Weakly Symmetric Mixed Finite Elements for Linear Elasticity

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Abstract The approximation of the equations of linear elasticity by so-called weakly symmetric mixed methods is considered. It is shown that the technique of mesh dependent norms yields a natural and elementary error analysis of the methods. The technique is applied to several families of methods.

1 Introduction

During the last decade the theory of mixed finite element methods have been recast with the aid of differential geometry, cf. [6]. This was first done for methods for scalar second order elliptic equation, i.e., the Raviart-Thomas-Nédélec [29, 30] and the Brezzi-Douglas-Marini-Duran-Fortin [14, 15] families. Lately, the theory has been extended to methods for linear elasticity [17]. Both methods with a symmetric approximation for the stress tensor [5, 19, 24, 33] and methods where the symmetry is imposed weakly [1, 4, 6, 18, 20, 21, 31], have been analyzed.

The purpose of this paper is to highlight an alternative and more elementary way of analysis, which, nevertheless, gives optimal error estimates. The approach is that of using mesh dependent norms, first used by Babuška, Osborn and Pitkäranta [11]. In this, the norm used for the "stress" variable is the L_2-norm, which has the physical meaning of energy. For the "displacement" variable the broken H^1-norm (now well-known from Discontinuous Galerkin Methods) is used. The stability of the methods follows directly from local scaling arguments. The second ingredient is the so-called "equilibrium condition," which the methods fulfill. Using these, the quasi-optimal error estimate for the stress follows by the classical saddle point theory.

Our exposition follows the talk at the conference, and we consider mixed methods for weakly symmetric elasticity elements. We write down the continuous form of the problem, for which we first prove the uniqueness. Then we prove the stability in the natural energy norms. Then we turn to the mixed method, and we follow exactly the same approach to show the uniqueness and stability of the

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method. In this we use only elementary finite element techniques, but nevertheless, we obtain the optimal error estimates.

For the displacement the analysis yields a superconvergence result for the distance between the L_2 projection onto the discrete space and the finite element solution. This is utilized to postprocess the displacement yielding an approximation of two polynomial degrees higher, with an optimal convergence rate.

2 The Equations of Elasticity

We consider the equations of linear elasticity for which we, for simplicity, assume a unit Young's modulus $E = 1$, and a vanishing Poisson ratio $\nu = 0$. The unknown are the symmetric stress tensor $\sigma \in \mathbb{R}^{d\times d}$, and the displacement $u \in \mathbb{R}^d$.

The displacement gradient is the sum of its symmetric and skew-symmetric parts, the strain and the rotation tensors:

$$\nabla u = \varepsilon(u) + \omega(u)$$

with

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$$

and

$$\omega(u) = \frac{1}{2}(\nabla u - \nabla u^T).$$

Note that symmetric and skew-symmetric tensors are orthogonal. The linear elasticity problem in mixed form is then [23]: Find σ and u such that

$$\begin{aligned} \sigma - \varepsilon(u) &= 0, \\ \operatorname{div}\sigma + f &= 0 \ \text{ in } \Omega, \\ u &= 0 \ \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

Mixed finite elements based on this formulation are rather complicated to construct, cf. [5, 19, 24, 33]. The simplest elements are composite, and the requirement of pure polynomials lead to elements of high degree [2, 7].

It is, however, possible to design simpler elements if one treats the symmetry of the stress tensor as an independent equation [4, 18, 31]. Mechanically, this is natural, since the symmetry of the stress tensor is the condition of moment equilibrium [23].

With one more equation, an additional unknown is needed, and this is the skew-symmetric rotation, which we denote by ρ. The system of equations is then

$$\begin{aligned} \sigma + \rho - \nabla u &= 0, \\ \sigma - \sigma^T &= 0, \\ \operatorname{div} \sigma + f &= 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{2}$$

Comparing (1) and (2) we see that they are equivalent with $\rho = \omega(u)$.

The weak formulation is: Find $(\sigma, \rho, u) \in [L^2(\Omega)]^{d\times d} \times [L^2(\Omega)]^{d\times d}_{\text{skw}} \times [H^1_0(\Omega)]^d$ such that

$$\begin{aligned} (\sigma, \tau) + (\rho, \tau) - (\nabla u, \tau) &= 0 \quad \forall \tau \in [L^2(\Omega)]^{d\times d} \\ (\sigma, \eta) &= 0 \quad \forall \eta \in [L^2(\Omega)]^{d\times d}_{\text{skw}} \\ (\sigma, \nabla v) &= (f, v) \quad \forall v \in [H^1_0(\Omega)]^d. \end{aligned} \tag{3}$$

The first thing to check, is the uniqueness of solution to these equations.

