Stabilized Galerkin for Linear Advection of Vector Fields

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Abstract We present a stabilized Galerkin method for linear advection of vector fields and prove, for sufficiently smooth solutions, optimal a priori error estimates for $\boldsymbol{H}(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}(\mathrm{Div}, \Omega)$-conforming approximation spaces.

1 Introduction

The focus of this article is the following linear advection problem for a vector field $\mathbf{A}$:

$$\begin{aligned} \alpha\mathbf{A} + \mathbf{curl}\,\mathbf{A} \times \boldsymbol{\beta} + \mathbf{grad}\,(\mathbf{A}\cdot\boldsymbol{\beta}) = \mathbf{f} \quad \text{in } \Omega\,, \\ \mathbf{A}_{|\Gamma_{\text{in}}} = \mathbf{A}_0 \quad \text{on } \Gamma_{\text{in}}\,. \end{aligned} \tag{1}$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded domain with inflow boundary $\Gamma_{\text{in}} \subset \partial\Omega$, and $\boldsymbol{\beta} = \boldsymbol{\beta}(\boldsymbol{x})$ and $\alpha = \alpha(\boldsymbol{x})$ are given parameters of which we assume $\boldsymbol{\beta} \in \boldsymbol{W}^{1,\infty}(\Omega)$ and $\alpha \in L^\infty(\Omega)$.

This advection problem is an important model problem for devising reliable numerical methods for problems in electromagnetics and fluid dynamics, when vector fields such as electromagnetic fields or vorticity are advected in some flow. Since the natural space of such quantities is either $\boldsymbol{H}(\mathbf{curl}, \Omega)$ or $\boldsymbol{H}(\mathrm{Div}, \Omega)$, it is desirable to have stabilized methods for appropriate conforming finite element spaces.

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Problem (1) owes its name to the following consideration: Let $\mathbf{X}_t(\mathbf{x})$ denote the flow associated with the given vector field $\boldsymbol{\beta}(\mathbf{x})$, C a one-dimensional manifold and $\mathbf{X}_t(C)$ the image of C under the flow. The transformation rule for line integrals yields

$$\frac{d}{dt}\int_{\mathbf{X}_t(C)} \mathbf{A}(\mathbf{y})\cdot d\mathbf{S}(\mathbf{y}) = \frac{d}{dt}\int_C D\mathbf{X}_t^T(\mathbf{x})\mathbf{A}(\mathbf{X}_t(\mathbf{x}))\cdot d\mathbf{S}(\mathbf{x})\,,$$

and a lengthy calculation verifies:

$$\frac{d}{dt}\left(D\mathbf{X}_t^T(\mathbf{x})\mathbf{A}(\mathbf{X}_t(\mathbf{x}))\right)_{|t=0} = \mathbf{curl}\,\mathbf{A}(\mathbf{x})\times\boldsymbol{\beta}(\mathbf{x}) + \mathbf{grad}\,(\mathbf{A}(\mathbf{x})\cdot\boldsymbol{\beta}(\mathbf{x}))\ .$$

Hence, the first order differential operator in (1) is a generalization of the material derivative of scalar functions u that are integrated over volumes M, i.e.

$$\frac{d}{dt}\int_{\mathbf{X}_t(M)} u(\mathbf{y})d\mathbf{y} = \frac{d}{dt}\int_M \det(D\mathbf{X}_t(\mathbf{x}))\,u(\mathbf{X}_t(\mathbf{x}))d\mathbf{x} \tag{2}$$

and

$$\frac{d}{dt}\left(\det(D\mathbf{X}_t(\mathbf{x}))\,u(\mathbf{X}_t(\mathbf{x}))\right)_{|t=0} = \mathrm{Div}(\boldsymbol{\beta}(\mathbf{x})u(\mathbf{x}))\,.$$

It is the framework of differential forms [4] that embeds this advection idea in a general setting, *the Lie derivative formalism*, and we would like to refer to [2, 9, 17] for recent applications of this formalism in devising new numerical methods.

