Chapter 10
Implementation of Hadamard Matrices for Image Processing

Leonid Mironovsky and Valery Slaev

Abstract The image quality influences the accuracy of obtained results. In the chapter, the application of the strip-method for noise-immune storage and transmission of images is analyzed. At the same time, before transmitting the matrix transformation of an original image has to be done, when the image fragments are mixed up and superimposed each other. The transformed image is transmitted over a communication channel, where it is distorted with a pulse noise, the latter being, for example, a possible reason for a complete loss of separate image fragments. After the signal transmission to the receiving end, an inverse transformation is performed. During this transformation, the reconstruction of the image takes place. If it is possible to provide a uniform distribution of the pulse noise over the whole area, which the image occupies without any changes of its energy, then a noticeable decrease of noise amplitude will take place and an acceptable quality of all fragments of the image are reconstructed. The tasks of the chapter are the consideration the versions of the two-sided strip-transformation of images and the choice of optimal transformation matrices. A great attention has been paid to the implementation of Hadamard matrices and matrices close to them such as Hadamard-Mersenne, Hadamard-Fermat, and Hadamard-Euler matrices.

Keywords Image quality · Strip-method · Image transmission · Matrix transformations · Pulse noise · Inverse transformation · Hadamard matrix · Two-levels M-matrix · Three-levels M-matrix · Multi-levels M-matrices

L. Mironovsky (✉)
St. Petersburg State University of Aerocosmic Instrumentation, 67 Bol.Morskaya st., St. Petersburg 19000, Russian Federation
e-mail: miron@aanet.ru

V. Slaev
D.I. Mendeleyev Research Institute for Metrology, 19 Moskovsky av., St. Petersburg 190005, Russian Federation
e-mail: v.a.slaev@vniim.ru

M.N. Favorskaya and L.C. Jain (eds.), *Computer Vision in Control Systems-1*, Intelligent Systems Reference Library 73, DOI 10.1007/978-3-319-10653-3_10

10.1 Introduction

Many problems of information transforming and data studying are connected with the images processing and transmitting. For example, it is possible to name the remote sensing of the Earth surface by satellites, rentgenography and its application in medicine, investigations of biological and chemical processes, among others. The accuracy of the obtained results depends from the quality of images.

In this chapter, the application of the strip-method [1–3] for storage and noise-immune transmission of images is analyzed [4]. At the same time, before transmitting the matrix transformation of an original image is fulfilled, when the image fragments are mixed up and superimposed each other. The transformed image is transmitted over a communication channel, where it is distorted by pulse noises, the latter being for example a possible reason for a complete loss of separate image fragments. After the signal transmission to the receiving end, an inverse transformation is performed. During this transformation, the reconstruction of the image takes place. If it is possible to provide a uniform distribution of the pulse noise over the whole area, which the image occupies without any changes of its energy, then a noticeable decrease of noise amplitude will take place and the acceptable quality of all fragments of the reconstructed image is achieved.

Section 10.2 provides the related work. Strip-method of image transformation is developed in Sect. 10.3. The Hadamard matrices and matrices closed to them are represented in Sect. 10.4. Conclusion is situated in Sect. 10.5.

10.2 Related Work

It is reasonable that the strip-method is merely one of the methods used for increasing the accuracy of signal and image transmission over communication channels. A great number of publications are devoted to issues of raising the noise-resistance of information transmission systems [3, 5–8], and others. It is also necessary to mention some works in the adjacent fields of activities such as the works in the cluster systems of message transmission and linear predistortion of signals, which were maden by Russian researches Ageyev, Babanov, Lebedev, Marigodov, Suslonov, Tsibakov, Yaroslavsky, and others; the works connecting with the method of redundant variables; the works in the linear transformation and block coding of signals and images, which were done by American researches Costas, Lang, Leith, Pierce, Upatnieks [9–17], and others. Thus, the noise control based on introduction of pre-distortions at the stage of signal transmission and on optimal processing at the stage of signal reception is widely used in information transmission systems.

