Stochastic Orders for Fuzzy Random Variables

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Abstract. The comparison of random variables can be made by means of stochastic orders such as expected utility or statistical preference. One possible model when the random variables are imprecisely observed is to consider fuzzy random variables, so that the images become fuzzy sets. This paper proposes two comparison methods for fuzzy random variables: one based on fuzzy rankings and another one that uses the extensions of stochastic orders to an imprecise framework. The particular case where the images of the fuzzy random variables are triangular fuzzy numbers is investigated.We illustrate our results by means of a decision making problem.

Keywords: Fuzzy random variables, stochastic orders, expected utility, statistical preference, possibility measures.

1 Introduction

A decision making problem under uncertainty requires the choice between several alternatives that are usually modeled by means of random variables; the choice between them is made by means of stochastic orders [11]. When we have imprecise information about the consequences of the different alternatives, we need to consider a more general model, such as sets of random variables, random sets, or, as we do in this paper, fuzzy random variables [6], where the images are fuzzy sets instead of real numbers. In order to extend stochastic orderings to this case, we follow in this paper two different avenues. On the one hand, based on the idea behind statistical preference, we compare fuzzy random variables by means of a choice model over their images, using fuzzy rankings, where by *fuzzy ranking* we refer to a method for the comparison of fuzzy sets. On the other hand, and similarly to expected utility, we can also compare fuzzy random variables in terms of their expectations. Since the expectation of a fuzzy random variable can be modeled by a possibility measure, we shall use the methods established in [9,10] for the comparison of imprecise probability models.

The paper is organized as follows: Section 2 introduces the main notions about fuzzy random variables and stochastic orders defined under imprecision. Then we discuss the two approaches mentioned above for the comparison of fuzzy random variables, and in Section 4 we investigate the particular case where the images of the fuzzy random variables are triangular fuzzy numbers. Finally, Section 5 illustrates our methods in a decision making problem. The paper concludes with

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some additional remarks and a discussion of other approaches to this problem. Proofs are omitted because of space limitations.

2 Preliminary Notions

2.1 Fuzzy Random Variables

Fuzzy random variables were introduced simultaneously by Kruse and Meyer [6] and Puri and Ralescu [12]. In this paper, we follow the epistemic approach considered in [6]. Let $\mathcal{F}(\mathbb{R})$ denote the set of all fuzzy sets on $\mathbb{R}$.

Definition 1 ([6]). *A fuzzy random variable is a map* $\widetilde{X} : \Omega \to \mathcal{F}(\mathbb{R})$ *such that the* α*-cuts* $\widetilde{X}_\alpha$ *are strongly measurable multi-valued mappings.*

Following Kruse and Meyer, fuzzy random variables can be used to model the imprecise knowledge about an unknown random variable U_0. For any $\omega \in \Omega, \omega' \in \mathbb{R}$, $\widetilde{X}(\omega)(\omega')$ can be interpreted as the acceptability degree of the proposition "$U_0(\omega) = \omega'$". With a similar reasoning, it is possible to define a fuzzy set on the class of measurable functions from Ω to $\mathbb{R}$, $\mu_{\widetilde{X}}$, that associates the value

$$\mu_{\widetilde{X}}(U) = \inf\{\widetilde{X}(\omega)(U(\omega)) : \omega \in \Omega\}$$

for any measurable function $U : \Omega \to \mathbb{R}$. Then, according to [6] this value can be understood as the acceptability degree of the proposition "$U = U_0$". Using the fuzzy set $\mu_{\widetilde{X}}$ it is possible to define the expectation of a fuzzy random variable as the fuzzy set $E_{\widetilde{X}}$ with membership function:

$$E_{\widetilde{X}}(r) = \sup\{\mu_{\widetilde{X}}(U) : E(U) = r\}. \tag{1}$$

$E_{\widetilde{X}}(r)$ can be interpreted as the acceptability degree of the proposition "$E(U_0) = r$". This membership function can also be seen as a possibility distribution, and as a consequence this expectation can be regarded as a possibility measure.

