Homogenization of the One-Dimensional Wave Equation

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Abstract We present a method for two-scale model derivation of the periodic homogenization of the one-dimensional wave equation in a bounded domain. It allows for analyzing the oscillations occurring on both microscopic and macroscopic scales. The novelty reported here is on the asymptotic behavior of high frequency waves and especially on the boundary conditions of the homogenized equation. Numerical simulations are reported.

1 Introduction

The paper is devoted to the periodic homogenization of the wave equation in a one-dimensional open bounded domain where the time-independent coefficients are $\varepsilon-$periodic with small period $\varepsilon > 0$. Corrector results for the low frequency waves have been published in [2, 7]. These works were not taking into account fast time oscillations, so the models reflect only a part of the physical solution. In [3], an homogenized model has been developed to cover the time and space oscillations occurring both at low and high frequencies. It is comprised with a second order microscopic equation with quasi-periodic boundary conditions but also with a first order macroscopic equation which boundary condition was missing. Therefore, establishing the boundary conditions of the homogenized model is critical and is the goal of the present work. A generalization of the wave equation posed in $\mathbb{R}^n$ has also been considered in [4] but taking into account only ε-periodic oscillations in the space variables resulting in periodic conditions in the microscopic problem. Periodic homogenization of the wave equation have been derived for other asymptotic regime, for instance for long time in [5, 6, 8, 10].

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To this end, the wave equation is written under the form of a first order formulation and the modulated two-scale transform W_k^ε is applied to the solution U^ε as in [3]. For $n \in \mathbb{N}^*$ and $k \in \mathbb{R}$, the nth eigenvalue λ_n^k of the Bloch wave problem with k-quasi-periodic boundary conditions satisfies $\lambda_n^k = \lambda_n^{-k}$, in addition $\lambda_m^k = \lambda_n^k$ for $k \in \mathbb{Z}/2$, so the corresponding waves are oscillating with the same frequency. The homogenized model is thus derived for pairs of fibers $\{-k, k\}$ if $k \neq 0$ and for fiber $\{0\}$ otherwise which allows to derive the expected boundary conditions. The weak limit of $\sum_{\sigma \in I^k} W_\sigma^\varepsilon U^\varepsilon$ includes low and high frequency waves, the former being solution of the homogenized model derived in [2, 7] and the latter are associated to Bloch wave expansions. Numerical results comparing solutions of the wave equation with solution of the two-scale model for fixed ε and k are reported in the last section.

2 The Physical Problem and Elementary Properties

The physical problem We consider $I = (0, T) \subset \mathbb{R}^+$ a finite time interval and $\Omega = (0, \alpha) \subset \mathbb{R}^+$ a space interval, which boundary is denoted by $\partial\Omega$. Here, as usual $\varepsilon > 0$ denotes a small parameter intended to go to zero. Two functions $(a^\varepsilon, \rho^\varepsilon)$ are assumed to obey a prescribed profile $a^\varepsilon := a\left(\frac{x}{\varepsilon}\right)$ and $\rho^\varepsilon := \rho\left(\frac{x}{\varepsilon}\right)$ where $\rho \in L^\infty(\mathbb{R})$, $a \in W^{1,\infty}(\mathbb{R})$ are both Y−periodic where $Y = (0, 1)$. Moreover, they are required to satisfy the standard uniform positivity and ellipticity conditions, $0 < \rho^0 \leq \rho \leq \rho^1$ and $0 < a^0 \leq a \leq a^1$, for some given strictly positive numbers ρ^0, ρ^1, a^0 and a^1. We consider $u^\varepsilon(t, x)$ solution to the wave equation with the source term $f^\varepsilon \in L^2(I \times \Omega)$, initial conditions $u_0^\varepsilon \in H^1(\Omega)$, $v_0^\varepsilon \in L^2(\Omega)$ and homogeneous Dirichlet boundary conditions,

$$\begin{aligned}
&\rho^\varepsilon \partial_{tt} u^\varepsilon - \partial_x\left(a^\varepsilon \partial_x u^\varepsilon\right) = f^\varepsilon \text{ in } I \times \Omega,\\
&u^\varepsilon(t = 0, .) = u_0^\varepsilon \text{ and } \partial_t u^\varepsilon(t = 0, .) = v_0^\varepsilon \text{ in } \Omega,\\
&u^\varepsilon = 0 \text{ on } I \times \partial\Omega.
\end{aligned} \tag{1}$$

