On the Local Mesh Size of Nitsche's Method for Discontinuous Material Parameters

Mika Juntunen

Abstract We propose Nitsche's method for discontinuous parameters that takes the local mesh sizes of the non-matching meshes carefully into account. The method automatically adapts to the changing material parameters and mesh sizes. With continuous parameters, the method compares to the classical Nitsche's method. With large discontinuity, the method approaches assigning Dirichlet boundary conditions with Nitsche's method.

1 Introduction

Suppose the computational domain is divided along the material edges yielding material parameters that are discontinuous over the subdomain interfaces. If the discontinuity in the material parameters is moderate, the Nitsche's method in [3] applies. Some of the problems with large parameter discontinuities are avoided using the harmonic average of the material parameters to create a weighted average flux over the interface [1, 2, 5, 7–9, 11, 13, 14].

In this article we propose Nitsche's method that takes both the material parameters and the mesh sizes carefully into account in the bilinear form. Both the average flux and the stabilizing term are modified to depend on the material parameters and the mesh sizes, similar to [2]. As a result, the proposed method automatically adapts to the material parameters and mesh sizes. If there is no discontinuity over the interface and the mesh sizes are of the same order, we have the classical Nitsche's method. If the mesh sizes or material parameters have very large contrast over the interface, the method reduces to assigning Dirichlet boundary conditions with Nitsche's method.

M. Juntunen (✉)
Department of Mathematics and Systems Analysis, Aalto University School of Science, P.O. Box 11100, 00076 Aalto, Finland
e-mail: mika.juntunen@aalto.fi

A. Abdulle et al. (eds.), *Numerical Mathematics and Advanced Applications - ENUMATH 2013*, Lecture Notes in Computational Science and Engineering 103, DOI 10.1007/978-3-319-10705-9_5

2 Model Problem

Consider a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with a piecewise smooth boundary $\partial\Omega$. Assume that the domain is divided into two non-overlapping subdomains Ω_1 and Ω_2. The subdomains cover the whole domain $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ and they share an interface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. We solve the Poisson problem such that

$$\begin{aligned} -\nabla \cdot k_i \nabla u_i &= f \quad \text{in } \Omega_i, \quad i = 1, 2, && (1) \\ u_i &= 0 \quad \text{on } \partial\Omega, \\ u_1 - u_2 &= 0 \quad \text{on } \Gamma, && (2) \\ k_1 \frac{\partial u_1}{\partial n_1} + k_2 \frac{\partial u_2}{\partial n_2} &= 0 \quad \text{on } \Gamma, && (3) \end{aligned}$$

in which $k_i \in \mathbb{R}$, $0 < k_{\min} < k_i < k_{\max}$, $i = 1, 2$, are the material parameters and $f \in L^2(\Omega)$ is the load function. We denote with $k_i \frac{\partial u_i}{\partial n_i} = k_i \nabla u_i \cdot \mathbf{n}_i$ the normal flux and with $\mathbf{n}_1$ and $\mathbf{n}_2$ the outward normals of the subdomains. We also use $\mathbf{n} = \mathbf{n}_1 = -\mathbf{n}_2$ and

$$\frac{\partial u_1}{\partial n} = \frac{\partial u_1}{\partial n_1} \quad \text{and} \quad \frac{\partial u_2}{\partial n} = \frac{\partial u_2}{\partial n_1} = -\frac{\partial u_2}{\partial n_2}.$$

Let the subdomains be divided into sets of non-overlapping elements denoted by $\mathscr{T}_1^h$ and $\mathscr{T}_2^h$, in which h denotes the maximum diameter of elements. Let $\mathscr{E}_1^h$ and $\mathscr{E}_2^h$ denote the edges or faces of the meshes $\mathscr{T}_1^h$ and $\mathscr{T}_2^h$, respectively. Let h_K denote the diameter of an element $K \in \mathscr{T}_i^h$ and h_E the diameter of $E \in \mathscr{E}_i^h$.

