Steady Mixed Convection in a Heated Lid-Driven Square Cavity Filled with a Fluid-Saturated Porous Medium

Bengisen Pekmen and Munevver Tezer-Sezgin

Abstract Steady mixed convection flow in a porous square cavity with moving side walls is studied numerically using the dual reciprocity boundary element method (DRBEM). The equations governing the two-dimensional, steady, laminar mixed convection flow of an incompressible fluid are solved for various values of parameters as Darcy (Da), Grashof (Gr), and Prandtl (Pr) numbers. The results are given in terms of vorticity contours, streamlines and isotherms. Further, average Nusselt number variations with respect to the problem parameters are also presented. The fluid flows slowly as Da decreases since the permeability of the medium decreases, and the increase in Grashof number causes the flow to pass to the natural convective behavior. DRBEM has the advantage of using considerably small number of grid points due to the boundary only nature of the method. This provides the numerical procedure computationally cheap and efficient.

1 Introduction

In many fundamental heat transfer analyses, convective flows in porous media have received much attention and played the central role due to the important applications as in packed sphere beds, insulation for buildings, grain storage, chemical catalytic reactors, and geophysical problems. The underground spread of pollutants, solar power collectors, and geothermal energy systems include porous media.

The theoretical and analytical details of heat transfer in porous medium may be found in the books [5, 6]. Also, a lot of numerical studies concerning heat

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B. Pekmen (✉)
Department of Mathematics, Atılım University, 06836, Ankara, Turkey

Institute of Applied Mathematics, Middle East Technical University, 06800, Ankara, Turkey
e-mail: e146303@metu.edu.tr

M. Tezer-Sezgin
Department of Mathematics, Institute of Applied Mathematics, Middle East Technical University, 06800, Ankara, Turkey
e-mail: munt@metu.edu.tr

A. Abdulle et al. (eds.), *Numerical Mathematics and Advanced Applications - ENUMATH 2013*, Lecture Notes in Computational Science and Engineering 103, DOI 10.1007/978-3-319-10705-9_68

transfer in a porous medium are reported in the last decade. Among these, numerical solutions obtained by DRBEM [9], finite element method (FEM) [2], penalty FEM with biquadratic elements [10], finite volume method (FVM) [1, 11], and the finite difference method (FDM) [4, 7] may be mentioned.

In this study, steady mixed convection flow in a porous square cavity with differentially heated and moving side walls is studied numerically using the DRBEM. An isotropic, homogeneous porous medium saturated with an incompressible, viscous fluid is considered. The thermal and physical properties of the fluid are assumed to be constant, but the fluid density varies according to Boussinesq approximation. The fluid and the solid particles are also assumed to be in local thermal equilibrium. Viscous dissipation, and Forchheimer terms (quadratic drag terms) in the momentum equations are neglected.

The two-dimensional, steady, laminar mixed convection flow of an incompressible fluid is taken into account. The non-dimensional governing equations in terms of stream function ψ-temperature T-vorticity w are [2]

$$\nabla^2\psi = -w \tag{1a}$$

$$\frac{1}{\epsilon_p Re}\nabla^2 w = \frac{1}{\epsilon_p^2}\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}\right) - \frac{Gr}{Re^2}\frac{\partial T}{\partial x} + \frac{1}{Da\,Re}w \tag{1b}$$

$$\frac{1}{Pr\,Re}\nabla^2 T = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \tag{1c}$$

where ϵ_p is the porosity of the porous medium, $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$, $w = \partial v/\partial x - \partial u/\partial y$. Non-dimensional physical parameters are Reynolds, Grashof, Darcy and Prandtl numbers, respectively, given as

$$Re = \frac{U_0 L}{\nu_e}, \quad Gr = \frac{g\beta\Delta T L^3}{\nu_e^2}, \quad Da = \frac{\kappa}{L^2}, \quad Pr = \frac{\nu_e}{\alpha_e}, \tag{2}$$

with characteristic velocity U_0, characteristic length L, gravitational acceleration g, effective kinematic viscosity ν_e, permeability of the porous medium κ, thermal expansion coefficient β, temperature difference $\Delta T = T_h - T_c$, effective thermal diffusivity α_e of the porous medium.