By setting: $U^\varepsilon := (\sqrt{a^\varepsilon}\partial_x u^\varepsilon, \sqrt{\rho^\varepsilon}\partial_t u^\varepsilon)$, $A^\varepsilon = \begin{pmatrix} 0 & \sqrt{a^\varepsilon}\partial_x\left(\frac{1}{\sqrt{\rho^\varepsilon}}.\right) \\ \frac{1}{\sqrt{\rho^\varepsilon}}\partial_x\left(\sqrt{a^\varepsilon}.\right) & 0 \end{pmatrix}$, $U_0^\varepsilon := (\sqrt{a^\varepsilon}\partial_x u_0^\varepsilon, \sqrt{\rho^\varepsilon} v_0^\varepsilon)$ and $F^\varepsilon := (0, f^\varepsilon/\sqrt{\rho^\varepsilon})$, we reformulate the wave equation (1) as an equivalent system: $(\partial_t - A^\varepsilon)U^\varepsilon = F^\varepsilon$ in $I \times \Omega$, $U^\varepsilon(t = 0) = U_0^\varepsilon$ in Ω and $U_2^\varepsilon = 0$ on $I \times \partial\Omega$ where U_2^ε is the second component of U^ε. From now on, this system will be referred to as the physical problem and taken in the distributional sense,

$$\int_{I\times\Omega} F^\varepsilon \cdot \Psi + U^\varepsilon \cdot (\partial_t - A^\varepsilon)\Psi dtdx + \int_\Omega U_0^\varepsilon \cdot \Psi(t = 0)\, dx = 0, \tag{2}$$

for all the admissible test functions $\Psi \in H^1(I \times \Omega)^2$ such that $\Psi(t,.) \in D(A^\varepsilon)$ for a.e. $t \in I$ where the domain $D(A^\varepsilon) := \{(\varphi,\phi) \in L^2(\Omega)^2 | \sqrt{a^\varepsilon}\varphi \in H^1(\Omega),\ \phi/\rho \in H_0^1(\Omega)\}$. As proved in [3], the operator iA^ε with the domain $D(A^\varepsilon)$ is self-adjoint on $L^2(\Omega)^2$. We assume that the data are bounded $\|f^\varepsilon\|_{L^2(I\times\Omega)} + \|\partial_x u_0^\varepsilon\|_{L^2(\Omega)} + \|v_0^\varepsilon\|_{L^2(\Omega)} \le c_0$, then U^ε is uniformly bounded in $L^2(I \times \Omega)$.

Bloch waves We introduce the dual $Y^* = \left(-\frac{1}{2}, \frac{1}{2}\right)$ of Y. For any $k \in Y^*$, we define the space of $k-$quasi-periodic functions $L_k^2 := \{u \in L_{loc}^2(\mathbb{R}) \mid u(x+\ell) = u(x)e^{2i\pi k\ell}$ a.e. in $\mathbb{R}$ for all $\ell \in \mathbb{Z}\}$ and set $H_k^s := L_k^2 \cap H_{loc}^s(\mathbb{R})$ for $s \ge 0$. The periodic functions correspond to $k = 0$. For a given $k \in Y^*$, we denote by $(\lambda_n^k, \phi_n^k)_{n\in\mathbb{N}^*}$ the Bloch wave eigenelements that are solution to $\mathscr{P}(k) : -\partial_y\left(a\partial_y\phi_n^k\right) = \lambda_n^k\rho\phi_n^k$ in Y with $\phi_n^k \in H_k^2(Y)$ and $\|\phi_n^k\|_{L^2(Y)} = 1$. The asymptotic spectral problem $\mathscr{P}(k)$ is also restated as a first order system by setting $A_k := \begin{pmatrix} 0 & \sqrt{a}\partial_y\left(\frac{1}{\sqrt{\rho}}.\right) \\ \frac{1}{\sqrt{\rho}}\partial_y\left(\sqrt{a}.\right) & 0 \end{pmatrix}$, $n_{A_k} = \frac{1}{\sqrt{\rho}}\begin{pmatrix} 0 & \sqrt{a}n_Y \\ \sqrt{a}n_Y & 0 \end{pmatrix}$ and $e_n^k := \frac{1}{\sqrt{2}}\begin{pmatrix} -is_n/\sqrt{\lambda_{|n|}^k}\sqrt{a}\partial_y\left(\phi_{|n|}^k\right) \\ \sqrt{\rho}\phi_{|n|}^k \end{pmatrix}$ where s_n and n_Y denote the sign of $n \in \mathbb{Z}^*$ and the outer unit normal of ∂Y respectively. As proved in [3], iA_k is self-adjoint on the domain $D(A_k) := \{(\varphi,\phi) \in L^2(Y)^2 | \sqrt{a}\varphi \in H_k^1(Y), \phi/\sqrt{\rho} \in H_k^1(Y) \subset L^2(Y)^2\}$. The Bloch wave spectral problem $\mathscr{P}(k)$ is equivalent to finding pairs $\left(\lambda_{|n|}^k, e_n^k\right)$ indexed by $n \in \mathbb{Z}^*$ solution to $\mathscr{Q}(k) : A_k e_n^k = is_n\sqrt{\lambda_{|n|}^k}e_n^k$ in Y with $e_n^k \in H_k^1(Y)^2$. We pose $M_n^k := \{m\in\mathbb{Z}^* | \lambda_{|m|}^k = \lambda_{|n|}^k$ and $s_m = s_n\}$ and introduce the coefficients $b(k,n,m) = \int_Y \rho\phi_{|n|}^k \cdot \phi_{|m|}^k dy$ and $c(k,n,m) = is_n/\left(2\sqrt{\lambda_{|n|}^k}\right)\int_Y \phi_{|n|}^k \cdot a\partial_y\phi_{|m|}^k - a\partial_y\phi_{|n|}^k \cdot \phi_{|m|}^k dy$ for $n,m \in M_n^k$.