2.2 Stochastic Orders under Imprecision

Stochastic orders are methods for the comparison of random quantities. Here we shall use *expected utility*, given by $X \succeq_{\text{EU}} Y \Leftrightarrow E(X) \geq E(Y)$, and *statistical preference* [2,3], that is based on a probabilistic relation. A probabilistic relation on a set of alternatives $\mathcal{A}$ is a map defined from $\mathcal{A}^2$ to $[0,1]$ such that $Q(a,b) + Q(b,a) = 1$ for any $(a,b) \in \mathcal{A}^2$, where $Q(a,b)$ measures the strength of the preference of a over b. Statistical preference considers a set of alternatives formed by random variables, and defines a probabilistic relation by $Q(X,Y) = P(X > Y) + \frac{1}{2}P(X = Y)$. Then, X is statistically preferred to Y, denoted by $X \succeq_{\text{SP}} Y$, if $Q(X,Y) \geq \frac{1}{2}$. In what remains we will use a well-known alternative expression for statistical preference: $X \succeq_{\text{SP}} Y$ if and only if $P(X \geq Y) \geq P(Y \geq X)$.

In a context of imprecision, it may be necessary to choose between *sets* of random variables, instead of single ones. This problem was studied in some detail in [9,10], and a number of extensions of a given stochastic order to the imprecise case were considered. In the next definition, $\succeq$ denotes a stochastic order that could be either the expected utility or statistical preference, as we shall use in this paper, or any other stochastic order.

Definition 2 ([10, Def. 5]). *Consider two sets of random variables $\mathcal{X}, \mathcal{Y}$ and a stochastic order $\succeq$. We say that:*

- *$\mathcal{X}$ is $\succeq_1$-preferred to $\mathcal{Y}$ if $U \succeq V$ for any $U \in \mathcal{X}$ and $V \in \mathcal{Y}$.*
- *$\mathcal{X}$ is $\succeq_2$-preferred to $\mathcal{Y}$ if there is $U \in \mathcal{X}$ such that $U \succeq V$ for any $V \in \mathcal{Y}$.*
- *$\mathcal{X}$ is $\succeq_3$-preferred to $\mathcal{Y}$ if for any $V \in \mathcal{Y}$ there is $U \in \mathcal{X}$ such that $U \succeq V$.*
- *$\mathcal{X}$ is $\succeq_4$-preferred to $\mathcal{Y}$ if there are $U \in \mathcal{X}$ and $V \in \mathcal{Y}$ such that $U \succeq V$.*
- *$\mathcal{X}$ is $\succeq_5$-preferred to $\mathcal{Y}$ if there is $V \in \mathcal{Y}$ such that $U \succeq V$ for any $U \in \mathcal{X}$.*
- *$\mathcal{X}$ is $\succeq_6$-preferred to $\mathcal{Y}$ if for any $U \in \mathcal{X}$ there is $V \in \mathcal{Y}$ such that $U \succeq V$.*

When the extended stochastic order is either expected utility or statistical preference, we shall use the notation $\succeq_{\mathrm{EU_i}}$ or $\succeq_{\mathrm{SP_i}}$, respectively.

Some stochastic orders, such as expected utility, compare two random variables by means of their associated probability distributions. For those, the definitions above can be used to compare sets of probability distributions, also called *credal sets*. This allows us to compare imprecise probability models, such as possibility measures. Indeed, the credal set associated with a possibility measure Π is given by:

$$\mathcal{M}(\Pi) = \{P \text{ probability} \mid P \leq \Pi\}.$$

Then, we can compare two possibility measures Π_{X} and Π_{Y} by means of their associated credal sets. Our next result considers the extensions of expected utility, and uses $\Pi_{\mathrm{X}} \succeq_{\mathrm{EU_i}} \Pi_{\mathrm{Y}}$ to denote $\mathcal{M}(\Pi_{\mathrm{X}}) \succeq_{\mathrm{EU_i}} \mathcal{M}(\Pi_{\mathrm{Y}})$ for $i = 1, \ldots, 6$. Recall also that the conjugate function N of a possibility measure Π, given by $N(A) = 1 - \Pi(A^c)$ for every A, is usually named *necessity measure*.

