Space-Time Discontinuous Galerkin Method for the Problem of Linear Elasticity

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Abstract The subject of this paper is the numerical solution of the problem of dynamic linear elasticity by several time-discretization techniques based on the application of the discontinuous Galerkin (DG) method in space. In the formulation of the numerical scheme, the nonsymmetric, symmetric and incomplete versions of the discretization of the elasticity term and the interior and boundary penalty are used. The DG space discretization is combined with the backward-Euler, second-order backward-difference formula and DG time discretization. Finally, we present some test problems.

1 Introduction

This paper is concerned with the application of the discontinuous Galerkin (DG) method to the solution of dynamic linear elasticity problem. (For a survey of DG techniques, see, e.g., [2, 4].) The DG space discretization is combined with the time discretization by the backward Euler (BEDG), second-order BDF (BDFDG) or DG scheme in time (STDG).

In [3], the method using the DG technique in time, but conforming finite elements in space is analyzed in the case of a linear wave equation. Here we are not interested in the computation of wave propagation in an elastic body, but our future goal will be to apply the developed method, which is different from the scheme analyzed in [3], to the solution of the interaction of a fluid and an elastic body.

We describe the mentioned methods and apply them to a test problem in order to compare their quality. Numerical experiments show that the STDG method is

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the most promising. Our further work will be oriented to a deeper analysis of the developed method and its applications to fluid-structure interaction (FSI) problems.

2 Formulation of the Dynamic Elasticity Problem

We consider an elastic body represented by a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary formed by two disjoint parts: $\partial\Omega = \Gamma_D \cup \Gamma_N$. By $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x}) : [0, T] \times \Omega \to \mathbb{R}^2$ we denote the displacement of the body. The symbol $\nabla \boldsymbol{u} = \left(\partial u_i / \partial x_j\right)_{i,j=1}^2$ denotes the gradient of the function $\boldsymbol{u}$. The dynamic elasticity problem is defined as follows: we seek for the displacement function $\boldsymbol{u}$ such that

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + c_M \rho \frac{\partial \boldsymbol{u}}{\partial t} - \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) - c_K \frac{\partial}{\partial t} \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } (0, T) \times \Omega, \tag{1}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{in } (0, T) \times \Gamma_D, \quad \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{g}_N \quad \text{in } (0, T) \times \Gamma_N, \tag{2}$$

$$\boldsymbol{u}(0, x) = \boldsymbol{u}_0(x), \quad \frac{\partial \boldsymbol{u}}{\partial t}(0, x) = z_0(x), \quad \text{in } \Omega. \tag{3}$$

Here $\boldsymbol{f} : (0, T) \times \Omega \to \mathbb{R}^2$ is the outer volume force, $\boldsymbol{u}_D : (0, T) \times \Gamma_D \to \mathbb{R}^2$ is the boundary displacement, $\boldsymbol{g}_N : (0, T) \times \Gamma_N \to \mathbb{R}^2$ is the boundary normal stress, $\boldsymbol{u}_0 : \Omega \to \mathbb{R}^2$ is the initial displacement, $z_0 : \Omega \to \mathbb{R}^2$ is the initial displacement velocity, $T > 0$ is the time interval length and $\rho > 0$ is the constant material density. The expressions $c_M \rho \frac{\partial \boldsymbol{u}}{\partial t}$ and $c_K \frac{\partial}{\partial t} \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u})$ with $c_M,\ c_K \geq 0$ represent structural and viscous damping terms. We assume that the material is *isotropic* and *homogeneous* and that the *stress tensor* $\boldsymbol{\sigma}(\boldsymbol{u})$ depends on the *infinitesimal strain tensor* $\boldsymbol{e}(\boldsymbol{u})$ by the relation

$$\boldsymbol{\sigma}(\boldsymbol{u}) := \lambda \operatorname{tr}(\boldsymbol{e}(\boldsymbol{u}))\boldsymbol{I} + 2\mu \boldsymbol{e}(\boldsymbol{u}), \quad \boldsymbol{e}(\boldsymbol{u}) := \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right). \tag{4}$$

We assume that the *Lamè parameters* λ and μ are constant. For most solid materials it holds that $\lambda, \mu > 0$. Finally, $\operatorname{tr}(\boldsymbol{e}(\boldsymbol{u}))$ denotes the trace of the tensor $\boldsymbol{e}(\boldsymbol{u})$.

