Variational Principles for Eigenvalues of Nonlinear Eigenproblems

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Abstract Variational principles are very powerful tools when studying self-adjoint linear operators on a Hilbert space $\mathscr{H}$. Bounds for eigenvalues, comparison theorems, interlacing results and monotonicity of eigenvalues can be proved easily with these characterizations, to name just a few. In this paper we consider generalization of these principles to families of linear, self-adjoint operators depending continuously on a scalar in a real interval.

1 Introduction

Let A be a self-adjoint operator on a Hilbert space $\mathscr{H}$ with scalar product $\langle\cdot,\cdot\rangle$, and denote by $\lambda_1 \le \lambda_2 \le \dots$ those eigenvalues of A (if there are any), which are smaller than the minimum of the essential spectrum $\sigma_{\text{ess}}(A)$, each counted according to its multiplicity. Then λ_j can be characterized by three fundamental variational principles [28], namely by Rayleigh's principle [19]

$$\lambda_j = \min\{R(x) : \langle x, x_i\rangle = 0,\ i = 1,\dots,j-1\} \tag{1}$$

where $R(x) := \langle Ax, x\rangle / \langle x, x\rangle$ is the Rayleigh quotient and $x_1,\dots,x_{j-1}$ is a set of orthogonal eigenvectors of A (x_i corresponding to λ_i), the minmax characterization by Poincaré [18]

$$\lambda_j = \min_{\dim V = j}\ \max_{x\in V, x\neq 0} R(x), \tag{2}$$

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and the maxmin principle due to Courant [5], Fischer [9] and Weyl [30]

$$\lambda_j = \max_{\dim V = j-1} \min_{x \in V^\perp, x \neq 0} R(x) \tag{3}$$

where $V^\perp := \{x \in \mathscr{H} \,:\, \langle v, x\rangle = 0 \text{ for every } v \in V\}$.

The purpose of this paper is to survey generalizations of these principles to the nonlinear eigenvalue problem

$$T(\lambda)x = 0 \tag{4}$$

and to trace the history of these generalizations. Here $T(\lambda)$, $\lambda \in J$, is a family of linear self-adjoint and bounded operator on $\mathscr{H}$, and J is a real open interval which may be unbounded. As in the linear case $T(\lambda) := \lambda I - A$ we call $\lambda \in J$ an eigenvalue of $T(\cdot)$ if Eq. (4) has a nontrivial solution $x \neq 0$ and the solution x is called a corresponding eigenelement.

We stress the fact that we are only concerned with real eigenvalues in J although $T(\cdot)$ may be defined on a larger subset of $\mathbb{C}$, and $T(\cdot)$ may have additional eigenvalues in $\mathbb{C} \setminus J$.

2 Overdamped Problems

To receive generalizations of the variational principles to the nonlinear eigenvalue problem (4) the Rayleigh quotient $R(x)$ of a linear problem $Ax = \lambda x$ has to be replaced with some functional. We assume that for every $x \in J$, $x \neq 0$ the real equation $f(\lambda; x) := \langle T(\lambda)x, x\rangle = 0$ has at most one solution in J denoted by $p(x)$. This defines the so called Rayleigh functional p which obviously generalizes the Rayleigh quotient for the linear case.

If the Rayleigh functional p is defined on the entire space $\mathscr{H} \setminus \{0\}$ then the eigenproblem (4) is called overdamped. This term is motivated by the finite dimensional quadratic eigenvalue problem

$$T(\lambda)x = \lambda^2 Mx + \lambda Cx + Kx = 0 \tag{5}$$

governing the damped free vibrations of a system where $M, C, K \in \mathbb{R}^{n\times n}$ are symmetric and positive definite matrices corresponding to the mass, the damping and the stiffness of the system, respectively.

Assume that the damping $C = \alpha\tilde{C}$ depends on a parameter $\alpha \geq 0$. Then for $\alpha = 0$ the system has purely imaginary eigenvalues corresponding to harmonic vibrations of the system. Increasing α the eigenvalues move into the left half plane as conjugate complex pairs corresponding to damped vibrations. Finally they reach the negative real axis as double eigenvalues where they immediately split and move into opposite directions.

