On Fuzzy Equations with 2×2 Matrices

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Abstract. The question of probability of a system of fuzzy equations solvability in a max–t-norm fuzzy algebra for several t-norms and 2×2 matrices is considered. We derive that the probability of solving such a system is very low, namely $\frac{1}{10}$ for Gödel norm, $\frac{11}{60}$ for Łukasiewicz norm, $\frac{5}{36}$ for product norm and zero for drastic norm. These results are surprising compared to the case of a finite vector space, where the probability is one.

Keywords: t-norms, fuzzy relation equations, fuzzy algebras.

1 Introduction

Since the pioneer work [1] of fuzzy relation equations, many results on finding minimal and maximal elements and solvability of such systems have been developed (e.g. [2,3,4,5,6,7]). However, to the author's knowledge, there is no result describing something like probability of solving such a system. By probability is understood a situation, when one picks up uniformly randomly a matrix A together with a right-hand side b with coefficients from $[0; 1]$, then how often will the composed system $A \otimes x = b$ have a solution with respect to x?

This paper aims to make a first step in answering such kind of questions. To do this, we have restricted ourselves only to 2×2 matrices and four fuzzy algebras with t-norms minimum, Łukasiewicz, product and drastic. The derivation of conditional probabilities in these cases is presented. However, generalization to higher dimensions is the task of our future work.

After introduction, we continue by definitions of t-norms, fuzzy algebras and formulating rigorously the question of our concern. Then we answer the question in the case of finite vector spaces to have a classical result to which the results of fuzzy cases can be compared. The main section starts with results common for all t-norms. Then the respective cases are studied. The last section summarizes obtained results.

* This work was supported by grant SGS13/PřF/2014 of the University of Ostrava.

P. Grzegorzewski et al. (eds.), *Strengthening Links between Data Analysis & Soft Computing*,
Advances in Intelligent Systems and Computing 315, DOI: 10.1007/978-3-319-10765-3_12

2 Preliminaries

In this section, basic definitions, problem formulation and result for classical case are given.

Definition 1. *A mapping* $\mathrm{T} : [0;1] \times [0;1] \mapsto [0;1]$ *is called a t-*norm*, if the following conditions are satisfied for all* $a, b, c \in [0;1]$*:*

1. $a \,\mathrm{T}(b \,\mathrm{T}\, c) = (a \,\mathrm{T}\, b) \,\mathrm{T}\, c$ *(associativity),*
2. $a \,\mathrm{T}\, b = b \,\mathrm{T}\, a$ *(commutativity),*
3. *if* $a \le b$*, then* $a \,\mathrm{T}\, c \le b \,\mathrm{T}\, c$ *(monotonicity),*
4. $a \,\mathrm{T}\, 1 = a$ *(1 as neutral element).*

From all continuous t-norms (w.r.t. usual definition of continuity of a mapping), the following three play a prominent role, since every other continuous t-norm is their ordinal sum [9]:

$$a \,\mathrm{T}_G\, b := \min\{a, b\},\ a \,\mathrm{T}_{\mathrm{L}}\, b := \max\{0, a + b - 1\},\ a \,\mathrm{T}_{\Pi}\, b := ab.$$

The fourth one studied in this paper is the drastic norm:

$$a \,\mathrm{T}_D\, b = \begin{cases} 0 & \text{if } \max(a,b) < 1, \\ \min(a,b) & \text{otherwise,} \end{cases}$$

which is not continuous, but plays an important theoretical role, since it is the smallest possible t-norm.

Definition 2. *Given the unit interval* $[0;1]$ *together with a t-norm* T*, by* max–t-norm algebra *we understand an algebra* $\mathcal{A} = ([0;1], \max, \mathrm{T}, 0, 1)$ *of a type* $(2, 2, 0, 0)$.

Fuzzy algebras are usually viewed as complete residuated lattices (as e.g. in [8]). The chosen definition 2 is fully sufficient in what follows, since only the properties of t-norms are utilized.

We denote $\oplus := \max$ and $\otimes := \mathrm{T}$. The respective t-norm will be clear from the context. In notation, multiplication $\otimes$ is given precedence over addition $\oplus$, i.e., $a \otimes b \oplus c$ means the same as $(a \otimes b) \oplus c$.