Theorem 1 *The solution to (3) is unique.*

Proof We have to prove that $f = 0$ implies $u = 0$, $\rho = 0$ and $\sigma = 0$. To this end, we first choose $\eta = \sigma - \sigma^T$ in the second equation. The orthogonality of symmetric and skew-symmetric tensors then yields

$$0 = (\sigma, \eta) = (\sigma, \sigma - \sigma^T) = \frac{1}{2}\|\sigma - \sigma^T\|_0^2$$

implying

$$\sigma = \sigma^T.$$

Next, choosing $\tau = \sigma$ in the first equation and $v = u$ in the third, we get (as $f = 0$), again using the orthogonality,

$$\|\sigma\|_0^2 - (\varepsilon(u), \sigma) = 0 \text{ and } (\sigma, \nabla u) = (\sigma, \varepsilon(u)) = 0.$$

Hence

$$\sigma = 0.$$

The first equation now reduces to

$$(\rho, \tau) - (\nabla u, \tau) = 0,$$

the symmetric and skew-symmetric parts of which are

$$(\tau_{\mathrm{sym}}, \varepsilon(u)) = 0 \quad \text{and} \quad (\rho - \omega(u), \tau_{\mathrm{skw}}) = 0.$$

Choosing $\tau_{\mathrm{sym}} = \varepsilon(u)$ and $\tau_{\mathrm{skw}} = \rho - \omega(u)$ then yields

$$\varepsilon(u) = 0 \ \text{ and } \ \rho - \omega(u) = 0.$$

Hence, the deflection u is a rigid displacement, and since it vanishes on the boundary, it vanishes in the whole of the domain. From above, it then follows that also the rotation ρ vanishes.

For the analysis we need stronger results, i.e., stability in proper norms, and these we will choose as energy type norms.

Theorem 2 *There exists a positive constant C such that*

$$\|\sigma\|_0 + \|\varepsilon(u)\|_0 + \|\rho - \omega(u)\|_0 \leq C\|f\|_* \tag{4}$$

and

$$\|u\|_1 + \|\rho\|_0 \leq C\|f\|_{-1}, \tag{5}$$

with

$$\|f\|_* = \sup_{v \in [H_0^1(\Omega)]^d} \frac{(f, v)}{\|\varepsilon(v)\|_0}.$$

Proof Define the bilinear form

$$B(\sigma, \rho, u; \tau, \eta, v) = (\sigma, \tau) - (\rho - \nabla u, \tau) - (\eta - \nabla v, \sigma)$$

so that the variational form is

$$B(\sigma, \rho, u; \tau, \eta, v) + (f, v) = 0 \quad \forall (\tau, \eta, v) \in [L^2(\Omega)]^{d\times d} \times [L^2(\Omega)]^{d\times d}_{\mathrm{skw}} \times [H_0^1(\Omega)]^d.$$

It now holds

$$B(\sigma, \rho, u; \sigma, -\rho, -u) = \|\sigma\|_0^2.$$

Define

$$\tau = \rho - \nabla u.$$

By the orthogonality $(\rho - \omega(u), \varepsilon(u)) = 0$ we get

$$\|\tau\|_0^2 = \|\rho - \omega(u)\|_0^2 + \|\varepsilon(u)\|_0^2$$

and

$$B(\sigma, \rho, u, \tau, 0, 0) = (\sigma, \tau) + \|\rho - \omega(u)\|_0^2 + \|\varepsilon(u)\|_0^2.$$

Since $|ab| \leq (a^2 + b^2)/2,\ a, b \in \mathbb{R}$, Schwarz inequality gives

$$\begin{aligned}(\sigma, \tau) + \|\rho - \omega(u)\|_0^2 + \|\varepsilon(u)\|_0^2 &\geq -\|\sigma\|_0\|\tau\|_0 + \|\rho - \omega(u)\|_0^2 + \|\varepsilon(u)\|_0^2 \\ &\geq -\frac{1}{2}\|\sigma\|_0^2 - \frac{1}{2}\|\tau\|_0^2 + \|\rho - \omega(u)\|_0^2 + \|\varepsilon(u)\|_0^2 \\ &= -\frac{1}{2}\|\sigma\|_0^2 + \frac{1}{2}(\|\rho - \omega(u)\|_0^2 + \|\varepsilon(u)\|_0^2).\end{aligned}$$