When eventually all eigenvalues have become real, and all eigenvalues going to the right are right of all eigenvalues moving to the left the system is called overdamped. In this case the two solutions

$$p_{\pm}(x) = (-\alpha\langle\tilde{C}x, x\rangle \pm \sqrt{\alpha^2\langle\tilde{C}x, x\rangle^2 - 4\langle Mx, x\rangle\langle Kx, x\rangle})/(2\langle Mx, x\rangle).$$

of the quadratic equation

$$\langle T(\lambda)x, x\rangle = \lambda^2\langle Mx, x\rangle + \lambda\alpha\langle\tilde{C}x, x\rangle + \langle Kx, x\rangle = 0 \quad (6)$$

are real, and they satisfy $\sup_{x\neq 0} p_-(x) < \inf_{x\neq 0} p_+(x)$.

Hence, for $J_- := (-\infty, \inf_{x\neq 0} p_+(x))$ Eq. (6) defines the Rayleigh functional p_-, and for $J_+ := (\sup_{x\neq 0} p_-(x), 0)$ it defines the Rayleigh functional p_+.

Duffin [6] proved that all eigenvalues $\lambda_1^- \leq \dots \lambda_n^-$ and $\lambda_1^+ \leq \cdots \leq \lambda_n^+$ are maxmin values of the functionals p_- and p_+, respectively, and Rogers [20] generalized it to the finite dimensional overdamped case.

Theorem 1 *Let $T(\lambda) \in \mathbb{R}^{n\times n}$, $\lambda \in J$ be an overdamped family of symmetric matrices depending continuously differentiable on $\lambda \in J$ such that $\langle T'(p(x))x, x\rangle > 0$ for every $x \neq 0$. Then there are exactly n eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ of $T(\lambda)x = 0$ in J, and it holds*

$$\lambda_j = \min_{\dim V=j} \max_{x\in V, x\neq 0} p(x), \quad j = 1, \dots, n. \quad (7)$$

Infinite dimensional overdamped problems were considered first for quadratic problems $(A - \lambda^2 B - \lambda I)x = 0$ where A and B are bounded, positive definite and compact by Turner [22] and Weinberger [27] who proved all three types of variational characterization by linearization (i.e. taking advantage of the fact that the quadratic problem is equivalent to a linear self-adjoint eigenproblem), and by Langer [15] who proved minmax and maxmin characterizations for the quadratic problem $(\lambda^2 A + \lambda B + C)x = 0$ taking advantage of the theory of J-self-adjoint operators.

The general overdamped problem was considered by Hadeler [11] who proved the following minmax and maxmin theorem:

Theorem 2 *Let $T(\lambda) : \mathscr{H} \to \mathscr{H}$, $\lambda \in J$ be a family of linear self-adjoint and bounded operators such that (4) is over-damped, and assume that for $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that $T(\lambda) + \nu(\lambda)I$ is compact.*

Let $T(\cdot)$ be continuously differentiable and suppose that

$$\langle T'(p(x))x, x\rangle > 0 \quad \textit{for ever } x \neq 0. \quad (8)$$

Let the eigenvalues λ_n of $T(\lambda)x = 0$ be numbered in non-decreasing order regarding their multiplicities. Then they can be characterized by the following two variational principles

$$\lambda_n = \min_{\dim V=n} \max_{x\in V,\ x\neq 0} p(x)$$
$$= \max_{\dim V=n-1} \min_{x\in V^{\perp},\ x\neq 0} p(x).$$

Moreover, Hadeler [11] generalized Rayleigh's principle for overdamped problems proving that the eigenvectors are orthogonal with respect to the generalized scalar product

$$[x, y] := \begin{cases} \dfrac{\langle (T(p(x)) - T(p(y)))x, y\rangle}{p(x) - p(y)}, & \text{if } p(x) \neq p(y) \\ \langle T'(p(x))x, y\rangle, & \text{if } p(x) = p(y) \end{cases} \tag{9}$$

which is symmetric, definite and homogeneous, but in general it is not bilinear.

Further generalizations of the minmax and maxmin characterizations were proved for certain overdamped polynomial eigenproblems by Turner [23], and for general overdamped problems by Rogers [21], Werner [29], Abramov [1], and Hadeler [12] who relaxed the compactness conditions on $T(\cdot)$.

Markus [16] and Hasanov [13] (with a completely different proof) considered nonoverdamped problems which depended only continuously on the parameter and they replaced assumption (8) with the condition that $\langle T(\lambda)x, x\rangle$ is increasing at the point $p(x)$ given in condition (A_2) of the next section

3 Nonoverdamped Problems

We consider the nonlinear eigenvalue problem (4), where $T(\lambda) : \mathscr{H} \to \mathscr{H}, \lambda \in J$, is a family of self-adjoint and bounded operators depending continuously on the parameter λ.

