1.18}$$

where $Q(t) = \int_{\alpha}^{t} q(s)ds$.

Remark 4.1.5 *Note that if $\mathbb{T} = \mathbb{N}$, then $\mu(t) = 1$ and Eq. (4.1.12) (when $r(t) = 1$) becomes*

$$\Delta^2 y(n) + q(n)y(n+1) = 0, \tag{4.1.19}$$

and the result in Corollary 4.1.2 reduces to the following result.

Corollary 4.1.4 *Suppose y is a solution of Eq. (4.1.19). If $y(\alpha) = \Delta y(\beta) = 0$, then*

$$\sqrt{2}\left(\sum_{n=\alpha}^{\beta-1} (Q(n))^2\,(n-\alpha)\right)^{\frac{1}{2}} + \sup_{\alpha\le n\le\beta}|Q(n)| > 1,$$

where $Q(n) = \sum_{s=n}^{\beta-1} q(s)$. *If instead* $\Delta y(\alpha) = y(\beta) = 0$, *then*

$$\sqrt{2}\left(\sum_{n=\alpha}^{\beta-1} (Q(n))^2 (\beta - n)\right)^{\frac{1}{2}} + \sup_{\alpha\le n\le\beta} |Q(n)| > 1,$$

where $Q(n) = \sum_{s=\alpha}^{n-1} q(s)$.

Theorem 4.1.7 *Suppose that* y *is a nontrivial solution of (4.1.12). If* $y(\alpha) = y^{\Delta}(\beta) = 0$, *then*

$$\sqrt{2}\left(\sup_{\alpha\le t\le\beta} \frac{Q^2(t)}{r(t)} \int_{\alpha}^{\beta} \frac{1}{r(t)}\left(\int_{\alpha}^{t} \frac{\Delta u}{r(u)}\right)\Delta t\right)^{\frac{1}{2}} + \sup_{\alpha\le t\le\beta}\left|\frac{Q(t)}{r(t)}\right| \mu(t) \ge 1, \tag{4.1.20}$$

where $Q(t) = \int_t^{\beta} q(s)ds$. *If instead* $y^{\Delta}(\alpha) = y(\beta) = 0$, *then*

$$\sqrt{2}\left(\sup_{\alpha\le t\le\beta} \frac{Q^2(t)}{r(t)} \int_{\alpha}^{\beta} \frac{1}{r(t)}\left(\int_{t}^{\beta} \frac{\Delta u}{r(u)}\right)\Delta t\right)^{\frac{1}{2}} + \sup_{\alpha\le t\le\beta}\left|\frac{Q(t)}{r(t)}\right| \mu(t) \ge 1, \tag{4.1.21}$$

where $Q(t) = \int_{\alpha}^{t} q(s)ds$.

Proof. We prove (4.1.20). Multiplying (4.1.12) by y^{σ} and integrating by parts and following the proof of Theorem 4.1.6, we have

$$\int_{\alpha}^{\beta} r(t)\left(y^{\Delta}(t)\right)^2 \Delta t = \int_{\alpha}^{\beta} q(t)\left(y^{\sigma}(t)\right)^2 \Delta t = -\int_{\alpha}^{\beta} Q^{\Delta}(t)\left(y^{\sigma}(t)\right)^2 \Delta t.$$

Integrating by parts the right-hand side and using the fact that $y(\alpha) = 0 = Q(\beta)$, we see that

$$\begin{aligned}\int_{\alpha}^{\beta} r(t)\left(y^{\Delta}(t)\right)^2 \Delta t &\le \int_{\alpha}^{\beta} |Q(t)|\,|y(t)+y^{\sigma}(t)|\,\left|y^{\Delta}(t)\right| \Delta t \\ &\le \sup_{\alpha\le t\le\beta}\left|\frac{Q(t)}{r(t)}\right| \int_{\alpha}^{\beta} r(t)\,|y(t)+y^{\sigma}(t)|\,\left|y^{\Delta}(t)\right| \Delta t.\end{aligned}$$

Applying the inequality (3.1.37) with (3.1.38) and cancelling the term $\int_{\alpha}^{\beta} r(t)\left(y^{\Delta}(t)\right)^2 \Delta t$, we get the desired inequality (4.1.20). The proof of (4.1.21) is similar to the proof of (4.1.20) by using integration by parts and Corollary 3.1.4 instead of Corollary 3.1.3. The proof is complete. ■

As a special case of Theorem 4.1.7, when $r(t) = 1$, we have the following result.

Corollary 4.1.5 *Suppose that* y *is a nontrivial solution of (4.1.1). If* $y(\alpha) = y^{\Delta}(\beta) = 0$, *then*

$$\sup_{\alpha \le t \le \beta} |Q(t)|\,(\beta - \alpha) + \sup_{\alpha \le t \le \beta} |Q(t)|\,\mu(t) \ge 1,$$

where $Q(t) = \int_t^\beta q(s)ds$. *If instead* $y^{\Delta}(\alpha) = y(\beta) = 0$, *then*

$$\sup_{\alpha \le t \le \beta} |Q(t)|\,(\beta - \alpha) + \sup_{\alpha \le t \le \beta} |Q(t)|\,\mu(t) \ge 1,$$

where $Q(t) = \int_\alpha^t q(s)ds$.

As special case of Corollary 4.1.5, when $\mathbb{T} = \mathbb{R}$, (note that in this case $\mu(t) = 0$), we have the following result due to Harris and Kong [73] for the second order differential equation (4.1.16).

Corollary 4.1.6 *Suppose that* y *is a nontrivial solution of (4.1.16). If* $y(\alpha) = y^{'}(\beta) = 0$, *then*

$$(\beta - \alpha) \max_{\alpha \le t \le \beta} \left| \left(\int_t^\beta q(s)ds \right) \right| \ge 1. \tag{4.1.22}$$

If instead $y^{'}(\alpha) = y(\beta) = 0$, *then*

$$(\beta - \alpha) \max_{\alpha \le t \le \beta} \left| \int_\alpha^t q(s)ds \right| \ge 1. \tag{4.1.23}$$

As a special case of Corollary 4.1.5, when $\mathbb{T} = \mathbb{N}$ (note that in this case $\mu(t) = 1$), we have the following result for the second order difference equation (4.1.19).

Corollary 4.1.7 *Suppose* y *is a solution of Eq. (4.1.19). If* $\Delta y\,(\alpha) = y\,(\beta) = 0$, *then*

$$\sup_{\alpha \le n \le \beta} |Q(n)|\,(\beta + 1 - \alpha) > 1,$$

where $Q(n) = \sum_{s=n}^{\beta-1} q(s)$. *If instead* $y\,(\alpha) = \Delta y\,(\beta) = 0$, *then*

$$\sup_{\alpha \le n \le \beta} |Q(n)|\,(\beta + 1 - \alpha) > 1 > 1,$$

where $Q(n) = \sum_{s=\alpha}^{n-1} q(s)$.

Remark 4.1.6 *The above results yield sufficient conditions for the disfocality of (4.1.12), i.e., sufficient conditions so that there does not exist a nontrivial solution* y *satisfying either* $y(\alpha) = y^{\Delta}(\beta) = 0$ *or* $y^{\Delta}(\alpha) = y(\beta) = 0$.

Now, we assume that there exists a unique $h \in [\alpha, \beta]_{\mathbb{T}}$ such that

$$\int_\alpha^h \frac{\Delta t}{r(t)} = \int_h^\beta \frac{\Delta t}{r(t)}. \tag{4.1.24}$$

Note that when $r(t) = 1$, we see that $(h - \alpha) = (\beta - h)$, so that the unique solution of (4.1.24) is given by $h = (\alpha + \beta)/2$.

Theorem 4.1.8 *Assume that (4.1.24) holds and $Q^\Delta(t) = q(t)$. Suppose that y is a nontrivial solution of (4.1.12). If $y(\alpha) = y(\beta) = 0$, then*

$$\left[\sqrt{2}\left(\int_\alpha^\beta \frac{Q^2(t)}{r(t)}\left(\int_\alpha^h \frac{\Delta u}{r(u)}\right)\Delta t\right)^{\frac{1}{2}} + \sup_{\alpha \le t \le \beta} \mu(t)\left|\frac{Q(t)}{r(t)}\right|\right] \ge 1. \tag{4.1.25}$$

Proof. As in the proof of Theorem 4.1.6 by multiplying (4.1.12) by $y^\sigma(t)$, integrating by parts and using $y(\alpha) = y(\beta) = 0$, we have that

$$\int_\alpha^\beta r(t)\left|y^\Delta(t)\right|^2 dt \le \int_\alpha^\beta |Q(t)|\,|y(t) + y^\sigma(t)|^\gamma \left|y^\Delta(t)\right| dt. \tag{4.1.26}$$

Then

$$\int_\alpha^\beta r(t)\left|y^\Delta(t)\right|^2 dt \le K(\alpha, \beta)\int_\alpha^\beta r(t)\left|y^\Delta(t)\right|^2 dt,$$

where $K(\alpha, \beta)$ is defined as in (4.2.10). From the last inequality, after cancelling the term $\int_\alpha^\beta r(t)\left|y^\Delta(t)\right|^2 \Delta t$, we get the desired inequality (4.1.25). This completes the proof. ■

When $r(t) = 1$, (note that in this case $h = (\alpha + \beta)/2)$), we have the following result for Eq. (4.1.1).

Theorem 4.1.9 *Assume that $Q^\Delta(t) = q(t)$. Suppose that y is a nontrivial solution of (4.1.1). If $y(\alpha) = y(\beta) = 0$, then*

$$\left[\sqrt{\beta - \alpha}\left(\int_\alpha^\beta Q^2(t)\Delta t\right)^{\frac{1}{2}} + \sup_{\alpha \le t \le \beta}(\mu(t)\,|Q(t)|)\right] \ge 1.$$

Remark 4.1.7 *The results in Theorems 4.1.8 and 4.1.9 yield sufficient conditions for the disconjugacy of (4.3.1), i.e., sufficient conditions so that there does not exist a nontrivial solution y satisfying $y(\alpha) = y(\beta) = 0$.*

As a special case of Theorem 4.1.9, when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$, we have the following results for the second order differential equation (4.1.16) and the second order difference equation (4.1.19).

Corollary 4.1.8 *Assume that $Q^{'}(t) = q(t)$. Suppose that y is a nontrivial solution of (4.1.16). If $y(\alpha) = y(\beta) = 0$, then*

$$\int_\alpha^\beta \left(\int_\alpha^t q(u)du\right)^2 dt \ge \frac{1}{\beta - \alpha}. \tag{4.1.27}$$

Corollary 4.1.9. *Assume that* $\Delta Q(n) = q(n)$*. Suppose that* y *is a nontrivial solution of (4.1.19). If* $y(\alpha) = y(\beta) = 0$*, then*

$$\left[\sqrt{\beta-\alpha}\left(\sum_{n=\alpha}^{n-1} Q^2(n)\right)^{\frac{1}{2}} + \sup_{\alpha\le n\le\beta} |Q(n)|\right] \ge 1.$$

4.2 Second Order Half-Linear Equation

In this section, we consider some second order half-linear dynamic equations on time scales and establish Lyapunov inequalities. First we consider the second-order half-linear dynamic equation of the form

$$\left(r(t)\varphi(x^{\Delta})\right)^{\Delta} + p(t)\varphi(x^{\sigma}(t)) = 0, \tag{4.2.1}$$

on an arbitrary time scale $\mathbb{T}$, where $\varphi(u) = |u|^{\gamma-1}u$, $\gamma > 0$ is a positive constant, r and p are real rd-continuous positive functions defined on $\mathbb{T}$ with $r(t) \neq 0$. The results for (4.2.1) are adapted from [130].