The modulated two-scale transform Let us assume from now that the domain Ω is the union of a finite number of entire cells of size ε or equivalently that the sequence ε is exactly $\varepsilon_n = \frac{\alpha}{n}$ for $n \in \mathbb{N}^*$. For any $k \in Y^*$, we define $I^k = \{-k,k\}$ if $k \neq 0$ and $I^0 = \{0\}$. By choosing $\Lambda = (0,1)$ as a time unit cell, we introduce the operator $W_k^\varepsilon : L^2(I \times \Omega)^2 \to L^2(I \times \Lambda \times \Omega \times Y)^2$ acting in all time and space variables, $W_k^\varepsilon := \left(1 - \sum_{n\in\mathbb{Z}^*}\Pi_n^k\right)S_k^\varepsilon + \sum_{n\in\mathbb{Z}^*}T^{\varepsilon\alpha_n^k}\overset{k}{\Pi}_n S_k^\varepsilon$ where the time and space two-scale transforms $T^{\varepsilon\alpha_n^k}$ and S_k^ε, and the orthogonal projector Π_n^k onto e_n^k are defined in [3], see pages 11, 15 and 17, with $\alpha_n^k = 2\pi/\sqrt{\lambda_{|n|}^k}$, and where it is proved that $\|W_k^\varepsilon u\|_{L^2(I\times\Lambda\times\Omega\times Y)}^2 = \|u\|_{L^2(I\times\Omega)}^2$.

We define $(\mathfrak{B}_n^k v)(t,x) = v(t, \frac{t}{\varepsilon\alpha_n^k}, x, \frac{x}{\varepsilon})$ the operator that operates on functions $v(t,\tau,x,y)$ defined in $I \times \mathbb{R}\times\Omega \times \mathbb{R}$. The notation $O(\varepsilon)$ refers to numbers or functions tending to zero when $\varepsilon \to 0$ in a sense made precise in each case. The next lemma shows that $\mathfrak{B}_n^k$ is an approximation of $T^{\varepsilon\alpha_n^k *}S_k^{\varepsilon *}$ for a function which is periodic in τ and $k-$quasi-periodic in y, where $T^{\varepsilon\alpha_n^k *} : L^2(I \times \Lambda) \to L^2(I)$ and $S_k^{\varepsilon *} : L^2(\Omega \times Y) \to L^2(\Omega)$ are adjoint of $T^{\varepsilon\alpha_n^k}$ and S_k^ε respectively.

Lemma 1 *Let* $v \in C^1(I \times \Lambda \times \Omega \times Y)$ *a periodic function in* τ *and* $k-$*quasi-periodic in* y, *then* $T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} v = \mathfrak{B}_n^k v + O(\varepsilon)$ *in the* $L^2(I \times \Omega)$ *sense. Consequently, for any sequence* u^ε *bounded in* $L^2(I \times \Omega)$ *such that* $T^{\varepsilon\alpha_n^k} S_k^\varepsilon u^\varepsilon$ *converges to* u *in* $L^2(I \times \Lambda \times \Omega \times Y)$ *weakly when* $\varepsilon \to 0$,

$$\int_{I\times\Omega} u^\varepsilon \cdot \mathfrak{B}_n^k v \, dtdx \to \int_{I\times\Lambda\times\Omega\times Y} u \cdot v \, dtd\tau dxdy \ \textit{ when } \varepsilon \to 0. \tag{3}$$