We consider the problem geometry consisting of the cross-section of a unit square cavity which has the moving lids on the left and right walls (Fig. 1). The boundary conditions are as follows. The velocity $v = 1$ on the vertical walls with $u = \psi = 0$; and $u = v = \psi = 0$ on the horizontal walls. The right wall is the hot ($T_h = 1$), and the left wall is the cold ($T_c = 0$) wall while the top and bottom walls are adiabatic ($\partial T/\partial n = 0$). Vorticity boundary conditions are unknown, and are going to be derived with the help of DRBEM coordinate matrix during the iterative solution procedure.

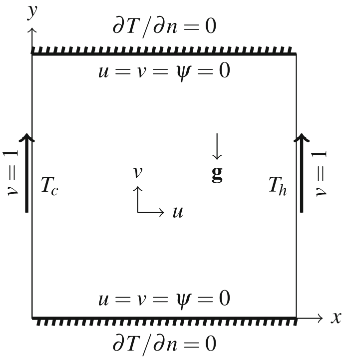

Fig. 1 Problem configuration

2 DRBEM Application

DRBEM treats all the right hand side terms of Eqs. 1 as inhomogeneity, and an approximation for this inhomogeneous term is proposed [8] as

$$b \approx \sum_{j=1}^{N+L} \alpha_j f_j = \sum_{j=1}^{N+L} \alpha_j \nabla^2 \hat{u}_j \tag{3}$$

where N is the number of boundary nodes, L is the number of internal collocation points, α_j's are sets of initially unknown coefficients, and the f_j's are approximating functions which are related to particular solutions $\hat{u}_j$ with $\nabla^2 \hat{u}_j = f_j$. The radial basis functions f_j's are usually chosen as polynomials of radial distance $r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ as $f_{ij} = 1 + r_{ij} + r_{ij}^2 + \ldots + r_{ij}^n$ where i and j correspond to the source(fixed) (x_i, y_i) and the field(variable) (x_j, y_j) points, respectively.

DRBEM transforms differential equations defined in a domain Ω to integral equations on the boundary Γ. For this, differential equation is multiplied by the fundamental solution $u^* = -ln(r)/(2\pi)$ of Laplace equation and integrated over the domain. In Eqs. 1, the right hand sides are approximated using Eq. 3 giving Laplacian terms on both sides. Using Divergence theorem for the Laplacian terms on both sides of the equation, domain integrals are transformed to boundary integrals as follows

$$c_i u_i + \int_\Gamma u q^* d\Gamma - \int_\Gamma q u^* d\Gamma = \sum_{j=1}^{N+L} \alpha_j \left(c_i \hat{u}_{ij} + \int_\Gamma q^* \hat{u}_{ij} d\Gamma - \int_\Gamma u^* \hat{q}_{ij} d\Gamma \right), \tag{4}$$

where $c_i = 0.5$ if the boundary Γ is a straight line and $i \in \Gamma$, and $c_i = 1$ when node i is inside, $\hat{q}_{ij} = \partial \hat{u}_{ij}/\partial n$ with the outward unit normal $\mathbf{n}$ to Γ.

Discretizing the boundary Γ by using N linear elements, evaluating integrals over each element, and then performing assembly procedure for all elements result in a system of equations for each of the Eqs. 1 as

$$Hu - Gu_q = \left(H\hat{U} - G\hat{Q} \right) F^{-1} b, \tag{5}$$

where H and G are BEM matrices contain integral values of fundamental solution u^* and its normal derivative over the boundary elements, respectively. F is the coordinate matrix formed from the radial basis functions f_j's. $\hat{U}$ and $\hat{Q}$ matrices are of size $(N + L) \times (N + L)$, and are built from particular solution $\hat{u}$ and its normal derivative $\hat{q} = \partial\hat{u}/\partial n$ at the $(N + L)$ source and field points. The vector b is formed from the right hand sides of Eqs. 1.