Theorem 4.2.1 *Let* $x(t)$ *be a positive solution of (4.2.1) on* $\mathbb{T}$ *satisfying* $x(a) = x(b) = 0$*,* $x(t) \neq 0$ *for* $t \in (a,b)$ *and* $x(t)$ *has a maximum at a point* $c \in (a,b)$*. Then*

$$\left(\int_a^b r^{\frac{-1}{\gamma}}(t)\Delta t\right)^{\gamma} \int_a^b p(t)\Delta t \ge 2^{\gamma+1}. \tag{4.2.2}$$

Proof. Let

$$M = |x(c)| = \left|\int_a^c x^{\Delta}(t)\Delta t\right| = \left|\int_c^b x^{\Delta}(t)\Delta t\right|. \tag{4.2.3}$$

From (4.2.3), we observe that

$$2M = \left|\int_a^c x^{\Delta}(t)\Delta t\right| + \left|\int_c^b x^{\Delta}(t)\Delta t\right| \le \int_a^c \left|x^{\Delta}(t)\right|\Delta t + \int_c^b \left|x^{\Delta}(t)\right|\Delta t.$$

This implies that

$$2M \le \int_a^b \left|x^{\Delta}(t)\right|\Delta t = \int_a^b r^{\frac{-1}{\gamma+1}}(t)(r^{\frac{1}{\gamma+1}}(t)\left|x^{\Delta}(t)\right|)\Delta t. \tag{4.2.4}$$

From this we get

$$(2M)^{\gamma+1} \le \left(\int_a^b r^{\frac{-1}{\gamma+1}}(t)(r^{\frac{1}{\gamma+1}}(t)\left|x^{\Delta}(t)\right|)\Delta t\right)^{\gamma+1}. \tag{4.2.5}$$

Applying the Hölder inequality with $f(t) = r^{\frac{-1}{\gamma+1}}(t)$, $g(t) = r^{\frac{1}{\gamma+1}}(t)\left|x^{\Delta}(t)\right|$, $p = \gamma + 1$ and $q = \frac{\gamma+1}{\gamma}$, we obtain

$$\begin{aligned}
&\int_a^b r^{\frac{-1}{\gamma+1}}(t)(r^{\frac{1}{\gamma+1}}(t)\left|x^{\Delta}(t)\right|)\Delta t \\
&\leq \left(\int_a^b \left(r^{\frac{-1}{\gamma+1}}(t)\right)^{\frac{\gamma+1}{\gamma}}\Delta t\right)^{\frac{\gamma}{\gamma+1}} \left(\int_a^b \left(r^{\frac{1}{\gamma+1}}(t)\left|x^{\Delta}(t)\right|\right)^{\gamma+1}\Delta t\right)^{\frac{1}{\gamma+1}} \\
&= \left(\int_a^b r^{\frac{-1}{\gamma}}(t)\Delta t\right)^{\frac{\gamma}{\gamma+1}} \left(\int_a^b \left(r^{\frac{1}{\gamma+1}}(t)\left|x^{\Delta}(t)\right|\right)^{\gamma+1}\Delta t\right)^{\frac{1}{\gamma+1}} \\
&= \left(\int_a^b r^{\frac{-1}{\gamma}}(t)\Delta t\right)^{\frac{\gamma}{\gamma+1}} \left(\int_a^b r(t)(\left|x^{\Delta}(t)\right|)^{\gamma+1}\Delta t\right)^{\frac{1}{\gamma+1}}.
\end{aligned}$$

Thus

$$\left(\int_a^b r^{\frac{-1}{\gamma+1}}(t)(r^{\frac{1}{\gamma+1}}(t)\left|x^{\Delta}(t)\right|)\Delta t\right)^{\gamma+1} \leq \left(\int_a^b r^{\frac{-1}{\gamma}}(t)\Delta t\right)^{\gamma}\left(\int_a^b r(t)(\left|x^{\Delta}(t)\right|)^{\gamma+1}\Delta t\right). \tag{4.2.6}$$

Substituting (4.2.6) in (4.2.5), we have

$$(2M)^{\gamma+1} \leq \left(\int_a^b r^{\frac{-1}{\gamma}}(t)\Delta t\right)^{\gamma}\left(\int_a^b r(t)(\left|x^{\Delta}(t)\right|)^{\gamma+1}\Delta t\right). \tag{4.2.7}$$

Using integration by parts we see that (note $x(a) = x(b) = 0$)

$$\begin{aligned}
\int_a^b r(t)(\left|x^{\Delta}(t)\right|)^{\gamma+1}\Delta t &= \int_a^b x^{\Delta}(t)\left(r(t)(\left|x^{\Delta}(t)\right|)^{\gamma-1}x^{\Delta}(t)\right)\Delta t \\
&= -\int_a^b \left[r(t)(\left|x^{\Delta}(t)\right|)^{\gamma-1}x^{\Delta}(t)\right]^{\Delta} x^{\sigma}(t)\Delta t. \qquad (4.2.8)
\end{aligned}$$

Now (4.2.1) implies that

$$\int_a^b r(t)(\left|x^{\Delta}(t)\right|)^{\gamma+1}\Delta t = \int_a^b p(t)\left(x^{\sigma}(t)\right)^{\gamma+1}\Delta t.$$

This and (4.2.7) imply that

$$\begin{aligned}
(2M)^{\gamma+1} &\leq \left(\int_a^b r^{\frac{-1}{\gamma}}(t)\Delta t\right)^{\gamma}\left(\int_a^b p(t)\left(x^{\sigma}(t)\right)^{\gamma+1}\Delta t\right) \\
&\leq M^{\gamma+1}\left(\int_a^b r^{\frac{-1}{\gamma}}(t)\Delta t\right)^{\gamma}\left(\int_a^b p(t)\Delta t\right).
\end{aligned}$$

Now, dividing by $M^{\gamma+1}$, we have

$$\left(\int_a^b r^{\frac{-1}{\gamma}}(t)\Delta t\right)^{\gamma}\left(\int_a^b p(t)\Delta t\right) \geq 2^{\gamma+1},$$

which is the desired inequality (4.2.2). The proof is complete. ∎

Remark 4.2.1 *Note the inequality with $\gamma = 1$ and $r(t) = 1$, reduces to the inequality*

$$\int_a^b p(t)\Delta t > \frac{4}{b-a}. \tag{4.2.9}$$

Now, we consider the half-linear delay dynamic equation

$$(r(t)(\varphi(x^{\Delta}(t)))^{\Delta} + p(t)(\varphi(x(\tau(t))) = 0, \tag{4.2.10}$$

on an arbitrary time scale $\mathbb{T}$, where $\gamma > 0$ is a positive constant, r and p are real rd-continuous positive functions defined on $\mathbb{T}$ with $r(t) \neq 0$, $\tau : \mathbb{T} \to \mathbb{T}$, $\tau(t) \leq t$ for all $t \in \mathbb{T}$, $\lim_{t\to\infty}\tau(t) = \infty$, and

$$\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta t = \infty. \tag{4.2.11}$$

Note that when the condition (4.2.11) holds, then the positive solution $x(t)$ of (4.2.10) satisfies $x^{\Delta}(t) > 0$. Under this condition, we see, since $\tau(t) \leq t$, that $x(\tau(t))/x^{\sigma}(t) \leq 1$. Using this claim we have the following result for (4.2.10).

Corollary 4.2.1 *Assume that (4.2.11) holds and let $x(t)$ be a positive solution of (4.2.10) on $\mathbb{T}$ satisfying $x(a) = x(b) = 0$, $x(t) \neq 0$ for $t \in (a, b)$ and $x(t)$ has a maximum at a point $c \in (a, b)$. Then*

$$\left(\int_a^b r^{\frac{-1}{\gamma}}(t)\Delta t\right)^{\gamma}\int_a^b p(t)\Delta t \geq 2^{\gamma+1}.$$

Proof. We proceed as in the proof of Theorem 4.2.1, to get

$$(2M)^{\gamma+1} \leq \left(\int_a^b r^{\frac{-1}{\gamma}}(t)\Delta t\right)^{\gamma}\left(\int_a^b r(t)(\left|x^{\Delta}(t)\right|)^{\gamma+1}\Delta t\right).$$

Using integration by parts we see that (note $x(a) = x(b) = 0$)

$$\int_a^b r(t)(\left|x^{\Delta}(t)\right|)^{\gamma+1}\Delta t = -\int_a^b \left[r(t)(\left|x^{\Delta}(t)\right|)^{\gamma-1}x^{\Delta}(t)\right]^{\Delta} x^{\sigma}(t)\Delta t.$$

Now (4.2.10) implies that

$$\int_a^b r(t)(|x^\Delta(t)|)^{\gamma+1}\Delta t = \int_a^b p(t)\left(\frac{x(\tau(t))}{x^\sigma(t)}\right)^\gamma (x^\sigma(t))^{\gamma+1}\,\Delta t.$$

Using the above claim, since $x(\tau(t))/x^\sigma(t) \le 1$, we have

$$\int_a^b r(t)(|x^\Delta(t)|)^{\gamma+1}\Delta t \le \int_a^b p(t)\,(x^\sigma(t))^{\gamma+1}\,\Delta t.$$

The remainder of the proof is similar to the proof in Theorem 4.2.1 and hence is omitted. ■

In the following, we establish some sufficient conditions for the disconjugacy of (4.2.1).

Theorem 4.2.2 *Let r and p satisfy*

$$\int_a^b p(t)\Delta t < \begin{cases} \dfrac{r^{\gamma+1}(a)}{r^\gamma(b)}\dfrac{(b-c)^\gamma+(c-a)^\gamma}{(c-a)^\gamma(b-c)^\gamma}, & \textit{if } r(t) \textit{ is increasing,} \\ \dfrac{r^{\gamma+1}(b)}{r^\gamma(a)}\dfrac{(b-c)^\gamma+(c-a)^\gamma}{(c-a)^\gamma(b-c)^\gamma}, & \textit{if } r(t) \textit{ is decreasing.} \end{cases} \tag{4.2.12}$$

Then (4.2.1) is disconjugate in $\mathbb{T}$.

Proof. Suppose that (4.2.12) holds and assume for the sake of contradiction that (4.2.1) is not disconjugate. Then there exists a nontrivial solution x with $x(a) = x(b) = 0$. Using this x, and integrate by parts to see that (note $x(a) = x(b) = 0$)

$$\begin{aligned}\int_a^b r(t)(|x^\Delta(t)|)^{\gamma+1}\Delta t &= \int_a^b x^\Delta(t)\left(r(t)(|x^\Delta(t)|)^{\gamma-1}x^\Delta(t)\right)\Delta t \\ &= -\int_a^b \left[r(t)(|x^\Delta(t)|)^{\gamma-1}x^\Delta(t)\right]^\Delta x(t)\Delta t.\end{aligned}$$

Now (4.2.1) implies that

$$\int_a^b r(t)(|x^\Delta(t)|)^{\gamma+1}\Delta t = \int_a^b p(t)\,(x(t))^{\gamma+1}\,\Delta t$$

Then, we have

$$\begin{aligned}M^{\gamma+1}\int_a^b p(t)\Delta t \ge \int_a^b p(t)\,|x(t)|^{\gamma+1}\,\Delta t &\ge \int_a^b r(t)\left|x^\Delta(t)\right|^{\gamma+1}\Delta t \\ &= \int_a^c r(t)\left|x^\Delta(t)\right|^{\gamma+1}\Delta t + \int_c^b r(t)\left|x^\Delta(t)\right|^{\gamma+1}\Delta t, \end{aligned} \tag{4.2.13}$$

where M is defined as in Theorem 4.2.1. Now, since

$$\int_a^c r(t)\left|x^\Delta(t)\right|\Delta t=\int_a^c r^{\frac{\gamma}{\gamma+1}}(t)\left(r^{\frac{1}{\gamma+1}}(t)\left|x^\Delta(t)\right|\right)\Delta t,$$

we have after applying the Hölder inequality with $f(t)=r^{\frac{\gamma}{\gamma+1}}(t)$, $g(t)=r^{\frac{1}{\gamma+1}}(t)\left|x^\Delta(t)\right|$, $p=\gamma+1$ and $q=\frac{\gamma+1}{\gamma}$, that

$$\begin{aligned}&\int_a^c r^{\frac{\gamma}{\gamma+1}}(t)\left(r^{\frac{1}{\gamma+1}}(t)\left|x^\Delta(t)\right|\right)\Delta t\\ &\leq\left(\int_a^c\left(r^{\frac{\gamma}{\gamma+1}}(t)\right)^{\frac{\gamma+1}{\gamma}}\Delta t\right)^{\frac{\gamma}{\gamma+1}}\left(\int_a^c\left(r^{\frac{1}{\gamma+1}}(t)\left|x^\Delta(t)\right|\right)^{\gamma+1}\Delta t\right)^{\frac{1}{\gamma+1}}\\ &=\left(\int_a^c r(t)\Delta t\right)^{\frac{\gamma}{\gamma+1}}\left(\int_a^c r(t)\left|x^\Delta(t)\right|^{\gamma+1}\Delta t\right)^{\frac{1}{\gamma+1}}.\end{aligned}$$

Then

$$\begin{aligned}&\left(\int_a^c r(t)\Delta t\right)^{\gamma}\left(\int_a^c r(t)\left|x^\Delta(t)\right|^{\gamma+1}\Delta t\right)\\ &\geq\left(\int_a^c r^{\frac{\gamma}{\gamma+1}}(t)\left(r^{\frac{1}{\gamma+1}}(t)x^\Delta(t)\right)\Delta t\right)^{\gamma+1}=\left(\int_a^c r(t)\left|x^\Delta(t)\right|\Delta t\right)^{\gamma+1}.\end{aligned}$$