We assume that

(A_1) *For every fixed $x \in \mathscr{H}$, $x \neq 0$ the real equation*

$$f(\lambda; x) := \langle T(\lambda)x, x\rangle = 0 \tag{10}$$

has at most one solution $\lambda =: p(x) \in J$.

which defines the Rayleigh functional p of (4) with respect to J, and we denote by $\mathscr{D}(p) \subset \mathscr{H}$ the domain of definition of p.

Generalizing the definiteness requirement for linear pencils $T(\lambda) = \lambda B - A$ we further assume that $\langle T(\lambda)x, x\rangle$ is increasing at the point $p(x)$, i.e.

(A_2) *For every $x \in \mathscr{D}(p)$ and every $\lambda \in J$ with $\lambda \neq p(x)$ it holds that*

$$(\lambda - p(x)) f(\lambda; x) > 0. \tag{11}$$

The key to the variational principle in the nonoverdamped case is an appropriate enumeration of the eigenvalues. In general, the natural enumeration i.e. the first eigenvalue is the smallest one, followed by the second smallest one etc. is not reasonable. Instead, the number of an eigenvalue λ of the nonlinear problem (4) is inherited from the location of the eigenvalue 0 in the spectrum of the operator $T(\lambda)$ based on the following consideration (cf. [26]).

For $j \in \mathbb{N}$ and $\lambda \in J$ let

$$\mu_j(\lambda) := \sup_{V \in S_j} \min_{v \in V, v \neq 0} \frac{\langle T(\lambda)v, v\rangle}{\langle v, v\rangle} \tag{12}$$

where S_j is the set of all j dimensional subspaces of $\mathscr{H}$. We assume that

(A_3) *If $\mu_n(\lambda) = 0$ for some $n \in \mathbb{N}$ and some $\lambda \in J$, then for $j = 1, \ldots, n$ the supremum in $\mu_j(\lambda)$ is attained, and $\mu_1(\lambda) \geq \mu_2(\lambda) \geq \cdots \geq \mu_n(\lambda)$ are the n largest eigenvalues of the linear operator $T(\lambda)$. Conversely, if $\mu = 0$ is an eigenvalue of the operator $T(\lambda)$, then $\mu_n(\lambda) = 0$ for some $n \in \mathbb{N}$.*

Definition 1 $\lambda \in J$ is an nth eigenvalue of $T(\cdot)$ if $\mu_n(\lambda) = 0$ for $n \in \mathbb{N}$.

Condition (A_3) is satisfied for example if for every $\lambda \in J$ the supremum of the essential spectrum of $T(\lambda)$ is less than 0. The following stronger condition that for every $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that $T(\lambda) + \nu(\lambda)I$ is a compact operator was used in [11].

The following Lemma proved in [25] (and in [26] for $T(\lambda)$ depending differentiable on λ) relates the supremum of p on a subspace V to the sign of the Rayleigh quotient of $T(\lambda)$ on V.

Lemma 1 *Under the conditions (A_1), (A_2) and (A_3) let $\lambda \in J$, and assume that V is a finite dimensional subspace of $\mathscr{H}$ such that $V \cap \mathscr{D}(p) \neq \emptyset$. Then*

$$\lambda \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \sup_{x \in V \cap \mathscr{D}(p)} p(x) \quad \Leftrightarrow \quad \min_{x \in V} \langle T(\lambda)x, x\rangle \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 \tag{13}$$

Proof [25], Lemma 2.4

Theorem 3 *Assume that the conditions (A_1), (A_2) and (A_3) are satisfied. Then the nonlinear eigenvalue problem $T(\lambda)x = 0$ has at most a countable set of eigenvalues in J, and it holds that:*

(i) For every $n \in \mathbb{N}$ there exists at most one nth eigenvalue, and the following characterization holds:

$$\lambda_n = \min_{\substack{V \in S_n \\ V \cap \mathcal{D}(p) \neq \emptyset}} \sup_{v \in V \cap \mathcal{D}(p)} p(v). \tag{14}$$

(ii) If

$$\lambda_n := \inf_{\substack{V \in S_n \\ V \cap \mathcal{D}(p) \neq \emptyset}} \sup_{v \in V \cap \mathcal{D}(p)} p(v) \in J, \tag{15}$$

then λ_n is an nth eigenvalue of (4), and the infimum is attained, i.e. the characterization (14) holds.