Note that for $k = 0$, the convergence (3) regarding each variable corresponds to the definition of two-scale convergence in [1]. The proof is carried out in three steps. First the explicit expression of $T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} v$ is derived, second the approximation of $T^{\varepsilon\alpha_n^k *} S_k^{\varepsilon *} v$ is deduced, finally the convergence (3) follows. For a function $v(t, \tau, x, y)$ defined in $I \times \Lambda \times \Omega \times Y$, we observe that

$$A^\varepsilon \mathfrak{B}_n^k v = \mathfrak{B}_n^k \left(\left(\frac{A_k}{\varepsilon} + B \right) v \right) \text{ and } \partial_t \left(\mathfrak{B}_n^k v \right) = \mathfrak{B}_n^k \left(\left(\frac{\partial_\tau}{\varepsilon \alpha_n^k} + \partial_t \right) v \right), \tag{4}$$

where the operator B is defined as the result of the formal substitution of $x-$derivatives by $y-$derivatives in A_k.

3 Homogenized Results and Their Proof

For $k \in Y^*$, we decompose

$$\frac{\alpha k}{\varepsilon} = h_\varepsilon^k + l_\varepsilon^k \text{ with } h_\varepsilon^k = \left[\frac{\alpha k}{\varepsilon} \right] \text{ and } l_\varepsilon^k \in [0, 1), \tag{5}$$

and assume that the sequence ε is varying in a set $E_k \subset \mathbb{R}^{+*}$ so that

$$l_\varepsilon^k \to l^k \text{ when } \varepsilon \to 0 \text{ and } \varepsilon \in E_k \text{ with } l^k \in [0, 1). \tag{6}$$

After extraction of a subsequence, we introduce the weak limits of the relevant projections along e_n^k for any $n \in \mathbb{Z}^*$,

$$F_n^k := \lim_{\varepsilon \to 0} \int_{\Lambda \times Y} T^{\varepsilon\alpha_n^k} S_k^\varepsilon F^\varepsilon \cdot e^{2i\pi s_n \tau} e_n^k dyd\tau \text{ and } U_{0,n}^k := \lim_{\varepsilon\to 0} \int_Y S_k^\varepsilon U_0^\varepsilon \cdot e_n^k dy. \tag{7}$$

The next lemmas state the microscopic equation for each mode and the corresponding macroscopic equation.

Lemma 2 *For* $k \in Y^*$ *and* $n \in \mathbb{Z}^*$, *let* U^ε *be a bounded solution of (2), there exists at least a subsequence of* $T^{\varepsilon\alpha_n^k} S_k^\varepsilon U^\varepsilon$ *converging weakly towards a limit* U_n^k

in $L^2(I \times \Lambda \times \Omega \times Y)^2$ when ε tends to zero. Then U_n^k is a solution of the weak formulation of the microscopic equation

$$\left(\frac{\partial_\tau}{\alpha_n^k} - A_k\right) U_n^k = 0 \text{ in } I \times \Lambda \times \Omega \times Y \tag{8}$$

and is periodic in τ and $k-$quasi-periodic in y. Moreover, it can be decomposed as

$$U_n^k(t,\tau,x,y) = \sum\nolimits_{p \in \mathbb{M}_n^k} u_p^k(t,x)\, e^{2i\pi s_p \tau} e_p^k(y) \text{ with } u_p^k \in L^2(I \times \Omega). \tag{9}$$

Lemma 3 *In the condition of Lemma 2, for each $k \in Y^*$, $n \in \mathbb{Z}^*$, $\varepsilon \in E_k$, for each $\sigma \in I^k$ and $q \in M_n^\sigma$, the macroscopic equation is stated by*

$$\begin{aligned} &\sum\nolimits_{p \in M_n^\sigma} \left(b(\sigma,p,q)\, \partial_t u_p^\sigma - c(\sigma,p,q)\, \partial_x u_p^\sigma\right) = F_q^\sigma \text{ in } I \times \Omega, \\ &\sum\nolimits_{p \in M_n^\sigma} b(\sigma,p,q)\, u_p^\sigma(t=0) = U_{0,q}^\sigma \text{ in } \Omega, \end{aligned} \tag{10}$$

with the boundary conditions in case where there exists $p \in M_n^k$ such that $c(k,p,q) \neq 0$ and $\phi_{|p|}^k(0) \neq 0$,

$$\sum\nolimits_{\sigma \in I^k, p \in M_n^\sigma} u_p^\sigma \phi_{|p|}^\sigma(0)\, e^{sign(\sigma) 2i\pi \frac{l^k x}{\alpha}} = 0 \text{ on } I \times \partial\Omega. \tag{11}$$