The advection problem (1) can be regarded as the hyperbolic limit case of an advection-diffusion type problem, where a **curl curl**-operator doubles for the diffusion. Such models appear for electromagnetic problems within a quasi-magnetostatic setting. This was the main motivation in [11] to define and analyse stabilized Galerkin methods for the linear advection problem (1) that rely on $\boldsymbol{H}(\mathbf{curl},\Omega)$-conforming finite element spaces. The theoretical convergence theory in [11] yields the same approximation results as a more classical stabilized Discontinuous Galerkin method for Friedrichs' operators [5, 6, 16], which employs approximation functions that are discontinuous across element interfaces. In contrast, the functions of $\boldsymbol{H}(\mathbf{curl},\Omega)$-conforming finite element spaces, sometimes called *edge elements* or *Whitney forms* [1, 12, 18, 19], have continuous tangential components and discontinuous normal components at the element interfaces. The classical stabilized methods that work with globally continuous finite element functions, add certain stabilization terms to the standard Galerkin variational formulations, that enhance the stability but do not destroy the consistency of the methods [14, 15]. The stabilization effect of the method in [11] does not rely on such additional stabilization terms, but uses the upwinding idea of the Discontinuous Galerkin method. We consider this to be a remarkable advantage of $\boldsymbol{H}(\mathbf{curl}, \Omega)$-conforming approximation spaces for linear advection of vector fields: *a similar simple stabilization as in Discontinuous Galerkin methods, but fewer degrees of freedom.*

In light of this point of view, it appears reasonable to ask for stabilized Galerkin methods for (1) that use $\boldsymbol{H}(\mathrm{Div}, \Omega)$-conforming finite element spaces since the functions of these spaces have discontinuous tangential and continuous normal components at the element interfaces. Besides this conceptual motivation we also emphasize that the advection operator $(\boldsymbol{\beta} \cdot \mathbf{grad})\mathbf{u}$ in linearized Navier-Stokes problems can be rephrased in terms of the advection operator in (1). And, $\boldsymbol{H}(\mathrm{Div}, \Omega)$-conforming finite element spaces are frequently used for such kind of problems [3, 7].

In the next section, Sect. 2 we present the method and state stability and consistency. This is followed, in Sect. 3, by a short summary on previous convergence result. In Sect. 4 we state and prove the main result.

2 Stabilized Galerkin

Standard well-posedness results for (1) (see e.g. [10, Section 3]) require the following assumption.

Assumption 1 *We assume that $\alpha \in L^{\infty}(\Omega)$ and $\boldsymbol{\beta} \in \boldsymbol{W}^{1,\infty}(\Omega)$ are such that $\lambda_{\min}\left\{(2\alpha - \mathrm{Div}\,\boldsymbol{\beta})\mathbf{I}_3 + D\boldsymbol{\beta} + (D\boldsymbol{\beta})^T\right\} \geq \alpha_0$, almost everywhere in Ω for some $\alpha_0 > 0$. $D\boldsymbol{\beta}$ is the Jacobi matrix and λ_{min} the smallest eigenvalue.*

Let us first introduce some notation that is similar the notation used in Discontinuous Galerkin methods.

Let $\mathscr{T}$ be a regular partition of Ω into tetrahedral elements T; h_T is the diameter of T, and $h = \max_{T \in \mathscr{T}} h_T$. The boundary of each element is decomposed into four triangles, called *facets*. We assume that each facet f has a distinguished normal $\mathbf{n}_f$. If a facet f is contained in the boundary of some element T then either $\mathbf{n}_f = \mathbf{n}_{\partial T|_f}$ or $\mathbf{n}_f = -\mathbf{n}_{\partial T|_f}$. Then, if $\mathbf{u}$ is a piecewise smooth vector field on $\mathscr{T}$, $\mathbf{u}^+$ and $\mathbf{u}^-$ denote the two different restrictions of $\mathbf{u}$ to f, e.g. $\mathbf{u}^+ := \mathbf{u}_{|_{T^+}}$ where element T^+ has outward normal $\mathbf{n}_f$. With these restrictions we define also the jump $[\mathbf{u}]_f = \mathbf{u}^+ - \mathbf{u}^-$ and the average $\{\mathbf{u}\}_f = \frac{1}{2}(\mathbf{u}^+ + \mathbf{u}^-)$. For $f \subset \partial\Omega$ we assume f to be oriented such that $\mathbf{n}_f$ points outwards. Let $\mathscr{F}^{\circ}$ and $\mathscr{F}^{\partial}$ be the set of interior and boundary facets. $\mathscr{F}^{\partial}_-, \mathscr{F}^{\partial}_+ \subset \mathscr{F}^{\partial}$ are the sets of facets on the inflow and outflow boundary, respectively.