However, the majority of works deal with the pre-distortion methods and the correction by using a root-mean-square criterion, whereas the methods satisfying the requirements for optimizing the information transmission systems with the help of a minimax criterion have been developed to a significantly lesser degree. Therefore, it would be more useful to develop and study new methods for blanking the pulse interference, which are supported by using the minimax criterion and modern computer processing for images.

10.3 Strip-Method of Image Transformation

In this section, the basis of two-dimensional strip-transformation (Sect. 10.3.1) and the choice of optimal transformation matrices (Sect. 10.3.2) will be discussed.

10.3.1 Two-Dimensional Strip-Transformation

The first stage of the strip-method for the transformation of one-dimensional signals consists in a "cutting" the original signal into n strips with equal duration and a forming from them the n-dimensional vector $\mathbf{X}$. At the second stage, this vector is fallen under the isometric transformation by its multiplying on the orthogonal matrix $\mathbf{A}$ of the dimension $n \times n$

$$\mathbf{Y} = \mathbf{AX}.$$

In the same way, the first stage of the strip-transformation of two-dimensional signals (images) consists in dividing the original image $\mathbf{P}$ into N rectangular fragments similar in size as it is shown in Fig. 10.1. Let the number of horizontal and vertical stripes, into which the image is conditionally "cut", be denoted as m and n; then $N = mn$. Further, a linear combination of the fragments is made. At that, there are two approaches such as the vector and the matrix ones [18, 19].

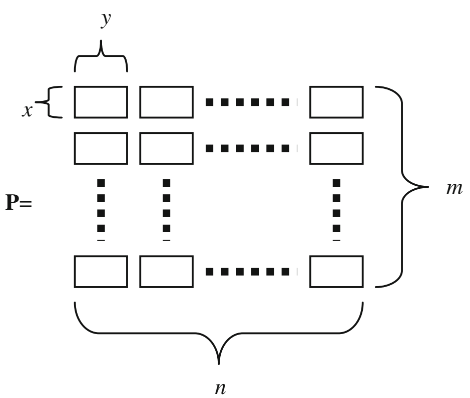


Fig. 10.1 The image presentation as the block-matrix **P**

According to the first (vector) approach, the obtained fragments are used to form an N-dimensional block-vector X that as in the one-dimensional case undergoes the isometric transformation by multiplying it on the orthogonal matrix **A** of the dimension $N \times N$: $\mathbf{Y} = \mathbf{AX}$. Let this version, entirely the same as in the one-dimensional case, be denoted as the one-sided strip-transformation. Its main weakness is too high dimensionality of the matrix **A** and corresponding calculation costs.

According to the second (matrix) approach, the original image divided into fragments, is considered as a block-matrix **X** of the dimension $m \times n$. Here three versions of isometric transformation of this matrix with the purpose to "mix" its fragments are possible [20]:

- The multiplication by the orthogonal $m \times m$ matrix **B** on the left (the left-sided matrix transformation)

$$\mathbf{Z}_1 = \mathbf{BX}.$$

- The multiplication by the orthogonal $n \times n$ matrix **A** on the right (the right-sided matrix transformation)

$$\mathbf{Z}_2 = \mathbf{XA}.$$

- The simultaneous multiplication by the matrix **B** on the left and by the matrix **A** on the right (the two-sided or bilateral matrix transformation)

$$\mathbf{Z}_3 = \mathbf{BXA}.$$

All versions listed above are shown in Fig. 10.2. It illustrates a chain of transformations of the original image **P**, which results into an image being transmitted over the communication channel.

The first and the last versions of transformation are of the main interest, since they provide the most complete "mixing" of the image fragments. Each fragment of the transformed image contains information about all $N = mn$ fragments of the original image **P**. In other two versions $\mathbf{Z}_1 = \mathbf{BX}$ and $\mathbf{Z}_2 = \mathbf{XA}$ only horizontal or only vertical stripes into which the original image has been "cut" are linear combined.