Matrix-vector form for Eqs. 1 are written as

$$H\psi^{m+1} - G\psi_q^{m+1} = -Sw^m \tag{6a}$$

$$(H - PrReSM)\,T^{m+1} - GT_q^{m+1} = 0 \tag{6b}$$

$$\left(H - \frac{Re}{\epsilon_p} SM - \frac{\epsilon_p}{Da} S \right) w^{m+1} - Gw_q^{m+1} = -\epsilon_p \frac{Gr}{Re} S \frac{\partial F}{\partial x} F^{-1} T^{m+1} \tag{6c}$$

where $S = (H\hat{U} - G\hat{Q})F^{-1}$, $u^{m+1} = (\partial F/\partial y)F^{-1}\psi^{m+1}, v^{m+1} = -(\partial F/\partial x)F^{-1}\psi^{m+1}$,
$M = \left([u]_d^{m+1} \frac{\partial F}{\partial x} F^{-1} + [v]_d^{m+1} \frac{\partial F}{\partial y} F^{-1} \right)$, the subscript d shows the diagonal matrix, and m is the iteration level.

Unknown vorticity boundary conditions are obtained from the definition of w as

$$w = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial F}{\partial x} F^{-1} v - \frac{\partial F}{\partial y} F^{-1} u, \tag{7}$$

with the help of coordinate matrix F. Also, all the space derivatives in b are computed by using DRBEM coordinate matrix F, i.e.

$$\frac{\partial T}{\partial x} = \frac{\partial F}{\partial x} F^{-1} T, \qquad \frac{\partial w}{\partial y} = \frac{\partial F}{\partial y} F^{-1} w. \tag{8}$$

Systems of Eqs. 6a–6c are solved iteratively for the unknowns ψ, T, w, and normal derivatives ψ_q, T_q, w_q. Initially, ψ, T and w are taken as zero except on the boundary. First, Eq. 6a is solved for stream function. Then, stream function is used to compute velocity components u and v inserting their boundary conditions.

The energy and vorticity transport equations are then solved by using u and v, respectively. The iterations continue until the criterion [4]

$$\frac{\|\psi^{m+1}-\psi^{m}\|_{\infty}}{\|\psi^{m+1}\|_{\infty}}+\frac{\|T^{m+1}-T^{m}\|_{\infty}}{\|T^{m+1}\|_{\infty}}+\frac{\|w^{m+1}-w^{m}\|_{\infty}}{\|w^{m+1}\|_{\infty}}<\epsilon \tag{9}$$

is satisfied where $\epsilon = 10^{-5}$ is the tolerance to stop the iterations.

In order to accelerate the convergence for large values of problem parameters a relaxation parameter $0 < \gamma \leq 1$ is used for the vorticity as $w^{m+1} \leftarrow \gamma w^{m+1} + (1-\gamma)w^{m}$. Further, average Nusselt number through the heated wall is computed by $\overline{Nu} = \int_0^1 (\partial T/\partial x)dy$.

3 Numerical Results

As a validation case, a non-porous unit square cavity with heated bottom, cold top wall, adiabatic left and right walls and moving top lid is considered. As is seen in Table 1, present results using considerably small number of grid points are in good agreement with the results in [10] where 57×57 grid points are used.

In the numerical computations of stream function, vorticity and temperature in a square cavity with heated and upwards moving vertical walls, radial basis function $f = 1 + r$, and 8-point Gaussian quadrature are used for the construction of F, H and G BEM matrices. $N = 96$, $L = 625$ are taken, and $Re = 100$ is fixed. Cavity contains a fluid saturated porous medium with $\epsilon_p \leq 1$. Mixed convection flow behavior in this porous medium is depicted in terms of streamlines, isotherms, and vorticity contours for various values of Da, Gr and Pr.

