Connecting Interval-Valued Fuzzy Sets with Imprecise Probabilities

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Abstract. We study interval-valued fuzzy sets as a model for the imprecise knowledge of the membership function of a fuzzy set. We compare three models for the probabilistic information about this membership function: the set of distributions of the measurable selections, the upper and lower probabilities of the associated random interval, and its p-box. We give sufficient conditions for the equality between these sets, and establish a connection with the notion of probability induced by an intuitionistic fuzzy set. An alternative approach to the problem by means of sets of finitely additive distributions is also considered.

Keywords: Interval-valued fuzzy sets, random intervals, measurable selections, upper and lower probabilities, p-boxes.

1 Introduction

Interval-valued fuzzy sets [18] (IVF-sets, for short) were introduced as an extension of fuzzy sets [16] to model situations in which the "true" membership function is in some sense unknown. Then, instead of providing a precise membership degree, IVF-sets assign an interval of possible membership degrees. Thus, given an universe Ω, an IVF-set A is defined, for any $\omega \in \Omega$, by $[l_A(\omega), u_A(\omega)]$, and it is given the epistemic interpretation that all we know about the "true" membership degree of ω is that it belongs to that interval.

A related extension of fuzzy sets is given by *intuitionistic fuzzy sets* [1,2] (IF-sets, for short). For them, the interpretation is slightly different: they assign a membership and a non-membership degree to any element of the possibility space. Thus, an IF-set A is defined by two functions $\mu_{\mathrm{A}}, \nu_{\mathrm{A}} : \Omega \to [0,1]$, so that for any $\omega \in \Omega$, $\mu_{\mathrm{A}}(\omega)$ and $\nu_{\mathrm{A}}(\omega)$ model the degree in which ω satisfies and does not satisfy the notion encompassed by the fuzzy set A, with the restriction $\mu_{\mathrm{A}}(\omega) + \nu_{\mathrm{A}}(\omega) \leq 1$. In this sense, they constitute an instance of *bipolar* models [8]. Although IF-sets and IVF-sets model different situations, they are mathematically equivalent [4].

In this work, we shall assume that the IVF-set is defined on a probability space and that the unknown membership function is measurable, and shall investigate the probabilistic information about its associated distribution. In Section 3, we

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compare three possible models, from the most to the least informative: the set of distributions of the measurable selections, those bounded between the upper and lower probabilities of the IVF-set, and those determined by the associated p-box. The advantage of these less informative models is that they are determined by a set and a point function, respectively.

We shall establish sufficient conditions for the equality between these three models, and give examples showing that they are not equivalent in general. Our results shall provide moreover a connection with the notion of probability induced by an intuitionistic fuzzy set from [10]. Finally, in Section 4 we shall investigate the problem without the assumption of measurability. In that case, we shall work with sets of finitely additive probabilities and with the theory of coherent lower previsions of Walley [15]. We shall see that the equivalences above do not always hold in this case. We conclude the paper with some additional remarks in Section 5. Due to the space restrictions, proofs have been omitted.

2 Preliminary Concepts

2.1 Random Sets

Since in this paper we shall deal with the probabilistic information of IVF-sets, it is interesting to recall a few notions of the sets of probabilities associated with multi-valued mappings. Given a probabilistic space $(\Omega, \mathcal{A}, P)$ and a measurable space $(\Omega', \mathcal{A}')$, a *random set* [6] is a multi-valued mapping $\Gamma : \Omega \to \mathcal{P}(\Omega')$ such that $\Gamma^*(A) = \{\omega \in \Omega : \Gamma(\omega) \cap A \neq \emptyset\} \in \mathcal{A}$ for any $A \in \mathcal{A}'$.

A random set Γ can be used to model the imprecise knowledge about a random variable X, in the sense that for every $\omega \in \Omega$ all we know about $X(\omega)$ is that it belongs to $\Gamma(\omega)$. Then, X belongs to the set of *measurable selections* of Γ:

$$S(\Gamma) = \{U : \Omega \to \Omega' \text{ random variable} \mid U(\omega) \in \Gamma(\omega)\ \forall \omega \in \Omega\}, \tag{1}$$

and the probability measure it induces on $\mathcal{A}'$ belongs to

$$\mathcal{P}(\Gamma) = \{P_U : U \in S(\Gamma)\}. \tag{2}$$

Another way of summarizing the information given by a random set is by means of its associated upper and lower probabilities:

Definition 1 ([6]). *Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(\Omega', \mathcal{A}')$ a measurable space and $\Gamma : \Omega \to \mathcal{P}(\Omega')$ a random set. Then its* upper *and* lower probabilities *$P^*_\Gamma, P_{*\Gamma} : \mathcal{A}' \to [0,1]$ are given by:*

$$P^*_\Gamma(A) = P(\{\omega : \Gamma(\omega) \cap A \neq \emptyset\}) \text{ and } P_{*\Gamma}(A) = P(\{\omega : \Gamma(\omega) \subseteq A\})\ \forall A \in \mathcal{A}'. \tag{3}$$

These upper and lower probabilities define a credal set $\mathcal{M}(P^*_\Gamma)$ by:

$$\mathcal{M}(P^*_\Gamma) = \{P \text{ probability} : P_{*\Gamma}(A) \leq P(A) \leq P^*_\Gamma(A)\ \forall A \in \mathcal{A}'\}.$$

It is easy to see that $\mathcal{P}(\Gamma) \subseteq \mathcal{M}(P^*_\Gamma)$, and that the two sets do not coincide in general. The equality between them was investigated in [11] for the particular case of random closed intervals we shall consider later on.

2.2 P-Boxes

In these notes, we shall also work with one particular imprecise probability model: p-boxes.

Definition 2 ([9]). *A* distribution function *on* $\Omega = [0,1]$ *is a monotone mapping* $F : [0,1] \to [0,1]$ *that is right-continuous and satisfies* $F(1) = 1$*. Given two monotone functions* $\underline{F}, \overline{F} : [0,1] \to [0,1]$ *satisfying* $\underline{F}(1) = \overline{F}(1) = 1$ *and* $\underline{F} \leq \overline{F}$*, its associated* probability box *(p-box, for short)* $(\underline{F}, \overline{F})$ *is the set of distribution functions bounded between* $\underline{F}$ *and* $\overline{F}$*.*

The assumption of right-continuity of distribution functions guarantees that they are in a one-to-one correspondence with σ-additive probability measures. More generally, a monotone and normalized function $F : [0,1] \to [0,1]$ represents the cumulative probabilities associated with an infinite number of different finitely additive probability measures. See [14] for a study of p-boxes from the point of view of finitely additive probability measures.

The *credal set* associated with the p-box $(\underline{F}, \overline{F})$ is given by

$$\mathcal{M}(\underline{F}, \overline{F}) := \{P \text{ probability } : \underline{F} \leq F_P \leq \overline{F}\},$$

where F_P denotes the distribution function of P.

3 Probabilistic Information of Interval-Valued Fuzzy Sets

In this section, we detail a number of ways in which IVF-sets can be related to imprecise probability models.

3.1 IVFS as Random Intervals

As we mentioned in the introduction, an interval-valued fuzzy set can be regarded as a model for the imprecise knowledge of the membership function of a fuzzy set. In this section, we shall assume that the IVF-set is defined on the probability space $([0,1], \beta_{[0,1]}, \lambda_{[0,1]})$, and that the multi-valued mapping $\Gamma_A : [0,1] \to \mathcal{P}([0,1])$, given by

$$\Gamma_A(\omega) := [l_A(\omega), u_A(\omega)] \tag{4}$$

is a random set. This means [11, Theorem 3.1] that the mappings $l_A, u_A : [0,1] \to [0,1]$ must be $\beta_{[0,1]} - \beta_{[0,1]}$-measurable.

If we assume that the 'true' membership function imprecisely specified by means of the IVF-set is $\beta_{[0,1]} - \beta_{[0,1]}$-measurable, then it must belong to the set $S(\Gamma_A)$ given by Eq. (1), and its associated probability measure will belong to the set $\mathcal{P}(\Gamma_A)$ given by Eq. (2). As we have seen in Section 2.1, $\mathcal{P}(\Gamma_A)$ is included in the set $\mathcal{M}(P^*_{\Gamma_A})$ of probability measures that are dominated by $P^*_{\Gamma_A}$, but the two sets do not coincide in general. The equality between these two sets for random closed intervals was studied in [11]. Using the results from that paper, it is easy to establish the following:

Proposition 1. $\mathcal{M}(P^*_{\Gamma_{\mathrm{A}}}) = \mathcal{P}(\Gamma_{\mathrm{A}})$ *under any of the following conditions:*

(C1) l_A, u_A *are increasing.*
(C2) $l_A(\omega) = 0$ *for any* $\omega \in [0,1]$.
(C3) l_A, u_A *are strictly comonotone*[1].