Therefore, only two versions of transformation will be considered below:

- The one-sided strip-transformation provided by Eq. 10.1, where **X** is the block-vector of the dimension $mn \times 1$, **A** is the orthogonal matrix of the order mn.

$$\mathbf{Y} = \mathbf{AX} \tag{10.1}$$

- The two-sided strip-transformation by Eq. 10.2, where **X** is the block-matrix of the dimension $m \times n$, **B** and **A** are the orthogonal matrices of the orders m and n.

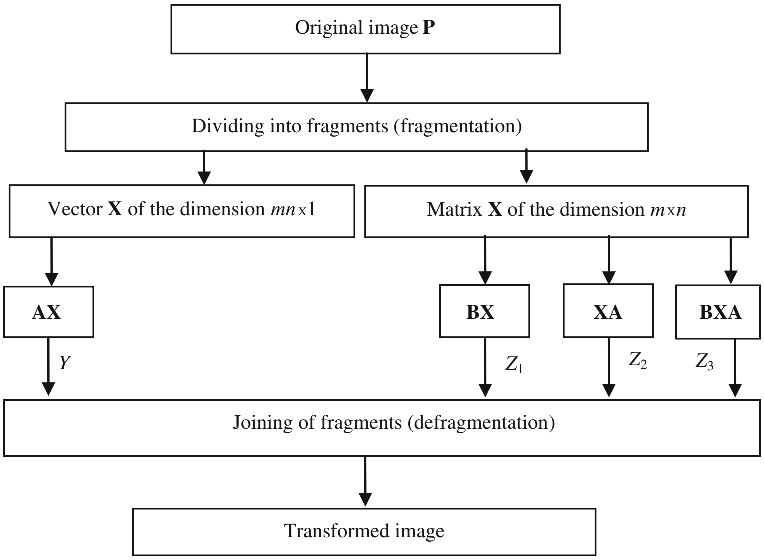


Fig. 10.2 The strip-transformation of two-dimensional signals

$$\mathbf{Z} = \mathbf{BXA} \tag{10.2}$$

Correspondingly, the inverse transformations, when the image is reconstructed at the receiving end of the communication channel, are described by Eq. 10.3 for the one-sided transformation and Eq. 10.4 for the two-sided transformation.

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y} \tag{10.3}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{Z}\mathbf{B}^{-1} \tag{10.4}$$

Let these two versions be described in more details.

The image transmission with using the one-sided strip-transformation. Let an original and reconstructed images be denoted as **P** and **P′**, and a straight and an inverse operators, realizing fragmentation and defragmentation of the image, as S_1 and $S_1{}' = S_1{}^{-1}$.

In the communication channel to the vector $\mathbf{Y} = \mathbf{AX}$, a pulse noise signal $\boldsymbol{\Delta}$ is added. As a result, at the output of the channel we get an image-vector $\mathbf{Y}' = \mathbf{Y} + \boldsymbol{\Delta}$. At the receiving end, the inverse one-sided strip-transformation is performed aimed at obtaining a vector $\mathbf{X}'$. This transformation is described by Eq. 10.5.

$$\mathbf{X}' = \mathbf{A}^{-1}\mathbf{Y}' = \mathbf{A}^{-1}(\mathbf{Y} + \boldsymbol{\Delta}) = \mathbf{A}^{-1}\mathbf{Y} + \mathbf{A}^{-1}\boldsymbol{\Delta} = \mathbf{X} + \mathbf{A}^{-1}\boldsymbol{\Delta} \tag{10.5}$$

The obtained vector $\mathbf{X}'$ is represented in the form of the sum of the vector $\mathbf{X}$ and noise vector $\mathbf{\Delta}$, which have experienced the inverse transformation. At the last stage the vector $\mathbf{X}'$ is transformed into the matrix $m \times n$, describing the reconstructed image $\mathbf{P}$ with a noise $\mathbf{\Delta}' = \mathbf{A}^{-1}\mathbf{\Delta}$ added to it.

As it has already been shown, the main disadvantage of the one-sided strip-transformation is too large dimension of the matrix $\mathbf{A}$ equal to $mn \times mn$ (the number of entries of this matrix is equal to the squared number of fragments, into which the image is divided). The matrices $\mathbf{B}$ and $\mathbf{A}$ used in the two-sided strip-transformation have significantly smaller dimensions (at $m = n$ a total number of their elements is equal to the doubled number of image fragments). This facilitates their formation and storage.