We define the bilinear mapping, $(\mathbf{u}, \mathbf{v})_{f,\boldsymbol{\beta}} := \int_f (\boldsymbol{\beta} \cdot \mathbf{n}_f)(\mathbf{u} \cdot \mathbf{v})\,\mathrm{d}S$, the advection operator $\mathsf{L}_{\boldsymbol{\beta}}\,\mathbf{u} := \mathbf{grad}(\boldsymbol{\beta} \cdot \mathbf{u}) + \mathbf{curl}\,\mathbf{u} \times \boldsymbol{\beta}$ and its formal adjoint $\mathscr{L}_{\boldsymbol{\beta}}\,\mathbf{u} := \mathbf{curl}(\boldsymbol{\beta} \times \mathbf{u}) - \boldsymbol{\beta}\,\mathrm{Div}\,\mathbf{u}$. Hence, for smooth $\mathbf{u}$ and $\mathbf{v}$ we have $\left(\mathsf{L}_{\boldsymbol{\beta}}\,\mathbf{u}, \mathbf{v}\right)_{\Omega} - \left(\mathbf{u}, \mathscr{L}_{\boldsymbol{\beta}}\,\mathbf{v}\right)_{\Omega} = (\mathbf{u}, \mathbf{v})_{\partial\Omega,\boldsymbol{\beta}}$. In the following $\mathbf{V}_h$ denotes some space of piecewise polynomial vector fields that are continuous on each element of the mesh $\mathscr{T}$. For the moment we do not specify, which components are continuous and which components are discontinuous at the element interfaces. The *stabilized Galerkin method* reads as follows:

Find $\mathbf{u} \in \mathbf{V}_h$, such that:

$$\mathsf{a}(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega - \sum_{f \in \mathscr{F}^\partial_-} (\mathbf{g}, \mathbf{v})_{f,\boldsymbol{\beta}}\,, \quad \forall \mathbf{v} \in \mathbf{V}_h\,, \tag{3}$$

with

$$\begin{aligned}\mathsf{a}(\mathbf{u}, \mathbf{v}) = {} & (\alpha\mathbf{u}, \mathbf{v})_\Omega + \sum_T (\mathbf{curl}\,\mathbf{u} \times \boldsymbol{\beta}, \mathbf{v})_T - (\mathbf{u}, \boldsymbol{\beta}\mathrm{Div}\,\mathbf{v})_T \\ & + \sum_{f \in \mathscr{F}^\circ} \int_f \boldsymbol{\beta} \cdot \{\mathbf{u}\}_f [\mathbf{v}]_f \cdot \mathbf{n}_f \,\mathrm{d}S - \int_f \left([\mathbf{u}]_f \times \mathbf{n}_f\right) \cdot \left(\{\mathbf{v}\}_f \times \boldsymbol{\beta}\right) \mathrm{d}S \\ & + \sum_{f \in \mathscr{F}^\circ} \int_f c_f \boldsymbol{\beta} \cdot [\mathbf{u}]_f [\mathbf{v}]_f \cdot \mathbf{n}_f \,\mathrm{d}S + \int_f c_f \left([\mathbf{u}]_f \times \mathbf{n}_f\right) \cdot \left([\mathbf{v}]_f \times \boldsymbol{\beta}\right) \mathrm{d}S \\ & + \sum_{f \in \mathscr{F}^\partial \setminus \mathscr{F}^\partial_-} \int_f (\boldsymbol{\beta} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{n}_f) \,\mathrm{d}S - \sum_{f \in \mathscr{F}^\partial_-} \int_f \left(\mathbf{u} \times \mathbf{n}_f\right) \cdot (\mathbf{v} \times \boldsymbol{\beta}) \,\mathrm{d}S\,,\end{aligned} \tag{4}$$

We refer to [11, Section 2] for a detailed derivation of this method. There too, it is shown that the method is consistent and stable in the mesh dependent norm (with $\|\cdot\|^2_{f,\boldsymbol{\beta}} := (\mathbf{u}, \mathbf{u})_{f,\boldsymbol{\beta}}$)

$$\|\mathbf{u}\|_h^2 := \|\mathbf{u}\|^2_{L^2(\Omega)} + \sum_{f \in \mathscr{F}^\circ} \left\| [\mathbf{u}]_f \right\|^2_{f,c_f\boldsymbol{\beta}} + \sum_{f \in \mathscr{F}^\partial \setminus \mathscr{F}^\partial_-} \|\mathbf{u}\|^2_{f,\frac{1}{2}\boldsymbol{\beta}} + \sum_{f \in \mathscr{F}^\partial_-} \|\mathbf{u}\|^2_{f,-\frac{1}{2}\boldsymbol{\beta}}\,,$$

when the parameter c_f fulfills the following positivity condition.