With $(\varphi, \eta, v) = (\sigma + \tau, -\rho, -u)$ we now get

$$B(\sigma, \rho, u; \varphi, \eta, v) \geq \frac{1}{2}\big(\|\sigma\|_0^2 + \|\rho - \omega(u)\|_0^2 + \|\varepsilon(u)\|_0^2\big).$$

Hence, it holds

$$\|\sigma\|_0^2 + \|\rho - \omega(u)\|_0^2 + \|\varepsilon(u)\|_0^2 \leq 2|(f, v)| \leq 2\|f\|_*\|\varepsilon(v)\|_0 = 2\|f\|_*\|\varepsilon(u)\|_0.$$

The arithmetic-geometric mean inequality gives

$$2\|f\|_*\|\varepsilon(u)\|_0 \leq 2\|f\|_*^2 + \frac{1}{2}\|\varepsilon(u)\|_0^2.$$

Combining the two inequalities above gives

$$\|\sigma\|_0^2 + \|\rho - \omega(u)\|_0^2 + \frac{1}{2}\|\varepsilon(u)\|_0^2 \leq 2\|f\|_*^2,$$

and the stability estimate (4) follows. Korn's inequality and the standard definition of $\|\cdot\|_{-1}$ then gives (5).

Remark 1 In the theorem we proved that the stability of the bilinear form B follows from the L_2-ellipticity of the bilinear form $(\cdot, \cdot)$ and the "inf-sup" condition (which we also verified)

$$\sup_{\tau \in [L^2(\Omega)]^{d\times d}} \frac{(\tau, \eta - \nabla v)}{\|\tau\|_0} \geq C(\|\eta - \omega(v)\|_0 + \|\varepsilon(v)\|_0) \quad \forall (\eta, v) \in [L^2(\Omega)]^{d\times d}_{\mathrm{skw}} \times [H^1_0(\Omega)]^d,$$

i.e., the main result of the Babuška-Brezzi [8–10, 13] theory applied for this particular problem.

3 Mixed Finite Element Methods

The mixed method is not directly based on the variational formulation of the previous section. Instead, a formulation in which the integration by parts

$$(\nabla v, \tau) = -(\operatorname{div} \tau, u) + \langle \tau n, v \rangle_{\partial\Omega}$$

is used. The stress is sought in

$$H(\operatorname{div}:\Omega) = \{\ \tau \mid \tau \in [L^2(\Omega)]^{d\times d},\ \operatorname{div} \tau \in [L^2(\Omega)]^d\ \}$$

and the displacement in $[L^2(\Omega)]^d$. The Dirichlet boundary condition for the displacement now becomes a natural boundary condition, and the mixed formulation is then: Find $(\sigma_h, \rho_h, u_h) \in S_h \times K_h \times V_h \subset H(\operatorname{div}:\Omega) \times [L^2(\Omega)]^{d\times d}_{\text{skw}} \times [L^2(\Omega)]^d$ such that

$$\begin{aligned} (\sigma_h, \tau) + (\rho_h, \tau) + (\operatorname{div} \tau, u_h) &= 0 \quad \forall \tau \in S_h, \\ (\sigma_h, \eta) &= 0 \quad \forall \eta \in K_h, \\ (\operatorname{div} \sigma_h, v) + (f, v) &= 0 \quad \forall v \in V_h. \end{aligned} \tag{6}$$

By defining

$$b(\tau; v, \eta) = (\operatorname{div} \tau, v) + (\tau, \eta).$$

the problem in saddle point form is: Find $(\sigma_h, \rho_h, u_h) \in S_h \times K_h \times V_h \subset H(\operatorname{div}:\Omega) \times [L^2(\Omega)]^{d\times d}_{\text{skw}} \times [L^2(\Omega)]^d$ such that

$$\begin{aligned} (\sigma_h, \tau) + b(\tau; u_h, \rho_h) &= 0 \quad \forall \tau \in S_h, \\ b(\sigma_h; v, \eta) + (f, v) &= 0 \quad \forall (v, \eta) \in V_h \times K_h. \end{aligned}$$

By the Babuška-Brezzi theory the stability is a consequence of an "inf-sup" condition

$$\sup_{\tau \in S_h} \frac{b(\tau; v, \eta)}{\|\tau\|} \geq C \|v, \eta\| \quad \forall (v, \eta) \in V_h \times K_h$$

for some norms. The traditional approach is to use the $H(\operatorname{div}:\Omega)$ norm for the stress and the L_2 norm for both the displacement and rotation. With this choice, a direct application of the Babuška-Brezzi theory does not give optimal error estimate. Furthermore, when posing the stability in these norms, Korn's inequality has been used, and hence the discrete stability cannot be proved by local scaling arguments.