Note, that $([0;1], \oplus, 0)$ and $([0;1], \otimes, 1)$ constitute commutative monoids and that T is distributive w.r.t. max: $a \otimes b \oplus a \otimes c = a \otimes (b \oplus c)$.

We extend the algebra $\mathcal{A}$ to a vector space-like structure $\mathcal{A}^n$ by formally replacing operations in standard matrix multiplication (matrix addition and multiplication by elements from a ring) by our operations $\oplus$ and $\otimes$. For example,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} \otimes x_1 \oplus a_{12} \otimes x_2 \\ a_{21} \otimes x_1 \oplus a_{22} \otimes x_2 \end{pmatrix}.$$

Our question can be formulated as follows: *When randomly picking up a square matrix* $A \in \mathcal{A}^{n \times n}$ *and a column* $b \in \mathcal{A}^n$*, what is the* probability*, that the equation*

$$A \otimes x = b$$

is solvable in $x \in \mathcal{A}^n$*?*

By *probability* is meant the ratio of the volume of all solvable pairs (A, b) to the volume of all pairs. Answers for $n = 2$ and four t-norms are given in section 3, but before proceeding to fuzzy algebras, we answer our question for the case of finite vector spaces over a field $\mathbb{R}$:

– for a regular matrix every system of linear equations (SLE) is solvable,
– determinant of such a matrix is nonzero,
– non-regular matrices satisfy equation $\det A = 0$,
– the set of all non-regular matrices thus compose a set of a zero measure in $\mathbb{R}^{n^2}$,
– then the set of pairs (A, b) of SLE with A non-regular is of a zero measure in $\mathbb{R}^{n^2+n}$, too.

Thence, in this case, the answer is $P_{\mathbb{R}} = 1$.

3 max–*t*-norm Algebras

We restrict ourselves to 2×2 dimensional squares A

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{1}$$

from $\mathcal{A}^{2\times 2}$. Notation

$$\begin{pmatrix} \Box_1 & \Box_2 \\ \Box_3 & \Box_4 \end{pmatrix}_{\Box_5}$$

represents the set of all such pairs (A, b) for which

$$a_{11}\Box_1 b_1,\ a_{12}\Box_2 b_1,\ a_{21}\Box_3 b_2,\ a_{22}\Box_4 b_2 \text{ and finally } b_1\Box_5 b_2,$$

where $b = (b_1, b_2)^T$ is the right-hand side and $\Box_i \in \{<, >\}$ for $i = 1, 2, \ldots 5$. We consider only strict inequalities, since coefficients are uniformly distributed and the probability of obtaining just one exact value is zero. When $\Box_5$ is omitted, there is no relation between b_1 and b_2.

The following cases are either unsolvable due to the monotonicity of t-norms, i.e., when $a, b \in [0; 1],\ a < b$ then $a \otimes x < b$ for all $x \in [0; 1]$, or their solvable parts have a zero measure:

$$\begin{pmatrix} < & < \\ < & < \end{pmatrix}, \qquad V = \tfrac{1}{9},$$

$$\begin{pmatrix} > & < \\ > & < \end{pmatrix}, \begin{pmatrix} < & > \\ < & > \end{pmatrix}, \qquad V = \tfrac{1}{36},$$

$$\begin{pmatrix} > & > \\ < & < \end{pmatrix}, \begin{pmatrix} < & < \\ > & > \end{pmatrix}, \qquad V = \tfrac{1}{9},$$

$$\begin{pmatrix} > & < \\ < & < \end{pmatrix}_>, \begin{pmatrix} < & > \\ < & < \end{pmatrix}_>, \begin{pmatrix} < & < \\ > & < \end{pmatrix}_<, \begin{pmatrix} < & < \\ < & > \end{pmatrix}_<, V = \tfrac{1}{90},$$

$$\begin{pmatrix} > & < \\ < & < \end{pmatrix}_<, \begin{pmatrix} < & > \\ < & < \end{pmatrix}_<, \begin{pmatrix} < & < \\ > & < \end{pmatrix}_>, \begin{pmatrix} < & < \\ < & > \end{pmatrix}_>, V = \tfrac{4}{90}.$$

V denotes the volume of such a set, e.g.