This implies that

$$\left(\int_a^c r(t)\left|x^\Delta(t)\right|^{\gamma+1}\Delta t\right)\geq\frac{\left(\int_a^c r(t)\left|x^\Delta(t)\right|\Delta t\right)^{\gamma+1}}{\left(\int_a^c r(t)\Delta t\right)^{\gamma}}. \tag{4.2.14}$$

Also we see that

$$\left(\int_c^b r(t)\left|x^\Delta(t)\right|^{\gamma+1}\Delta t\right)\geq\frac{\left(\int_c^b r(t)\left|x^\Delta(t)\right|\Delta t\right)^{\gamma+1}}{\left(\int_c^b r(t)\Delta t\right)^{\gamma}}. \tag{4.2.15}$$

Substituting (4.2.14) and (4.2.15) into (4.2.13), we have

$$\begin{aligned}&M^{\gamma+1}\int_a^b p(t)\Delta t\\ &\geq\frac{\left(\int_a^c r(t)\left|x^\Delta(t)\right|\Delta t\right)^{\gamma+1}}{\left(\int_a^c r(t)\Delta t\right)^{\gamma}}+\frac{\left(\int_c^b r(t)\left|x^\Delta(t)\right|\Delta t\right)^{\gamma+1}}{\left(\int_c^b r(t)\Delta t\right)^{\gamma}}\end{aligned}$$

$$\geq \begin{cases} \dfrac{\left(r(a)\int_a^c \left|x^\Delta(t)\right|\Delta t\right)^{\gamma+1}}{\left(\int_a^c r(t)\Delta t\right)^\gamma} + \dfrac{\left(r(a)\int_c^b \left|x^\Delta(t)\right|\Delta t\right)^{\gamma+1}}{\left(\int_c^b r(t)\Delta t\right)^\gamma}, & \text{if } r(t) \text{ is increasing,} \\ \dfrac{\left(r(b)\int_a^c \left|x^\Delta(t)\right|\Delta t\right)^{\gamma+1}}{\left(\int_a^c r(t)\Delta t\right)^\gamma} + \dfrac{\left(r(b)\int_c^b \left|x^\Delta(t)\right|\Delta t\right)^{\gamma+1}}{\left(\int_c^b r(t)\Delta t\right)^\gamma}, & \text{if } r(t) \text{ is decreasing,} \end{cases}$$

$$\geq \begin{cases} \dfrac{r^{\gamma+1}(a)M^{\gamma+1}}{r^\gamma(b)(c-a)^\gamma} + \dfrac{r^{\gamma+1}(a)M^{\gamma+1}}{r^\gamma(b)(b-c)^\gamma}, & \text{if } r(t) \text{ is increasing,} \\ \dfrac{r^{\gamma+1}(b)M^{\gamma+1}}{r^\gamma(a)(c-a)^\gamma} + \dfrac{r^{\gamma+1}(b)M^{\gamma+1}}{r^\gamma(a)(b-c)^\gamma}, & \text{if } r(t) \text{ is decreasing.} \end{cases}$$

Dividing by $M^{\gamma+1}$, we have

$$\int_a^b p(t)\Delta t \geq \begin{cases} \dfrac{r^{\gamma+1}(a)}{r^\gamma(b)} \dfrac{(b-c)^\gamma + (c-a)^\gamma}{(c-a)^\gamma(b-c)^\gamma}, & \text{if } r(t) \text{ is increasing,} \\ \dfrac{r^{\gamma+1}(b)}{r^\gamma(a)} \dfrac{(b-c)^\gamma + (c-a)^\gamma}{(c-a)^\gamma(b-c)^\gamma}, & \text{if } r(t) \text{ is decreasing,} \end{cases}$$

which is a contradiction with (4.2.12) and hence completes the proof. ■

As a consequence from Theorem 4.2.2, by using the fact that

$$\left(\frac{x_1^\gamma + x_2^\gamma}{2}\right)^{\frac{1}{\gamma}} \geq \frac{2x_1x_2}{x_1+x_2}, \text{ for } x_1 = c-a \text{ and } x_2 = b-c,$$

we have the following result.

Theorem 4.2.3 . *If r and p satisfy*

$$\int_a^b p(t)\Delta t < \begin{cases} \dfrac{r^{\gamma+1}(a)}{r^\gamma(b)} \dfrac{2^{\gamma+1}}{(b-a)^\gamma}, & \textit{if } r(t) \textit{ is increasing,} \\ \dfrac{r^{\gamma+1}(b)}{r^\gamma(a)} \dfrac{2^{\gamma+1}}{(b-a)^\gamma}, & \textit{if } r(t) \textit{ is decreasing.} \end{cases} \tag{4.2.16}$$

Then (4.2.1) is disconjugate in $\mathbb{T}$.

We end this section by applying Opial type inequalities to establish some Lyapunov type inequalities for the second order half-linear dynamic equation

$$(r(t)(y^\Delta(t))^\gamma)^\Delta + q(t)\,(y^\sigma(t))^\gamma = 0, \quad \text{on } [a,b]_\mathbb{T}, \tag{4.2.17}$$

where $\mathbb{T}$ is an arbitrary time scale. The results are adapted from [133]. For Eq. (4.2.17), we assume that $0 < \gamma \leq 1$ is a quotient of odd positive integers, r and q are real rd-continuous functions defined on $\mathbb{T}$ with $r(t) > 0$. We obtain lower bounds for the spacing $\beta - \alpha$ where y is a solution of (4.2.17) satisfying some conditions at α and β.

To simplify the presentation of the results, we define

$$\begin{aligned} M(\beta) &: = \sup_{\alpha\leq t\leq\beta}\mu^{\gamma}(t)\frac{|Q(t)|}{r(t)}, \text{ where } Q(t)=\int_t^{\beta}q(s)\Delta s, \\ M(\alpha) &: = \sup_{\alpha\leq t\leq\beta}\mu^{\gamma}(t)\frac{|Q(t)|}{r(t)}, \text{ where } Q(t)=\int_{\alpha}^{t}q(s)\Delta s. \end{aligned}$$

Note that when $\mathbb{T}=\mathbb{R}$, we have $M(\alpha)=0=M(\beta)$, and when $\mathbb{T}=\mathbb{Z}$, we have

$$M(\beta)=\sup_{\alpha\leq t\leq\beta}\frac{\left|\sum_{s=t}^{\beta-1}q(s)\right|}{r(t)}, \text{ and } M(\alpha)=\sup_{\alpha\leq t\leq\beta}\frac{\left|\sum_{s=\alpha}^{t-1}q(s)\right|}{r(t)}. \tag{4.2.18}$$

Theorem 4.2.4 *Suppose that y is a nontrivial solution of (4.2.17) and y^{Δ} does not change sign in $(\alpha,\beta)_{\mathbb{T}}$. If $y(\alpha)=y^{\Delta}(\beta)=0$, then*

$$\frac{2}{(\gamma+1)^{\frac{\gamma}{\gamma+1}}}\times\left(\int_{\alpha}^{\beta}\frac{|Q(x)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(x)}\left(\int_{\alpha}^{x}\frac{\Delta t}{r^{\frac{1}{\gamma}}(t)}\right)^{\gamma}\Delta x\right)^{\frac{\gamma}{\gamma+1}}+2^{1-\gamma}M(\beta)\geq 1, \tag{4.2.19}$$

where $Q(t)=\int_t^{\beta}q(s)\Delta s$. If $y^{\Delta}(\alpha)=y(\beta)=0$, then

$$\frac{2}{(\gamma+1)^{\frac{\gamma}{\gamma+1}}}\left(\int_{\alpha}^{\beta}\frac{|Q(x)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(x)}\left(\int_{x}^{\beta}\frac{\Delta t}{r^{\frac{1}{\gamma}}(t)}\right)^{\gamma}\Delta x\right)^{\frac{\gamma}{\gamma+1}}+2^{1-\gamma}M(\alpha)\geq 1, \tag{4.2.20}$$

where $Q(t)=\int_{\alpha}^{t}q(s)\Delta s$.

Proof. We prove (4.2.19). Without loss of generality we may assume that $y(t)>0$ in $[\alpha,\beta]_{\mathbb{T}}$. Multiplying (4.2.17) by y^{σ} and integrating by parts, we have

$$\begin{aligned} \int_{\alpha}^{\beta}\left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}y^{\sigma}(t)\Delta t &= r(t)\left(y^{\Delta}(t)\right)^{\gamma}y(t)\Big|_{\alpha}^{\beta} \\ &\quad -\int_{\alpha}^{\beta}r(t)\left(y^{\Delta}(t)\right)^{\gamma+1}\Delta t \\ &= -\int_{\alpha}^{\beta}q(t)\left(y^{\sigma}(t)\right)^{\gamma+1}\Delta t. \end{aligned}$$

Using the assumptions that $y(\alpha)=y^{\Delta}(\beta)=0$ and $Q(t)=\int_t^{\beta}q(s)\Delta s$, we have

$$\int_{\alpha}^{\beta}r(t)\left(y^{\Delta}(t)\right)^{\gamma+1}\Delta t=\int_{\alpha}^{\beta}q(t)\left(y^{\sigma}(t)\right)^{\gamma+1}\Delta t=-\int_{\alpha}^{\beta}Q^{\Delta}(t)\left(y^{\sigma}(t)\right)^{\gamma+1}\Delta t. \tag{4.2.21}$$

Integrating by parts the right-hand side we see that

$$\int_{\alpha}^{\beta} r(t)\left(y^{\Delta}(t)\right)^{\gamma+1}\Delta t = -\left.Q(t)(y(t))^{\gamma+1}\right|_{\alpha}^{\beta} + \int_{\alpha}^{\beta} Q(t)\left(y^{\gamma+1}(t)\right)^{\Delta}\Delta t.$$

Again using the facts that $y(\alpha) = 0 = Q(\beta)$, we obtain

$$\int_{\alpha}^{\beta} r(t)\left(y^{\Delta}(t)\right)^{\gamma+1} dt = \int_{\alpha}^{\beta} Q(t)\left(y^{\gamma+1}(t)\right)^{\Delta} dt. \tag{4.2.22}$$

Applying the chain rule formula and the inequality (3.3.2), we see that

$$\begin{aligned}
\left|\left(y^{\gamma+1}(t)\right)^{\Delta}\right| &\le (\gamma+1)\int_{0}^{1}\left|hy^{\sigma}(t)+(1-h)y(t)\right|^{\gamma} dh\left|y^{\Delta}(t)\right| \\
&\le (\gamma+1)\left|y^{\Delta}(t)\right|\int_{0}^{1}\left|hy^{\sigma}(t)\right|^{\gamma} dh \\
&\quad +(\gamma+1)\left|y^{\Delta}(t)\right|\int_{0}^{1}\left|(1-h)y(t)\right|^{\gamma} dh \\
&= \left|y^{\Delta}(t)\right|\left|y^{\sigma}(t)\right|^{\gamma} + \left|y^{\Delta}(t)\right|\left|y(t)\right|^{\gamma} \\
&\le 2^{1-\gamma}\left|y^{\sigma}(t)+y(t)\right|^{\gamma}\left|y^{\Delta}(t)\right|. \qquad (4.2.23)
\end{aligned}$$

This and (4.2.22) imply that

$$\int_{\alpha}^{\beta} r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t \le 2^{1-\gamma}\int_{\alpha}^{\beta}\left|Q(t)\right|\left|y(t)+y^{\sigma}(t)\right|^{\gamma}\left|y^{\Delta}(t)\right|\Delta t.$$

Applying the inequality (3.3.22) with $s(t) = |Q(t)|$, $p = \gamma$ and $q = 1$, we have

$$\int_{\alpha}^{\beta} r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t \le 2^{1-\gamma}K_1(\alpha,\beta,\gamma,1)\int_{\alpha}^{\beta} r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t, \tag{4.2.24}$$

where

$$\begin{aligned}
K_1(\alpha,\beta,\gamma,1) &= M(\beta) + 2^{\gamma}\left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma+1}} \\
&\quad \times\left(\int_{\alpha}^{\beta}\left|Q(x)\right|^{\frac{\gamma+1}{\gamma}} r^{-\frac{1}{\gamma}}(x)\left(\int_{\alpha}^{x} r^{\frac{-1}{\gamma}}(t)\Delta t\right)^{\gamma}\Delta x\right)^{\frac{\gamma}{\gamma+1}}.
\end{aligned}$$

Then, we have from (4.2.24) after cancelling the term $\int_{\alpha}^{\beta} r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t$, that

$$2^{1-\gamma}M(\beta) + \frac{2}{(\gamma+1)^{\frac{1}{\gamma+1}}}\times\left(\int_{\alpha}^{\beta}\frac{\left|Q(x)\right|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(x)}\left(\int_{\alpha}^{x}\frac{\Delta t}{r^{\frac{1}{\gamma}}(t)}\right)^{\gamma}\Delta x\right)^{\frac{\gamma}{\gamma+1}} \ge 1,$$

which is the desired inequality (4.2.19). The proof of (4.2.20) is similar to (4.2.19) by using integration by parts and (3.3.29) of Theorem 3.3.5 and (3.3.30) instead of (3.3.23). The proof is complete. ■

As a special case of Theorem 4.2.4, when $r(t) = 1$, we have the following result.