3 Discretization

In order to introduce the discrete problem we rewrite problem (1)–(3) as a couple of first-order equations in time: find functions $\boldsymbol{u}$ and $z : [0, T] \times \Omega \to \mathbb{R}^2$ such that

$$\varrho \frac{\partial z}{\partial t} + c_M \varrho z - \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) - c_K \operatorname{div} \boldsymbol{\sigma}(z) = \boldsymbol{f}, \tag{5}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} - z = 0 \quad \text{in } (0, T) \times \Omega, \tag{6}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{in } (0,T)\times\Gamma_D, \quad \boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{n} = \boldsymbol{g}_N \quad \text{in } (0,T)\times\Gamma_N, \tag{7}$$

$$\boldsymbol{u}(0,x) = \boldsymbol{u}_0(x), \quad z(0,x) = z_0(x) \quad \text{in } \Omega. \tag{8}$$

3.1 Notation

Let us assume that the computational domain Ω is polygonal. By $\mathscr{T}_h$ we denote a triangulation of the domain Ω with triangular elements $K \in \mathscr{T}_h$ having standard properties from the finite element method, cf. [1].

We say that the elements $K, K' \in \mathscr{T}_h$ are *neighbours*, if the set $\partial K \cap \partial K'$ has positive 1-dimensional measure. We say that $\Gamma \subset \partial K$ is a *face* of K, if it is a maximal connected open subset of either $\partial K \cap \partial K'$, where K' is a neighbour of K or of $\partial K \cap \Gamma_D$ or of $\partial K \cap \Gamma_N$. By $\mathscr{F}_h$ we denote the system of all faces of all elements $K \in \mathscr{T}_h$. Further, we define the set of all boundary, "Dirichlet", "Neumann" and inner faces by

$$\mathscr{F}_h^B = \{\Gamma \in \mathscr{F}_h; \Gamma \subset \partial\Omega\}, \quad \mathscr{F}_h^D = \{\Gamma \in \mathscr{F}_h; \Gamma \subset \Gamma_D\},$$
$$\mathscr{F}_h^N = \{\Gamma \in \mathscr{F}_h; \Gamma \subset \Gamma_N\}, \quad \mathscr{F}_h^I = \mathscr{F}_h \backslash \mathscr{F}_h^B,$$

respectively. We put $\mathscr{F}_h^{ID} = \mathscr{F}_h^I \cup \mathscr{F}_h^D$. For each $\Gamma \in \mathscr{F}_h$ we define a unit normal vector $\boldsymbol{n}_\Gamma$. We assume that for $\Gamma \in \mathscr{F}_h^B$ the normal $\boldsymbol{n}_\Gamma$ has the same orientation as the outer normal to $\partial\Omega$. For each $\Gamma \in \mathscr{F}_h^I$ the orientation of $\boldsymbol{n}_\Gamma$ is arbitrary, but fixed.

We define the finite dimensional space

$$S_{hp} = \left\{v \in L^2(\Omega); v|_K \in P^p(K), K \in \mathscr{T}_h\right\},$$

where $p \geq 1$ is an integer and $P^p(K)$ denotes the space of all polynomials on K of degree $\leq p$. It is easy to show that $\dim S_{hp} = N_{\mathscr{T}_h}(p+1)(p+2)/2$, where $N_{\mathscr{T}_h}$ is the number of elements in $\mathscr{T}_h$.