The image transmission with using of the two-sided transformation. The image $\mathbf{Z} = \mathbf{BXA}$ obtained as a result of the two-sided strip-transformation of the original image $\mathbf{P}$ is transmitted through the communication channel. A pulse noise signal Δ (a block-matrix of the dimension $m \times n$) is added to the image transmitted into the channel. As a result at the output of the channel we have an image $\mathbf{Z}' = \mathbf{Z} + \mathbf{\Delta}$. At the receiving end the image $\mathbf{Z}'$ is affected by the inverse two-sided transformation for getting a matrix of a resulting image $\mathbf{P}'$. It is described by Eq. 10.6.

$$\mathbf{P}' = \mathbf{A}^{-1}\mathbf{Z}'\mathbf{B}^{-1} = \mathbf{A}^{-1}(\mathbf{Z} + \Delta)\mathbf{B}^{-1} = \mathbf{A}^{-1}\mathbf{Z}\mathbf{B}^{-1} + \mathbf{A}^{-1}\Delta\mathbf{B}^{-1} = \mathbf{P} + \mathbf{A}^{-1}\Delta\mathbf{B}^{-1} \tag{10.6}$$

In accordance with Eq. 10.6, the recipient will see the original image $\mathbf{P}$ with the noise added to it in the channel and changed by the inverse two-sided transformation. To make this method more appropriate in practice, matrices $\mathbf{A}$ and $\mathbf{B}$ have the equal sizes that simplify calculations and save memory. Then Eq. 10.2 takes the form of Eq. 10.7, where $\mathbf{A}$ is an orthonormal matrix.

$$\mathbf{Z} = \mathbf{AXA} \tag{10.7}$$

The Eq. 10.6 will simplify in a following manner (Eq. 10.8).

$$\mathbf{P}' = \mathbf{P} + \mathbf{A}^{\mathrm{T}}\Delta\mathbf{A}^{\mathrm{T}} \tag{10.8}$$

For further simplifying of the transformation, it is useful to apply a symmetrical matrix $\mathbf{A}$. In this case the inverse transformation will coincide with the straight one and the need to store and calculate separately the inverse matrix will disappear. The noise at the output of the system will be determined by the formula $\Delta' = \mathbf{A}\Delta\mathbf{A}$.

Mathematical formalism. In Eqs. 10.2–10.8 and other formulae of this part of the chapter, the multiplication of usual numeric matrices $\mathbf{A}$ and $\mathbf{B}$ by the block matrices $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, the elements of which are the image fragments, is performed. At the same time the following rules are used.


Fig. 10.3 Original image

Summation of blocks (*fragments*). Separate blocks (fragments) of image matrices are summed up by adding the corresponding block elements. This operation is similar to summation of two matrices of the same sizes.

Multiplication of a fragment by a number. The operation is performed by multiplying each fragment pixel by a number. At the same time the brightness of the fragment changes as a whole. The operation is similar to the multiplication of the matrix by a number.

Multiplication of the block matrix by the numeric matrix. Such a multiplication is performed in the same way as it is done in multiplying the numeric matrices according to the rule "a row by a column" taking into account the manner of two first operations.

Example. Loss of a unit (a single noise). Let the test image given in Fig. 10.3 be analyzed. The original image has definite boundaries and is characterized by the presence of both large objects and small details. Let a message (Fig. 10.3) be divided into $8 \times 8 = 64$ units, which are in series transmitted over a communication channel. By a single noise they imply a distortion or loss of one from 64 units. An example of such a noise is shown in Fig. 10.4.

Now let the two-sided transformation of the original image with the orthonormalized Hadamard matrix of the 8th order be done (this is done before the transmission). The image obtained without transformation is shown in Fig. 10.4, and the image obtained with the strip-transformation is shown in Fig. 10.5.