Corollary 4.2.2 *Suppose that y is a nontrivial solution of*

$$\left(\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + q(t)\left(y^{\sigma}(t)\right)^{\gamma} = 0, \quad t \in [\alpha, \beta]_{\mathbb{T}}, \tag{4.2.25}$$

and y^{Δ} does not change sign in $(\alpha, \beta)_{\mathbb{T}}$. If $y(\alpha) = y^{\Delta}(\beta) = 0$, then

$$\frac{2}{(\gamma+1)^{\frac{1}{\gamma+1}}} \times \left[\int_{\alpha}^{\beta} |Q(t)|^{\frac{1+\gamma}{\gamma}} (t-\alpha)^{\gamma} \Delta t\right]^{\frac{\gamma}{\gamma+1}} + 2^{1-\gamma} \sup_{\alpha \le t \le \beta} \left(\mu^{\gamma}(t) |Q(t)|\right) \ge 1, \tag{4.2.26}$$

where $Q(t) = \int_t^{\beta} q(s)\Delta s$. If $y^{\Delta}(\alpha) = y(\beta) = 0$, then

$$\frac{2}{(\gamma+1)^{\frac{1}{\gamma+1}}} \left[\int_{\alpha}^{\beta} |Q(t)|^{\frac{1+\gamma}{\gamma}} (\beta - t)^{\gamma} \Delta t\right]^{\frac{\gamma}{\gamma+1}} + 2^{1-\gamma} \sup_{\alpha \le t \le \beta} \left(\mu^{\gamma}(t) |Q(t)|\right) \ge 1, \tag{4.2.27}$$

where $Q(t) = \int_{\alpha}^{t} q(s)\Delta s$.

Corollary 4.2.3 *Suppose that y is a nontrivial solution of (4.2.25) and y^{Δ} does not change sign in $(\alpha, \beta)_{\mathbb{T}}$, and $\gamma \le 1$ is a quotient of odd positive integers. If $y(\alpha) = y^{\Delta}(\beta) = 0$, then*

$$\frac{2(\beta-\alpha)^{\gamma}}{(\gamma+1)} \max_{\alpha \le t \le \beta} \left|\int_t^{\beta} q(s)\Delta s\right| + 2^{1-\gamma} \sup_{\alpha \le t \le \beta} \left(\mu^{\gamma}(t) \left|\int_t^{\beta} q(s)\Delta s\right|\right) \ge 1, \tag{4.2.28}$$

whereas if $y^{\Delta}(\alpha) = y(\beta) = 0$, then

$$\frac{2(\beta-\alpha)^{\gamma}}{(\gamma+1)} \max_{\alpha \le t \le \beta} \left|\int_{\alpha}^{t} q(s)\Delta s\right| + 2^{1-\gamma} \sup_{\alpha \le t \le \beta} \left(\mu^{\gamma}(t) \left|\int_{\alpha}^{t} q(s)\Delta s\right|\right) \ge 1. \tag{4.2.29}$$

As a special when $\mathbb{T} = \mathbb{R}$, we have $M(\alpha) = M(\beta) = 0$ and we consider the second order half-linear differential equation

$$\left((y'(t))^{\gamma}\right)' + q(t)(y(t))^{\gamma} = 0, \ \ \alpha \le t \le \beta, \tag{4.2.30}$$

where $\gamma \le 1$ is a quotient of odd positive integers.

Corollary 4.2.4 *Assume that $\gamma \leq 1$ is a quotient of odd positive integers. Suppose that y is a nontrivial solution of (4.2.30) and y' does not change sign in (α, β). If $y(\alpha) = y'(\beta) = 0$, then*

$$\frac{2}{(\gamma+1)}(\beta-\alpha)^{\gamma} \sup_{\alpha \leq t \leq \beta} \left| \int_t^{\beta} q(s)ds \right| \geq 1. \tag{4.2.31}$$

If instead $y'(\alpha) = y(\beta) = 0$, then

$$\frac{2}{(\gamma+1)}(\beta-\alpha)^{\gamma} \sup_{\alpha \leq t \leq \beta} \left| \int_{\alpha}^{t} q(s)ds \right| \geq 1. \tag{4.2.32}$$

As a special when $\mathbb{T} = \mathbb{Z}$, we see that $M(\alpha)$ and $M(\beta)$ are defined as in (4.2.18) and we consider the second order half-linear difference equation

$$\Delta\left((\Delta y(n))^{\gamma}\right) + q(n)(y(n+1))^{\gamma} = 0, \;\; \alpha \leq n \leq \beta, \tag{4.2.33}$$

where $\gamma \leq 1$ is a quotient of odd positive integers.

Corollary 4.2.5 *Suppose that y is a nontrivial solution of (4.3.17) and $\Delta y(n)$ does not change sign in $(\alpha, \beta)_{\mathbb{T}}$, and $\gamma \leq 1$ is a quotient of odd positive integers. If $y(\alpha) = \Delta y(\beta) = 0$, then*

$$\frac{2(\beta-\alpha)^{\gamma}}{(\gamma+1)} \max_{\alpha \leq n \leq \beta} \left| \sum_{s=n}^{\beta-1} q(s) \right| + 2^{1-\gamma} \sup_{\alpha \leq n \leq \beta} \left(\left| \sum_{s=n}^{\beta-1} q(s) \right| \right) \geq 1,$$

whereas if $\Delta y(\alpha) = y(\beta) = 0$, then

$$\frac{2(\beta-\alpha)^{\gamma}}{(\gamma+1)} \max_{\alpha \leq n \leq \beta} \left| \sum_{s=\alpha}^{n-1} q(s) \right| + 2^{1-\gamma} \sup_{\alpha \leq n \leq \beta} \left(\left| \sum_{s=\alpha}^{n-1} q(s) \right| \right) \geq 1.$$

Remark 4.2.2 *The above results yield sufficient conditions for the disfocality of (4.3.1), i.e., sufficient conditions so that there does not exist a nontrivial solution y satisfying either $y(\alpha) = y^{\Delta}(\beta) = 0$, or $y^{\Delta}(\alpha) = y(\beta) = 0$.*

Next we employ Theorem 3.3.6 to determine a lower bound for the distance between consecutive zeros of solutions of (4.2.17). Note that the applications of the above results allow the use of arbitrary anti-derivative Q in the above arguments. In the following, we assume that $Q^{\Delta}(t) = q(t)$ and there exists $h \in (\alpha, \beta)$ which is the unique solution of the equation

$$K_1(\alpha, \beta) = K_1(\alpha, \beta, h) = K_1(\alpha, h, \beta) < \infty, \tag{4.2.34}$$

where

$$K_1(\alpha, \beta, h) = \frac{2^{\gamma}}{(\gamma+1)^{\frac{1}{\gamma+1}}} \times \left(\int_{\alpha}^{\beta} \frac{|Q(x)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(x)} \left(\int_{\alpha}^{h} \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} \right)^{\gamma} \Delta x \right)^{\frac{\gamma}{\gamma+1}},$$

and

$$K_1(\alpha, h, \beta) = \frac{2^\gamma}{(\gamma+1)^{\frac{1}{\gamma+1}}} \left(\int_\alpha^\beta \frac{|Q(x)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(x)} \left(\int_h^\beta \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} \right)^\gamma \Delta x \right)^{\frac{\gamma}{\gamma+1}}.$$

Theorem 4.2.5 *Assume that $Q^\Delta(t) = q(t)$. Suppose y is a nontrivial solution of (4.2.17) and $y^\Delta(t)$ does not change sign in (α, β). If $y(\alpha) = y(\beta) = 0$, then*

$$K_1(\alpha, \beta) \geq 1, \tag{4.2.35}$$

where $K_1(\alpha, \beta)$ is defined as in (4.2.34).

Proof. Multiply (4.2.17) by $y^\sigma(t)$, and proceed as in Theorem 4.2.4 and use $y(\alpha) = y(\beta) = 0$, to get

$$\int_\alpha^\beta r(t) \left(y^\Delta(t)\right)^{\gamma+1} \Delta t = \int_\alpha^\beta q(t) \left(y(t)\right)^{\gamma+1} \Delta t = \int_\alpha^\beta Q^\Delta(t) \left(y^\sigma(t)\right)^{\gamma+1} \Delta t.$$

Integrating by parts the right-hand side, we see that

$$\int_\alpha^\beta r(t) \left(y^\Delta(t)\right)^{\gamma+1} \Delta t = Q(t)(y(t))^{\gamma+1}\Big|_\alpha^\beta + \int_\alpha^\beta (-Q(t)) \left(y^{\gamma+1}(t)\right)^\Delta \Delta t.$$

Again using the facts that $y(\alpha) = 0 = y(\beta)$, we obtain

$$\int_\alpha^\beta r(t) \left|y^\Delta(t)\right|^{\gamma+1} \Delta t \leq \int_\alpha^\beta |Q(t)| \, |y(t) + y^\sigma(t)|^\gamma \left|y^\Delta(t)\right| \Delta t.$$

Applying the inequality (3.3.31) with $s(t) = |Q(t)|$, $p = \gamma$ and $q = 1$, we have

$$\int_\alpha^\beta r(t) \left|y^\Delta(t)\right|^{\gamma+1} dt \leq 2^{1-\gamma} K_1(\alpha, \beta) \int_\alpha^\beta r(t) \left|y^\Delta(t)\right|^{\gamma+1} \Delta t.$$

From this inequality, after cancelling $\int_\alpha^\beta \left|y^\Delta(t)\right|^{\gamma+1} \Delta t$, we get the desired inequality (4.2.35). This completes the proof. ■

4.3 Second Order Equations with Damping Terms

In this section we consider the second-order half-linear dynamic equation with a damping term

$$\left(r(t) \left(x^\Delta(t)\right)^\gamma\right)^\Delta + p(t) \left(x^\Delta(t)\right)^\gamma + q(t) \left(x^\sigma(t)\right)^\gamma = 0, \quad t \in [\alpha, \beta]_{\mathbb{T}}, \tag{4.3.1}$$

where $\mathbb{T}$ is an arbitrary time scale and $\sigma(t)$ is the forward jump operator on $\mathbb{T}$ which is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$.

We say that a solution x of (4.3.1) has a generalized zero at t if $x(t) = 0$, and has a generalized zero in $(t, \sigma(t))$ in the case $x(t)x^\sigma(t) < 0$ and $\mu(t) > 0$.

Equation (4.3.1) is disconjugate on the interval $[t_0, b]_{\mathbb{T}}$, if there is no nontrivial solution of (4.3.1) with two (or more) generalized zeros in $[t_0, b]_{\mathbb{T}}$. We say that (4.3.1) is right disfocal (left disfocal) on $[\alpha, \beta]_{\mathbb{T}}$ if the solutions of (4.3.1) such that $x^{\Delta}(\alpha) = 0$ ($x^{\Delta}(\beta) = 0$) have no generalized zeros in $[\alpha, \beta]_{\mathbb{T}}$. For Eq. (4.3.1) the point $\beta > \alpha$ is called a right focal point of α if the solution of (4.3.1) with initial conditions $x(\alpha) \neq 0$, $x^{\Delta}(\alpha) = 0$ satisfies $x(\beta) = 0$. The left focal point is defined similarly.

We will assume that $\gamma \geq 1$ is a quotient of odd positive integers, r, p and q are real rd-continuous functions defined on $\mathbb{T}$ with $r(t) > 0$ and $\mu(t)\,|p(t)| \leq r(t)/c$ where c is a positive constant such that $c \geq 1$. We also assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[a, b]_{\mathbb{T}}$ by $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$. To simplify the presentation of the results, we define

$$\begin{aligned}
\Lambda(\beta) \ &: = \sup_{\alpha\leq t\leq\beta} \mu^{\gamma}(t)\frac{|Q(t)|}{r(t)}, \text{ where } Q(t) = \int_t^{\beta} q(s)\Delta s, \\
\Lambda(\alpha) \ &: = \sup_{\alpha\leq t\leq\beta} \mu^{\gamma}(t)\frac{|Q(t)|}{r(t)}, \quad \text{where} \quad Q(t) = \int_{\alpha}^{t} q(s)\Delta s, \\
R_{\alpha}(t) \ &: = \int_{\alpha}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}, \quad \text{and} \quad R_{\beta}(t) := \int_t^{\beta} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}.
\end{aligned}$$

Note that when $\mathbb{T} = \mathbb{R}$, we have $\Lambda(\alpha) = 0 = \Lambda(\beta)$ and when $\mathbb{T} = \mathbb{Z}$, we have

$$\Lambda(\beta) = \sup_{\alpha\leq t\leq\beta} \frac{\left|\sum_{s=t}^{\beta-1} q(s)\right|}{r(t)}, \text{ and } \Lambda(\alpha) = \sup_{\alpha\leq t\leq\beta} \frac{\left|\sum_{s=\alpha}^{t-1} q(s)\right|}{r(t)}. \tag{4.3.2}$$

Now, we are ready to state and prove the main results.