(iii) If there is an m-th and an n-th eigenvalue λ_m and λ_n in J with $m < n$, then J contains a k-th eigenvalue λ_k, $m < k < n$ as well, and

$$\inf J < \lambda_m \leq \lambda_{m+1} \leq \cdots \leq \lambda_n < \sup J.$$

Proof (i) If λ_n is an n-th eigenvalue, then $\mu_n(\lambda_n) = 0$, and

$$\mu_n(\lambda_n) = \max_{\dim V = n} \min_{x \in V,\ \|x\|=1} \langle T(\lambda_n)x, x\rangle = \min_{x \in \tilde{V},\ \|x\|=1} \langle T(\lambda_n)x, x\rangle$$

for the invariant subspace $\tilde{V}$ corresponding to the n largest eigenvalues of $T(\lambda_n)$

Hence, $\min_{x \in V, \|x\|=1} \langle T(\lambda_n)x, x\rangle \leq 0$ for every V with $\dim V = n$, and (13) implies

$$\sup_{x \in V \cap \mathcal{D}} p(x) \geq \lambda_n = \sup_{x \in \tilde{V} \cap \mathcal{D}} p(x).$$

Hence, λ_n is a minmax value of p.

(ii) Was proved in [26] under the condition that $T(\lambda)$ depends differentiable on λ. But the proof uses only the fact that $\mathcal{D}(p)$ is an open set (which follows also from (A_1) and (A_2) considered here; cf. Lemma 2.3 in [25]) and the analogue of Lemma 1. So the proof holds also for the continuous case considered here.

(iii) Follows from the continuity of $\mu_k(\lambda)$ in J (cf. [7]).

Remark 1 We only considered the case that for every $\lambda \in J$ the supremum of the essential spectrum of $T(\lambda)$ is less than 0. In the same way we obtain for the case that for every $\lambda \in J$ the infimum of $T(\lambda)$ exceeds 0 a maxinf characterization of the eigenvalues of $T(\cdot)$ in J if we replace (A_2) with

(A_2') $(\lambda - p(x)) f(\lambda; x) < 0$ *for every $x \in \mathcal{D}(p)$ and $\lambda \in J$ such that $\lambda \neq p(x)(p)$*

and (A_3) with

(A_3') *If* $\nu_m(\lambda) := \inf_{V \in S_m} \max_{x \in V, x \neq 0} \langle T(\lambda)x, x\rangle / \langle x, x\rangle = 0$ *for some* $m \in \mathbb{N}$ *and some* $\lambda \in J$, *then for* $j = 1, \ldots, m$ *the supremum in* $\nu_j(\lambda)$ *is attained, and* $\nu_1(\lambda) \leq \nu_2(\lambda) \leq \cdots \leq \nu_m(\lambda)$ *are the* m *smallest eigenvalues of the linear operator* $T(\lambda)$. *Conversely, if* $\nu = 0$ *is an eigenvalue of the operator* $T(\lambda)$, *then* $\nu_m(\lambda) = 0$ *for some* $m \in \mathbb{N}$.

If the eigenvalues of $T(\cdot)$ are now enumerated in decreasing order, i.e. $\lambda \in J$ is an mth eigenvalue of $T(\cdot)$ if $\nu_m(\lambda) = 0$ for $m \in \mathbb{N}$, then λ_m can be characterized as

$$\lambda_m = \max_{\substack{V \in S_m \\ V \cap \mathscr{D}(p) \neq \emptyset}} \inf_{v \in V \cap \mathscr{D}(p)} p(v).$$

In the following we consider only problem (4) under the conditions (A_1), (A_2) and (A_3), although the analogue results also hold under the conditions (A_1), (A_2') and (A_3') with the modified enumeration given above.

If the extreme eigenvalue λ_1 is contained in J, then the enumeration based on (A_3) is the natural ordering. For this case Barston [3] proved the minmax characterization for some extreme real eigenvalues for the finite dimensional quadratic eigenvalue problem. Abramov [2] and Hasanov [14] derived the minmax and maxmin characterizations for the extreme eigenvalues for pencils of waveguide type, which are certain quadratic eigenvalues problems depending on two parameters.