As Da decreases (Fig. 2), permeability decreases and causes a force opposite to the flow direction which tends to resist the flow. This means that the fluid flows slowly. While the center of streamlines is in the direction of moving lids, they cluster along the left and right boundaries forming boundary layers, and the effects of moving walls almost disappear. Isotherms become almost perpendicular to the top and bottom walls pointing to the increase in conduction dominated effect. Circulation in the vorticity through the upper corners due to the effect of moving

Table 1 $Re = 500$, $\gamma = 0.1$, $\overline{Nu}$ comparison with various Pr numbers

		[10]	Present		
Pr	Gr	$\overline{Nu}$	$\overline{Nu}$	N,L	CPU(sec.)
0.01	10^4	1.0431	1.0372	136,529	139.9
0.01	10^5	1.0721	1.0733	136,529	129.3
0.1	10^4	2.3815	2.3711	96,529	110.7
0.1	10^5	2.8704	2.8731	96,576	143.9
1	10^4	5.5695	5.5661	96,729	256.2
1	10^5	6.3313	6.3242	96,900	591.7

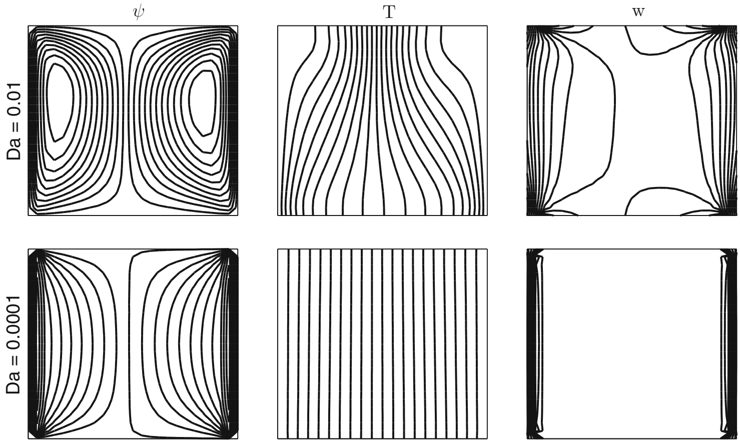

Fig. 2 $Pr = 0.71,\ Gr = 10^3,\ \epsilon_p = 1$

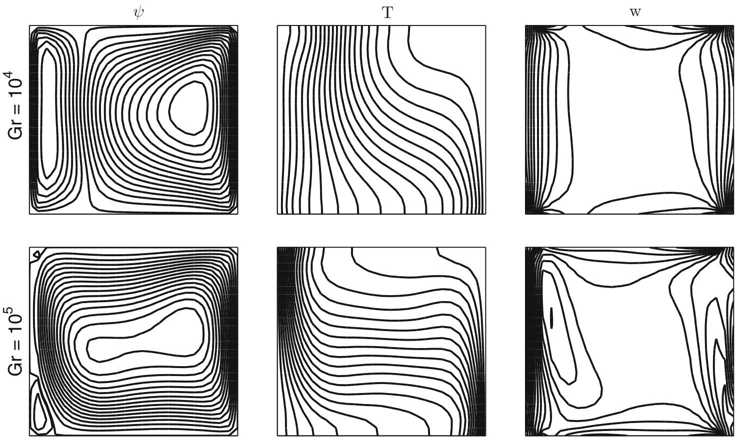

Fig. 3 $Pr = 0.71,\ Da = 0.01,\ \epsilon_p = 1$

lids diminishes, and strong boundary layers are formed through the right and left walls leaving a stagnant region at the center.

As Gr increases, the left counter-clockwise secondary cell starts to be squeezed through the left wall, and the clockwise primary cell is centered. Buoyancy effect is pronounced due to the increase in $Ri = Gr/Re^2$. That is, natural convection is high. Actually, this can be seen in isotherms at $Gr = 10^5$. While the isotherms pronounce the forced convection with $Gr = 10^3,\ Da = 0.01(Ri = 0.1)$ in Fig. 2, they cluster through the left and right walls forming strong temperature gradients for $Gr = 10^5$ (Fig. 3). Even though there is a Darcy effect with strength $Da = 0.01$, one is able to observe the characteristics of mixed convection flow in a non-porous medium in