Suppose the solutions u_i belong to V_i such that

$$V_i = \left\{ v \in H^1(\Omega_i) \, : \, \frac{\partial v_i}{\partial n_i}|_\Gamma \in L^2(\Gamma), \, v|_{\partial\Omega} = 0 \right\}, \quad i = 1, 2.$$

Let the finite element spaces be

$$V_i^h = \{ v \in H^1(\Omega_i) \, : \, v|_{\partial\Omega} = 0, \;\; v|_K \in \mathscr{P}^p(K) \; \forall K \in \mathscr{T}_i^h \}, \quad i = 1, 2,$$

in which $\mathscr{P}^p$ denotes the polynomials of degree $\leq p$. We assume $p \geq 1$. We use the notation $V = V_1 \times V_2$ and $V^h = V_1^h \times V_2^h$. Respectively, we use $v = (v_1, v_2) \in V$ and $v^h = (v_1^h, v_2^h) \in V^h$ to denote the pair of functions defined in the subdomains.

3 The Proposed Method

In this section we propose Nitsche's method that takes the possible discontinuity of the material parameters and mesh sizes into account. The method is similar to [2].

For simplicity of notation we define the functions $h_i : \bar{\Omega}_i \to \mathbb{R}$, $i = 1, 2$ such that

$$h_i(x) = \begin{cases} h_K & \text{if } x \in K,\ K \in \mathscr{T}_i^h, \\ h_E & \text{if } x \in E,\ E \in \mathscr{E}_i^h. \end{cases}$$

At the interface Γ, for $v \in V$ we use $[\![v]\!] = v_1 - v_2$ to denote the jump over the interface.

Let $(\cdot,\cdot)_G$ denote the L^2-inner product over a domain G. Multiplying Eq. (1) with a test function $v \in V_i$ and integrating by parts gives

$$(k_i \nabla u_i, v_i)_{\Omega_i} - \left(k_i \frac{\partial u_i}{\partial n_i}, v_i\right)_{\Gamma} = (f, v_i)_{\Omega_i} .$$

By Eq. (3) it holds that

$$\left(\frac{k_1 h_2}{k_2 h_1 + k_1 h_2}\left(-k_1 \frac{\partial u_1}{\partial n} + k_2 \frac{\partial u_2}{\partial n}\right), v_1\right)_{\Gamma} = 0,$$
$$\left(\frac{k_2 h_1}{k_2 h_1 + k_1 h_2}\left(k_1 \frac{\partial u_1}{\partial n} - k_2 \frac{\partial u_2}{\partial n}\right), -v_2\right)_{\Gamma} = 0,$$

and by Eq. (2) it holds that

$$\left(\frac{k_1 k_2}{k_2 h_1 + k_1 h_2}[\![u]\!], [\![v]\!]\right)_{\Gamma} = 0.$$

Adding the equations above and introducing a stability parameter $\gamma > 0$ gives the weak form of the proposed Nitsche's method: Find $u^h \in V^h$ such that

$$\mathscr{B}^h(u^h, v^h) = \mathscr{F}(v^h) \quad \forall v^h \in V^h.$$

The bilinear form is

$$\mathscr{B}^h(w, v) = \sum_{i=1}^{2}(k_i \nabla w_i, \nabla v_i)_{\Omega_i} - \left(\left\{\!\!\left\{ k \frac{\partial w}{\partial n} \right\}\!\!\right\}, [\![v]\!]\right)_{\Gamma} - \left(\left\{\!\!\left\{ k \frac{\partial v}{\partial n} \right\}\!\!\right\}, [\![w]\!]\right)_{\Gamma}$$
$$+ \gamma \left(\frac{k_1 k_2}{k_2 h_1 + k_1 h_2}[\![w]\!], [\![v]\!]\right)_{\Gamma} ,$$

in which

$$\left\{\!\!\left\{ k\frac{\partial v}{\partial n} \right\}\!\!\right\} = \alpha_1\, k_1 \frac{\partial v_1}{\partial n} + \alpha_2\, k_2 \frac{\partial v_2}{\partial n},$$

$$\alpha_1 = \frac{k_2 h_1}{k_2 h_1 + k_1 h_2}, \qquad \alpha_2 = \frac{k_1 h_2}{k_2 h_1 + k_1 h_2},$$

denotes the weighted average flux. The linear functional is simply

$$\mathscr{F}(v) = \sum_{i=1}^{2} (f, v_i)_{\Omega_i} \,.$$

By the derivation above it is clear that the proposed method is consistent with the strong form.