An analysis of the image shows that without strip-transformation this image loses its scenario (Fig. 10.4). When using the strip-transformation with the Hadamard matrices of the 8th order (Fig. 10.5) the image is obtained without any significant

Fig. 10.4 The image with the loss of one unit without strip-transformation

Fig. 10.5 The image obtained using the strip-transformation with the Hadamard matrix of the 8th order

distortions. Having estimated the images obtained by the subjective way, it is possible to note that the quality of the obtained image is quite acceptable.

Image scaling. In practice an image transmitted is represented in the form of a matrix consisted from separate pixels (brightness values). As a rule, the number of luminance range is taken as 256, that corresponds to eight binary digits.

After two-sided strip-transformation the image can transgress the bounds of a digit plane, and it is necessary to return it into acceptable bounds by means of dividing the image value by a definite scale parameter.

The most unfavorable case is typical for a purely white image. As a consequence of the strip-transformation with the Hadamard matrix of the n order all elements of such image become zero (black) ones apart from one element, the value of which will exceed the permissible range by n times.

It is obvious that the introduction of too great scale parameter can adversely affect the quality of the image restored. Therefore, a problem of choosing a minimal value of this scale parameter arises. It is possible to indicate the following versions of solving the problem. The simplest one of them is to use a fixed scale parameter n at the transmitting point and the coefficient $1/n$ at the receiving end. Shortcomings of this version are evident. A more flexible way may consist in introduction of an adaptive scale parameter that is specially calculated for each image and transmitted with it over a communication channel.

It is also possible to use a threshold filter (an amplitude detector) for limiting maximum values of the signal transmitted. With all this going on, it is possible to decrease the contrast of the image being received. For example, a white-black image of the "chess-board" type can turn into a black-grey one. The presence of prior statistical information concerning properties of the images transmitted can also help to solve the problem of scale operation.

The above is related to the case of white and black images. Technically such images are represented in the form of a matrix that consists from a number of pixels (brightness values). Just this matrix is subjected to fragmentation in the process of the strip-transformation. As to the color images, the situation is somewhat more complicated. One of the standard methods for presenting the color images is the application of three-layered matrix Red–Green–Blue. In this case, each of the three layers of the image matrix is exposed to the strip-transformation action.

10.3.2 Choice of Optimal Transformation Matrices

As a consequence of dividing the original image into fragments, shown in Fig. 10.1, a block-matrix containing $m \times n$ blocks is obtained. The entries of this matrix are rectangular and have dimensions $x \times y$. All fragments of this matrix are of the same dimensions. In those cases, when the number of pixels in a row or column of the original image matrix cannot be divided by m or n, giving an integer, it is necessary to add pixels from the right or from the bottom of the image. They should not distort the image or excessively contrast it.

The separation of the image into fragments permits to decrease significantly the calculation costs. The larger fragments mean the smaller dimension of the transformation matrix **A**. The image fragment dimension should be chosen on the basis of an expected duration of the pulse noise, i.e. the linear dimensions of the distorted image segment. The best version will be the choice of fragment dimension equal to maximal noise duration. This will allow the noise to be distributed over the image at the output of the system in the most uniform way. The chosen fragment dimension will determine the dimensions of the transformation matrix.

To attenuate the amplitude of pulse noise as much as possible, it is necessary to secure the uniform distribution of the noise over the image by applying the inverse transformation at the receiving end of the communication channel. This makes it possible to reconstruct information about distorted and "lost" fragments. Moreover, this arises a need to determine the type of a transformation matrix **A** that will minimize the noise amplitude in the reconstructed image.

In case of the one-sided strip transformation, the level of noise $\boldsymbol{\Delta}'$ in the reconstructed image is determined by Eq. 10.5. If the matrix **A** is symmetrical and orthogonal, then Eq. 10.9 will take place.

$$\Delta' = \mathbf{A}\Delta \tag{10.9}$$

Similar equation can be derived from Eq. 10.8 for the two-sided transformation in a view of Eq. 10.10.

$$\Delta' = \mathbf{A}\Delta\mathbf{A} \tag{10.10}$$

Let us assume that the noise in the communication channel distorts only one fragment of the image (a single noise pulse). This means that only one of the block-vector components, $\boldsymbol{\Delta}_i$ from Eq. 10.9 or that of the block-matrix, $\boldsymbol{\Delta}_i$ from Eq. 10.10 can be non-zero.