First in the next section we will consider specific finite element spaces. Now we will only assume that they are piecewise polynomials on the finite element mesh $\mathscr{C}_h$. The collection of edges/faces on the mesh is denoted by Γ_h.

In our approach we proceed in analogy to (4). The norm chosen for the stress is a mesh dependent L_2 norm

$$\|\tau\|_{0,h}^2 = \|\tau\|_0^2 + \sum_{E\in\Gamma_h} h_E \|\tau n\|_{0,E}^2.$$

This is paired with the broken norm:

$$\|v,\eta\|_h^2 = \sum_{K\in\mathscr{C}_h} \|\varepsilon(v)\|_{0,K}^2 + \sum_{E\in\Gamma_h} h_E^{-1} \|[\![v]\!]\|_{0,E}^2 + \sum_{K\in\mathscr{C}_h} \|\eta - \omega(v)\|_{0,K}^2.$$

Here $[\![v]\!]$ denotes the jump of v when E is in the interior of the domain, and the value of v when $E \subset \partial\Omega$.

Lemma 1 *It holds*

$$|b(\tau; v,\eta)| \le \|\tau\|_{0,h}\|v,\eta\|_h \quad \forall(\tau,\eta,v) \in S_h \times K_h \times V_h.$$

Proof Integrating by parts on each element and using Schwarz inequality yields

$$\begin{aligned}
b(\tau; v,\eta) &= (\operatorname{div}\tau, v) + (\tau,\eta)\\
&= \sum_{K\in\mathscr{C}_h} (\operatorname{div}\tau, v)_K + (\tau,\eta)\\
&= -\sum_{K\in\mathscr{C}_h} (\tau,\nabla v)_K + \sum_{E\in\Gamma_h} \langle \tau n, [\![v]\!]\rangle_E + (\tau,\eta)\\
&= -\sum_{K\in\mathscr{C}_h} (\tau,\varepsilon(v))_K + \sum_{E\in\Gamma_h} \langle \tau n, [\![v]\!]\rangle_E + \sum_{K\in\mathscr{C}_h} (\tau,\eta-\omega(v))_K\\
&\le \sum_{K\in\mathscr{C}_h} \|\tau\|_{0,K}\|\varepsilon(v)\|_{0,K} + \sum_{E\in\Gamma_h} \|\tau n\|_{0,E}\|[\![v]\!]\|_{0,E} + \sum_{K\in\mathscr{C}_h} \|\tau\|_{0,K}\|\eta-\omega(v)\|_{0,K}\\
&\le \|\tau\|_{0,h}\|v,\eta\|_h.
\end{aligned}$$

The stability condition we require for the method is hence

$$\sup_{\tau\in S_h} \frac{b(\tau; v,\eta)}{\|\tau\|_{0,h}} \ge C\|v,\eta\|_h \quad \forall(v,\eta) \in V_h \times K_h. \tag{7}$$

With the broken H^1 norm

$$\|v\|_{1,h}^2 = \sum_{K\in\mathscr{C}_h} \|\nabla v\|_{0,K}^2 + \sum_{E\in\Gamma_h} h_E^{-1}\|[\![v]\!]\|_{0,E}^2, \tag{8}$$

the following discrete Korn's inequality holds in the subspace V_h [12, 28]:

$$\sum_{K\in\mathscr{C}_h} \|\varepsilon(v)\|_{0,K}^2 + \sum_{E\in\Gamma_h} h_E^{-1}\|[\![v]\!]\|_{0,E}^2 \geq C\,\|v\|_{1,h}^2. \tag{9}$$

The triangle inequality then gives

$$\|v,\eta\|_h \geq C(\|v\|_{1,h} + \|\eta\|_0). \tag{10}$$

The stability condition can thus be written

$$\sup_{\tau\in S_h} \frac{b(\tau; v,\eta)}{\|\tau\|_{0,h}} \geq C(\|v\|_{1,h} + \|\eta\|_0) \quad \forall (v,\eta)\in V_h\times K_h. \tag{11}$$

In addition to the stability condition the discrete spaces have to satisfy the equilibrium condition.

$$\operatorname{div} S_h \subset V_h. \tag{12}$$

For the L_2 projection $P_h : [L_2(\Omega)]^d \to V_h$, it then holds

$$(\operatorname{div}\tau, v - P_h v) = 0, \quad \forall \tau\in S_h,\ \forall v\in [L_2(\Omega)]^d. \tag{13}$$

We then have the following error estimate.