Assumption 2 *Assume the parameters c_f in the definition (3) satisfy for all faces f the positivity condition $c_f \boldsymbol{\beta} \cdot \mathbf{n}_f > K|\boldsymbol{\beta} \cdot \mathbf{n}_f|$ for positive $K \in \mathbb{R}$.*

Lemma 1 *Let Assumptions 1 and 2 hold. Then we have for all $\mathbf{u} \in \mathbf{V}_h$:*

$$\mathsf{a}(\mathbf{u}, \mathbf{u}) \geq \min(\frac{1}{2}\alpha_0, 1)\|\mathbf{u}\|_h^2\,.$$

3 Previous Results

If we choose $\mathbf{V}_h$ in (3) to be a space of vector fields that have neither continuous tangential nor continuous normal components at the element interfaces our method coincides with the Discontinuous Galerkin method for Friedrichs' operators [11, Section 3]. The choice $c_f = \frac{\boldsymbol{\beta} \cdot \mathbf{n}_f}{|\boldsymbol{\beta} \cdot \mathbf{n}_f|}$ yields the classical upwind methods, and we can cite the following convergence result from [5, Theorem 4.6 & Corollary 4.7], [16, Theorem 50 & Corollary 12] or [11, Theorem 3.1]

Theorem 1 *Let Assumptions 1 and 2 hold. Let* $\mathbf{V}_h$ *be the finite element space of discontinuous piecewise polynomial vector fields:*

$$\mathbf{V}_h = \mathbf{V}^r_{\text{dis}} := \{\mathbf{v} \in \boldsymbol{L}^2(\Omega),\ \mathbf{v}_{|T} \in (P_r(T))^3\ ,\ T \in \mathscr{T}\}\ , \tag{5}$$

where P_r*,* $r \geq 0$ *is the space of polynomials of degree r or less. Let* $\mathbf{u} \in \boldsymbol{H}^{r+1}(\Omega)$ *and* $\mathbf{u}_h \in \mathbf{V}_h$ *be the solutions to the advection problem (1) and its variational formulation (3). We get with* $C > 0$ *depending only on* α*,* $\boldsymbol{\beta}$*,* K*, the polynomial degree and the shape regularity*

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq C h^{r+\frac{1}{2}} \|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(\Omega)}\ .$$

Surprisingly, the same rate of convergence can be shown, if $\mathbf{V}_h$ in (3) is a space of vector fields that have continuous tangential but discontinuous normal components at the element interfaces [11, Theorem 4.2].

Theorem 2 *Let Assumptions 1 and 2 hold.* P_r*,* $r \geq 0$ *is the space of polynomials of degree r or less. Let then* $\mathbf{V}_h$ *be a finite element space of* $\boldsymbol{H}(\mathbf{curl}, \Omega)$*-conforming piecewise polynomial vector fields of degree r or less:*

$$\mathbf{V}_h = \mathbf{V}^r_{\text{cnf},1} := \{\mathbf{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega)\ ,\ \mathbf{v}_{|T} \in (P_r(T))^3\ ,\ T \in \mathscr{T}\}\ ,$$

such that best approximation estimates

$$\min_{\mathbf{w}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{w}_h\|_{\boldsymbol{H}^s(T)} \leq C h^{r+1-s} \|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(T)}\ ,\ s = 0, 1\ , \forall \mathbf{u} \in \boldsymbol{H}^{r+1}(\Omega)$$

hold with constants depending only on shape regularity of the mesh, e.g., $\mathbf{V}_h$ *can belong to one of the two families of spaces proposed in [18] and [19]. Let* $\mathbf{u}$ *and* $\mathbf{u}_h \in \mathbf{V}_h$ *be the solutions to the advection problem (1) and its discrete variational formulation (3). Then, with* $C > 0$ *depending only on* α*,* $\boldsymbol{\beta}$*,* K *the polynomial degree and shape regularity, we get*

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq C h^{r+\frac{1}{2}} \|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(\Omega)}\ ,$$

provided that h is sufficiently small.