Proposition 1. *For any two possibility measures Π_{X} and Π_{Y}, with conjugate necessity measures N_{X} and N_{Y}, respectively, it holds that:*

- $\Pi_{\mathrm{X}} \succeq_{\mathrm{EU_1}} \Pi_{\mathrm{Y}} \Leftrightarrow (C)\int id \mathrm{d}\Pi_{\mathrm{X}} \geq (C)\int id \mathrm{d}N_{\mathrm{Y}}$;
- $\Pi_{\mathrm{X}} \succeq_{\mathrm{EU_2}} \Pi_{\mathrm{Y}} \Leftrightarrow \Pi_{\mathrm{X}} \succeq_{\mathrm{EU_3}} \Pi_{\mathrm{Y}} \Leftrightarrow (C)\int id \mathrm{d}N_{\mathrm{X}} \geq (C)\int id \mathrm{d}N_{\mathrm{Y}}$;
- $\Pi_{\mathrm{X}} \succeq_{\mathrm{EU_4}} \Pi_{\mathrm{Y}} \Leftrightarrow (C)\int id \mathrm{d}N_{\mathrm{X}} \geq (C)\int id \mathrm{d}\Pi_{\mathrm{Y}}$;
- $\Pi_{\mathrm{X}} \succeq_{\mathrm{EU_5}} \Pi_{\mathrm{Y}} \Leftrightarrow \Pi_{\mathrm{X}} \succeq_{\mathrm{EU_6}} \Pi_{\mathrm{Y}} \Leftrightarrow (C)\int id \mathrm{d}\Pi_{\mathrm{X}} \geq (C)\int id \mathrm{d}\Pi_{\mathrm{Y}}$;

where $(C)\int f \mathrm{d}\mu$ denotes the Choquet integral of f with respect to the non-additive measure μ, and id denotes the identity function $id(x) = x$.

3 Comparison of Fuzzy Random Variables

As we mentioned in Section 2.2, two possible ways of comparing two random variables X, Y are expected utility and statistical preference, given by:

$$X \succeq_{\text{EU}} Y \Leftrightarrow E(X) \geq E(Y). \tag{2}$$

$$X \succeq_{\text{SP}} Y \Leftrightarrow P(\{\omega : X(\omega) \geq Y(\omega)\}) \geq P(\{\omega : Y(\omega) \geq X(\omega)\}). \tag{3}$$

In this section, we extend these two orders to fuzzy random variables. In the case of expected utility, the comparison of the expectations leads us to the comparison of possibility measures; concerning statistical preference, the comparison of the images of fuzzy random variables motivates the use of fuzzy rankings.

3.1 Comparison by Means of Fuzzy Rankings

Fuzzy rankings are methods for the comparison of quantities modeled by means of fuzzy sets, in that they measure to what extent one fuzzy set is larger than the other. Consider two fuzzy random variables, $\widetilde{X}, \widetilde{Y}$ modeling the imprecise knowledge of respective random variables X, Y. Then for every ω in the initial space $\widetilde{X}(\omega)$ and $\widetilde{Y}(\omega)$ are the fuzzy sets that represent the degree of acceptability of the propositions "$X(\omega) = \omega'$" and "$Y(\omega) = \omega'$", for any $\omega' \in \mathbb{R}$. In order to compare the fuzzy random variables $\widetilde{X}$ and $\widetilde{Y}$, we can compare the fuzzy sets $\widetilde{X}(\omega)$ and $\widetilde{Y}(\omega)$ for every $\omega \in \Omega$. This leads at once to the following definition:

Definition 3. *Let $\widetilde{X}, \widetilde{Y} : \Omega \to \mathcal{F}(\mathbb{R})$ be two fuzzy random variables on a probability space $(\Omega, \mathcal{A}, P)$, and let $\succsim$ be a fuzzy ranking. We say that $\widetilde{X}$ is $\succsim$-*statistically preferred *to $\widetilde{Y}$, and denote it $\widetilde{X} \succsim^P \widetilde{Y}$, when*

$$P(\{\omega \in \Omega : \widetilde{X}(\omega) \succsim \widetilde{Y}(\omega)\}) \geq P(\{\omega \in \Omega : \widetilde{Y}(\omega) \succsim \widetilde{X}(\omega)\}).$$

When the fuzzy ranking $\succsim$ is complete, that is, if it allows the comparison of every pair of fuzzy sets, we obtain the following result.

Proposition 2. *Let $\succsim$ be a complete fuzzy ranking, and define:*

$$Q(\widetilde{X}, \widetilde{Y}) = P(\{\omega : \widetilde{X}(\omega) \succ \widetilde{Y}(\omega)\}) + \frac{1}{2} P(\{\omega : \widetilde{X}(\omega) \sim \widetilde{Y}(\omega)\}).$$

Then $Q(\widetilde{X}, \widetilde{Y}) + Q(\widetilde{Y}, \widetilde{X}) = 1 \ \forall \widetilde{X}, \widetilde{Y}$, and $\widetilde{X}$ is $\succsim$-statistically preferred to $\widetilde{Y}$ if and only if $Q(\widetilde{X}, \widetilde{Y}) \geq \frac{1}{2}$. Moreover, if $\succsim$ extends the natural order on $\mathbb{R}$, then $\succsim$-statistical preference is an extension of statistical preference given by Eq. (3).

3.2 Comparison by Means of Stochastic Orders

Another way of comparing fuzzy random variables is by extending appropriately the order associated with expected utility, given by Eq. (2). Consider two fuzzy random variables $\widetilde{X}$ and $\widetilde{Y}$, and let $E_{\widetilde{X}}, E_{\widetilde{Y}}$ be their respective fuzzy expectations, given by Eq. (1). These expectations are fuzzy sets, or, equivalently, possibility measures. It leads to the following definition.

Definition 4. *We say that* $\widetilde{X}$ *is preferred to* $\widetilde{Y}$ *with respect to the i-th extension of expected utility, and denote it* $\widetilde{X} \succeq_{\mathrm{EU_i}} \widetilde{Y}$, *when* $E_{\widetilde{X}} \succeq_{\mathrm{EU_i}} E_{\widetilde{Y}}$, *where* $\succeq_{\mathrm{EU_i}}$ *is given in Definition 2.*

This result, together with Proposition 1, reduces the comparison of fuzzy random variables to the comparison of appropriate Choquet integrals. For a thorough discussion of the interpretation behind the different extensions $\succeq_{\mathrm{EU_i}}$, for $i = 1, \ldots, 6$, we refer to [9,10].

4 Particular Case: Triangular Fuzzy Random Variables

In this section we study the particular case where the images of $\widetilde{X}$ and $\widetilde{Y}$ are triangular fuzzy numbers. Recall that $A = (a_1, a_2, a_3)$ is a *triangular fuzzy number* when its membership function is given by:

$$A(\omega) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{for } a_1 < x \leq a_2. \\ \frac{a_3-x}{a_3-a_2} & \text{for } a_2 < x \leq a_3. \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

4.1 Fuzzy Rankings on Triangular Fuzzy Random Variables

Fuzzy rankings usually take a simple expression when applied to triangular fuzzy numbers. Here we consider two well-known fuzzy rankings, introduced by Dubois and Prade in [5].

Definition 5 ([5]). *Let* A, B *be two fuzzy numbers, and define:*

- **Possibility of Dominance:** $PD(A, B) = \sup_{x \geq y}(\min(A(x), B(y)))$.
- **Necessity of Strict Dominance:** $NSD(A, B) = 1 - \sup_{x \leq y}(\min(A(x), B(y)))$.