Theorem 3 *Suppose that the stability condition* (11) *and the equilibrium condition* (12) *are valid. Then there exists a positive constant C such that*

$$\|\sigma-\sigma_h\|_{0,h} + \|\rho-\rho_h\|_0 + \|P_h u - u_h\|_{1,h} \leq C\big(\inf_{\tau\in S_h}\|\sigma-\tau_h\|_{0,h} + \inf_{\eta\in K_h}\|\rho-\eta\|_0\big).$$

Proof Define the bilinear form

$$B(\sigma,\rho,u;\tau,\eta,v) = (\sigma,\tau) + b(\tau;\rho,u) + b(\sigma;\eta,v).$$

Let $(\tau,\eta)\in S_h\times K_h$. The stability implies that exits $(\varphi,\gamma,z)\in S_h\times K_h\times V_h$, with

$$\|\varphi\|_{0,h} + \|\gamma\|_0 + \|z\|_{1,h} \leq C,$$

such that

$$\|\sigma_h-\tau\|_{0,h} + \|\rho_h-\eta\|_0 + \|u_h - P_h u\|_{1,h} \leq B(\sigma_h-\tau, \rho_h-\eta, u_h - P_h u; \varphi,\gamma,z).$$

The discrete and exact solutions satisfy

$$
\begin{aligned}
B(\sigma_h - \tau, \rho_h - \eta, u_h - P_h u; \varphi, \gamma, z) &= -(f, z) - B(\tau, \eta, P_h u; \varphi, \gamma, z) \\
&= B(\sigma, \rho, u; \varphi, \gamma, z) - B(\tau, \eta, P_h u; \varphi, \gamma, z) \\
&= B(\sigma - \tau, \rho - \eta, u - P_h u; \varphi, \gamma, z) \\
&= (\sigma - \tau, \varphi) + b(\varphi; \rho - \eta, u - P_h u) + b(\sigma - \tau; \gamma, z) \\
&= (\sigma - \tau, \varphi) + (\varphi; \rho - \eta) + (\operatorname{div} \varphi, u - P_h u) + b(\sigma - \tau; \gamma, z) \\
&= (\sigma - \tau, \varphi) + (\varphi; \rho - \eta) + b(\sigma - \tau; \gamma, z),
\end{aligned}
$$

where we in the last step used (13). By the Schwarz inequality we have

$$
\begin{aligned}
(\sigma - \tau, \varphi) + (\varphi, \rho - \eta) + b(\sigma - \tau; \gamma, z) \ &\leq (\|\sigma - \tau\|_{0,h} + \|\rho - \eta\|_0)\|\varphi\|_0 \\
&\quad + \|\sigma - \tau\|_{0,h}(\|\gamma\|_0 + \|z\|_{1,h}),
\end{aligned}
$$

and by combining the above inequalities, the assertion is proved.

4 Finite Element Families

In this section we discuss concrete families of elements. In all of them the elements are triangles or tetrahedra. We start with the one introduced by us in 1988.

The Stenberg family [31].

We will partly use different notation as in [31]. For $K \in \mathscr{C}_h$ we define the bubble function $b_K \in P_{d+1}(K)$ by

$$
b_K = \prod_{i=0}^{d} \lambda_i,
$$

where $\lambda_0, \ldots, \lambda_d$, are the barycentric coordinates on K. For a vector valued function z in $\mathbb{R}^3$ we define $\operatorname{curl} z = \nabla \times z$ and for a scalar function z we let

$$
\operatorname{curl} z = \left(\frac{\partial z}{\partial x_2}, -\frac{\partial z}{\partial x_1}\right).
$$

Define

$$
\begin{aligned}
S_{k+d-1}(K) = \{\, \tau = \{\tau_{ij}\},\ i, j = 1, \ldots, d \mid (\tau_{i1}, \ldots, \tau_{id}) = \operatorname{curl}(w^i b_K) \\
w^i \in [P_{k-1}(K)]^3 \text{ for } d = 3, \text{ and } w^i \in P_{k-1}(K) \ \text{ for } d = 2,\ i = 1, \ldots, d \,\}.
\end{aligned}
$$

Here the index $k+d-1$ is equal to the polynomial degree of the space (the degree of the bubble b_K is $d+1$, w^i is of degree $k-1$, and taking the curl lowers the degree by one). Note that these "stabilizing" degrees of freedom satisfy

$$\operatorname{div}\tau = 0 \text{ on } K, \text{ and } \tau n = 0 \text{ on } \partial K, \ \forall \tau \in S_{k+d-1}(K). \tag{14}$$