3.1 A Priori Analysis

Following [3, 10, 12, 13] we use the mesh dependent norms in the analysis. Let $\|\cdot\|_G$ denote the L^2 norm over a domain G. The parameters k_1 and k_2 are explicitly shown in the norms on V:

$$\|v\|_{1,h}^2 = \sum_{i=1}^{2} \left\| k_i^{\frac{1}{2}} \nabla v_i \right\|_{\Omega_i}^2 + \left\| \left(\frac{k_1 k_2}{k_2 h_1 + k_1 h_2} \right)^{\frac{1}{2}} [\![v]\!] \right\|_{\Gamma}^2,$$

$$|\!|\!| v |\!|\!|_{1,h}^2 = \|v\|_{1,h}^2 + \sum_{i=1}^{2} \left\| (h_i k_i)^{\frac{1}{2}} \frac{\partial v_i}{\partial n} \right\|_{\Gamma}^2.$$

Clearly $\|v\|_{1,h} \le |\!|\!| v |\!|\!|_{1,h}$ for all $v \in V$. The converse, $|\!|\!| v^h |\!|\!|_{1,h} \le C \left\| v^h \right\|_{1,h}$ with a $C > 0$, holds for any $v^h \in V^h$. This follows using the trace inequality [4, 6]

$$\left\| (h_i k_i)^{\frac{1}{2}} \frac{\partial v_i^h}{\partial n} \right\|_{\partial K}^2 \le C_I \left\| k_i^{\frac{1}{2}} \nabla v_i^h \right\|_K^2 \qquad \forall K \in \mathscr{T}_i^h,\ v_i^h \in V_i^h, \tag{4}$$

for a $C_I > 0$. Consequently, the norms are equivalent in V^h independent of the mesh sizes h_i and the parameters k_i: There exists $c, C > 0$ such that

$$c \left\| v^h \right\|_{1,h} \le |\!|\!| v^h |\!|\!|_{1,h} \le C \left\| v^h \right\|_{1,h} \quad \forall v^h \in V^h.$$

Theorem 1 *The bilinear form $\mathscr{B}^h$ is continuous in V with the norm $|||\cdot|||_{1,h}$ and, assuming the stability parameter satisfies $\gamma > 2C_I$, the bilinear form $\mathscr{B}^h$ is coercive in V^h with the norm $|||\cdot|||_{1,h}$.*

Proof Recall the definition of the weighted average flux and observe that

$$\begin{aligned}
&\left(\left\{\!\!\left\{ k\frac{\partial w}{\partial n}\right\}\!\!\right\}, [\![v]\!]\right)_\Gamma = \left(\alpha_1 k_1 \frac{\partial w_1}{\partial n} + \alpha_2 k_2 \frac{\partial w_2}{\partial n}, [\![v]\!]\right)_\Gamma \\
&= \left((\alpha_1 k_1 h_1)^{\frac{1}{2}} \frac{\partial w_1}{\partial n} + (\alpha_2 k_2 h_2)^{\frac{1}{2}} \frac{\partial w_2}{\partial n}, \left(\frac{k_1 k_2}{k_2 h_1 + k_1 h_2}\right)^{\frac{1}{2}} [\![v]\!] \right)_\Gamma \\
&\le \left(\left\| (\alpha_1 k_1 h_1)^{\frac{1}{2}} \frac{\partial w_1}{\partial n} \right\|_\Gamma + \left\| (\alpha_2 k_2 h_2)^{\frac{1}{2}} \frac{\partial w_2}{\partial n} \right\|_\Gamma \right) \left\| \left(\frac{k_1 k_2}{k_2 h_1 + k_1 h_2}\right)^{\frac{1}{2}} [\![v]\!] \right\|_\Gamma \\
&\le \left(\left\| (k_1 h_1)^{\frac{1}{2}} \frac{\partial w_1}{\partial n} \right\|_\Gamma + \left\| (k_2 h_2)^{\frac{1}{2}} \frac{\partial w_2}{\partial n} \right\|_\Gamma \right) \left\| \left(\frac{k_1 k_2}{k_2 h_1 + k_1 h_2}\right)^{\frac{1}{2}} [\![v]\!] \right\|_\Gamma \qquad (5)
\end{aligned}$$

for all $w, v \in V$. Using (5) it is easy to see that the bilinear form $\mathscr{B}^h$ is continuous.