Then we denote $A \succeq_{\mathrm{PD}} B$ when $PD(A, B) \geq PD(B, A)$ (and similarly for NSD). In case of triangular fuzzy numbers, these definitions can be simplified:

Lemma 1. *Let* $A = (a_1, a_2, a_3)$ *and* $B = (b_1, b_2, b_3)$ *be two triangular fuzzy numbers. It holds that* $A \succeq_{\mathrm{PD}} B \Leftrightarrow A \succeq_{\mathrm{NSD}} B \Leftrightarrow a_2 \geq b_2$.

Proof. This is a consequence of Eq. (4) and Definition 5.

Using this result, we can simplify Definition 3 for these fuzzy rankings.

Proposition 3. *Given two triangular fuzzy random variables* $\widetilde{X}$ *and* $\widetilde{Y}$ *such that* $\widetilde{X}(\omega) = (a_1^\omega, a_2^\omega, a_3^\omega)$ *and* $\widetilde{Y}(\omega) = (b_1^\omega, b_2^\omega, b_3^\omega)$ $\forall \omega \in \Omega$,

$$\widetilde{X} \succsim_{\mathrm{PD}}^{P} \widetilde{Y} \Leftrightarrow \widetilde{X} \succsim_{\mathrm{NSD}}^{P} \widetilde{Y} \Leftrightarrow P(\{\omega \in \Omega : a_2^\omega \geq b_2^\omega\}) \geq P(\{\omega \in \Omega : b_2^\omega \geq a_2^\omega\}).$$

Note also that both PD and NSD are complete fuzzy rankings, and then Proposition 2 can be applied.

4.2 Stochastic Orders on Triangular Fuzzy Random Variables

We now turn on the comparison of triangular fuzzy random variables by means of the generalizations of expected utility. We begin by showing a well-known result that easily allows to compute the expectation of a triangular fuzzy number.

Proposition 4 ([4,7]). *Consider a fuzzy random variable* $\widetilde{X}$ *such that* $\widetilde{X}(\omega)$ *is a triangular fuzzy number* $(a_1^\omega, a_2^\omega, a_3^\omega)$ *for any* ω. *Consider the functions* $f_1(\omega) = a_1^\omega$, $f_2(\omega) = a_2^\omega$ *and* $f_3(\omega) = a_3^\omega$, *for any* $\omega \in \Omega$. *Then,* $E_{\widetilde{X}} = (e_1, e_2, e_3)$ *is also a triangular fuzzy number, where* $e_1 = E(f_1)$, $e_2 = E(f_2)$ *and* $e_3 = E(f_3)$.

Next we show that Definition 4 can be simplified in this case. The proof follows by considering the interpretations of Definition 2 in the case of expected utility (see for instance [9, Remark 3]), and the formulas of the 'best' and 'worst' alternatives in the credal set associated with a possibility measure in the particular case of triangular fuzzy numbers.

Proposition 5. *Consider two possibility measures* Π_X *and* Π_Y *whose associated fuzzy sets are the triangular fuzzy numbers* (a_1, a_2, a_3) *and* (b_1, b_2, b_3), *respectively. Then:*

- $\Pi_\mathrm{X} \succeq_{\mathrm{EU}_1} \Pi_\mathrm{Y} \Leftrightarrow a_1 + a_2 \geq b_2 + b_3$.
- $\Pi_\mathrm{X} \succeq_{\mathrm{EU}_2} \Pi_\mathrm{Y} \Leftrightarrow \Pi_\mathrm{X} \succeq_{\mathrm{EU}_3} \Pi_\mathrm{Y} \Leftrightarrow a_2 + a_3 \geq b_2 + b_3$.
- $\Pi_\mathrm{X} \succeq_{\mathrm{EU}_4} \Pi_\mathrm{Y} \Leftrightarrow a_2 + a_3 \geq b_1 + b_2$.
- $\Pi_\mathrm{X} \succeq_{\mathrm{EU}_5} \Pi_\mathrm{Y} \Leftrightarrow \Pi_\mathrm{X} \succeq_{\mathrm{EU}_6} \Pi_\mathrm{Y} \Leftrightarrow a_1 + a_2 \geq b_1 + b_2$.