4 $\boldsymbol{H}(\mathrm{Div}, \boldsymbol{\Omega})$-Conforming Approximation

In this section we prove the main result, the optimal convergence of our method (3) when $\mathbf{V}_h$ is a space of $\boldsymbol{H}(\mathrm{Div}, \Omega)$-conforming vector fields. The proof relies on the so-called *averaging interpolation operators* mapping piecewise polynomial non-conforming vector fields to piecewise polynomial $\boldsymbol{H}(\mathbf{curl}, \Omega)$-conforming or $\boldsymbol{H}(\mathrm{Div}, \Omega)$-conforming vector fields. Similar to $\mathbf{V}^r_{\mathrm{cnf},1}$ in Theorem 2 we introduce $\mathbf{V}^r_{\mathrm{cnf},2} := \{\mathbf{v} \in \boldsymbol{H}(\mathrm{Div}, \Omega),\ \mathbf{v}_{|T} \in (P_r(T))^3,\ T \in \mathscr{T}\}$, the space of $\boldsymbol{H}(\mathrm{Div}, \Omega)$-conforming finite elements.

Proposition 1 *Let $\mathbf{u} \in \mathbf{V}^r_{\mathrm{dis}}$. Then there exist $\mathbf{u}^{c,1} \in \mathbf{V}^r_{\mathrm{cnf},1}$ and $\mathbf{u}^{c,2} \in \mathbf{V}^r_{\mathrm{cnf},2}$ such that*

$$\left\|\mathbf{u} - \mathbf{u}^{c,1}\right\|^2_{L^2(\Omega)} \leq C_1 \sum_{f \in \mathscr{F}^\circ} h_f \int_f \left|[\mathbf{u}]_f \times \mathbf{n}_f\right|^2 \,\mathrm{d}S \tag{6}$$

and

$$\left\|\mathbf{u} - \mathbf{u}^{c,2}\right\|^2_{L^2(\Omega)} \leq C_2 \sum_{f \in \mathscr{F}^\circ} h_f \int_f \left|[\mathbf{u}]_f \cdot \mathbf{n}_f\right|^2 \,\mathrm{d}S\,, \tag{7}$$

where h_f is the diameter of facet f and C_1 and C_2 depend only on the shape-regularity and the polynomial degree r, and, in particular, are independent of the mesh size.

The proof of (6) can be found in [13, Proposition 4.5] and the second assertion follows by similar arguments (see also [8, Proposition 4.1.2]).

Theorem 3 *Let Assumptions 1 and 2 hold. P_r, $r \geq 0$ is the space of polynomials of degree r or less. Let then $\mathbf{V}_h$ be a finite element space of $\boldsymbol{H}(\mathrm{Div}, \Omega)$-conforming piecewise polynomial vector fields of degree r or less:*

$$\mathbf{V}_h = \mathbf{V}^r_{\mathrm{cnf},2} := \{\mathbf{v} \in \boldsymbol{H}(\mathrm{Div}, \Omega),\ \mathbf{v}_{|T} \in (P_r(T))^3,\ T \in \mathscr{T}\}\,,$$

such that best approximation estimates

$$\min_{\mathbf{w}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{w}_h\|_{\boldsymbol{H}^s(T)} \leq C h^{r+1-s} \|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(T)}\,,\ s = 0, 1\,, \forall \mathbf{u} \in \boldsymbol{H}^{r+1}(\Omega)$$

hold with constants depending only on shape regularity of the mesh. Let $\mathbf{u}$ and $\mathbf{u}_h \in \mathbf{V}_h$ be the solutions to the advection problem (1) and its discrete variational formulation (3). Then, with $C > 0$ depending only on α, $\boldsymbol{\beta}$, K the polynomial degree and shape regularity, we get

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq C h^{r+\frac{1}{2}} \|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(\Omega)}\,,$$

provided that h is sufficiently small.