For the general $T(\cdot)$ it can be shown that the eigenspaces corresponding to eigenvalues in J are contained in $\mathscr{D}(p) \cup \{0\}$. Hence the minmax characterization obtains the following form:

Theorem 4 *Let the conditions* (A_1), (A_2) *and* (A_3) *be satisfied, and assume that* $\lambda_1 = \inf_{x \in \mathscr{D}(p)} p(x) \in J$, *and* $\lambda_n \in J$ *for some* $n \in \mathbb{N}$.

If $j \in \{1, \ldots, n\}$ *and* $V \in S_j$ *such that* $\lambda_j = \sup_{x \in V \cap \mathscr{D}(p)} p(x)$, *then* $V \subset \mathscr{D}(p) \cup \{0\}$, *and the characterization of* λ_j *can be replaced with*

$$\lambda_n = \min_{\substack{V \in S_j \\ V \subset \mathscr{D}(p) \cup \{0\}}} \sup_{v \in V \cap \mathscr{D}(p)} p(v). \tag{16}$$

The generalization of the maxmin characterization of Courant, Fischer and Weyl is based on the following Lemma which was proved in [24]:

Lemma 2 *Let* $\lambda \in J$, *and let* V *be a finite dimensional subspace of* $\mathscr{H}$ *such that* $V^{\perp} \cap \mathscr{D} \neq \emptyset$. *Then it holds that*

$$\lambda \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \inf_{x \in V^{\perp} \cap \mathscr{D}(p)} p(x) \qquad \Leftrightarrow \qquad \max_{x \in V^{\perp},\, \|x\|=1} \langle T(\lambda)x, x\rangle \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0$$

Theorem 5 *Assume that the conditions* (A_1)*,* (A_2) *and* (A_3) *are satisfied. If there exists an* n*-th eigenvalue* $\lambda_n \in J$ *of* $T(\lambda)x = 0$*, then*

$$\lambda_n = \max_{\substack{V \in S_{n-1} \\ V^\perp \cap \mathscr{D} \neq \emptyset}} \inf_{v \in V^\perp \cap \mathscr{D}} p(v),$$

and the maximum is attained by $W := span\{u_1, \dots, u_{n-1}\}$ *where* u_j *denotes an eigenvector corresponding to the* j*-largest eigenvalue* $\mu_j(\lambda_n)$ *of* $T(\lambda_n)$*.*

Essentially the same variational characterizations of Poincaré and of Courant-Fischer-Weyl type were derived by Mel'nik and Nazarov [17], where $T(\lambda)$ is a set of bounded self-adjoint operators depending continuously differentiable on λ, by Griniv and Mel'nik [10] for $T(\lambda) = A(\lambda) - I$, where $A(\lambda)$ is self-adjoint, and compact, and by Binding, Eschwé and H. Langer [4] for general bounded and self-adjoint $T(\lambda)$ depending continuously on λ. Eschwé and M. Langer [8] obtained these variational characterizations for unbounded operators. In all of these papers the natural enumeration of the eigenvalues is used, but the dimension of the subspace in the characterizations is shifted by the number of the largest eigenvalue of $T(\lambda_1)$.

Hadeler [11] proved Rayleigh's principle for differentiable overdamped problems. For the continuous case the generalized scalar product (9) has to be modified for the case $p(x) = p(y)$ setting $[x, y] := \langle x, y \rangle$. Then the generalized scalar product $[\cdot, \cdot]$ becomes discontinuous for $p(x) = p(y)$, but the continuity is not needed in the proof of Rayleigh's principle which obtains the following form:

Theorem 6 *Under the conditions* (A_1)*,* (A_2)*,* (A_3) *assume that* J *contains* $n \geq 1$ *eigenvalues* $\lambda_1 \leq \cdots \leq \lambda_n$ *(where* λ_i *is an* i*th eigenvalue) with corresponding* $[\cdot, \cdot]$ *orthogonal eigenvectors* $x_1, \dots, x_n$*.*

If there exists $x \in \mathscr{D}(p)$ *with* $[x_i, x] = 0$ *for* $i = 1, \dots, n$ *then* J *contains an* $(n + 1)$*th eigenvalue, and*

$$\lambda_{n+1} = \inf\{p(x) : [x_j, x] = 0, \ i = 1, \dots, n\}. \tag{17}$$

References

1. Y.S. Abramov, Variational principles for nonlinear eigenvalue problems. Funct. Anal. Appl. **7**, 317–318 (1973)
2. Y.S. Abramov, Linear operators and spectral theory: pencils of waveguide type and related extremal problems. J. Math. Sci. **64**, 1278–1288 (1993)
3. E.M. Barston, A minimax principle for nonoverdamped systems. Int. J. Eng. Sci. **12**, 413–421 (1974)
4. P. Binding, D. Eschwé, H. Langer, Variational principles for real eigenvalues of self-adjoint operator pencils. Integr. Equ. Oper. Theory **38**, 190–206 (2000)
5. R. Courant, Über die Eigenwerte bei den Differentialgleichungen der mathematischen Physik. Math. Z. **7**, 1–57 (1920)
6. R.J. Duffin, A minimax theory for overdamped networks. J. Rat. Mech. Anal. **4**, 221–233 (1955)