The low frequency part U_H^0 relates to the weak limit in $L^2(I \times \Omega \times Y)^2$ of the kernel part of S_k^ε in the definition of W_k^ε. It has been treated completely, in [2, 3]. Here, we focus on the non-kernel part of S_k^ε, it relates to the high frequency waves and microscopic and macroscopic scales. In order to obtain the solution of the model, we analyze the asymptotic behaviour of each mode through $T^{\varepsilon \alpha_n^k} S_k^\varepsilon$ as in Lemmas 2 and 3. Then the full solution is the sum of all modes. The main Theorem states as follows.

Theorem 1 *For a given $k \in Y^*$, let U^ε be a solution of (2) bounded in $L^2(I \times \Omega)$, for $\varepsilon \in E_k$, as in (5,6), the limit G^k of any weakly converging extracted subsequence of $\sum_{\sigma \in I^k} W_\sigma^\varepsilon U^\varepsilon$ in $L^2(I \times \Lambda \times \Omega \times Y)^2$ can be decomposed as*

$$G^k(t,\tau,x,y) = \chi_0(k)\, U_H^0(t,x,y) + \sum_{\sigma \in I^k, n \in \mathbb{Z}^*} u_n^\sigma(t,x)\, e^{2i\pi s_n \tau} e_n^\sigma(y) \tag{12}$$

where $\left(u_n^\sigma\right)_{n,\sigma}$ are solutions of the macroscopic equation, and the characteristic function $\chi_0(k) = 1$ if $k = 0$ and $= 0$ otherwise.

Thus, the physical solution U^ε is approximated by two-scale modes

$$U^\varepsilon(t,x) \simeq \chi_0(k)\, U_H^k\left(t, x, \frac{x}{\varepsilon}\right) + \sum_{\sigma \in I^k, n \in \mathbb{Z}^*} u_n^\sigma(t,x)\, e^{is_n \sqrt{\lambda_n^\sigma} t/\varepsilon} e_n^\sigma\left(\frac{x}{\varepsilon}\right). \tag{13}$$

The remain of this section provides the proofs of results.

Proof of Lemma 2 The test functions of the weak formulation (2) are chosen as $\Psi^\varepsilon = \mathfrak{B}_n^k \Psi(t,x)$ for $k \in Y^*$, $n \in \mathbb{Z}^*$ where $\Psi \in C^\infty(I \times \Lambda \times \Omega \times Y)^2$ is periodic in τ and $k-$quasi-periodic in y. From (4) multiplied by ε, since $\left(\frac{\partial_\tau}{\alpha_n^k} - A_k\right)\Psi$ is periodic in τ and $k-$quasi-periodic in y and $T^{\varepsilon\alpha_n^k} S_k^\varepsilon U^\varepsilon \to U_n^k$ in $L^2(I \times \Lambda \times \Omega \times Y)^2$ weakly, Lemma 1 allows to pass to the limit in the weak formulation, $\int_{I\times\Lambda\times\Omega\times Y} U_n^k \cdot \left(\frac{\partial_\tau}{\alpha_n^k} - A_k\right)\Psi dt d\tau dx dy = 0$. Using the assumption $U_n^k \in D(A_k) \cap L^2\left(I \times \Omega \times Y; H^1(\Lambda)\right)$ and applying an integration by parts, $\int_{I\times\Lambda\times\Omega\times Y} \left(-\frac{\partial_\tau}{\alpha_n^k} + A_k\right) U_n^k \cdot \Psi dt d\tau dx dy + \int_{I\times\partial\Lambda\times\Omega\times Y} U_n^k \cdot \Psi dt d\tau dx dy - \int_{I\times\Lambda\times\Omega\times\partial Y} U_n^k \cdot n_{A_k} \Psi dt d\tau dx dy = 0$. Choosing $\Psi \in L^2(I \times \Omega; H_0^1 \Lambda \times Y)$ comes the strong form (8). Since the product of a periodic function by a $k-$quasi-periodic function is $k-$quasi-periodic then $n_{A_k}\Psi$ is $k-$quasi-periodic in y. Therefore, U_n^k is periodic in τ and $k-$quasi-periodic in y. Moreover, (9) is obtained, by projection.