Because of the time discretization we introduce a uniform partition $0 = t_0 < \cdots < t_M = T$ of the time interval $[0,T]$ with a constant time step $\tau = t_m - t_{m-1}$, $m = 1,\ldots,M$. Let $p \geq 1$, $q \geq 0$ be integers. By $S_{hp}^{\tau q}$ we denote the space of piecewise polynomial functions

$$S_{hp}^{\tau q} = \left\{ v \in L^2((0,T)\times\Omega); v|_{I_m} = \sum_{i=0}^{q} t^i \varphi_i \text{ with } \varphi_i \in S_{hp}, m = 1,\ldots,M \right\},$$

where $I_m = (t_{m-1}, t_m)$, $m = 1,\ldots,M$. The space $S_{hp}^{\tau q}$ consists of all polynomials of degree less or equal to q in time with coefficients in S_{hp}. As we see, functions from $S_{hp}^{\tau q}$ are, in general, discontinuous on faces $\Gamma \in \mathscr{F}^I$ and at time instants $t_m,\ m = 1, \ldots, M-1$. The dimension of the space $S_{hp}^{\tau q}$ equals $M(q+1)\dim S_{hp}$.

For each $\Gamma \in \mathscr{F}_h^I$ there exist two neighbouring elements $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathscr{T}_h$ such that $\Gamma \subset \partial K_\Gamma^{(L)} \cap \partial K_\Gamma^{(R)}$. We use the convention that $\boldsymbol{n}_\Gamma$ is the outer normal to $\partial K_\Gamma^{(L)}$ and the inner normal to $\partial K_\Gamma^{(R)}$. For $\boldsymbol{v} \in [S_{hp}]^2$ or $[S_{hp}^{\tau q}]^2$ we introduce the following notation:

$$\boldsymbol{v}|_\Gamma^{(L)} = \text{the trace of } \boldsymbol{v}|_{K_\Gamma^{(L)}} \text{ on } \Gamma,\ \boldsymbol{v}|_\Gamma^{(R)} = \text{the trace of } \boldsymbol{v}|_{K_\Gamma^{(R)}} \text{ on } \Gamma,$$
$$\langle \boldsymbol{v} \rangle_\Gamma = \tfrac{1}{2}\left(\boldsymbol{v}|_\Gamma^{(L)} + \boldsymbol{v}|_\Gamma^{(R)}\right), \qquad [\boldsymbol{v}]_\Gamma = \boldsymbol{v}|_\Gamma^{(L)} - \boldsymbol{v}|_\Gamma^{(R)},$$

where $\Gamma \in \mathscr{F}_h^I$. For $\Gamma \in \mathscr{F}_h^B$ there exists an element $K_\Gamma^{(L)} \in \mathscr{T}_h$ such that $\Gamma \subset K_\Gamma^{(L)} \cap \partial\Omega$. Then for $\boldsymbol{v} \in [S_{hp}]^2$ we introduce the following notation:

$$\boldsymbol{v}|_\Gamma^{(L)} = \text{the trace of } \boldsymbol{v}|_{K_\Gamma^{(L)}} \text{ on } \Gamma, \quad \langle \boldsymbol{v} \rangle_\Gamma = [\boldsymbol{v}]_\Gamma = \boldsymbol{v}|_\Gamma^{(L)}.$$

In case that $[\cdot]_\Gamma$, $\langle \cdot \rangle_\Gamma$ and $\boldsymbol{n}_\Gamma$ appear in integrals $\int_\Gamma \ldots\ \mathrm{d}S$, $\Gamma \in \mathscr{F}_h$, we omit the subscript Γ and simply write $[\cdot]$, $\langle \cdot \rangle$ and $\boldsymbol{n}$, respectively.

Finally, by $T : S$ we shall denote the tensor inner product, defined by

$$T : S = \sum_{i=1}^{2}\sum_{j=1}^{2} T_{ij}S_{ij} = \mathrm{tr}\left(T^T S\right),\ S,\ T \in \mathbb{R}^{2\times 2}.$$

3.2 Space Discretization

We begin with the space discretization of the dynamic elasticity problem. An approximate solution of problem (5)–(8), i.e., the approximations of the functions $\boldsymbol{u}$, $\boldsymbol{z}$ will be sought in the space $\mathscr{V} := [S_{hp}]^2$ in the finite-difference based schemes or $\mathscr{V} := [S_{hp}^{\tau q}]^2$ in the space-time discontinuous Galerkin method.