Proof Let $\bar{\mathbf{u}}_h$ denote the global L^2-projection of $\mathbf{u}$ onto $\mathbf{V}_h$ and define $\boldsymbol{\eta} := \mathbf{u} - \bar{\mathbf{u}}_h$ and $\boldsymbol{\gamma}_h := \mathbf{u}_h - \bar{\mathbf{u}}_h$. With this, the error $\|\mathbf{u} - \mathbf{u}_h\|_h$ is bounded by two terms:

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq \|\eta\|_h + \|\boldsymbol{\gamma}_h\|_h\,.$$

For the first term, by the assumptions of the theorem, we have

$$\|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(T)} \le C h^{r+1} \|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(T)} ,$$

and for the second term, by stability, consistency and $\boldsymbol{\gamma}_h \in \mathbf{V}^r_{\text{cnf},2}$:

$$\min(\frac{1}{2}\alpha_0, 1)\|\boldsymbol{\gamma}_h\|_h^2 \le \mathsf{a}(\boldsymbol{\eta}, \boldsymbol{\gamma}_h) .$$

Next, we add and subtract the Lie-derivative with respect to a piecewise constant velocity field $\boldsymbol{\beta}_h \in \mathbf{V}^0_{\text{dis}}$ that is the L^2-projection of $\boldsymbol{\beta}$ onto $\mathbf{V}^0_{\text{dis}}$:

$$\begin{aligned} \mathsf{a}(\boldsymbol{\eta}, \boldsymbol{\gamma}_h) = &\ (\alpha\boldsymbol{\eta}, \boldsymbol{\gamma}_h)_\Omega + \sum_T \left(\boldsymbol{\eta}, (\mathscr{L}_{\boldsymbol{\beta}} - \mathscr{L}_{\boldsymbol{\beta}_h})\boldsymbol{\gamma}_h\right)_T + \left(\boldsymbol{\eta}, \mathscr{L}_{\boldsymbol{\beta}_h}\boldsymbol{\gamma}_h\right)_T \\ &+ \sum_{f\in\mathscr{F}^\partial\setminus\mathscr{F}^\partial_-} (\boldsymbol{\eta}, \boldsymbol{\gamma}_h)_{f,\boldsymbol{\beta}} + \sum_{f\in\mathscr{F}^\circ} \left(\{\boldsymbol{\eta}\}_f, [\boldsymbol{\gamma}_h]_f\right)_{f,\boldsymbol{\beta}} + \left(c_f\,[\boldsymbol{\eta}]_f, [\boldsymbol{\gamma}_h]_f\right)_{f,\boldsymbol{\beta}} . \end{aligned}$$

Yet, as $\sum_T \left(\boldsymbol{\eta}, \mathscr{L}_{\boldsymbol{\beta}_h}\boldsymbol{\gamma}_h\right)_T \neq 0$, the difficult part is now to show

$$\left|\sum_T (\boldsymbol{\eta}, \mathbf{curl}(\boldsymbol{\gamma}_h \times \boldsymbol{\beta}_h) + \boldsymbol{\beta}_h \operatorname{Div}\boldsymbol{\gamma}_h)_T\right| \le C h^{-\frac{1}{2}} \|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{\gamma}_h\|_h . \tag{8}$$

Let $\mathbf{w}^{c,1} \in \mathbf{V}^r_{\text{cnf},1}$ and $\mathbf{w}^{c,2} \in \mathbf{V}^r_{\text{cnf},2}$ be the conforming approximations of $\boldsymbol{\beta}_h \times \boldsymbol{\gamma}_h \in \mathbf{V}^r_{\text{dis}}$ and $\boldsymbol{\beta}_h \operatorname{Div}\boldsymbol{\gamma}_h \in \mathbf{V}^r_{\text{dis}}$. Since $\boldsymbol{\eta} = \mathbf{u} - \bar{\mathbf{u}}_h$ and both $\mathbf{curl}\,\mathbf{w}^{c,1} \in \mathbf{V}^r_{\text{cnf},2}$ and $\mathbf{w}^{c,2} \in \mathbf{V}^r_{\text{cnf},2}$ we find

$$\left|\sum_T (\boldsymbol{\eta}, \mathbf{curl}(\boldsymbol{\gamma}_h \times \boldsymbol{\beta}_h))_T\right| \le C_0 h^{-1} \|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)} \left\|\boldsymbol{\gamma}_h \times \boldsymbol{\beta}_h - \mathbf{w}^{c,1}\right\|_{\boldsymbol{L}^2(\Omega)}$$

and

$$\left|\sum_T (\boldsymbol{\eta}, \boldsymbol{\beta}_h \operatorname{Div}\boldsymbol{\gamma}_h)_T\right| \le \|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)} \left\|\boldsymbol{\beta}_h \operatorname{Div}\boldsymbol{\gamma}_h - \mathbf{w}^{c,2}\right\|_{\boldsymbol{L}^2(\Omega)} .$$