7. J. Eisenfeld, E.H. Rogers, Elementary localization theorems for nonlinear eigenproblems. J. Math. Anal. Appl. **48**, 325–340 (1974)
8. D. Eschwé, M. Langer. Variational principles of eigenvalues of self-adjoint operator functions. Integr. Equ. Oper. Theory **49**, 287–321 (2004)
9. E. Fischer, Über quadratische Formen mit reellen Koeffizienten. Monatshefte für Math. und Phys. **16**, 234–249 (1905)
10. R.O. Griniv, T.A. Mel'nik, On the singular Rayleigh functional. Math. Notes **60**, 97–100 (1996)
11. K.P. Hadeler, Variationsprinzipien bei nichtlinearen Eigenwertaufgaben. Arch. Ration. Mech. Anal. **30**, 297–307 (1968)
12. K.P. Hadeler, Nonlinear eigenvalue problems, in *Numerische Behandlung von Differentialgleichungen* ed. by R. Ansorge, L. Collatz, G. Hämmerlin, W. Törnig, ISNM 27 (Birkhäuser, Stuttgart, 1975), pp. 111–129
13. M. Hasanov, An approximation method in the variational theory of the spectrum of operator pencils. Acta Appl. Math. **71**, 117–126 (2002)
14. M. Hasanov, The spectra of two–parameter quadratic operator pencils. Math. Comput. Model. **54**, 742–755 (2011)
15. H. Langer, Über stark gedämpfte Scharen im Hilbertraum. J. Math. Mech. **17**, 685–705 (1968)
16. A.S. Markus, *Introduction to the Spectral Theory of Polynomial Operator Pencils* (AMS Translations of Mathematical Monographs, Providence, 1988)
17. T.A. Mel'nik, S.A. Nazarov, The asymptotics of the solution to the Neumann spectral problem in a domain of the "dense-comb" type. J. Math. Sci. **85**, 2326–2345 (1997)
18. H. Poincaré, Sur les equations aux dérivées partielles de la physique mathématique. Am. J. Math. **12**, 211–294 (1890)
19. L. Rayleigh, Some general theorems relating to vibrations. Proc. Lond. Math. Soc. **4**, 357–368 (1873)
20. E.H. Rogers, A minimax theory for overdamped systems. Arch. Ration. Mech. Anal. **16**, 89–96 (1964)
21. E.H. Rogers, Variational properties of nonlinear spectra. J. Math. Mech. **18**, 479–490 (1968)
22. R.E.L. Turner, Some variational principles for nonlinear eigenvalue problems. J. Math. Anal. Appl. **17**, 151–160 (1967)
23. R.E.L. Turner, A class of nonlinear eigenvalue problems. J. Func. Anal. **7**, 297–322 (1968)
24. H. Voss, A maxmin principle for nonlinear eigenvalue problems with application to a rational spectral problem in fluid–solid vibration. Appl. Math. **48**, 607–622 (2003)
25. H. Voss, A minmax principle for nonlinear eigenproblems depending continuously on the eigenparameter. Numer. Linear Algebra Appl. **16**, 899–913 (2009)
26. H. Voss, B. Werner, A minimax principle for nonlinear eigenvalue problems with applications to nonoverdamped systems. Math. Meth. Appl. Sci. **4**, 415–424 (1982)
27. H.F. Weinberger, On a nonlinear eigenvalue problem. J. Math. Anal. Appl. **21**, 506–509 (1968)
28. A. Weinstein, W.F. Stenger, *Methods of Intermediate Problems for Eigenvalues* (Academic, New York, 1972)
29. B. Werner, Das Spektrum von Operatorenscharen mit verallgemeinerte Rayleighquotienten. Arch. Ration. Mech. Anal. **42**, 223–238 (1971)
30. H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung). Math. Ann. **71**, 441–479 (1912)