In the first step, we multiply Eqs. (5)–(6) by test functions $\boldsymbol{v}$ and $\boldsymbol{w} \in \mathscr{V}$, respectively, integrate the resulting equations over $K \in \mathscr{T}_h$, sum the resulting equations over all $K \in \mathscr{T}_h$ and use the following relations. Using Green's theorem, we obtain the equality

$$-\sum_{K\in\mathscr{T}_h}\int_K \mathrm{div}\,\boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{v}\,\mathrm{d}x = \sum_{K\in\mathscr{T}_h}\int_K \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{v})\,\mathrm{d}x$$
$$-\sum_{\Gamma\in\mathscr{F}_h^{ID}}\int_\Gamma (\langle\boldsymbol{\sigma}(\boldsymbol{u})\rangle\cdot\boldsymbol{n})\cdot[\boldsymbol{v}]\,\mathrm{d}S - \sum_{\Gamma\in\mathscr{F}_h^{N}}\int_\Gamma \boldsymbol{g}_N\cdot\boldsymbol{v}\,\mathrm{d}S.$$

The interior and boundary penalty methods incorporate the fact that for the exact solution we have $[\boldsymbol{u}]_\Gamma = 0$ for each $\Gamma \in \mathscr{F}_h^I$ and $\boldsymbol{u}$ satisfies the Dirichlet condition in (7). Hence,

$$\sum_{\Gamma \in \mathscr{F}_h^{ID}} \int_\Gamma \frac{C_W}{h_\Gamma}[\boldsymbol{u}] \cdot [\boldsymbol{v}] \,\mathrm{d}S = \sum_{\Gamma \in \mathscr{F}_h^{D}} \int_\Gamma \frac{C_W}{h_\Gamma}\boldsymbol{u}_D \cdot \boldsymbol{v} \,\mathrm{d}S$$

for each $\boldsymbol{v} \in \mathscr{V}$, where $C_W > 0$ is a given parameter and h_Γ represents the "magnitude" of Γ as, for example, the length of Γ of $h_\Gamma = \left(h_{K_\Gamma}^{(L)} + h_{K_\Gamma}^{(R)}\right)/2$.

Finally, for the exact solution $\boldsymbol{u}$ and arbitrary $\boldsymbol{v} \in \mathscr{V}$ we have

$$\sum_{\Gamma \in \mathscr{F}_h^{ID}} \int_\Gamma (\langle \boldsymbol{\sigma}(\boldsymbol{v}) \rangle \cdot \boldsymbol{n}) \cdot [\boldsymbol{u}] \,\mathrm{d}S = \sum_{\Gamma \in \mathscr{F}_h^{D}} \int_\Gamma (\boldsymbol{\sigma}(\boldsymbol{v}) \cdot \boldsymbol{n}) \cdot \boldsymbol{u}_D \,\mathrm{d}S.$$

We now define the forms $a_h(\boldsymbol{u},\boldsymbol{v}) : \mathscr{V} \times \mathscr{V} \to \mathbb{R}$, $\ell_h^s(\boldsymbol{v}) : \mathscr{V} \to \mathbb{R}$ and $\ell_h^e(\boldsymbol{v}) : \mathscr{V} \to \mathbb{R}$ by

$$\begin{aligned} a_h(\boldsymbol{u},\boldsymbol{v}) = & \sum_{K \in \mathscr{T}_h} \int_K \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{v}) \,\mathrm{d}x - \sum_{\Gamma \in \mathscr{F}^{ID}} \int_\Gamma (\langle \boldsymbol{\sigma}(\boldsymbol{u}) \rangle \cdot \boldsymbol{n}) \cdot [\boldsymbol{v}] \,\mathrm{d}S \\ & - \theta \sum_{\Gamma \in \mathscr{F}^{ID}} \int_\Gamma (\langle \boldsymbol{\sigma}(\boldsymbol{v}) \rangle \cdot \boldsymbol{n}) \cdot [\boldsymbol{u}] \,\mathrm{d}S + \sum_{\Gamma \in \mathscr{F}^{ID}} \int_\Gamma \frac{C_W}{h_\Gamma} [\boldsymbol{u}] \cdot [\boldsymbol{v}] \,\mathrm{d}S, \end{aligned} \tag{9}$$