This result is interesting because it allows us to summarize the available information about the distribution of the membership function (the set of probability measures $\mathcal{P}(\Gamma_A)$) by means of the set function $P^*_{\Gamma_A}$. The conditions above may be interpreted in the following way:

(C1) As ω increases in [0,1], the evidence in favor of ω satisfying A increases.
(C2) There is no evidence supporting that any element satisfies A.
(C3) The intervals associated with the elements are ordered. In particular, this holds when the length of the intervals is constant.

On the other hand, the equality $\mathcal{P}(\Gamma) = \mathcal{M}(P^*_\Gamma)$ does not necessarily hold for all random closed intervals Γ [11, Example 3.3]. It is easy to adapt this example to our context and deduce that there are IVF-sets where the information about the membership function is not completely determined by the upper probability $P^*_{\Gamma_{\mathrm{A}}}$: it would suffice to take $\Gamma_{\mathrm{A}} : [0,1] \to \mathcal{P}([0,1])$ given by

$$\Gamma_{\mathrm{A}}(\omega) = \left[0.5 - \frac{\omega}{2}, 0.5 + \frac{\omega}{2}\right] \quad \text{for every } \omega \in [0,1]. \tag{5}$$

3.2 P-Boxes Induced by an IVF-Set

Now we take one step forward and study under which conditions the upper and lower probabilities $P^*_{\Gamma_A}, P_{*\Gamma_A}$ of the random interval associated with the IVF-set A can be summarized by means of two point functions: its lower and upper distribution functions $\underline{F}_{\mathrm{A}}, \overline{F}_{\mathrm{A}} : [0,1] \to [0,1]$, given by

$$\underline{F}_{\mathrm{A}}(x) := P_{*\Gamma_{\mathrm{A}}}([0,x]) = P_{u_A}([0,x]) \text{ and } \overline{F}_{\mathrm{A}}(x) := P^*_{\Gamma_{\mathrm{A}}}([0,x]) = P_{l_A}([0,x]). \tag{6}$$

We shall refer to $(\underline{F}_{\mathrm{A}}, \overline{F}_{\mathrm{A}})$ as the *p-box associated with the IVF-set* A. The credal set associated with this p-box is given by:

$$\mathcal{M}(\underline{F}_{\mathrm{A}}, \overline{F}_{\mathrm{A}}) := \{Q : \beta_{[0,1]} \to [0,1] : \underline{F}_{\mathrm{A}}(x) \leq F_{\mathrm{Q}}(x) \leq \overline{F}_{\mathrm{A}}(x)\ \forall x \in [0,1]\},$$

where F_{Q} is the distribution function associated with the probability measure Q. It is immediate to see that the set $\mathcal{M}(\underline{F}_{\mathrm{A}}, \overline{F}_{\mathrm{A}})$ includes $\mathcal{M}(P^*_{\Gamma_{\mathrm{A}}})$. However, the two sets do not coincide in general, and as a consequence the use of the lower and upper distribution functions may produce a loss of information. This was shown in [5, Example 3.3] for arbitrary random sets. Next, we give an example with random closed intervals associated with an IVF-set:

[1] We say that two functions $A, B : [0,1] \to [0,1]$ are *strictly comonotone* when $(A(\omega) - A(\omega')) \geq 0 \Leftrightarrow (B(\omega) - B(\omega')) \geq 0$ for any $\omega, \omega' \in [0,1]$.

Example 1. Consider the random set of Eq. (5), and let $(\underline{F}_A, \overline{F}_A)$ be its associated p-box. Given the distribution function F defined by:

$$F(x) = \begin{cases} \overline{F}_{\mathrm{A}}(x) & \text{if } x \leq \frac{1}{4}, \\ \frac{1}{2} & \text{if } x \in \left(\frac{1}{4}, \frac{3}{4}\right], \\ \underline{F}_{\mathrm{A}}(x) & \text{if } x > \frac{3}{4}, \end{cases}$$

its associated probability, P_F, belongs to $\mathcal{M}(\underline{F}_{\mathrm{A}}, \overline{F}_{\mathrm{A}})$. However, P_F does not belong to $\mathcal{M}(P^*_{\Gamma_{\mathrm{A}}})$, because

$$P_F\left(\left[\frac{1}{4}, \frac{3}{4}\right]\right) = 0 < P_{*\Gamma_{\mathrm{A}}}\left(\left[\frac{1}{4}, \frac{3}{4}\right]\right) = \frac{1}{2}. \quad \blacklozenge$$