The family of [31] is then defined for the degree $k \geq 2$ by

$$\begin{aligned} S_h &= \{\, \tau \in H(\operatorname{div}:\Omega) \mid \tau|_K \in [P_k(K)]^{d\times d} + S_{k+d-1}(K) \ \forall K \in \mathscr{C}_h \,\}, \\ K_h &= \{\, v \in [L^2(\Omega)]^{d\times d}_{\text{skw}} \mid v|_K \in [P_k(K)]^d_{\text{skw}} \ \forall K \in \mathscr{C}_h \,\}, \\ V_h &= \{\, v \in [L^2(\Omega)]^d \mid v|_K \in [P_{k-1}(K)]^d \ \forall K \in \mathscr{C}_h \,\}. \end{aligned} \tag{15}$$

Note that the stress space consists of d copies of the BDM/BDDF space augmented by the space $S_{k+d-1}(K)$ on each element.

We now in analogy with the uniqueness proof of Theorem 1, prove the uniqueness of the finite element solution.

Theorem 4 *The solution of* (6) *with the finite element spaces* (15) *is unique.*

Proof The uniqueness of the stress σ_h is immediate as for the continuous case. Hence, as in the proof of Theorem 1 we have to verify that the condition

$$b(\tau; u_h, \rho_h) = 0 \quad \forall \tau \in S_h$$

first implies that u_h is a rigid body motion and that $\rho_h = \omega(u_h)$, and then by the boundary conditions that both of them vanish. To this end, let K be arbitrary and choose τ such that $\tau = 0$ in $\Omega \setminus K$ and $\tau|_K \in S_{k+d-1}(K)$. Due to (14) it then holds

$$b(\tau; u_h, \rho_h) = (\rho_h, \tau) + (\operatorname{div}\tau, u_h) = (\rho_h, \tau)_K.$$

We proceed slightly differently in two and three dimensions. For $d = 2$ let

$$\rho_h = \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}.$$

We then choose

$$(\tau_{i1}, \tau_{i2}) = \operatorname{curl}\big(\frac{\partial z}{\partial x_i} b_K\big).$$

Integrating by parts, this gives

$$(\rho_h, \tau)_K = \int_K \big[-\frac{\partial}{\partial x_1}(b_K \frac{\partial z}{\partial x_1}) - \frac{\partial}{\partial x_2}(b_K \frac{\partial z}{\partial x_2})\big] z = \int_K b_K |\nabla z|^2.$$

The condition $b(\tau; u_h, \rho_h) = 0$ then implies that z equals a constant. Hence, on each K the rotation is constant ρ_K. Let $R(K)$ be the space of rigid body motions on K. For each K there is a $r_K \in R(K)$ such that

$$\rho_K = \omega(r_K) = \nabla r_K. \tag{16}$$

For $d = 3$ we choose

$$(\tau_{i1}, \tau_{i2}, \tau_{i3}) = \operatorname{curl}(\operatorname{curl} \rho_h^i \, b_K),$$

where ρ_h^i is the i-th row of ρ_h. Integrating by parts gives

$$0 = (\tau, \rho_h)_K = \sum_{i=1}^{3} \int_K b_K |\operatorname{curl} \rho_h^i|^2,$$

showing that $\operatorname{curl} \rho_h^i = 0$ on K. Since ρ_h is skew-symmetric, this implies that it is constant. Hence, we conclude that also for $d = 3$ there is a $r_K \in R(K)$ such that (16) holds.

In the sequel we use the subspace of S_h consisting of the Raviart-Thomas-Nédélec spaces

$$S_h^{RTN} = \{ \tau \in H(\operatorname{div}:\Omega) \mid \tau|_K \in [P_{k-1}(K)^d \oplus z\tilde{P}_{k-1}(K)]^d \ \forall K \in \mathscr{C}_h \}$$

where $\tilde{P}_{k-1}(K)$ denotes homogeneous polynomials of degree $k-1$.

The degrees of freedom for this spaces are

$$\langle \tau n, z \rangle_E \qquad \forall z \in P_{k-1}(E)^d, \ E \subset \partial K, \tag{17}$$

$$(\tau, z)_K \qquad \forall z \in [P_{k-2}(K)]^{d \times d}, \tag{18}$$

for each $K \in \mathscr{C}_h$.