$$\ell_h^s(\boldsymbol{v}) = \sum_{K \in \mathscr{T}_h} \int_K \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x, \tag{10}$$

$$\begin{aligned} \ell_h^e(\boldsymbol{v}) = & \sum_{\Gamma \in \mathscr{F}^{N}} \int_\Gamma \boldsymbol{g}_N \cdot \boldsymbol{v} \,\mathrm{d}S - \theta \sum_{\Gamma \in \mathscr{F}^{D}} \int_\Gamma (\boldsymbol{\sigma}(\boldsymbol{v}) \cdot \boldsymbol{n}) \cdot \boldsymbol{u}_D \,\mathrm{d}S \\ & + \sum_{\Gamma \in \mathscr{F}^{D}} \int_\Gamma \frac{C_W}{h_\Gamma} \boldsymbol{u}_D \cdot \boldsymbol{v} \,\mathrm{d}S. \end{aligned} \tag{11}$$

The parameter θ defines the symmetric ($\theta = 1$), incomplete ($\theta = 0$) and nonsymmetric ($\theta = -1$) variant of the interior penalty DG method.

Application of these formulas yields the system

$$\begin{aligned} \rho \left(\frac{\partial \boldsymbol{z}_h}{\partial t}, \boldsymbol{v}_h \right)_\Omega & + c_M \rho \, (\boldsymbol{z}_h, \boldsymbol{v}_h)_\Omega + a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + c_K a_h(\boldsymbol{z}_h, \boldsymbol{v}_h) \\ & = \ell_h^s(\boldsymbol{v}_h) + \ell_h^e(\boldsymbol{v}_h) + c_K \ell_h^{e,dt}(\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \mathscr{V}, \end{aligned} \tag{12}$$

$$\left(\frac{\partial \boldsymbol{u}_h}{\partial t}, \boldsymbol{w}_h \right)_\Omega - (\boldsymbol{z}_h, \boldsymbol{w}_h)_\Omega = 0 \quad \forall \boldsymbol{w}_h \in \mathscr{V}. \tag{13}$$

The term $\ell_h^{e,dt}(\boldsymbol{v}_h)$ is defined similarly as $\ell_h^e(\boldsymbol{v}_h)$ with the exception that the functions $\boldsymbol{u}_D$, $\boldsymbol{g}_N$ are replaced with $\partial \boldsymbol{u}_D/\partial t$, $\partial \boldsymbol{g}_N/\partial t$, respectively. By $(\cdot,\cdot)_\Omega$ we denote the $[L^2(\Omega)]^2$-scalar product.

3.3 Time Discretization

We consider two schemes based on finite-difference approximations in time. The process of the derivation of the full discretization is well-known and hence we shall only present the finite-difference approximations here. The backward-Euler (BE) scheme is based on the approximation

$$\frac{\partial \boldsymbol{u}}{\partial t}(t) \approx \frac{\boldsymbol{u}(t) - \boldsymbol{u}(t-\tau)}{\tau}.$$

The second finite-difference scheme is based on the second-order backward-difference formula

$$\frac{\partial \boldsymbol{u}}{\partial t}(t) \approx \frac{3\boldsymbol{u}(t) - 4\boldsymbol{u}(t-\tau) + \boldsymbol{u}(t-2\tau)}{2\tau}.$$

In order to define the space-time discontinuous Galerkin method, let us introduce the one-sided limits and the jump of a function $\boldsymbol{v} \in [S_{h\tau}^{pq}]^2$ at time t_m:

$$\boldsymbol{v}_m^+ = \lim_{s\to 0_+} \boldsymbol{v}(t_m + s), \quad \boldsymbol{v}_m^- = \lim_{s\to 0_+} \boldsymbol{v}(t_m - s), \quad \{\boldsymbol{v}\}_m = \boldsymbol{v}_m^+ - \boldsymbol{v}_m^-. \tag{14}$$