Theorem 4.3.1 *Suppose that x is a nontrivial solution of (4.3.1) and x^{Δ} does not change sign on $(\alpha, \beta)_{\mathbb{T}}$. If $x(\alpha) = x^{\Delta}(\beta) = 0$, then*

$$\begin{aligned}
&2^{2\gamma-2}\Lambda(\beta) + \frac{2^{3\gamma-2}}{(\gamma+1)^{\frac{1}{\gamma+1}}} \times \left(\int_{\alpha}^{\beta} \frac{|Q(t)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)} \left(R_{\alpha}(t)\right)^{\gamma} \Delta t\right)^{\frac{\gamma}{\gamma+1}} \\
&+ \left(\frac{\gamma}{1+\gamma}\right)^{\frac{\gamma}{\gamma+1}} \times \left(\int_{\alpha}^{\beta} \frac{|p(t)|^{\gamma+1}}{r^{\gamma}(t)} \left(R_{\alpha}(t)\right)^{\gamma} \Delta t\right)^{\frac{1}{\gamma+1}} \geq 1 - \frac{1}{c}, \qquad (4.3.3)
\end{aligned}$$

where $Q(t) = \int_t^{\beta} q(s)\Delta s$. If instead $x^{\Delta}(\alpha) = x(\beta) = 0$, then

$$\begin{aligned}
&2^{2\gamma-2}\Lambda(\alpha) + \frac{2^{3\gamma-2}}{(\gamma+1)^{\frac{1}{\gamma+1}}} \times \left(\int_{\alpha}^{\beta} \frac{|Q(t)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)} \left(R_{\beta}(t)\right)^{\gamma} \Delta t\right)^{\frac{\gamma}{\gamma+1}} \\
&+ \left(\frac{\gamma}{1+\gamma}\right)^{\frac{\gamma}{\gamma+1}} \times \left(\int_{\alpha}^{\beta} \frac{|p(t)|^{\gamma+1}}{r^{\gamma}(t)} \left(R_{\beta}(t)\right)^{\gamma} \Delta t\right)^{\frac{1}{\gamma+1}} \geq 1 - \frac{1}{c}, \qquad (4.3.4)
\end{aligned}$$

where $Q(t) = \int_{\alpha}^{t} q(s)\Delta s$.

Proof. We prove (4.3.3). Without loss of generality we may assume that $x(t) \geq 0$ in $[\alpha, \beta]_{\mathbb{T}}$. Multiplying (4.3.1) by x^σ and integrating by parts, we have

$$\begin{aligned}
&\int_\alpha^\beta \left(r(t)\left(x^\Delta(t)\right)^\gamma\right)^\Delta x^\sigma(t)\Delta t + \int_\alpha^\beta p(t)x^\sigma(t)\left(x^\Delta(t)\right)^\gamma \Delta t \\
= \; & r(t)\left(x^\Delta(t)\right)^\gamma x(t)\Big|_\alpha^\beta - \int_\alpha^\beta r(t)\left(x^\Delta(t)\right)^{\gamma+1}\Delta t \\
&+ \int_\alpha^\beta p(t)x^\sigma(t)\left(x^\Delta(t)\right)^\gamma \Delta t = -\int_\alpha^\beta q(t)\left(x^\sigma(t)\right)^{\gamma+1}\Delta t.
\end{aligned}$$

Using the assumption $x(\alpha) = x^\Delta(\beta) = 0$ we have

$$-\int_\alpha^\beta r(t)\left(x^\Delta(t)\right)^{\gamma+1}\Delta t + \int_\alpha^\beta p(t)x^\sigma(t)\left(x^\Delta(t)\right)^\gamma \Delta t = -\int_\alpha^\beta q(t)\left(x^\sigma(t)\right)^{\gamma+1}\Delta t.$$

This implies (note that $Q(t) = \int_t^\beta q(s)\Delta s$) that

$$\int_\alpha^\beta r(t)\left(x^\Delta(t)\right)^{\gamma+1}\Delta t = \int_\alpha^\beta p(t)x^\sigma(t)\left(x^\Delta(t)\right)^\gamma \Delta t - \int_\alpha^\beta Q^\Delta(t)\left(x^\sigma(t)\right)^{\gamma+1}\Delta t. \tag{4.3.5}$$

Integrating by parts the right-hand side, we see that

$$\begin{aligned}
\int_\alpha^\beta r(t)\left(x^\Delta(t)\right)^{\gamma+1}\Delta t \;=\; & \int_\alpha^\beta p(t)x^\sigma(t)\left(x^\Delta(t)\right)^\gamma \Delta t \\
& - Q(t)(x(t))^{\gamma+1}\Big|_\alpha^\beta + \int_\alpha^\beta Q(t)\left(x^{\gamma+1}(t)\right)^\Delta \Delta t.
\end{aligned}$$

Again using the assumptions $x(\alpha) = 0$ and $Q(\beta) = 0$, we obtain

$$\int_\alpha^\beta r(t)\left(x^\Delta(t)\right)^{\gamma+1}dt = \int_\alpha^\beta p(t)x^\sigma(t)\left(x^\Delta(t)\right)^\gamma \Delta t + \int_\alpha^\beta Q(t)\left(x^{\gamma+1}(t)\right)^\Delta \Delta t. \tag{4.3.6}$$

Applying the chain rule formula

$$\left(x^\lambda(t)\right)^\Delta = \lambda\int_0^1 \left[hx^\sigma(t) + (1-h)x(t)\right]^{\lambda-1} dh x^\Delta(t), \quad \text{for } \lambda > 0, \tag{4.3.7}$$

and the inequality

$$a^\lambda + b^\lambda \leq (a+b)^\lambda \leq 2^{\lambda-1}(a^\lambda + b^\lambda), \;\; \text{if} \;\; a, \; b \geq 0, \, \lambda \geq 1, \tag{4.3.8}$$

we see that

$$\begin{aligned}\left|\left(x^{\gamma+1}(t)\right)^{\Delta}\right| &\leq (\gamma+1)\int_0^1 |hx^{\sigma}(t)+(1-h)x(t)|^{\gamma}\,dh\,\left|x^{\Delta}(t)\right| \\ &\leq 2^{\gamma-1}(\gamma+1)\left|x^{\Delta}(t)\right|\int_0^1 |hx^{\sigma}(t)|^{\gamma}\,dh \\ &\quad +2^{\gamma-1}(\gamma+1)\left|x^{\Delta}(t)\right|\int_0^1 |(1-h)x(t)|^{\gamma}\,dh \\ &= 2^{\gamma-1}\left|x^{\Delta}(t)\right||x^{\sigma}(t)|^{\gamma}+2^{\gamma-1}\left|x^{\Delta}(t)\right||x(t)|^{\gamma} \\ &\leq 2^{\gamma-1}|x^{\sigma}(t)+x(t)|^{\gamma}\left|x^{\Delta}(t)\right|. \end{aligned} \tag{4.3.9}$$

This and (4.3.6) imply that

$$\begin{aligned}\int_{\alpha}^{\beta} r(t)\left|x^{\Delta}(t)\right|^{\gamma+1}\Delta t &\leq \int_{\alpha}^{\beta}|p(t)||x^{\sigma}(t)|\left|x^{\Delta}(t)\right|^{\gamma}\Delta t \\ &\quad +2^{\gamma-1}\int_{\alpha}^{\beta}|Q(t)||x(t)+x^{\sigma}(t)|^{\gamma}\left|x^{\Delta}(t)\right|\Delta t \end{aligned} \tag{4.3.10}$$

Applying the inequality (3.3.3) on the integral $\int_{\alpha}^{\beta}|Q(t)||x(t)+x^{\sigma}(t)|^{\gamma}$ $\left|x^{\Delta}(t)\right|\Delta t$, with $s(t)=|Q(t)|$, $p=\gamma$, $q=1$, we have

$$\int_{\alpha}^{\beta}|Q(t)||x(t)+x^{\sigma}(t)|^{\gamma}\left|x^{\Delta}(t)\right|\Delta t \leq K_1(\alpha,\beta,\gamma,1)\int_{\alpha}^{\beta} r(t)\left|x^{\Delta}(t)\right|^{\gamma+1}\Delta t, \tag{4.3.11}$$

where

$$K_1(\alpha,\beta,\gamma,1)=2^{2\gamma-2}\Lambda(\beta)+2^{3\gamma-2}\frac{1}{(\gamma+1)^{\frac{1}{\gamma+1}}}\left(\int_{\alpha}^{\beta}\frac{|Q(x)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(x)}(R_{\alpha}(x))^{\gamma}\,\Delta x\right)^{\frac{\gamma}{\gamma+1}}.$$

Using that fact that $x^{\sigma}=x(t)+\mu(t)x^{\Delta}(t)$, we see that

$$\begin{aligned}\int_{\alpha}^{\beta}|p(t)||x^{\sigma}(t)|\left|\left(x^{\Delta}(t)\right)^{\gamma}\right|\Delta t &= \int_{\alpha}^{\beta}|p(t)|\left|x(t)+\mu(t)x^{\Delta}(t)\right|\left|x^{\Delta}(t)\right|^{\gamma}\Delta t \\ &\leq \int_{\alpha}^{\beta}|p(t)||x(t)|\left|x^{\Delta}(t)\right|^{\gamma}\Delta t \\ &\quad +\int_{\alpha}^{\beta}\mu(t)|p(t)|\left|x^{\Delta}(t)\right|^{\gamma+1}\Delta t.\end{aligned}$$

Applying the inequality (3.2.33) on the integral $\int_{\alpha}^{\beta}|p(t)||x(t)|\left|x^{\Delta}(t)\right|^{\gamma}\Delta t$ with $s(t)=|p(t)|$, $p=1$ and $q=\gamma$, we see that

$$\int_{\alpha}^{\beta}|p(t)||x(t)|\left|x^{\Delta}(t)\right|^{\gamma}\Delta t \leq G_1(\alpha,\beta,1,\gamma)\int_{\alpha}^{\beta} r(t)\left|x^{\Delta}(t)\right|^{\gamma+1}\Delta t, \tag{4.3.12}$$

where

$$G_1(\alpha,\beta,1,\gamma)=\left(\frac{\gamma}{1+\gamma}\right)^{\frac{\gamma}{\gamma+1}}\times\left(\int_\alpha^\beta \frac{|p(t)|^{\gamma+1}}{(r(t))^\gamma}(R_\alpha(t))^\gamma\,\Delta t\right)^{\frac{1}{\gamma+1}}.$$

Using the assumption that $0\le p(t)\mu(t)\le r(t)/c$, we see that

$$\begin{aligned}\int_\alpha^\beta p(t)\left|x^\sigma(t)\right|\left|x^\Delta(t)\right|^\gamma\Delta t &\le G_1(\alpha,\beta,1,\gamma)\int_\alpha^\beta r(t)\left|x^\Delta(t)\right|^{\gamma+1}\Delta t\\ &\quad+\frac{1}{c}\int_\alpha^\beta r(t)\left|x^\Delta(t)\right|^{\gamma+1}\Delta t. \qquad (4.3.13)\end{aligned}$$

Substituting (4.3.11) and (4.3.13) into (4.3.10), we have

$$\begin{aligned}(1-\frac{1}{c})\int_\alpha^\beta r(t)\left|x^\Delta(t)\right|^{\gamma+1}\Delta t &\le K_1(\alpha,\beta,\gamma,1)\int_\alpha^\beta r(t)\left|x^\Delta(t)\right|^{\gamma+1}\Delta t\\ &\quad+G_1(\alpha,\beta,1,\gamma)\int_\alpha^\beta r(t)\left|x^\Delta(t)\right|^{\gamma+1}\Delta t. \qquad (4.3.14)\end{aligned}$$

Then, we have from (4.3.14) that

$$\begin{aligned}1-\frac{1}{c} &\le K_1(\alpha,\beta,\gamma,1)+G_1(\alpha,\beta,1,\gamma)\\ &= 2^{2\gamma-2}\Lambda(\beta)+\frac{2^{3\gamma-2}}{(\gamma+1)^{\frac{1}{\gamma+1}}}\left(\int_\alpha^\beta \frac{|Q(t)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)}(R_\alpha(t))^\gamma\,\Delta t\right)^{\frac{\gamma}{\gamma+1}}\\ &\quad+\left(\frac{\gamma}{1+\gamma}\right)^{\frac{\gamma}{\gamma+1}}\left(\int_\alpha^\beta \frac{|p(t)|^{\gamma+1}}{r^\gamma(t)}(R_\alpha(t))^\gamma\,\Delta t\right)^{\frac{1}{\gamma+1}},\end{aligned}$$

which is the desired inequality (4.3.3). The proof of (4.3.4) is similar to (4.3.3) using Theorems 3.2.9 and 3.3.2. The proof is complete. ■

In Theorem 4.3.1 if $r(t)=1$, then we have the following result.