$$V\left(\begin{pmatrix} < < \\ > < \end{pmatrix}_<\right) = \int_0^1 db_2 \int_0^{b_2} db_1 \int_0^{b_1} da_{11} \int_0^{b_1} da_{12} \int_{b_2}^1 da_{21} \int_0^{b_2} da_{22} =$$

$$= \int_0^1 db_2 \int_0^{b_2} db_1\, b_1^2\, b_2\,(1-b_2) = \frac{1}{90}.$$

We see, that the volume of unsolvable pairs must be strictly greater than

$$\frac{1}{9} + 2\cdot\frac{1}{36} + 2\cdot\frac{1}{9} + 4\cdot\frac{1}{90} + 4\cdot\frac{4}{90} = \frac{11}{18},$$

i.e., more than half of the whole volume of all pairs. On the other hand, the two following sets are always solvable for continuous t-norms (excluding thus also drastic norm):

$$\begin{pmatrix} > < \\ < > \end{pmatrix}, \begin{pmatrix} < > \\ > < \end{pmatrix}, \; V = \frac{1}{36}. \tag{2}$$

Thence, for continuous t-norms, the probability of system solvability with two equations and two variables is at least $\frac{1}{18}$. Solvability of other sets of pairs, namely

$$\begin{aligned}
&\text{I}: \begin{pmatrix} > > \\ > > \end{pmatrix}, && V = \tfrac{1}{9},\\
&\text{II}: \begin{pmatrix} < > \\ > > \end{pmatrix}_>, \begin{pmatrix} > < \\ > > \end{pmatrix}_>, \begin{pmatrix} > > \\ < > \end{pmatrix}_<, \begin{pmatrix} > > \\ > < \end{pmatrix}_<, && V = \tfrac{4}{90},\\
&\text{III}: \begin{pmatrix} < > \\ > > \end{pmatrix}_<, \begin{pmatrix} > < \\ > > \end{pmatrix}_<, \begin{pmatrix} > > \\ < > \end{pmatrix}_>, \begin{pmatrix} > > \\ > < \end{pmatrix}_>, && V = \tfrac{1}{90},
\end{aligned}$$

substantially depends on the chosen t-norm, as is shown in next four subsections.

3.1 $\max -T_G$ Algebra

Sets of type I and II are not solvable in $\max -T_G$ algebra, since in the first case one of b_i's is strictly greater than other, w.l.o.g. let $b_1 > b_2$, then solving this one $x_1 = b_1$ leads to $\min(x_1, a_{21}) > b_2$ in the second equation. Similarly for $x_2 = b_1$. Case II differs only in that there is just one possibility for choosing x_i such that the equation with greater right-hand side is solved. Consider for example

$$(A,b) \in \begin{pmatrix} > < \\ > > \end{pmatrix}_>,$$

then clearly $x_1 = b_1$ in order to solve the first equation. But then $\min(a_{21}, x_1) > b_2$ and the second equation can not hold.

Systems from sets III are solvable; for instance for a pair (A,b) from

$$(A,b) \in \begin{pmatrix} > > \\ < > \end{pmatrix}_>$$

take $x = (b_1, b_2)$. The first equation is not corrupted, because $b_2 < b_1$, and neither is the second, because $a_{21} < b_2 < b_1$ and thus $\min(a_{21}, x_1) = a_{21} < b_2$.

The overall probability is $P_G = \dfrac{1}{18} + 4\cdot\dfrac{1}{90} = \dfrac{1}{10}$, where $\frac{1}{18}$ comes from (2).

3.2 max $-T_Ł$ Algebra

In this algebra even systems from I and II may be solvable. Assume for example, that we would like to solve a system from I in variables a_{11} and a_{22}, i.e.,

$$x_1 = 1 + b_1 - a_{11},\ x_2 = 1 + b_2 - a_{22},$$

where $x_1, x_2 \leq 1$ because $a_{ii} > b_i$ for $i = 1, 2$. Following conditions ensure, that neither of the equations will be corrupted

$$a_{21} + b_1 - a_{11} \leq b_2,\ a_{12} + b_2 - a_{22} \leq b_1. \tag{3}$$

It is now convenient to divide case I into two parts $b_1 > b_2$ and $b_2 > b_1$. For the first part, the conditions (3) can be rewritten to the form