5 Example of Application in Decision Making

This section presents an application of the previous definitions to a decision making problem. We use the setting considered in [8]: a company operating in UK is considering the possibility of expanding to new markets. They consider four alternatives:

$\mathbf{A_1}$: Expand to the French market.
$\mathbf{A_2}$: Expand to the German market.
$\mathbf{A_3}$: Expand to the Italian market.
$\mathbf{A_4}$: Expand to the Spanish market.

The evaluation of the strategies depends on the economic situation for the next year, that may take four different values:

$\mathbf{S_1}$: Bad economic situation.
$\mathbf{S_2}$: Regular economic situation.
$\mathbf{S_3}$: Good economic situation.
$\mathbf{S_4}$: Very good economic situation.

The probabilities for each state are estimated as 0.1, 0.3, 0.3 and 0.3, respectively. Then, we can define the probability space $(\Omega, \mathcal{P}(\Omega), P)$, where $\Omega = \{S_1, S_2, S_3, S_4\}$, and model each alternative as a fuzzy random variable taking the following values, that represent the expected benefits:

	S_1	S_2	S_3	S_4
A_1	$(0.2, 0.3, 0.4)$	$(0.6, 0.7, 0.8)$	$(0.2, 0.3, 0.4)$	$(0.5, 0.6, 0.7)$
A_2	$(0.5, 0.5, 0.5)$	$(0.3, 0.4, 0.5)$	$(0.4, 0.5, 0.7)$	$(0.4, 0.5, 0.6)$
A_3	$(0.1, 0.2, 0.4)$	$(0.6, 0.8, 0.9)$	$(0.8, 0.9, 1)$	$(0.7, 0.8, 0.9)$
A_4	$(0.3, 0.4, 0.5)$	$(0.3, 0.4, 0.6)$	$(0.5, 0.5, 0.5)$	$(0.3, 0.4, 0.5)$

Since these alternatives are triangular fuzzy random variables, we can apply the results from Section 4. First of all, if we compare them pairwisely by means of PD and NSD, Lemma 1 assures that the two fuzzy rankings reduce to the comparison of the modal points of the triangular fuzzy numbers. The resulting preference degrees are summarized in the following table:

	A_1	A_2	A_3	A_4
A_1	$\cdot$	0.5	0.25	0.5
A_2	0.5	$\cdot$	0.25	0.75
A_3	0.75	0.75	$\cdot$	1
A_4	0.5	0.25	0	$\cdot$

Thus, we conclude that the best alternative is A_3, that is, to invest into the Italian market. If instead we compare these fuzzy random variables by means of the generalized expected utility, we deduce from Proposition 4 that the expectations of $A_1, \ldots, A_4$ are also triangular fuzzy numbers, and they are given by:

$$\begin{array}{ll} E_{A_1} = (0.41, 0.51, 0.61) & E_{A_2} = (0.38, 0.47, 0.59). \\ E_{A_3} = (0.64, 0.77, 0.88) & E_{A_4} = (0.38, 0.47, 0.59). \end{array}$$

Then, applying Propositions 4 and 5, we obtain the following results:

	A_1	A_2	A_3	A_4
A_1	$\cdot$	$\succeq_{\mathrm{EU}_{2,5}}$	$-$	$\succeq_{\mathrm{EU}_{2,5}}$
A_2	$-$	$\cdot$	$-$	$\succeq_{\mathrm{EU}_{2,5}}$
A_3	$\succeq_{\mathrm{EU}_1}$	$\succeq_{\mathrm{EU}_1}$	$\cdot$	$\succeq_{\mathrm{EU}_1}$
A_4	$-$	$-$	$-$	$\cdot$

Again A_3 seems to be the most adequate option, because it is preferable to the other alternatives with respect to the first extension of the expected utility (and as a consequence also with respect to any of the other extensions).