Now choose $\tau \in S_h^{RTN} \subset S_h$ such that $\tau = 0$ in $\Omega \setminus K$. Then it holds

$$\begin{aligned} b(\tau; u_h, \rho_h) &= (\rho_h, \tau) + (\operatorname{div} \tau, u_h) = (\rho_h, \tau)_K + (\operatorname{div} \tau, u_h)_K \\ &= (\rho_h, \tau)_K - (\tau, \nabla u_h)_K \\ &= (\nabla r_K, \tau)_K - (\tau, \nabla u_h)_K \\ &= (\nabla (r_K - u_h), \tau)_K. \end{aligned}$$

Now $\nabla(r_K - u_h)|_K \in [P_{k-2}(K)]^{d \times d}$, and hence the degrees of freedom (18) show that the condition $b(\tau; u_h, \rho_h) = 0$ implies that

$$\nabla(r_K - u_h) = 0. \tag{19}$$

The symmetric and skew-symmetric parts of this are

$$\varepsilon(u_h) = 0 \ \text{ and } \ \omega(u_h) = \omega(r_K) = \rho_h|_K.$$

Hence $u_h|_K$ is a rigid body motion and there is a constant vector c_K such that

$$u_h|_K = r_K + c_K.$$

For u_h and ρ_h it thus holds

$$b(\tau; u_h, \rho_h) = \sum_{E\in\Gamma_h} \langle \tau n, [\![u_h]\!]\rangle_E.$$

In the condition $b(\tau; u_h, \rho_h) = 0 \ \forall\tau \in S_h$, the degrees of freedom (17) for τ show that the jump $[\![u_h]\!]$ vanishes along interior edges, i.e. u_h is continuous and a global rigid body motion, and $\rho_h = \omega(u_h)$.

Finally, since, $[\![u_h]\!] = u_h$ on the boundary $\partial\Omega$, the degrees of freedom (17) show that $u_h = 0$, and hence also $\rho_h = 0$.

We note that in the uniqueness proof above, we have used local arguments element by element, and edge by edge. The stability can thus be built in the same manner and the norms and the bilinear form scales in the same way. Hence, we have essentially already proved the stability estimate.

Theorem 5 *There is a positive constant C such that*

$$\sup_{\tau\in S_h} \frac{b(\tau; v, \eta)}{\|\tau\|_{0,h}} \geq C\|v, \eta\|_h \quad \forall(v, \eta) \in V_h \times K_h. \tag{20}$$

Theorem 3 then gives the following error estimate.

Theorem 6 *There is a positive constant C such that*

$$\|\sigma - \sigma_h\|_{0,h} + \|\rho - \rho_h\|_0 + \|P_h u - u_h\|_{1,h} \leq Ch^{k+1}\big(\|\sigma\|_{k+1} + \|\rho\|_{k+1}\big).$$

Let us continue by discussing this family. First, for an implementation in mind, the sum of local spaces in the definition of the stress should be replaced by a direct sum. To this end, we define

$$\hat{S}_{k+d-1}(K) = \{\, \tau = \{\tau_{ij}\},\ i, j = 1, \ldots, d \mid (\tau_{i1}, \ldots, \tau_{id}) = \operatorname{curl}(w^i b_K),$$
$$w^i \in [\hat{P}_{k-1}(K)]^3 \text{ for } d{=}3, \text{ and } w^i \in \hat{P}_{k-1}(K) \text{ for } d{=}2,\ i{=}1,\ldots,d \,\},$$

with

$$\hat{P}_{k-1}(K) = \{\, v \in P_{k-1}(K) \mid (v, w)_K = 0\ \forall w \in P_{k-n}(K)\,\}.$$

Since all degrees of freedom in $S_{k+d-1}(K) \setminus \hat{S}_{k+d-1}(K)$ are contained in $[P_k(K)]^{d\times d}$, the stress space is

$$S_h = \{\, \tau \in H(\mathrm{div}:\Omega) \mid \tau|_K \in [P_k(K)]^{d\times d} \oplus \hat{S}_{k+d-1}(K)\ \forall K \in \mathscr{C}_h \,\}.$$

Next, we note from the proof that in the space $\hat{S}_{k+d-1}(K)$ there are more degrees of freedom that what is used to get the stability. We also note that this space contains polynomials of degree $k+1$ for $d=2$, and $k+2$ for $d=3$, and these do not contribute to the accuracy.

These drawbacks are fixed in the

The Gopalakrishnan-Guzmán family [22].