Applying the trace inequality (4) to (5) and using Young's inequality with a parameter $\epsilon > 0$ gives

$$\begin{aligned}
&\left(\left\{\!\!\left\{ k\frac{\partial w^h}{\partial n}\right\}\!\!\right\}, [\![v^h]\!]\right)_\Gamma \\
&\le \left(C_I^{\frac{1}{2}} \left\| k_1^{\frac{1}{2}} \nabla w_1^h \right\|_{\Omega_1} + C_I^{\frac{1}{2}} \left\| k_2^{\frac{1}{2}} \nabla w_2^h \right\|_{\Omega_2} \right) \left\| \left(\frac{k_1 k_2}{k_2 h_1 + k_1 h_2}\right)^{\frac{1}{2}} [\![v^h]\!] \right\|_\Gamma \\
&\le \frac{C_I}{2\epsilon} \left\| k_1^{\frac{1}{2}} \nabla w_1^h \right\|_{\Omega_1}^2 + \frac{C_I}{2\epsilon} \left\| k_2^{\frac{1}{2}} \nabla w_2^h \right\|_{\Omega_2}^2 + \epsilon \left\| \left(\frac{k_1 k_2}{k_2 h_1 + k_1 h_2}\right)^{\frac{1}{2}} [\![v^h]\!] \right\|_\Gamma^2
\end{aligned}$$

for $v^h, w^h \in V^h$. With this we get that

$$\mathscr{B}^h(v^h, v^h) \ge \left(1 - \frac{C_I}{\epsilon}\right) \sum_{i=1}^{2} \left\| k_i^{\frac{1}{2}} \nabla v_i^h \right\|_{\Omega_i}^2 + (\gamma - 2\epsilon) \left\| \left(\frac{k_1 k_2}{k_2 h_1 + k_1 h_2}\right)^{\frac{1}{2}} [\![v^h]\!] \right\|_\Gamma^2 .$$

By the assumption $\gamma > 2C_I$ we can choose $C_I < \epsilon < \gamma/2$, which shows that the bilinear form $\mathscr{B}^h$ is coercive. □

Let $u^h \in V^h$ denote the finite element solution and $u \in V$ the exact solution of the problem. The coercivity, consistency and continuity show that

$$\begin{aligned}\left\|u^h - v^h\right\|_{1,h}^2 \le C\mathscr{B}^h(u^h - v^h, u^h - v^h) &= C\mathscr{B}^h(u - v^h, u^h - v^h)\\ &\le C\left|\!\left|\!\left|u - v^h\right|\!\right|\!\right|_{1,h}\left\|u^h - v^h\right\|_{1,h}\end{aligned}$$

for any $v^h \in V^h$. Using the triangle inequality and the equivalence of norms we get

$$\left|\!\left|\!\left|u - u^h\right|\!\right|\!\right|_{1,h} \le C \inf_{v^h \in V^h} \left|\!\left|\!\left|u - v^h\right|\!\right|\!\right|_{1,h}.$$

Applying the interpolation results to the estimate above, we get the a priori result

$$\left|\!\left|\!\left|u - u^h\right|\!\right|\!\right|_{1,h} \le Ch^{s-1}\left(\sum_{i=1}^{2}\left\|k_i^{\frac{1}{2}} u_i\right\|_{s,\Omega_i}^2\right)^{\frac{1}{2}}$$

for $u_i \in H^s(\Omega_i)$ with $i = 1, 2$ and $2 \le s \le p + 1$.

4 Observations on the Method

The method adapts automatically and continuously with respect to the parameters. The relation between k_2h_1 and k_1h_2, or equivalently between k_1/h_1 and k_2/h_2, determines the behavior of the method.

Suppose that $k_2h_1 = k_1h_2$ at the interface Γ. Denoting $k/h = k_1/h_2 = k_2/h_2$, the bilinear form is

$$\begin{aligned}\mathscr{B}^h(w, v) = \sum_{i=1}^{2}(k_i\nabla w_i, \nabla v_i)_{\Omega_i} &- \left(\frac{1}{2}\left(k_1\frac{\partial w_1}{\partial n} + k_2\frac{\partial w_2}{\partial n}\right), [\![v]\!]\right)_\Gamma\\ &- \left(\frac{1}{2}\left(k_1\frac{\partial v_1}{\partial n} + k_2\frac{\partial v_2}{\partial n}\right), [\![w]\!]\right)_\Gamma + \gamma\left(\frac{k}{2h}[\![w]\!], [\![v]\!]\right)_\Gamma.\end{aligned}$$

In other words, the proposed method reduces to the method designed for continuous material parameters [3]. This indicates that the method in [3] should work for discontinuous material parameters too as long as the mesh sizes such that $k_2h_1 \approx k_1h_2$.