Our next result shows that the sufficient conditions we have established in Proposition 1 for the equality $\mathcal{M}(P^*_{\Gamma_A}) = \mathcal{P}(\Gamma_A)$ also guarantee the equality between $\mathcal{M}(P^*_{\Gamma_{\mathrm{A}}})$ and $\mathcal{M}(\underline{F}_{\mathrm{A}}, \overline{F}_{\mathrm{A}})$; thus, in those cases the p-box associated with the IVF-set keeps all the information about the probability distribution of the membership function.

Proposition 2. *Let A be a IVF-set on $([0,1], \beta_{[0,1]}, \lambda_{[0,1]})$, and let Γ_{A} be its associated random interval, given by Eq. (4). Then $\mathcal{P}(\Gamma_{\mathrm{A}}) = \mathcal{M}(\underline{F}_{\mathrm{A}}, \overline{F}_{\mathrm{A}}) = \mathcal{M}(P^*_{\Gamma_{\mathrm{A}}})$ under any of the following conditions:*

(C1) l_A, u_A are increasing.
(C2) $l_A(\omega) = 0$ for every $\omega \in [0,1]$.
(C3) l_A and u_A are strictly comonotone.

3.3 Probabilities Associated with IFS

Another connection between probability theory and intuitionistic fuzzy sets was established in [10] by means of the probabilities induced by IF-sets. Given a probability space $(\Omega, \mathcal{A}, P)$, the probability associated with an IF-set A is an element of the interval

$$\left[\int_\Omega \mu_{\mathrm{A}} \mathrm{d}P, \int_\Omega 1 - \nu_{\mathrm{A}} \mathrm{d}P\right]. \tag{7}$$

This definition generalizes an earlier one by Zadeh [17]. Using this notion, in [10] a link is established with probability theory by considering the appropriate operators in the spaces of real intervals and of intuitionistic fuzzy sets. Note that it is assumed that we have a structure of probability space on Ω and that the functions $\mu_{\mathrm{A}}, \nu_{\mathrm{A}}$ are measurable, as we have done in this paper. From [4], it is known that IF-sets and IVF-sets are mathematically equivalent. In fact, given an IF-set with membership and non-membership functions μ_{A} and ν_{A}, it defines an IVF-set by considering $[\mu_{\mathrm{A}}(\omega), 1 - \nu_{\mathrm{A}}(\omega)]$ for every $\omega \in \Omega$.

If we assume that $(\Omega, \mathcal{A}, P) = ([0,1], \beta_{[0,1]}, \lambda_{[0,1]})$ and consider the random interval associated with the intuitionistic fuzzy set A interpreted as an IVF-set, we can see that the interval in Eq. (7) corresponds simply to the set of expectations of the measurable selections of Γ_{A}: it follows from [12, Thm. 14] that the *Aumann integral* [3] of $(id \circ \Gamma_{\mathrm{A}})$ satisfies

$$\left[\inf(A) \int (\mathrm{id} \circ \Gamma_{\mathrm{A}}) \mathrm{d}P, \sup(A) \int (\mathrm{id} \circ \Gamma_{\mathrm{A}}) \mathrm{d}P\right] = \left[(C) \int \mathrm{id}\ \mathrm{d}P^*_{\Gamma_{\mathrm{A}}}, (C) \int \mathrm{id}\ \mathrm{d}P_{*\Gamma_{\mathrm{A}}}\right]$$

where (C) denotes the *Choquet* integral [7] with respect to the non-additive measures $P_{*\Gamma_{\mathrm{A}}}, P^*_{\Gamma_{\mathrm{A}}}$, respectively. Since on the other hand it is immediate to see that

$$\sup(A) \int (\mathrm{id} \circ \Gamma_{\mathrm{A}}) \mathrm{d}P = \int (1 - \nu_{\mathrm{A}}) \mathrm{d}P \text{ and } \inf(A) \int (\mathrm{id} \circ \Gamma_{\mathrm{A}}) \mathrm{d}P = \int \mu_{\mathrm{A}} \mathrm{d}P,$$

we deduce that the probabilistic information about the intuitionistic fuzzy set A can be determined in particular by the lower and upper probabilities of its associated random interval.