In two dimensions the family is obtained from our family by restricting the stabilizing degrees of freedom to those which are actually needed. From the proof above, we see that in $\mathbb{R}^2$ the additional degrees of freedom are

$$\hat{S}_{k+1}(K)=\{\, \tau=\{\tau_{ij}\},\ i,j=1,2 \mid (\tau_{i1},\tau_{i2}) = \mathrm{curl}\,(\frac{\partial z}{\partial x_i} b_K)\ i=1,2,\ z \in \tilde{P}_k(K) \,\}.$$

In three dimension they were able to reduce the degree of the additional degrees of freedom with one. They define the matrix bubble by

$$B_K = \sum_{l=0}^{3} \lambda_{l-3}\lambda_{l-2}\lambda_{l-1}(\nabla\lambda_l)^t\nabla\lambda_l,$$

where the index is modulo 4, and $\nabla\lambda_l$ is considered as a row vector. In [16] it is shown that this is symmetric and positively definite, and it can then be used as a weight for an inner product on tensors. Defining the curl of a tensor as the tensor in which each row is the curl of the corresponding row in the original tensor, the space is defined as

$$\hat{S}_{k+1}(K) = \{\, \tau \in [L^2(K)]^{d\times d} \mid \tau = \mathrm{curl}\,(\mathrm{curl}\,(\xi) B_K)\ \xi \in [\tilde{P}_k(K)]^d_{\mathrm{skw}} \,\}.$$

Since

$$(\mathrm{curl}\,(\mathrm{curl}\,(\xi)B_K), \xi)_K = (\mathrm{curl}\,(\xi)B_K, \mathrm{curl}\,(\xi))_K$$

these degrees of freedom can be used in the proof of the stability.

The family is then defined by

$$\begin{aligned} S_h &= \{\, \tau \in H(\mathrm{div}:\Omega) \mid \tau|_K \in [P_k(K)]^{d\times d} + \hat{S}_{k+1}(K)\ \forall K \in \mathscr{C}_h \,\}, \\ K_h &= \{\, v \in [L^2(\Omega)]^{d\times d}_{\mathrm{skw}} \mid v|_K \in [P_k(K)]^d_{\mathrm{skw}}\ \forall K \in \mathscr{C}_h \,\}, \\ V_h &= \{\, v \in [L^2(\Omega)]^d \mid v|_K \in [P_{k-1}(K)]^d\ \forall K \in \mathscr{C}_h \,\}. \end{aligned} \tag{21}$$

Theorem 7 *For $k \geq 2$ the family (21) is stable and the following error estimate holds*

$$\|\sigma - \sigma_h\|_{0,h} + \|\rho - \rho_h\|_0 + \|P_h u - u_h\|_{1,h} \leq Ch^{k+1}\big(\|\sigma\|_{k+1} + \|\rho\|_{k+1}\big).$$

Remark 2 The family of [16] is based on the Raviart-Thomas-Nédélec elements and not the Brezzi-Douglas-Fortin-Duran-Marini elements as above.

The third family to be considered is
The Arnold-Falk-Winther family [22].

In this the polynomial degree for the rotation is decreased by one. We then note that the there is no need to include additional degrees of freedom in order to obtain stability for the rotation, the degrees of freedom are already included in the stress space. The family is then the following.

$$\begin{aligned} K_h &= \{\, v \in [L^2(\Omega)]^{d\times d}_{\text{skw}} \mid v|_K \in [P_{k-1}(K)]^{d\times d} \;\; \forall K \in \mathscr{C}_h \,\} \\ V_h &= \{\, v \in L^2(\Omega)^d \mid v|_K \in [P_{k-1}(K)]^d \;\; \forall K \in \mathscr{C}_h \,\}, \\ S_h &= \{\, \tau \in H(\text{div}:\Omega) \mid \tau|_K \in [P_k(K)]^{d\times d} \;\; \forall K \in \mathscr{C}_h \,\}. \end{aligned} \tag{22}$$

The advantage of this space is that pure polynomial degrees of freedom are used. The disadvantage is, however, that the convergence rate is decreased by one and the full approximation power of the stress space is not achieved.

Theorem 8 *For $k \geq 2$ the family (22) is stable and the following error estimate holds*

$$\|\sigma - \sigma_h\|_{0,h} + \|\rho - \rho_h\|_0 + \|P_h u - u_h\|_{1,h} \leq Ch^{k}\big(\|\sigma\|_{k} + \|\rho\|_{k}\big).$$

In our analysis we have assumed that $k \geq 2$ so that the rigid body motions are included in the local displacement spaces. The method are, however, stable also for $k = 1$, but the analysis has to be modified. This case can be found in the recent paper [25].

Finally, let us remark that the estimates for $\|P_h u - u_h\|_{1,h}$ are superconvergence results that enables a local postprocessing of the displacement, cf. [3, 31, 32]. The postprocessed displacement is crucial for a posteriori estimates [26, 27].

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