The approximate space-time DG solution of problem (5)–(8) is defined as a couple $\boldsymbol{u}_{h\tau}, \boldsymbol{z}_{h\tau} \in [S_{h\tau}^{pq}]^2$ satisfying

$$\begin{aligned}
&\int_{I_m} \left(\rho\left(\frac{\partial \boldsymbol{z}_{h\tau}}{\partial t}, \boldsymbol{v}_{h\tau}\right)_\Omega + c_M \rho\,(\boldsymbol{z}_{h\tau}, \boldsymbol{v}_{h\tau})_\Omega + a_h(\boldsymbol{u}_{h\tau}, \boldsymbol{v}_{h\tau})\right. \\
&\qquad \left. + c_K a_h(\boldsymbol{z}_{h\tau}, \boldsymbol{v}_{h\tau})\right) \mathrm{d}t + (\{\boldsymbol{z}_{h\tau}\}_{m-1}, \boldsymbol{v}_{h\tau}(t_{m-1}+))_\Omega \\
&\qquad = \int_{I_m} \left(\ell_h^s(\boldsymbol{v}_{h\tau}) + \ell_h^e(\boldsymbol{v}_{h\tau}) + c_K \ell_h^{e,dt}(\boldsymbol{v}_{h\tau})\right) \mathrm{d}t \quad \forall \boldsymbol{v}_{h\tau} \in [S_{h\tau}^{sq}]^2, \\
&\int_{I_m} \left(\left(\frac{\partial \boldsymbol{u}_{h\tau}}{\partial t}, \boldsymbol{w}_{h\tau}\right)_\Omega - (\boldsymbol{z}_{h\tau}, \boldsymbol{w}_{h\tau})_\Omega\right) \mathrm{d}t + (\{\boldsymbol{u}_{h\tau}\}_{m-1}, \boldsymbol{w}_{h\tau}(t_{m-1}+))_\Omega = 0 \\
&\qquad \forall \boldsymbol{w}_{h\tau} \in [S_{h\tau}^{sq}]^2, \quad m = 1, \ldots, M.
\end{aligned} \tag{15}$$

The initial states $\boldsymbol{u}_h(0-)$, $\boldsymbol{z}_h(0-) \in [S_{hp}]^2$ are defined by $(\boldsymbol{u}_h(0-), \boldsymbol{v}_h)_\Omega = (\boldsymbol{u}_0, \boldsymbol{v}_h)_\Omega$, $(\boldsymbol{z}_h(0-), \boldsymbol{v}_h)_\Omega = (\boldsymbol{z}_0, \boldsymbol{v}_h)_\Omega$ for all $\boldsymbol{v}_h \in [S_{hp}]^2$.

In all three cases, the resulting linear systems are solved using the direct solver UMFPACK.

4 Numerical Experiments

Here we present numerical results for a simple model problem solved by the STDG method with $\theta = -1$ (nonsymmetric version of the space discretization). We assume that the domain Ω is represented by a rectangular elastic material, which is 2 cm long and 2 mm wide. We consider the following material properties: density $\varrho = 1{,}100\,\mathrm{kg.m^{-3}}$, Young's modulus $E = 10^5\,\mathrm{kg.m^{-1}.s^{-2}}$, Poisson's ratio $\nu = 0.4$. The Lamè parameters λ and μ can be computed from E and ν by relations $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, $\mu = \frac{E}{2(1+\nu)}$. These parameters correspond to a very soft, rubber-like material. The material is exposed to a horizontal surface force in the direction of the negative part of the x_1-axis for a short period of time. The lower left corner of the domain is at the point $[-0.001, -0.01]$ and the upper right corner is at the point $[0.001, 0.01]$. On the fixed part of the boundary, where $x_2 = -0.01$ and which is denoted by Γ_D, we prescribe the Dirichlet boundary condition (2) with $\boldsymbol{u}_D = \mathbf{0}$. The Neumann boundary condition (2) is prescribed on the rest of the boundary $\partial\Omega$, denoted by Γ_N, where we put $\boldsymbol{g}_N = (-20, 0)^T$ for $t < 0.02\,\mathrm{s}$, $x_1 = 0.001$ and $x_2 > 0.005$, and $\boldsymbol{g}_N = \mathbf{0}$ otherwise. Finally, we set $T = 0.5\,\mathrm{s}$, $c_M = 0.1\,\mathrm{s}^{-1}$ and $c_K = 0$.