Corollary 4.3.1 *Suppose that x is a nontrivial solution of (4.3.1) and x^Δ does not change sign in $(\alpha,\beta)_{\mathbb{T}}$. If $x(\alpha)=x^\Delta(\beta)=0$, then*

$$\begin{aligned}&2^{2\gamma-2}\Lambda(\beta)+\frac{2^{3\gamma-2}}{(\gamma+1)^{\frac{1}{\gamma+1}}}\times\left(\int_\alpha^\beta |Q(t)|^{\frac{\gamma+1}{\gamma}}(t-\alpha)^\gamma\,\Delta t\right)^{\frac{\gamma}{\gamma+1}}\\ &+\left(\frac{\gamma}{1+\gamma}\right)^{\frac{\gamma}{\gamma+1}}\times\left(\int_\alpha^\beta |p(t)|^{\gamma+1}(t-\alpha)^\gamma\,\Delta t\right)^{\frac{1}{\gamma+1}}\ge 1-\frac{1}{c},\end{aligned}$$

where $Q(t) = \int_t^\beta q(s)\Delta s$. *If instead* $x^\Delta(\alpha) = x(\beta) = 0$, *then*

$$2^{2\gamma-2}\Lambda(\alpha) + \frac{2^{3\gamma-2}}{(\gamma+1)^{\frac{1}{\gamma+1}}} \times \left(\int_\alpha^\beta |Q(t)|^{\frac{\gamma+1}{\gamma}} (\beta-t)^\gamma \Delta t\right)^{\frac{\gamma}{\gamma+1}}$$
$$+\left(\frac{\gamma}{1+\gamma}\right)^{\frac{\gamma}{\gamma+1}} \times \left(\int_\alpha^\beta |p(t)|^{\gamma+1} (\beta-t)^\gamma \Delta t\right)^{\frac{1}{\gamma+1}} \geq 1 - \frac{1}{c},$$

where $Q(t) = \int_\alpha^t q(s)\Delta s$.

As a special case of Theorem 4.3.1, when $\gamma = 1$, we have the following result.

Corollary 4.3.2 *Suppose that* x *is a nontrivial solution of (4.3.1) and* x^Δ *does not change sign in* $(\alpha,\beta)_{\mathbb{T}}$. *If* $x(\alpha) = x^\Delta(\beta) = 0$, *then*

$$\Lambda(\beta) + \sqrt{2}\left(\int_\alpha^\beta \frac{|Q(t)|^2}{r(t)} r_\alpha(t)\Delta t\right)^{\frac{1}{2}} + \frac{1}{\sqrt{2}}\left(\int_\alpha^\beta \frac{p^2(t)}{r(t)} R_\alpha(t)\Delta t\right)^{\frac{1}{2}} \geq 1 - \frac{1}{c},$$

where $R_\alpha(t) = \int_\alpha^t \frac{\Delta s}{r(s)}$ *and* $Q(t) = \int_t^\beta q(s)\Delta s$. *If instead* $x^\Delta(\alpha) = x(\beta) = 0$, *then*

$$\Lambda(\alpha) + \sqrt{2}\left(\int_\alpha^\beta \frac{|Q(t)|^2}{r(t)} r_\beta(t)\Delta t\right)^{\frac{1}{2}} + \frac{1}{\sqrt{2}}\left(\int_\alpha^\beta \frac{p^2(t)}{r(t)} R_\beta(t)\Delta t\right)^{\frac{1}{2}} \geq 1 - \frac{1}{c},$$

where $R_\beta(t) = \int_t^\beta \frac{\Delta s}{r(s)}$ *and* $Q(t) = \int_\alpha^t q(s)\Delta s$.

As a special case of Corollary 4.3.2, when $p(t) = 0$, we have the following result.

Corollary 4.3.3 *Suppose that* x *is a nontrivial solution of*

$$\left(r(t)x^\Delta(t)\right)^\Delta + q(t)x^\sigma(t) = 0, \quad t \in [\alpha,\beta]_{\mathbb{T}}, \tag{4.3.15}$$

and x^Δ *does not change sign in* $(\alpha,\beta)_{\mathbb{T}}$. *If* $x(\alpha) = x^\Delta(\beta) = 0$, *then*

$$\sqrt{2}\left(\int_\alpha^\beta \frac{|Q(t)|^2}{r(t)}\left(\int_\alpha^t \frac{\Delta t}{r(t)}\right)\Delta t\right)^{\frac{1}{2}} + \Lambda(\beta) \geq 1,$$

where $Q(t) = \int_t^\beta q(s)\Delta s$. *If instead* $x^\Delta(\alpha) = x(\beta) = 0$, *then*

$$\sqrt{2}\left(\int_\alpha^\beta \frac{|Q(t)|^2}{r(t)}\left(\int_t^\beta \frac{\Delta t}{r(t)}\right)\Delta t\right)^{\frac{1}{2}} + \Lambda(\alpha) \geq 1,$$

where $Q(t) = \int_\alpha^t q(s)\Delta s$.

Remark 4.3.1 *Theorem 4.3.1 yield sufficient conditions for the disfocality of (4.3.1), i.e., sufficient conditions so that there does not exist a nontrivial solution* x *satisfying* $x(\alpha) = x^{\Delta}(\beta) = 0$ *or* $x^{\Delta}(\alpha) = x(\beta) = 0$.

On a time scale $\mathbb{T}$, we note from the chain rule (4.3.7) that

$$\begin{aligned} \left((t-a)^{\lambda+\delta}\right)^{\Delta} &= (\lambda+\delta)\int_0^1 [h(\sigma(t)-a)+(1-h)(t-a)]^{\lambda+\delta-1}\,dh \\ &\geq (\lambda+\delta)\int_0^1 [h(t-a)+(1-h)(t-a)]^{\lambda+\delta-1}\,dh \\ &= (\lambda+\delta)(t-a)^{\lambda+\delta-1}. \end{aligned}$$

This implies that

$$\int_a^{\tau} (t-a)^{(\lambda+\delta-1)}\Delta t \leq \int_a^{\tau} \frac{1}{(\lambda+\delta)}\left((t-a)^{\lambda+\delta}\right)^{\Delta}\Delta t = \frac{(\tau-a)^{\lambda+\delta}}{(\lambda+\delta)}. \tag{4.3.16}$$

Now using the maximum of $|Q|$ and $|p|$ on $[\alpha,\beta]_{\mathbb{T}}$ and substituting (4.3.16) into the results of Corollary 4.3.1, we have the following result.

Corollary 4.3.4 *Suppose that* x *is a nontrivial solution of (4.3.1) and* x^{Δ} *does not change sign in* $(\alpha,\beta)_{\mathbb{T}}$. *If* $x(\alpha) = x^{\Delta}(\beta) = 0$, *then*

$$\begin{aligned} &\frac{2^{3\gamma-2}(\beta-\alpha)^{\gamma}}{(\gamma+1)} \max_{\alpha\leq t\leq\beta}\left|\int_t^{\beta} q(s)\Delta s\right| + \frac{\gamma^{\frac{\gamma}{\gamma+1}}}{\gamma+1}(\beta-\alpha)\max_{\alpha\leq t\leq\beta}|p(t)| \\ &+2^{2\gamma-2}\sup_{\alpha\leq t\leq\beta}\mu^{\gamma}(t)\left|\int_t^{\beta} q(s)\Delta s\right| \geq 1-\frac{1}{c}. \end{aligned}$$

If instead $x^{\Delta}(\alpha) = x(\beta) = 0$, *then*

$$\begin{aligned} &\frac{2^{3\gamma-2}(\beta-\alpha)^{\gamma}}{(\gamma+1)} \max_{\alpha\leq t\leq\beta}\left|\int_{\alpha}^{t} q(s)\Delta s\right| + \frac{\gamma^{\frac{\gamma}{\gamma+1}}}{\gamma+1}(\beta-\alpha)\max_{\alpha\leq t\leq\beta}|p(t)| \\ &+2^{2\gamma-2}\sup_{\alpha\leq t\leq\beta}\mu^{\gamma}(t)\left|\int_{\alpha}^{t} q(s)\Delta s\right| \geq 1-\frac{1}{c}, \end{aligned}$$

As a special when $\mathbb{T}=\mathbb{Z}$, we see that $\Lambda(\alpha)$ and $\Lambda(\beta)$ are defined as in (4.3.2) and we consider the second order half-linear difference equation

$$\Delta(\Delta x(n))^{\gamma} + p(n)(\Delta x(n))^{\gamma} + q(n)(x(n+1))^{\gamma} = 0, \quad \alpha\leq n\leq\beta, \tag{4.3.17}$$

where $\gamma \geq 1$ is a quotient of odd positive integers and $p(n) \leq 1/c$.

Corollary 4.3.5 *Suppose that x is a nontrivial solution of (4.3.17) and $\Delta x(n)$ does not change sign in $(\alpha,\beta)_{\mathbb{T}}$. If $x(\alpha)=\Delta x(\beta)=0$, then*

$$\begin{aligned} 1-\frac{1}{c} \quad &\leq \quad \frac{2^{3\gamma-2}(\beta-\alpha)^{\gamma}}{(\gamma+1)} \max_{\alpha\leq n\leq\beta}\left|\sum_{s=n}^{\beta-1} q(s)\right| + 2^{2\gamma-2}\sup_{\alpha\leq n\leq\beta}\left|\sum_{s=n}^{\beta-1} q(s)\right| \\ &\quad + \frac{\gamma^{\frac{\gamma}{\gamma+1}}}{\gamma+1}(\beta-\alpha)\max_{\alpha\leq n\leq\beta}|p(n)|. \end{aligned}$$

If instead $\Delta x(\alpha)=x(\beta)=0$, then

$$\begin{aligned} 1-\frac{1}{c} \quad &\leq \quad \frac{2^{3\gamma-2}(\beta-\alpha)^{\gamma}}{(\gamma+1)} \max_{\alpha\leq n\leq\beta}\left|\sum_{s=\alpha}^{n-1} q(s)\right| + 2^{2\gamma-2}\sup_{\alpha\leq n\leq\beta}\left|\sum_{s=\alpha}^{n-1} q(s)\right| \\ &\quad + \frac{\gamma^{\frac{\gamma}{\gamma+1}}}{\gamma+1}(\beta-\alpha)\max_{\alpha\leq n\leq\beta}|p(n)|. \end{aligned}$$

If we apply the inequality

$$|a+b|^{\lambda} \leq 2^{\lambda-1}\left(|a|^{\lambda}+|b|^{\lambda}\right), \text{ where } a,\ b \text{ are real numbers and } \lambda\geq 1,$$

with $a=x(t)$ and $b=\mu(t)hx^{\Delta}(t)$, then we have from (4.3.7) that

$$\begin{aligned} \left|\left(x^{\gamma+1}(t)\right)^{\Delta}\right| \quad &\leq \quad (\gamma+1)\left|x^{\Delta}(t)\right|\int_0^1 \left|x(t)+\mu(t)hx^{\Delta}(t)\right|^{\gamma} dh \\ &\leq \quad 2^{\gamma-1}(\gamma+1)\left|x^{\Delta}(t)\right|\int_0^1 |x(t)|^{\gamma}\, dh \\ &\quad + 2^{\gamma-1}(\gamma+1)\left|x^{\Delta}(t)\right|\int_0^1 \left|\mu(t)hx^{\Delta}(t)\right|^{\gamma} dh \\ &= \quad 2^{\gamma-1}(\gamma+1)\left|x^{\Delta}(t)\right||x(t)|^{\gamma} + 2^{\gamma-1}\mu(t)\left|x^{\Delta}(t)\right|^{\gamma+1}. \quad (4.3.18) \end{aligned}$$

Substituting (4.3.18) into (4.3.6), we have that

$$\begin{aligned} \int_{\alpha}^{\beta} r(t)\left|x^{\Delta}(t)\right|^{\gamma+1} dt \quad &\leq \quad \int_{\alpha}^{\beta} |p(t)|\,|x^{\sigma}(t)|\left|x^{\Delta}(t)\right|^{\gamma}\Delta t \\ &\quad + 2^{\gamma-1}(\gamma+1)\int_{\alpha}^{\beta} |Q(t|)\left|x^{\Delta}(t)\right||x(t)|^{\gamma}\Delta t \\ &\quad + 2^{\gamma-1}\int_{\alpha}^{\beta}\mu(t)\,|Q(t)|\left|x^{\Delta}(t)\right|^{\gamma+1}\Delta t. \quad (4.3.19) \end{aligned}$$

Using the inequality

$$\begin{aligned} \int_{\alpha}^{\beta} |p(t)|\,|x^{\sigma}(t)|\left|x^{\Delta}(t)\right|^{\gamma}\Delta t \leq &\int_{\alpha}^{\beta} |p(t)|\,|x(t)|\left|x^{\Delta}(t)\right|^{\gamma}\Delta t \\ &+ \int_{\alpha}^{\beta}\mu(t)\,|p(t)|\left|x^{\Delta}(t)\right|^{\gamma+1}\Delta t, \end{aligned}$$

we have from (4.3.19) that

$$\begin{aligned}\int_{\alpha}^{\beta} r(t)\left|x^{\Delta}(t)\right|^{\gamma+1} dt \leq & \int_{\alpha}^{\beta} |p(t)|\,|x(t)|\left|x^{\Delta}(t)\right|^{\gamma} \Delta t \\ & +2^{\gamma-1}(\gamma+1) \int_{\alpha}^{\beta} |Q(t|)\left|x^{\Delta}(t)\right| |x(t)|^{\gamma} \Delta t \\ & +\int_{\alpha}^{\beta} \mu(t)(|p(t)|+2^{\gamma-1}|Q(t)|\left|x^{\Delta}(t)\right|^{\gamma+1} \Delta t\end{aligned} \tag{4.3.20}$$

We now apply Opial inequalities to obtain results when the condition $\mu(t)$ $|p(t)| \leq r(t)/c$ is replaced by the new condition $\mu(t)(|p(t)|+2^{\gamma-1}|Q(t)|) \leq r(t)/c$.