The approximation results (6) and (7) give

$$\left\|\boldsymbol{\gamma}_h \times \boldsymbol{\beta}_h - \mathbf{w}^{c,1}\right\|^2_{\boldsymbol{L}^2(\Omega)} \le C_1 h \sum_{f\in\mathscr{F}^\circ} \left\|[\boldsymbol{\gamma}_h \times \boldsymbol{\beta}_h]_f \times \mathbf{n}_f\right\|^2_{\boldsymbol{L}^2(f)}$$

and

$$\left\|\boldsymbol{\beta}_h \operatorname{Div} \boldsymbol{\gamma}_h - \mathbf{w}^{c,2}\right\|^2_{\boldsymbol{L}^2(\Omega)} \le C_2 h \sum_{f \in \mathscr{F}^\circ} \left\| [\boldsymbol{\beta}_h \operatorname{Div} \boldsymbol{\gamma}_h]_f \cdot \mathbf{n}_f \right\|^2_{\boldsymbol{L}^2(f)}.$$

Inverse inequalities, approximation properties of $\boldsymbol{\beta}_h$, normal continuity of $\boldsymbol{\gamma}_h$ and tangential continuity yield for the right hand sides of the last two equations:

$$\begin{aligned}
\left\| [\boldsymbol{\gamma}_h \times \boldsymbol{\beta}_h]_f \times \mathbf{n}_f \right\|_{\boldsymbol{L}^2(f)} &\le \left\| [\boldsymbol{\gamma}_h \times (\boldsymbol{\beta}_h - \boldsymbol{\beta})]_f \times \mathbf{n}_f \right\|_{\boldsymbol{L}^2(f)} + \left\| [\boldsymbol{\gamma}_h \times \boldsymbol{\beta}]_f \times \mathbf{n}_f \right\|_{\boldsymbol{L}^2(f)} \\
&\le C_3 h^{\frac{1}{2}} \|\boldsymbol{\gamma}_h\|_{\boldsymbol{L}^2(T_1 \cup T_2)} + \left\| \boldsymbol{\beta} \cdot \mathbf{n}_f \, [\boldsymbol{\gamma}_h]_f - [\boldsymbol{\gamma}_h]_f \cdot \mathbf{n}_f \boldsymbol{\beta} \right\|_{\boldsymbol{L}^2(f)} \\
&\le C_3 h^{\frac{1}{2}} \|\boldsymbol{\gamma}_h\|_{\boldsymbol{L}^2(T_1 \cup T_2)} + \left\| \boldsymbol{\beta} \cdot \mathbf{n}_f \, [\boldsymbol{\gamma}_h]_f \right\|_{\boldsymbol{L}^2(f)}
\end{aligned}$$

and

$$\begin{aligned}
\left\| [\boldsymbol{\beta}_h \operatorname{Div} \boldsymbol{\gamma}_h]_f \cdot \mathbf{n}_f \right\|_{\boldsymbol{L}^2(f)} &\le \left\| [(\boldsymbol{\beta}_h - \boldsymbol{\beta}) \operatorname{Div} \boldsymbol{\gamma}_h]_f \cdot \mathbf{n}_f \right\|_{\boldsymbol{L}^2(f)} + \left\| \boldsymbol{\beta} \cdot \mathbf{n}_f \, [\operatorname{Div} \boldsymbol{\gamma}_h]_f \right\|_{\boldsymbol{L}^2(f)} \\
&\le C_4 h^{-\frac{1}{2}} \|\boldsymbol{\gamma}_h\|_{\boldsymbol{L}^2(T_1 \cup T_2)} + C_5 h^{-1} \left\| \boldsymbol{\beta} \cdot \mathbf{n}_f \, [\boldsymbol{\gamma}_h]_f \right\|_{\boldsymbol{L}^2(f)},
\end{aligned}$$

with constants C_3, C_4 and C_5 independent of h, and T_1 and T_2 those elements that share f. Hence we deduce (8), and the assertion follows.

We refer to [8, Section 4.1.4] for detailed numerical experiments for test cases with both smooth and non-smooth solutions.

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