Proof of Lemma 3 For $k \in Y^*$, let $\left(\lambda_{|p|}^\sigma, e_p^\sigma\right)_{p\in M_n^\sigma, \sigma\in I^k}$ be the Bloch eigenmodes of the spectral equation $\mathcal{Q}(\sigma)$ corresponding to the eigenvalue $\lambda_{|n|}^k$. We pose $\Psi^\varepsilon(t,x) = \sum_{\sigma\in I^k} \mathfrak{B}_n^\sigma \Psi_\varepsilon^\sigma \in H^1(I \times \Omega)^2$ as a test function in the weak formulation (2) with each $\Psi_\varepsilon^\sigma(t,\tau,x,y) = \sum_{q\in M_n^k} \varphi_{q,\varepsilon}^\sigma(t,x) e^{2i\pi s_q \tau} e_q^\sigma(y)$ where $\varphi_{q,\varepsilon}^\sigma \in H^1(I \times \Omega)$ and satisfies the boundary conditions $\sum_{\sigma\in I^k, q\in M_n^\sigma} e^{2i\pi s_q t/(\varepsilon\alpha_q^\sigma)} \varphi_{q,\varepsilon}^\sigma(t,x)$ $\phi_{|q|}^\sigma\left(\frac{x}{\varepsilon}\right) = O(\varepsilon)$ on $I \times \partial\Omega$. Note that this condition is related to the second component of Ψ^ε only. Since $\alpha_q^\sigma = \alpha_n^k$ and $s_q = s_n$ for all $q \in M_n^\sigma$ and $\sigma \in I^k$, so $e^{2i\pi s_q t/(\varepsilon\alpha_q^\sigma)} \neq 0$ can be eliminated. Extracting a subsequence $\varepsilon \in E_k$, using the $\sigma-$quasi-periodicity of $\phi_{|q|}^\sigma$ and (5,6), $\varphi_{q,\varepsilon}^\sigma$ converges strongly to some φ_q^σ in $H^1(I \times \Omega)$, then the boundary conditions are

$$\sum\nolimits_{\sigma\in I^k, q\in M_n^\sigma} \varphi_q^\sigma(t,x)\, \phi_{|q|}^\sigma(0)\, e^{sign(\sigma)2i\pi \frac{l^k x}{\alpha}} = 0 \text{ on } I \times \partial\Omega. \tag{14}$$

Applying (4) and since $\left(\frac{\partial_\tau}{\alpha_n^\sigma} - A_\sigma\right)\Psi^\sigma = 0$ for $\sigma \in I^k$, then in the weak formulation it remains $\sum\limits_{\sigma\in I^k} \int_{I\times\Omega} F^\varepsilon \cdot \mathfrak{B}_n^\sigma \Psi_\varepsilon^\sigma + U^\varepsilon \cdot \mathfrak{B}_n^\sigma(\partial_t - B)\Psi_\varepsilon^\sigma dt dx - \int_\Omega U_0^\varepsilon \cdot \mathfrak{B}_n^\sigma \Psi_\varepsilon^\sigma(t=0)\, dx = 0$. Since $(\partial_t - B)\Psi_\varepsilon^\sigma$ is $\sigma-$quasi-periodic, so passing to the limit thanks to Lemma 1, after using (7) and replacing the decomposition of U_n^σ, $\sum\limits_{\sigma\in I^k, \{p,q\}\in M_n^\sigma} \left(\int_{I\times\Omega} b(\sigma,p,q)\, u_p^\sigma \cdot \partial_t \varphi_q^\sigma - c(\sigma,p,q)\, u_p^\sigma \cdot \partial_x \varphi_q^\sigma - F_q^\sigma \cdot \varphi_q^\sigma dt dx - \int_\Omega U_{0,q}^\sigma \cdot \varphi_q^\sigma(t=0)\, dx\right) = 0$ for all $\varphi_q^\sigma \in H^1(I \times \Omega)$ fulfilling (14).

Moreover, if $u_q^\sigma \in H^1(I \times \Omega)$ then it satisfies the strong form of the internal equations (10) for each $\sigma \in I^k$, $q \in M_n^\sigma$ and the boundary conditions $\sum_{\sigma,p,q} c(\sigma,p,q)\, u_p^\sigma \overline{\varphi_q^\sigma} = 0$ on $I \times \partial\Omega$ for φ_q^σ satisfies (14).