Now, applying the inequality (3.2.33) on the term

$$\int_{\alpha}^{\beta} |Q(t)|\left|x^{\Delta}(t)\right| |x(t)|^{\gamma} \Delta t, \text{ with } s(t)=|Q(t)|,\ p=\gamma \text{ and } q=1,$$

we have

$$\int_{\alpha}^{\beta} |Q(t|)\,|x(t)|^{\gamma}\left|x^{\Delta}(t)\right| \Delta t \leq K_1^*(\alpha,\beta,\gamma,1) \int_{\alpha}^{\beta} r(t)\left|x^{\Delta}(t)\right|^{\gamma+1} \Delta t,$$

where

$$K_1^*(\alpha,\beta,\gamma,1)=\left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma+1}}\left(\int_{\alpha}^{\beta} \frac{|Q(t)|^{\frac{\gamma+1}{\gamma}}}{(r(t))^{\frac{1}{\gamma}}} R_{\alpha}^{\gamma}(t)\Delta t\right)^{\frac{\gamma}{\gamma+1}}.$$

Using the inequality

$$\int_{\alpha}^{\beta} |p(t)|\,|x(t)|\left|x^{\Delta}(t)\right|^{\gamma} \Delta t \leq G_1(\alpha,\beta,1,\gamma) \int_{\alpha}^{\beta} r(t)\left|x^{\Delta}(t)\right|^{\gamma+1} \Delta t,$$

where

$$G_1(\alpha,\beta,1,\gamma)=\left(\frac{\gamma}{1+\gamma}\right)^{\frac{\gamma}{\gamma+1}} \times\left(\int_{\alpha}^{\beta} \frac{|p(t)|^{\gamma+1}}{(r(t))^{\gamma}} R_{\alpha}^{\gamma}(t)\Delta t\right)^{\frac{1}{\gamma+1}}.$$

and proceeding as in the proof of Theorem 4.3.1, we obtain the following result.

Theorem 4.3.2 *Assume that $\mu(t)(|p(t)|+2^{\gamma-1}|Q(t)|) \leq r(t)/c$ where c is a positive constant such that $c \geq 1$. Suppose that x is a nontrivial solution of (4.3.1) and x^{Δ} does not change sign in $(\alpha,\beta)_{\mathbb{T}}$. If $x(\alpha)=x^{\Delta}(\beta)=0$, then*

$$\begin{aligned}& 2^{\gamma-1}(\gamma+1)^{\frac{\gamma}{\gamma+1}}\left(\int_{\alpha}^{\beta} \frac{|Q(t)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)} R_{\alpha}^{\gamma}(t)\Delta t\right)^{\frac{\gamma}{\gamma+1}} \\ & +\left(\frac{\gamma}{1+\gamma}\right)^{\frac{\gamma}{\gamma+1}}\left(\int_{\alpha}^{\beta} \frac{|p(t)|^{\gamma+1}}{r^{\gamma}(t)} R_{\alpha}^{\gamma}(t)\Delta t\right)^{\frac{1}{\gamma+1}} \geq 1-\frac{1}{c},\end{aligned}$$

where $Q(t) = \int_t^\beta q(s)\Delta s$*. If instead* $x^\Delta(\alpha) = x(\beta) = 0$*, then*

$$2^{\gamma-1}(\gamma+1)^{\frac{\gamma}{\gamma+1}}\left(\int_\alpha^\beta \frac{|Q(t)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)} R_\beta^\gamma(t)\Delta t\right)^{\frac{\gamma}{\gamma+1}} + \left(\frac{\gamma}{1+\gamma}\right)^{\frac{\gamma}{\gamma+1}}\left(\int_\alpha^\beta \frac{|p(t)|^{\gamma+1}}{r^\gamma(t)} R_\beta^\gamma(t)\Delta t\right)^{\frac{1}{\gamma+1}} \geq 1 - \frac{1}{c},$$

where $Q(t) = \int_\alpha^t q(s)\Delta s$*.*

Remark 4.3.2 *Note that when* $\mathbb{T} = \mathbb{R}$ *the condition* $\mu(t)(|p(t)| + 2^{\gamma-1}|Q(t)|) \leq r(t)/c$ *is removed since* $\mu(t) = 0$*.*

Next we apply Theorems 3.2.10 and 3.3.3 to determine a lower bound for the distance between consecutive generalized zeros of solutions of (4.3.1). In the following, we assume that $Q^\Delta(t) = q(t)$ and assume that there exists a unique $h \in (\alpha, \beta)_\mathbb{T}$, such that

$$R(h) := R_\alpha(h) = R_\beta(h). \tag{4.3.21}$$

Note that the best choice of h when $r(t) = 1$ is $h = (\beta + \alpha)/2$. In the following, we assume that

$$K^h(\alpha, \beta, \gamma, 1) = K_h(\alpha, \beta, \gamma, 1) < \infty, \tag{4.3.22}$$

where

$$K^h(\alpha,\beta,\gamma,1) = \frac{2^{3\gamma-2}}{(\gamma+1)^{\frac{1}{\gamma+1}}}\left(\int_\alpha^\beta \frac{|Q(t)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)} R_\alpha^\gamma(h)\Delta t\right)^{\frac{\gamma}{\gamma+1}} + 2^{2\gamma-2}\Lambda,$$

$$K_h(\alpha,\beta,\gamma,1) = \frac{2^{3\gamma-2}}{(\gamma+1)^{\frac{1}{\gamma+1}}}\left(\int_\alpha^\beta \frac{|Q(t)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)} R_\beta^\gamma(h)\Delta t\right)^{\frac{\gamma}{\gamma+1}} + 2^{2\gamma-2}\Lambda,$$

$$\Lambda := \sup_{\alpha \leq t \leq \beta} \mu^\gamma(t)\frac{|Q(t)|}{r(t)}, \quad \text{where } Q^\Delta(t) = q(t),$$

and

$$G^h(\alpha, \beta, 1, \gamma) = G_h(\alpha, \beta, 1, \gamma) < \infty, \tag{4.3.23}$$

where

$$G^h(\alpha,\beta,1,\gamma) = \left(\frac{\gamma}{1+\gamma}\right)^{\frac{\gamma}{\gamma+1}}\left(\int_\alpha^\beta \frac{|p(t)|^{\gamma+1}}{r^\gamma(t)} R_\alpha^\gamma(h)\Delta t\right)^{\frac{1}{\gamma+1}},$$

$$G_h(\alpha,\beta,1,\gamma) = \left(\frac{\gamma}{1+\gamma}\right)^{\frac{\gamma}{\gamma+1}}\left(\int_\alpha^\beta \frac{|p(t)|^{\gamma+1}}{r^\gamma(t)} R_\beta^\gamma(h)\Delta t\right)^{\frac{1}{\gamma+1}}.$$

Now, we assume that $K(\gamma,1)$ is the solution of the equation $K(\gamma,1) = K^h(\alpha,\beta,\gamma,1) = K_h(\alpha,\beta,\gamma,1)$ and given by

$$K(\gamma,1) = \frac{2^{3\gamma-2}}{(\gamma+1)^{\frac{1}{\gamma+1}}}\left(\int_\alpha^\beta \frac{|Q(t)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)} R^\gamma(h)\Delta t\right)^{\frac{\gamma}{\gamma+1}} + 2^{2\gamma-2}\Lambda, \qquad (4.3.24)$$

and similarly $G(1,\gamma)$ is given by

$$G(1,\gamma) = \left(\frac{\gamma}{1+\gamma}\right)^{\frac{\gamma}{\gamma+1}}\left(\int_\alpha^\beta \frac{|p(t)|^{\gamma+1}}{r^\gamma(t)} R^\gamma(h)\Delta t\right)^{\frac{1}{\gamma+1}}. \qquad (4.3.25)$$

Theorem 4.3.3 *Assume that $Q^\Delta(t) = q(t)$ and suppose x is a nontrivial solution of (4.3.1). If $x(\alpha) = x(\beta) = 0$, then*

$$K(\gamma,1) + G(1,\gamma) \geq 1 - \frac{1}{c}, \qquad (4.3.26)$$

where $K(\alpha,\beta)$ and $K(\alpha,\beta)$ are defined as in (4.3.24) and (4.3.25).

Proof. We multiply (4.3.1) by $x^\sigma(t)$ and proceed as in Theorem 4.3.1 to obtain

$$\int_\alpha^\beta r(t)\left(x^\Delta(t)\right)^{\gamma+1}\Delta t = \int_\alpha^\beta p(t)x^\sigma(t)\left(x^\Delta(t)\right)^\gamma \Delta t + \int_\alpha^\beta Q^\Delta(t)\left(x^\sigma(t)\right)^{\gamma+1}\Delta t.$$

Integrating by parts the right-hand side, we see that

$$\begin{aligned}\int_\alpha^\beta r(t)\left(x^\Delta(t)\right)^{\gamma+1}\Delta t &= \int_\alpha^\beta p(t)x^\sigma(t)\left(x^\Delta(t)\right)^\gamma \Delta t \\ &\quad + Q(t)(x(t))^{\gamma+1}\Big|_\alpha^\beta - \int_\alpha^\beta Q(t)\left(x^{\gamma+1}(t)\right)^\Delta \Delta t. \qquad (4.3.27)\end{aligned}$$

Using $x(\alpha) = 0 = x(\beta)$ we obtain

$$\int_\alpha^\beta r(t)\left|x^\Delta(t)\right|^{\gamma+1} dt \leq \int_\alpha^\beta |p(t)|\left|x^\sigma(t)\right|\left|x^\Delta(t)\right|^\gamma \Delta t + \int_\alpha^\beta |Q(t)|\left|\left(x^{\gamma+1}(t)\right)^\Delta\right| dt.$$

We proceed as in the proof of Theorem 4.3.1 to get

$$\int_\alpha^\beta |Q(t)|\left|\left(x^{\gamma+1}(t)\right)^\Delta\right|\Delta t \leq 2^{\gamma-1}\int_\alpha^\beta |Q(t)|\left|x(t)+x^\sigma(t)\right|^\gamma\left|x^\Delta(t)\right|\Delta t.$$

Applying the inequality (3.3.15) with $s(t) = |Q(t)|$, $p = \gamma$ and $q = 1$, we have

$$\int_\alpha^\beta |Q(t)|\left|x^{\gamma+1}(t)\right|^\Delta dt \leq 2^{\gamma-1}K(\gamma,1)\int_\alpha^\beta r(t)\left|x^\Delta(t)\right|^{\gamma+1}\Delta t.$$

Also, we obtain

$$\begin{aligned}&\int_\alpha^\beta |p(t)|\,|x^\sigma(t)|\,\left|x^\Delta(t)\right|^\gamma \Delta t\\ \leq\quad & G(1,\gamma)\int_\alpha^\beta r(t)\left|x^\Delta(t)\right|^{\gamma+1}\Delta t+\frac{1}{c}\int_\alpha^\beta r(t)\left|x^\Delta(t)\right|^{\gamma+1}\Delta t.\end{aligned}$$