4 A Non-measurable Approach

The previous developments assume that the IVF-set is defined on the probability space $([0,1], \beta_{[0,1]}, \lambda_{[0,1]})$ and that the functions $l_A, u_A : [0,1] \to [0,1]$, as well as the 'true' membership function of the fuzzy set modeled by A are $\beta_{[0,1]} - \beta_{[0,1]}$ measurable. Although this is a standard assumption when considering the probabilities associated with fuzzy events, it is arguably done for mathematical convenience only. In this section, we present an alternative approach where we get rid of the measurability assumptions by using finitely additive probabilities.

Consider thus a IVF-set A on $[0,1]$. Given its bounds l_A, u_A, we can define the multi-valued mapping $\Gamma_{\mathrm{A}} : [0,1] \to [0,1]$ by $\Gamma_{\mathrm{A}}(\omega) = [l_A(\omega), u_A(\omega)]\ \forall \omega$. Note that we are not assuming anymore that this multi-valued mapping is a random set. Our information about the 'true' membership function would be given by the set of functions

$$\{U : [0,1] \to [0,1] : l_A(\omega) \le U(\omega) \le u_A(\omega)\}. \tag{8}$$

Now, if we do not assume the measurability of l_A, u_A and consider then the field $\mathcal{P}([0,1])$ of all events in the initial space, we may not be able to model our uncertainty by means of a σ-additive probability measure. However, we can do so by means of a finitely additive probability measure. Moreover, the notions of lower and upper probabilities can be generalised to that case [13]. If for instance we consider a finitely additive probability P on $\mathcal{P}([0,1])$, then reasoning as in Section 3.1 we obtain that $P_U(C) \in [P_{\Gamma_{*A}}(C), P^*_{\Gamma_{\mathrm{A}}}(C)]$ for all $C \subseteq [0,1]$, where $P^*_{\Gamma_{\mathrm{A}}}, P_{*\Gamma_A}$ are defined by Eq. (3).

Then the information about P_U is given by the set of finitely additive probabilities dominated by $P^*_{\Gamma_{\mathrm{A}}}$, that coincides with the finitely additive probabilities induced by the elements of the set of Eq. (8). Hence, and in contradistinction to Section 3.1, when we work with finitely additive probabilities we do not need to make the distinction between $\mathcal{P}(\Gamma_{\mathrm{A}})$ and $\mathcal{M}(P^*_{\Gamma_{\mathrm{A}}})$.

The associated p-box is given now by the set of finitely additive distribution functions (that is, monotone and normalized) that lie between $\underline{F}_{\mathrm{A}}$ and $\overline{F}_{\mathrm{A}}$, where again $\underline{F}_{\mathrm{A}}, \overline{F}_{\mathrm{A}}$ are given by Eq. (6). Its associated set of finitely additive probability measures $\mathcal{M}(\underline{F}_A, \overline{F}_A)$ is determined by its lower envelope $\underline{E}_{(\underline{F},\overline{F})}$, that can be determined in the following way ([14]): if we denote by $\mathcal{H}$ the field of subsets of $[0,1]$ generated by the sets $\{[0,x],(x,1] : x \in [0,1]\}$, then any set in $\mathcal{H}$ is of the form

$$B_1 := [0,x_1] \cup (x_2,x_3] \cup \ldots (x_{2n},x_{2n+1}] \text{ or } B_2 := (x_1,x_2] \cup \ldots (x_{2n},x_{2n+1}]$$

for some $n \in \mathbb{N}, x_1 < x_2 < \cdots < x_n \in [0,1]$. It holds that

$$\underline{E}_{\underline{F},\overline{F}}(B_1) = \underline{F}_{\mathrm{A}}(x_1) + \textstyle\sum_{i=1}^n \max\{0, \underline{F}_{\mathrm{A}}(x_{2i+1}) - \overline{F}_{\mathrm{A}}(x_{2i})\},$$
$$\underline{E}_{\underline{F},\overline{F}}(B_2) = \textstyle\sum_{i=1}^n \max\{0, \underline{F}_{\mathrm{A}}(x_{2i}) - \overline{F}_{\mathrm{A}}(x_{2i-1})\},$$

and also $\underline{E}_{\underline{F},\overline{F}}(C) = \sup_{B\subseteq C, B\in\mathcal{H}} \underline{E}_{\underline{F},\overline{F}}(B)$ for any $C \subseteq [0,1]$.