Suppose now that $k_2h_1 \gg k_1h_2$ at the interface Γ due to k_2 being very large. At the limit $k_2 \to \infty$, the coefficients of the method become

$$\lim_{k_2\to\infty} \alpha_1 = 1, \qquad \lim_{k_2\to\infty} \frac{k_1k_2}{k_2h_1 + k_1h_2} = \frac{k_1}{h_1},$$

$$\lim_{k_2\to\infty} \alpha_2 = 0,$$

and the bilinear form is

$$\mathscr{B}^h(w,v) = \sum_{i=1}^{2}(k_i \nabla w_i, \nabla v_i)_{\Omega_i} - \left(k_1 \frac{\partial w_1}{\partial n}, [\![v]\!]\right)_\Gamma - \left(k_1 \frac{\partial v_1}{\partial n}, [\![w]\!]\right)_\Gamma + \gamma \left(\frac{k_1}{h_1}[\![w]\!], [\![v]\!]\right)_\Gamma .$$

The interpretation of the bilinear form above is: In the subdomain Ω_1, the method enforces continuity at the interface Γ using Nitsche's method.

References

1. C. Annavarapu, M. Hautefeuille, J.E. Dolbow, A robust Nitsche's formulation for interface problems. Comput. Methods Appl. Mech. Eng. **225–228**, 44–54 (2012)
2. N. Barrau, R. Becker, E. Dubach, R. Luce, A robust variant of NXFEM for the interface problem. Comptes Rendus Math. **350**(15–16), 789–792 (2012)
3. R. Becker, P. Hansbo, R. Stenberg, A finite element method for domain decomposition with non-matching grids. Math. Model. Numer. Anal. **37**(2), 209–225 (2003)
4. S.C. Brenner, L.R. Scott, *The Mathematical Theory of Finite Element Methods*. Texts in Applied Mathematics, vol. 15, 2nd edn. (Springer, New York, 2002)
5. E. Burman, P. Zunino, A domain decomposition method based on weighted interior penalties for advection-diffusion-reaction problems. SIAM J. Numer. Anal. **44**(4), 1612–1638 (2006)
6. D.A. Di Pietro, A. Ern, *Mathematical Aspects of Discontinuous Galerkin Methods*. Mathématiques et Applications, vol. 69 (Springer, Berlin/Heidelberg, 2012)
7. M. Dryja, On discontinuous Galerkin methods for elliptic problems with discontinuous coefficients. Comput. Methods Appl. Mech. Eng. **3**(1), 76–85 (2003)
8. A. Ern, A. Stephansen, A posteriori energy-norm error estimates for advection-diffusion equations approximated by weighted interior penalty methods. J. Comput. Math. **26**(4), 488–510 (2008)
9. A. Ern, A.F. Stephansen, P. Zunino, A discontinuous Galerkin method with weighted averages for advection-diffusion equations with locally small and anisotropic diffusivity. IMA J. Numer. Anal. **29**(2), 235–256 (2008)
10. M. Juntunen, R. Stenberg, Nitsche's method for general boundary conditions. Math. Comput. **78**(267), 1353–1374 (2009)
11. T.A. Laursen, M.A. Puso, J. Sanders, Mortar contact formulations for deformable–deformable contact: past contributions and new extensions for enriched and embedded interface formulations. Comput. Methods Appl. Mech. Eng. **205–208**, 3–15 (2012)
12. R. Stenberg, On some techniques for approximating boundary conditions in the finite element method. J. Comput. Appl. Math. **63**(1–3), 139–148 (1995)
13. ———, Mortaring by a method of J.A. Nitsche, in *Computational Mechanics*, Buenos Aires, 1998 (Centro Internac. Métodos Numér. Ing., Barcelona, 1998)
14. P. Zunino, L. Cattaneo, C.M. Colciago, An unfitted interface penalty method for the numerical approximation of contrast problems. Appl. Numer. Math. **61**(10), 1059–1076 (2011)