$$a_{21} \leq a_{11} + b_2 - b_1,\ a_{22} \geq a_{12} + b_2 - b_1$$

and $1 \geq a_{11} + b_2 - b_1 \geq b_2$ holds since $1 \geq a_{11} + \underbrace{b_2 - b_1}_{\leq 0}$ and $\underbrace{a_{11} - b_1}_{\geq 0} + b_2 \geq b_2$, similarly for a_{22}. Then the volume of solvable part can be computed as

$$V\left(\begin{pmatrix} (>) & > \\ > & (>) \end{pmatrix}_{>}\right) = \\ = \int_0^1 db_1 \int_0^{b_1} db_2 \int_{b_1}^1 da_{11} \int_{b_2}^{a_{11}+b_2-b1} da_{21} \int_{b_1}^1 da_{12} \int_{a_{12}+b_2-b_1}^1 da_{22} = \\ = \frac{1}{80}.$$

Parenthesis indicate elements solving corresponding equation. Similarly can be dealt with other cases. The volumes of solvable parts of respective cases are

$$\text{I'}: \begin{pmatrix} (>) & > \\ > & (>) \end{pmatrix}_{>}, \begin{pmatrix} > & (>) \\ (>) & > \end{pmatrix}_{>}, \begin{pmatrix} (>) & > \\ > & (>) \end{pmatrix}_{<}, \begin{pmatrix} > & (>) \\ (>) & > \end{pmatrix}_{<}, V = \frac{1}{80},$$

and $V(\text{II}) = \frac{1}{80}$, $V(\text{III}) = \frac{1}{144}$. The probability is then

$$P_Ł = \frac{1}{18} + 4 \cdot \frac{1}{80} + 4 \cdot \frac{1}{80} + 4 \cdot \frac{1}{144} = \frac{11}{60}.$$

3.3 max $-T_\Pi$ Algebra

This algebra has very similar properties as the previous one, just replace subtraction by division and addition by multiplication, e.g. $a_{11} + b_2 - b_1$ now reads $a_{11}\frac{b_2}{b_1}$. Then the volumes of solvable parts are

$$V\left(\begin{pmatrix} (>) & > \\ > & (>) \end{pmatrix}_{>}\right) =$$

$$= \int_0^1 db_1 \int_0^{b_1} db_2 \int_{b_1}^1 da_{11} \int_{b_2}^{a_{11}\frac{b_2}{b_1}} da_{21} \int_{b_1}^1 da_{12} \int_{a_{12}\frac{b_2}{b_1}}^1 da_{22} =$$

$$= \frac{1}{144},$$

$V(\mathrm{II}) = \dfrac{1}{180}$ and $V(\mathrm{III}) = \dfrac{1}{120}$. The probability in this case is

$$P_{\Pi} = \frac{1}{18} + 4 \cdot \frac{1}{144} + 4 \cdot \frac{1}{180} + 4 \cdot \frac{1}{120} = \frac{5}{36}.$$

3.4 max $-T_D$ Algebra

In this algebra the probability is zero, because in order to obtain a value b_i from a multiplication $A \otimes x$, there must be either a number 1 or b_i in the matrix A. However, the probability that this happens is zero.

4 Conclusions

In our contribution, we tried to make a first step in understanding what systems of linear equations (SLE) in classical vector spaces and fuzzy relation equations in fuzzy algebras have in common. We have shown computing with 2×2 matrices has completely different properties and from the point of view of solvability has nothing in common with SLE in vector spaces, since the probability of solvability of random pair (A, b) is very low in fuzzy algebras whilst in vector spaces it is certainly solvable. If we compare ordered t-norms with ordered obtained probabilities

$$\mathrm{T}_D \leq \mathrm{T}_{Ł} \leq \mathrm{T}_{\Pi} \leq \mathrm{T}_G,$$
$$P_D < P_G < P_{\Pi} < P_{Ł},$$
$$\frac{0}{180} < \frac{18}{180} < \frac{25}{180} < \frac{33}{180},$$

we can conjecture, that for two comparable continuous t-norms the less one (recall that $T_1 \leq T_2$ if for all $a, b \in [0; 1]$ holds that $a\ T_1\ b \leq a\ T_2\ b$) will have greater probability of solvability.

In our next research, we will aim to answer the question for a general dimension.

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