In order to find the boundary conditions of $\left(u_p^\sigma\right)_{\sigma,p}$, we distinguish between the two cases $k \neq 0$ and $k = 0$. First, for $k \neq 0$, $\lambda_{|n|}^k$ is simple so $M_n^k = \{n\}$.

Introducing $C = diag\,(c\,(\sigma, n, n))_\sigma$, $B = diag\,(b\,(\sigma, n, n))_\sigma$, $U = \left(u_n^\sigma\right)_\sigma$, $F = \left(F_n^\sigma\right)_\sigma$, $U_0 = \left(U_{0,n}^\sigma\right)_\sigma$, $\Psi = \left(\varphi_n^\sigma\right)_\sigma$, $\Phi = \left(\phi_{|n|}^\sigma\,(0)\, e^{sign(\sigma)2i\pi l^k x/\alpha}\right)_\sigma$, Eq. (10) states under matrix form $B\partial_t U + C\partial_x U = F$ in $I\times\Omega$ and $BU\,(t=0) = U_0$ in Ω which the boundary condition is rewritten as $CU\,(t,x)\,.\overline{\Psi}\,(t,x) = 0$ on $I\times\partial\Omega$ for all Ψ such that $\overline{\Phi}(x).\overline{\Psi}(t,x) = 0$ on $I\times\partial\Omega$. Equivalently, $CU\,(t,x)$ is collinear with $\overline{\Phi}(x)$ yielding the boundary condition $u_n^k\phi_{|n|}^k\,(0)\,e^{2i\pi\frac{l^k x}{\alpha}} + u_n^{-k}\phi_{|n|}^{-k}\,(0)\,e^{-2i\pi\frac{l^k x}{\alpha}} = 0$ on $I\times\partial\Omega$ after remarking that $c\,(k,n,n)\neq 0$ and $c\,(k,n,n) = -c\,(-k,n,n)$. Second, for $k = 0$, $\lambda_{|n|}^0$ is double $\lambda_{|n|}^0 = \lambda_{|m|}^0$ so $M_n^k = \{n,m\}$. With $C = (c\,(0,p,q))_{p,q}$, $B = (b\,(0,p,q))_{p,q}$, $U = \left(u_p^0\right)_p$, $F = \left(F_q^0\right)_q$, $U_0 = \left(U_{0,q}^0\right)_q$, $\Psi = \left(\varphi_q^0\right)_q$, $\Phi = \left(\phi_{|q|}^0\,(0)\right)_q$, the matrix form is still stated as above which the boundary condition is $u_n^0\phi_{|n|}^0\,(0) + u_m^0\phi_{|m|}^0\,(0) = 0$ on $I\times\partial\Omega$ after remarking that $c\,(0,p,p) = 0$ and $c\,(0,n,m)\neq 0$.

Proof of Theorem For a given $k\in Y^*$, let U^ε be solution of (2) which is bounded in $L^2(I\times\Omega)$, then $\|W_\sigma^\varepsilon U^\varepsilon\|_{L^2(I\times\Lambda\times\Omega\times Y)}$ is bounded for $\sigma\in I^k$. So there exists $G^k\in L^2\,(I\times\Lambda\times\Omega\times Y)^2$ such that, up to the extraction of a subsequence, $\sum_{\sigma\in I^k} W_\sigma^\varepsilon U^\varepsilon$ tends weakly to $G^k = \chi_0\,(k)\,U_H^0 + \sum_{\sigma\in I^k, n\in\mathbb{Z}^*} U_n^k$ in $L^2\,(I\times\Lambda\times\Omega\times Y)^2$. The high frequency part is based on the decomposition (9) and Lemma 3.

Remark 1 This method allows to complete the homogenized model of the wave equation in [3] for the one-dimensional case. Let $K\in\mathbb{N}^*$, we decompose $\frac{\alpha}{\varepsilon K} = \left[\frac{\alpha}{\varepsilon K}\right] + l_\varepsilon^1$ with $l_\varepsilon^1\in[0,1)$ and assume that the sequence ε is varying in a set $E_K\subset\mathbb{R}^{+*}$ so that $l_\varepsilon^1\to l^1$ when $\varepsilon\to 0$ with $l^1\in[0,1)$. For any $k\in L_K^*$, defined in [3], we denote $p_k = kK\in\mathbb{N}$, so $\frac{\alpha p_k}{\varepsilon K} = p_k\left[\frac{\alpha}{\varepsilon K}\right] + p_k l_\varepsilon^1$ and $p_k l_\varepsilon^1\to l^k := p_k l^1$ when $\varepsilon\to 0$ with the same sequence of $\varepsilon\in E_K$.