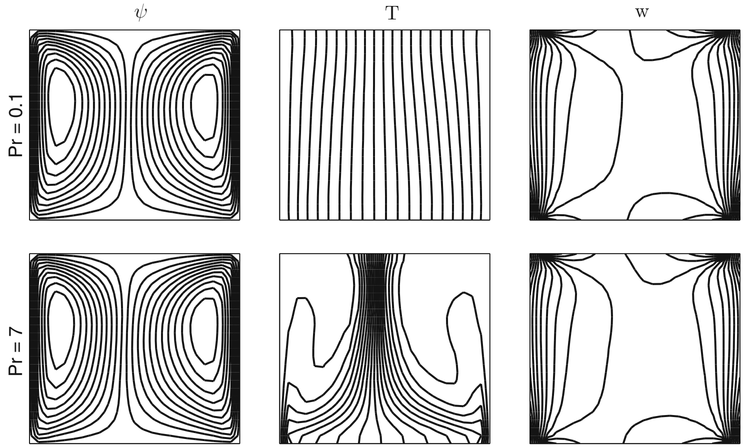

Fig. 4 $Da = 0.01,\ Gr = 10^3,\ \epsilon_p = 1$

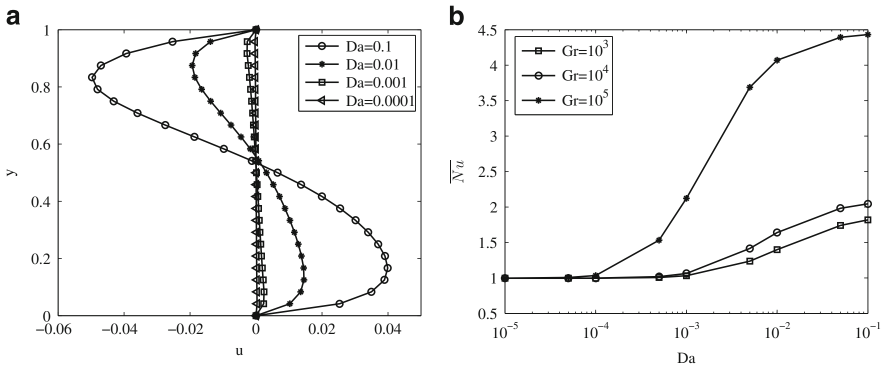

Fig. 5 Mid-u-velocity profile and average Nusselt Number on the heated wall. (**a**) $Gr = 10^3,\ Pr = 0.71,\ \epsilon_p = 1$. (**b**) $Pr = 0.71,\ \epsilon_p = 1$

the cavity [3]. Vorticity almost covers the cavity with new cells through the left and right walls, and spreads also along the top and bottom walls.

The increase in Pr only affects the isotherms (as is seen in Fig. 4) due to the dominance of convection terms in the temperature equation.

The decrease in the velocity of the fluid with the decrease in Da number is shown in Fig. 5a with the u-velocity profile through $x = 0.5$. The dominance of natural convection with high Gr is depicted in Fig. 5b. When Gr is increased, $\overline{Nu}$ values also increase. Average Nusselt number is almost the same for all values of Grashof number with $Da \leq 10^{-4}$ due to the dominance of conduction. However, $\overline{Nu}$ increases as Da increases showing the increase in the heat transfer.

Finally, we show how the heat transfer is affected by different values of porosity. As is seen in Fig. 6a ($Ri < 1$, forced convection is dominant), $\overline{Nu}$ increases at all ϵ_p