Next, we investigate the equality $\mathcal{M}(P^*_{\Gamma_A}) = \mathcal{M}(\underline{F}_A, \overline{F}_A)$ under the conditions (C1)–(C3) considered in Proposition 1. We begin by showing that the two sets may not coincide when condition (C1) is satisfied.

Example 2. Consider the random interval defined by:

$$\Gamma_{\mathrm{A}}(\omega) = \begin{cases} [\omega, 2\omega] & \text{if } \omega \in \left[0, \frac{1}{3}\right] \\ \left[\frac{1}{3}, \frac{2}{3}\right] & \text{if } \omega \in \left(\frac{1}{3}, \frac{2}{3}\right] \\ [2\omega - 1, \omega] & \text{if } \omega \in \left(\frac{2}{3}, 1\right] \end{cases}$$

where in the initial space $([0,1], \mathcal{P}([0,1]))$ we consider a finitely additive probability P that agrees with $\lambda_{[0,1]}$ on $\beta_{[0,1]}$. Then, $P_{*\Gamma_{\mathrm{A}}}\left(\left[\frac{1}{3}, \frac{2}{3}\right]\right) = \frac{1}{3}$. However, it holds that:

$$\underline{E}_{\underline{F}_{\mathrm{A}},\overline{F}_{\mathrm{A}}}\left(\left[\frac{1}{3}, \frac{2}{3}\right]\right) = \underline{E}_{\underline{F}_{\mathrm{A}},\overline{F}_{\mathrm{A}}}\left(\left(\frac{1}{3}, \frac{2}{3}\right]\right) = \underline{F}_{\mathrm{A}}\left(\frac{2}{3}\right) - \overline{F}_{\mathrm{A}}\left(\frac{1}{3}\right) = \frac{2}{3} - \frac{2}{3} = 0. \blacklozenge$$

With respect to condition (C2), we have proven the following:

Proposition 3. *Let A be an IVF-set on $([0,1], \mathcal{P}([0,1]), P)$ with $l_A = 0$, and let $P_{*\Gamma_A}, (\underline{F}_A, \overline{F}_A)$ be its associated lower probability and p-box. Then, $\underline{E}_{\underline{F}_{\mathrm{A}},\overline{F}_{\mathrm{A}}} = P_{*\Gamma_{\mathrm{A}}}$.*

Finally, the equality does not hold for condition (C3), as we show next:

Example 3. Consider the random interval Γ_{A} defined on $([0,1], \mathcal{P}([0,1]), P)$ by:

$$\Gamma_{\mathrm{A}}(\omega) = \begin{cases} \left[\frac{1}{2} - \omega, 1 - \omega\right] & \text{if } \omega \in \left[0, \frac{1}{4}\right] \\ \left[\frac{1}{4}, \frac{3}{4}\right] & \text{if } \omega \in \left(\frac{1}{4}, \frac{3}{4}\right] \\ \left[\omega - \frac{1}{2}, \omega\right] & \text{if } \omega \in \left(\frac{3}{4}, 1\right], \end{cases}$$

and where P is a finitely additive probability that agrees with $\lambda_{[0,1]}$ on $\beta_{[0,1]}$. Since $l_A(\omega) = u_A(\omega) - \frac{1}{2}$, we see that l_A and u_A are strictly comonotone. If we consider the set $\left[\frac{1}{4}, \frac{7}{8}\right]$, we observe that

$$\underline{E}_{\underline{F}_{\mathrm{A}},\overline{F}_{\mathrm{A}}}\left(\left[\frac{1}{4}, \frac{7}{8}\right]\right) = \underline{E}_{\underline{F}_{\mathrm{A}},\overline{F}_{\mathrm{A}}}\left(\left(\frac{1}{4}, \frac{7}{8}\right]\right) = \frac{1}{4} < \frac{3}{4} = P_{*\Gamma_{\mathrm{A}}}\left(\left[\frac{1}{4}, \frac{7}{8}\right]\right). \qquad \blacklozenge$$

5 Conclusions

Our results show that the probabilistic information that an IVF-set holds about the underlying membership function can be summarized under some conditions by means of its associated p-box, although not in all cases. However, the correspondence depends on the measurability assumption of this membership function, and does not hold when we work with finitely additive probabilities instead.

In the future, we intend to deepen in the study of the imprecise probability models associated with an IVF-set, and to generalize our results to other possibility spaces. It would also be interesting to explore the alternative approach where no probability structure is considered in the initial space, and our knowledge is given instead by the set of possibility measures associated with the selections.

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