6 Conclusions

Stochastic orders are methods for the comparison of random quantities. When the random variables to be compared are imprecisely described, they can be modeled by means of fuzzy random variables. This work presents a first approach to the extension of stochastic orders to the comparison of fuzzy random variables. We have considered two possibilities: the comparison of the images of the fuzzy random variables by means of a fuzzy ranking, and the comparison of the expectations by means of stochastic orders on possibility measures. We have investigated in more detail the particular case of fuzzy random variables whose images are triangular fuzzy numbers, and showed that the proposed methods can be simplified in that case. In addition, we have illustrated these methods in a decision making problem.

There are still several open lines of research on the comparison of fuzzy random variables. On the one hand, it is possible to extend other stochastic orders, such as stochastic dominance [11], to this context; on the other hand, we would like to deepen into the comparison of the properties of the different fuzzy rankings proposed in the literature with respect to this problem.

Finally, a different approach would be the comparison of fuzzy random variables by means of their α-cuts. In this case, the comparison is reduced to the comparison of random sets, and we can consider notions of *strong* or *weak* preference, depending on whether the comparison holds for every or any α-cut. Note also that the comparison of random sets can be made in two different ways: on the one hand, we can consider a stochastic order on random variables, and apply it to the sets of measurable selections by means of Definition 2 [9]; or we could also consider other stochastic orders for random sets, such as the ones considered in [1].

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References

1. Cascos, I., Molchanov, I.: A stochastic order for random vectors and random sets based on the Aumann expectation. Stat. Prob. Lett 63, 295–305 (2003)
2. De Schuymer, B., De Meyer, H., De Baets, B.: A fuzzy approach to stochastic dominance of random variables. In: De Baets, B., Kaynak, O., Bilgiç, T. (eds.) IFSA 2003. LNCS, vol. 2715, pp. 253–260. Springer, Heidelberg (2003)
3. De Schuymer, B., De Meyer, H., De Baets, B., Jenei, S.: On the cycle-transitivity of the dice model. Theory and Decision 54, 261–285 (2003)
4. Dengjie, Z.: The limit theorems for expectation of fuzzy random variables. Int. Journal of Pure and Applied Mathematics 54(4), 489–496 (2009)
5. Dubois, D., Prade, H.: Ranking fuzzy numbers in the setting of possibility theory. Information Sciences 30, 183–224 (1983)
6. Kruse, R., Meyer, D.R.: Statistics with vague data. Reidel Publishing Company (1987)
7. Loquin, K., Dubois, D.: Kriging and epistemic uncertainty: A critical discussion. In: Jeansoulin, R., Papini, O., Prade, H., Schockaert, S. (eds.) Methods for Handling Imperfect Spatial Information. STUDFUZZ, vol. 256, pp. 269–305. Springer, Heidelberg (2010)
8. Merigó, J.M., Casanovas, M., Yang, J.-B.: Group decision making with expertons and uncertain generalized probabilistic weighted aggregation operators. European Journal of Operational Research 235, 215–224 (2014)
9. Montes, I., Miranda, E., Montes, S.: Decision making with imprecise probabilities and utilities by means of statistical preference and stochastic dominance. European Journal of Operational Research 234(1), 209–220 (2014)
10. Montes, I., Miranda, E., Montes, S.: Stochastic dominance with imprecise information. Computational Statistics and Data Analysis 71, 868–886 (2014)
11. Müller, A., Stoyan, D.: Comparison Methods for Stochastic Models and Risks. Wiley (2002)
12. Puri, M.L., Ralescu, D.: Fuzzy random variables. J. Math. Anal. Appl. 114, 409–422 (1986)