4 Numerical Examples

We report simulations regarding comparison of physical solution and its approximation for $I = (0,1)$, $\Omega = (0,1)$, $\rho = 1$, $a = \frac{1}{3}\,(\sin(2\pi y) + 2)$, $f^\varepsilon = 0$, $v_0^\varepsilon = 0$, $\varepsilon = \frac{1}{10}$ and $k = 0.16$. Since $k\neq 0$, so the approximation (13) comes

$$U^\varepsilon\,(t,x)\simeq\sum\nolimits_{\sigma\in I^k, n\in\mathbb{Z}^*} u_n^\sigma\,(t,x)\,e^{is_n\sqrt{\lambda_{|n|}^\sigma}/\varepsilon}e_n^\sigma\left(\frac{x}{\varepsilon}\right). \tag{15}$$

The validation of the approximation is based on the modal decomposition of any solution $U^\varepsilon = \sum_{l\in\mathbb{Z}^*} R_l^\varepsilon\,(t)\,V_l^\varepsilon\,(x)$ where the modes V_l^ε are built from the solutions v_l^ε of the spectral problem $\partial_x\left(a^\varepsilon\partial_x v_l^\varepsilon\right) = \lambda_l^\varepsilon v_l^\varepsilon$ in Ω with $v_l^\varepsilon = 0$ on $\partial\Omega$. Moreover, in [9], two-scale approximations of modes have been derived on the form of linear

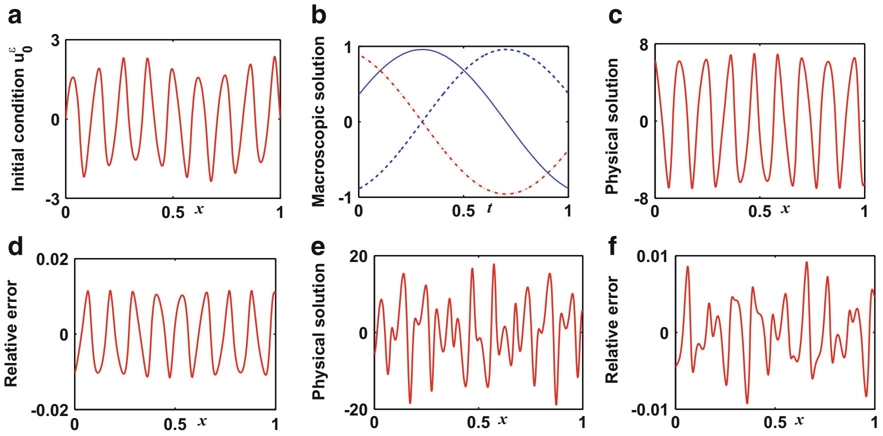

Fig. 1 Numerical results

combinations $\sum_{\sigma\in I^k}\theta_n^\sigma(x)\,\phi_{|n|}^\sigma\left(\frac{x}{\varepsilon}\right)$ of Bloch modes, so the initial conditions of the physical problem are taken on the form $u_0^\varepsilon(x) = \sum_{n\in\mathbb{N}^*,\sigma\in I^k}\theta_n^\sigma(x)\,\phi_n^\sigma\left(\frac{x}{\varepsilon}\right)$. Two simulations are reported, one for an initial condition u_0^ε spanned by the pair of Bloch modes corresponding to $n = 2$ when the other is spanned by three pairs $n \in \{2, 3, 4\}$. In the first case, the first component of U_0^ε approximates the first component of a single eigenvector V_l^ε approximated by (15) where all coefficients $u_n^\sigma = 0$ for $n \neq \pm 2$. Figure 1a shows the initial condition u_0^ε. Figure 1b presents the real part (solid line) and the imaginary part (dashed-dotted line) of the macroscopic solution u_n^k and also the real part (dotted line) and the imaginary part (dashed line) of u_n^{-k} at space step $x = 0.699$ when Fig. 1c, d plot the real part of the first component U_1^ε of physical solution and the relative error vector of U_1^ε with its approximation which $L^2(\Omega)$-norm is equal to 7e−3 at $t = 0.466$. For the second case where $u_n^\sigma = 0$ for $n \notin \{\pm 2, \pm 3, \pm 4\}$, the first component U_1^ε and the relative error vector of U_1^ε with its approximation which $L^2(\Omega)$-norm is 3.8e−3 are plotted in Fig. 1e, f. Finally, for the two cases the $L^2(I)$-relative errors at $x = 0.699$ on the first component are 8e−3 and 3.5e−3 respectively.

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