The rest of the proof is similar to that in the proof of Theorem 4.3.1. ■

4.4 Hamiltonian Systems

In this section we consider a linear matrix Hamiltonian dynamic system on time scales of the form

$$x^\Delta(t)=A(t)x^\sigma+B(t)u,\qquad u^\Delta(t)=-C(t)x^\sigma-A^*(t)u, \tag{4.4.1}$$

where A, B, and C are rd-continuous $n\times n$-matrix-valued functions on $\mathbb{T}$ such that $I-\mu(t)A(t)$ is invertible and $B(t)$ and $C(t)$ are positive semidefinite for all $t\in\mathbb{T}$. A corresponding quadratic functional is given by

$$\mathcal{F}(x,u)=\int_a^b \{u^*Bu-(x^\sigma)^*Cx^\sigma\}(t)\Delta t.$$

A pair (x,u) is called admissible if it satisfies the equation of motion

$$x^\Delta=A(t)x^\sigma+B(t)u.$$

Lemma 4.4.1 *If (x,u) solves (4.4.1) and if (y,v) is admissible, then*

$$\begin{aligned}\mathcal{F}(y,v)-\mathcal{F}(x,u)\quad=\quad&\mathcal{F}(y-x,v-u)\\ &+2\operatorname{Re}\left[(y-x)^*(b)u(b)-(y-x)^*(a)u(a)\right].\end{aligned}$$

Proof. Under the above assumption

$$\begin{aligned}&\mathcal{F}(y,v)-\mathcal{F}(x,u)-\mathcal{F}(y-x,v-u)\\ =\quad&\int_a^b\{v^*Bv-(y^\sigma)^*Cy^\sigma-u^*Bu+(x^\sigma)^*Cx^\sigma\\ &-[(v-u)^*B(v-u)-(y^\sigma-x^\sigma)^*C(y^\sigma-x^\sigma)]\}(t)\Delta t\\ =\quad&\int_a^b\{-2u^*Bu+v^*Bu+u^*Bv\\ &+2(x^\sigma)^*Cx^\sigma-(y^\sigma)^*Cx^\sigma-(x^\sigma)^*Cy^\sigma\}(t)\Delta(t)\\ =\quad&\int_a^b\{-2u^*Bu+2\operatorname{Re}[u^*Bv]+2(x^\sigma)^*Cx^\sigma-2\operatorname{Re}[(y^\sigma)^*Cx^\sigma]\}(t)\Delta(t)\end{aligned}$$

$$
\begin{aligned}
&= 2\,\mathrm{Re}\left(\int_a^b \{u^*(Bv - Bu) + [(x^\sigma)^* - (y^\sigma)^*]\,Cx^*\}\,(t)\Delta(t)\right) \\
&= 2\,\mathrm{Re}\left(\int_a^b \{u^*\ (y^\Delta - Ay^\sigma - x^\Delta + Ax^\sigma)\right. \\
&\quad \left. + [(x^\sigma)^* - (y^\sigma)^*] - u^\Delta - A^*u](t)\Delta(t)\right) \\
&= 2\,\mathrm{Re}\left(\int_a^b \left\{u^*(y^\Delta - x^\Delta) + (y^\sigma - x^\sigma)^*\ u^\Delta\right.\right. \\
&\quad \left. +2i\,\mathrm{Im}[u^*Ax^\sigma + (y^\sigma)^*A^*u]\ (t)\Delta(t)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\,\mathrm{Re}\left(\int_a^b \left\{u^*(y^\Delta - x^\Delta) + (y^\sigma - x^\sigma)^* u^\Delta\right\}(t)\Delta t\right) \\
&= 2\,\mathrm{Re}\left(\int_a^b \left\{u^*(y^\Delta - x^\Delta) + (u^\Delta)^*(y^\sigma - x^\sigma)\right\}(t)\Delta t\right) \\
&= 2\,\mathrm{Re}\left(\int_a^b \left\{[u^*(y - x)]^\Delta\right\}(t)\Delta t\right) \\
&= 2\,\mathrm{Re}\,\{u^*(b)[y(b) - x(b)] - u^*(a)[y(a) - x(a)]\}\,. \\
&= 2\,\mathrm{Re}\,\{[y - x]^*(b)u(b) - [y - x]^*(a)u(a)]\}\,,
\end{aligned}
$$

and we are finished. ■

For the remainder of this section we denote by $W(.,r)$ the unique solution of the initial value problem

$$
W^\Delta = -A^*(t)W, \qquad W(r) = I, \tag{4.4.2}
$$

where $r \in [a,b]$ is given. We also write

$$
F(s,r) = \int_r^s W^*(t,r)B(t)W(t,r)\Delta t. \tag{4.4.3}
$$

Observe that $W(t,r) \equiv I$ provided $A(t) \equiv 0$.

Lemma 4.4.2 *Let W and F be defined as in (4.4.2) and (4.4.3). If (y,v) is admissible and if $r,s \in \mathbb{T}$ with $a \le r < s \le b$ such that $F(s,r)$ is invertible, then*

$$
\int_r^s (v^*Bv)(t)\Delta t \ge [W^*(s,r)y(s) - y(r)]^*\,F^{-1}(s,r)\,[W^*(s,r)y(s) - y(r)]\,.
$$

Proof. Let

$$x(t) = W^{*-1}(t,r)\left\{y(r) + F(t,r)F^{-1}(s,r)\left[W^*(s,r)y(s) - y(r)\right]\right\}$$

and

$$u(t) = W(t,r)F^{-1}(s,r)[W^*(s,r)y(s) - y(r)].$$

Now

$$\begin{aligned} W(t,r)W^{-1}(\sigma(t),r) &= [W(\sigma(t),r) - \mu(t)W^{\Delta}(t,r)W^{-1}(\sigma(t),r) \\ &= I + \mu(t)A^*(t)W(t,r)W^{-1}(\sigma(t),r), \end{aligned}$$

and therefore $[I - \mu(t)A^*(t)]W(t,r)W^{-1}(\sigma(t),r) = I$, so that

$$[I - \mu(t)A(t)]x^{\Delta}(t) = A(t)x(t) + B(t)u(t),$$

and hence

$$\begin{aligned} x^{\Delta}(t) &= A(t)x(t) + \mu(t)A(t)x^2(t) + B(t)u(t) \\ &= A(t)x^{\sigma}(t) + B(t)u(t). \end{aligned}$$

Thus (x,u) solves the Hamiltonian system (4.4.1) with $C = 0$ and, we may apply Lemma 4.4.1 to $\mathcal{F}_0$ defined by

$$\mathcal{F}_0(x,u) = \int_r^s (u^*Bu)(t)\Delta t,$$

to obtain

$$\begin{aligned} \mathcal{F}_0(y,v) &= \mathcal{F}_0(x,u) + \mathcal{F}_0(y-x, v-u) \\ &\quad +2\,\mathrm{Re}\left\{u^*(s)y(s) - x(s) - u^*(r)[y(r) - x(r)]\right\} \\ &= \mathcal{F}_0(x,u) + \mathcal{F}_0(y-x,v-u) \geq \mathcal{F}_0(x,u) = \int_r^s (u^*Bu)(t)\Delta t \\ &= [W^*(s,r)y(s) - y(r)]^*F^{-1}(r,s)[W^*(s,r)y(s) - y(r)]. \end{aligned}$$

which shows our claim. ■

Remark 4.4.1 *The assumption in Lemma 4.4.2 that $F(s,r)$ is invertible if $r < s$ can be dropped if B is positive definite rather than positive semidefinite.*

We now may use Lemma 4.4.2 to derive a Lyapunov inequality for Hamiltonian systems.

Theorem 4.4.1 *Assume (4.4.1) has a solution (x,u) such that x is nontrivial and satisfies $x(a) = x(b) = 0$. With W and F introduced in (4.4.2)* *and (4.4.3), suppose that $F(b,c)$ and $F(c,a)$ are invertible, where $\|x(c)\| = \max_{t\in[a,b]\cap\mathbb{T}} \|x(t)\|$. Let λ be the biggest eigenvalue of*

$$F = \int_a^b W^*(t,c)B(t)W(t,c)\Delta t,$$

and let $v(t)$ be the biggest eigenvalue of $C(t)$. Then the Lyapunov inequality

$$\int_a^b v(t)\Delta t \geq \frac{4}{\lambda},$$

holds.

Proof. Suppose we are given a solution (x,u) of (4.4.1) such that $x(a) = x(b) = 0$. Lemma 4.4.1 then yields (using $y = v = 0$) that

$$\mathcal{F}(x,u) = \int_a^b \{u^*Bu - (x^\sigma)^*Cx^\sigma\}(t)\Delta t = 0.$$

Apply Lemma 4.4.2 twice (once with $r = a$ and $s = c$ and a second time with $r = c$ and $s = b$) to obtain

$$\begin{aligned}
&\int_a^b [(x^\sigma)^*Cx^\sigma](t)\Delta t \\
=\ & \int_a^b (u^*Bu)(t)\Delta t = \int_a^c (u^*Bu)(t)\Delta t + \int_a^b (u^*Bu)(t)\Delta t \\
\geq\ & x^*(c)W(c,a)F^{-1}(c,a)W^*(c,a)x(c) + x^*(c)F^{-1}(b,c)x(c) \\
=\ & x^*(c)[F^{-1}(b,c) - F^{-1}(a,c)]x(c) \geq 4x^*(c)F^{-1}x(c);
\end{aligned}$$

here we have used the relation $W(t,r)W(r,s) = W(t,s)$ and the inequality (see [34, Theorem 9 (i)]) and [120])

$$M^{-1} + N^{-1} \geq 4(M+N)^{-1},$$

Now, by applying the Rayleigh–Ritz Theorem (see [85, page 176]), we conclude

$$\begin{aligned}
\int_a^b v(t)\Delta t \ &\geq\ \int_a^b v(t)\frac{\|x^\sigma(t)\|^2}{\|x(c)\|^2}\Delta t \\
&=\ \frac{1}{\|x(c)\|^2}\int_a^b v(t)(x^\sigma(t))^*x^\sigma(t)\Delta t \geq \frac{1}{\|x(c)\|^2}\int_a^b (x^\sigma(t))^*C(t)x^\sigma(t)\Delta t \\
&\geq\ \frac{1}{\|x(c)\|^2}4x^*(c)F^{-1}x(c) \geq 4\min_{x\neq 0}\frac{x^*F^{-1}x}{x^*x} = \frac{4}{\lambda},
\end{aligned}$$

which is the desired inequality. The proof is complete. ∎

Remark 4.4.2 *If $A \equiv 0$, then $W \equiv I$ and $F = \int_a^b B(t)\Delta t$. If, in addition $B \equiv 1$, then $F = b - a$. Note the Lyapunov inequality $\int_a^b v(t)\Delta t \geq (4/\lambda)$ reduces to $\int_a^b p(t)\Delta t \geq (4/b-a)$ for the scalar case.*

We conclude with a result concerning the so-called right-focal boundary condition, i.e., $x(a) = u(b) = 0$.

Theorem 4.4.2 *Assume (4.4.1) has a solution (x, u) with x nontrivial and $x(a) = u(b) = 0$. With the notation as in Theorem 4.4.1, the Lyapunov inequality*

$$\int_a^b v(t)\Delta t \geq \frac{1}{\lambda},$$

holds.

Proof. Suppose (x, u) is a solution of (4.4.1) such that $x(a) = u(b) = 0$ with $a < b$. Choose the point c in $(a, b]$ where $\|x(t)\|$ is maximal. Applying Lemma 4.4.1 and we see

$$\int_a^b \left[(x^\sigma)^* C x^\sigma\right](t)\Delta t = \int_a^b (u^* B u)(t)\Delta t \geq \int_a^b (u^* B u)(t)\Delta t.$$

Using Lemma 4.4.2 with $r = a$ and $s = c$, we get

$$\begin{aligned}
\int_a^b (u^* B u)(t)\Delta t &\geq \left[W^*(c,a)x(c) - x(a)\right]^* F^{-1}(c,a)\left[W^*(c,a)x(c) - x(a)\right] \\
&= x^*(c)W(c,a)F^{-1}(c,a)W^*(c,a)x(c) \\
&= -x^*(c)F^{-1}(a,c)x(c) \\
&= x^*(c)\left(\int_a^b W^*(t,c)B(t)W(t,c)\Delta t\right)^{-1} x(c) \\
&\geq x^*(c)\left(\int_a^b W^*(t,c)B(t)W(t,c)\Delta t\right)^{-1} x(c) \\
&= x^*(c)F^{-1}x(c).
\end{aligned}$$

Hence,

$$\int_a^b [(x^\sigma)^* C x^\sigma](t)\Delta t \geq x^*(c)F^{-1}x(c),$$

and the same arguments as in the proof of Theorem 4.4.1 completes the proof.
■