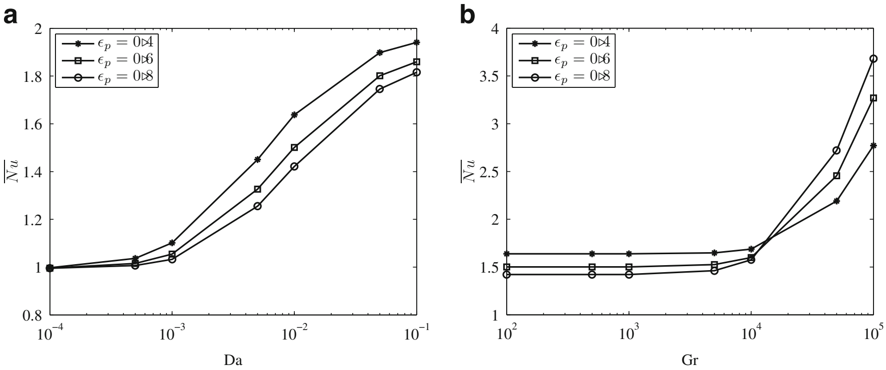

Fig. 6 Average Nusselt number variations with ϵ_p on the heated wall. (**a**) $Gr = 10^3, Pr = 0.71$. (**b**) $Da = 0.01, Pr = 0.71$

values as Da increases. High $\overline{Nu}$ values are obtained by small ϵ_p values which yields the increase in convective heat transfer. As the natural convective effect increases $Ri > 1$ (Fig. 6b), it is found that $\overline{Nu}$ takes larger values with $\epsilon_p = 0.8$ than the other ones. Namely, natural convection is pronounced with the increase in ϵ_p.

Conclusion

The two-dimensional, steady mixed convection flow in a square cavity with porous medium is numerically solved by dual reciprocity boundary element method. The space derivatives in inhomogeneous terms as well as unknown vorticity boundary conditions are easily computed by the coordinate matrix. For this Brinkmann-extended Darcy model, the decrease in Darcy number causes the fluid to flow slowly, and the heat to transfer in conductive mode. Natural convection is pronounced with the increase in Grashof number. In natural convection mode ($Ri > 1$), convective heat transfer increases in a high porosity of the medium.

References

1. M. Bhuvaneswari, S. Sivasankaran, Y.J. Kim, Effect of aspect ratio on convection in porous enclosure with partially active thermal walls. Comput. Math. Appl. **62**, 3844–3856 (2011)
2. S. Das, R.K. Sahoo, Effect of Darcy, fluid Rayleigh and heat generation parameters on natural convection in a porous square enclosure: a Brinkman-extended Darcy model. Int. Commun. Heat Mass Transf. **26**, 569–578 (1999)
3. R. Iwatsu, J.M. Hyun, K. Kuwahara, Mixed convection in a driven cavity with a stable vertical temperature gradient. Int. J. Heat Mass Transf. **36**, 1601–1608 (1993)

4. K.M. Khanafer, A.J. Chamkha, Mixed convection flow in a lid-driven enclosure filled with a fluid-saturated porous medium. Int. J. Heat Mass Transf. **42**, 2465–2481 (1999)
5. A. Narasimhan, *Essentials of Heat and Fluid Flow in Porous Media* (CRC/Taylor & Francis Group, Boca Raton, 2013)
6. D.A. Nield, A. Bejan, *Convection in Porous Media* (Springer, New York, 2006)
7. P. Nithiarasu, K.N. Seetharamu, T. Sundararajan, Natural convective heat transfer in a fluid saturated variable porosity medium. Int. J. Heat Mass Transf. **40**, 3955–3967 (1997)
8. P.W. Partridge, C.A. Brebbia, L.C. Wrobel, *The Dual Reciprocity Boundary Element Method* (Computational Mechanics Publications/Elsevier Science, Southampton, Boston/London, New York, 1992)
9. B. Pekmen, M. Tezer-Sezgin, DRBEM solution of free convection in porous enclosures under the effect of a magnetic field. Int. J. Heat Mass Transf. **56**, 454–468 (2013)
10. D. Ramakrishna, T. Basak, S. Roy, I. Pop, Numerical study of mixed convection within porous square cavities using Bejan's heatlines: effects of thermal aspect ratio and thermal boundary conditions. Int. J. Heat Mass Transf. **55**, 5416–5448 (2012)
11. E. Vishnuvardhanarao, M.K. Das, Laminar mixed convection in a parallel two-sided lid-driven differentially heated square cavity filled with a fluid-saturated porous medium. Numer. Heat Transf. A **53**, 88–110 (2007)