Figure 1 shows the model scheme and the time evolution of the computed displacement at several time instants. Figure 2 shows the evolution of the displacement at the fixed spatial point $[-0.001, 0.01]$ (upper left corner) obtained by the three presented numerical methods. Here STDGM 1 denotes the space-time discontinuous Galerkin method with the time polynomial degree $q = 1$. For all the computations the space polynomial degree p was set to 1, i.e. linear elements in space were used.

Fig. 1 Schema of the model problem (*left*) and the visualization of the evolution of the displacement function $\boldsymbol{u}$ at the time instants $t = 0, 0.01, 0.02, 0.03$ s

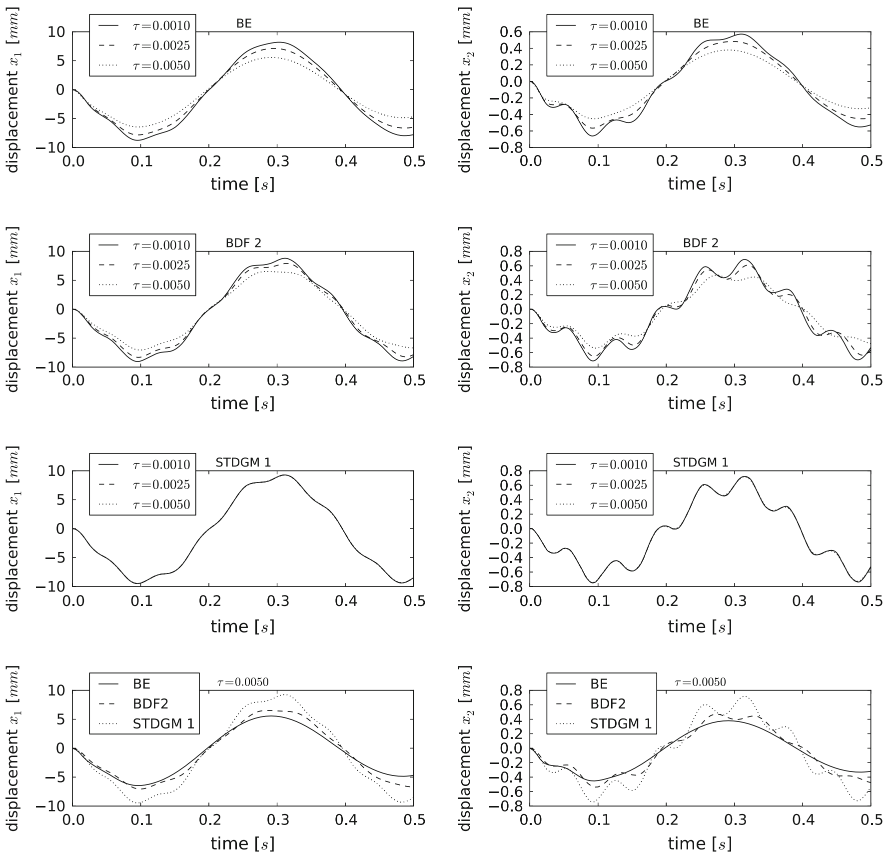

Fig. 2 The evolution of the displacement function $\boldsymbol{u}$ at the fixed point $[-0.001, 0.01]$

Conclusion

We have presented several different discretizations for the problem of dynamic linear elasticity based on the discontinuous Galerkin semi-discretization in space. A special attention was paid to the space-time discontinuous Galerkin method, which is based on the piecewise polynomial approximation of the sought function both in space and in time. The presented numerical example shows promising convergence properties of this method. For a given time step τ the error of the numerical solution obtained by the STDG method with $q = 1$ is lower than the error of the solution obtained by the method based on the second-order BDF method, which is of equal theoretical order of convergence. On the other hand, the STDG method is

(continued)

more expensive in terms of the computational time. This is caused by a larger system of linear algebraic equations, which has to be solved at each time level, and by the quadrature rules, which have to be applied not only in space but also in time.

The future work will be focused towards the analysis of the convergence of the STDG method and its comparison with other methods on more sophisticated test problems.

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