In both cases the noise level $\boldsymbol{\Delta}'$ in the signal reconstructed will be determined by a maximal entry module of the orthogonal matrix **A**. Indeed, if in Eq. 10.9 we assume that $\Delta_1 = 1, \Delta_2 = \cdots = \Delta_N = 0,$ then $\boldsymbol{\Delta}' = \mathbf{A}_1$, where $\mathbf{A}_1$ is the first column of the matrix **A**. Thus, the noise amplitude $\boldsymbol{\Delta}'$ will be equal to the maximal entry module of the first column of the matrix **A** (and in the general case to the whole matrix **A**).

In a similar manner, assuming, for example, that in Eq. 10.10 $\Delta_{11} = 1,$ and the remaining components are $\Delta_{ij} = 0$, the following equations will be obtained:

$$\Delta' = \mathbf{A}_1 \cdot \mathbf{A}_1^{\mathrm{T}} = \left[a_{1i} \cdot a_{1j}\right]_1^n.$$

Therefore, the maximal entry of the matrix $\boldsymbol{\Delta}'$ will be equal to $a_M^2,$ where a_M is the maximal entry module of the first column of the matrix **A**. At an arbitrary position of the non-zero entry in the matrix $\boldsymbol{\Delta}$, the maximal entry module a_M of the matrix **A** will be obtained.

Since the aim set is to attenuate to the limit the noise amplitude, then in both cases it is required to search such class of orthogonal matrices, the one the maximal entry module of which is minimal. The well-known decision of this task relates to the cases n, which are divided by four. Such matrices are the normalized Hadamard matrices. The less known decision for even n, which are not divided by four, is represented by so called Conference-matrices (C-matrices). They have a zero diagonal and their rest entries are equal to ±1.

The Hadamard matrices provide an ideally uniform distribution of a single noise pulse over the whole image area, decreasing its amplitude by n times (at $m \neq n$ by $\sqrt{mn}$ times). The C-matrices providing the noise attenuation by $(n - 1)$ times are only a little inferior to them. For odd n, the general solution of the problem is unknown for the authors. As a result of long term searches, the orthogonal matrices for $n = 3, 5, 7, 9, 11$, optimal in this sense, have been found. More detailed information about these and other matrices, closed to the Hadamard matrices, are given in the Sect. 10.4.

10.4 Hadamard Matrices and Matrices Closed to Them

The strip-method bases itself on isometric transformations of signals and images with the help of orthogonal matrices. One of the main requirements for these matrices is the most complete "mixing" of fragments of an original signal or image in the case of the straight transformation, as well as a uniform distribution of a pulse noise along the time duration of a reconstructed signal, or over the area of a reconstructed image at the inverse transformation.

In respect to mathematics, this means that the orthogonal matrices with entries closed in absolute value should be used. The classical representatives of such matrices are the Hadamard matrices. Moreover, the subject to a technical problem to be solved some additional requirements such as the matrices symmetry, the cycle structure (Toeplitz or Hankel matrices) can be set up.

Below the description and specific form of matrices, completely or partially meeting, are presented. First, there are the Hadamard and C-matrices, which provide the utmost degree "mixing" of signal and image fragments. Unfortunately, these matrices are far from being present in all cases. Therefore, the problem to find the orthogonal matrices similar to them with respect to their characteristics arises. Among the versions worthy of notice, there are matrices based on orthogonal systems of functions (trigonometric functions and polynomials), two-level D-matrices (matrices containing elements of only two types, e.g. $\pm a$, $\pm b$) and minimax matrices (M-matrices). Second, a generalization of Hadamard matrices for odd n, since their maximum modulo entries are minimal as compared to all other orthogonal matrices of the odd order, is considered.

The Hadamard matrices, the shortened Hadamard matrices, and the Conference matrices are represented in Sects. 10.4.1–10.4.3, respectively. Section 10.4.4 provides the optimal orthogonal matrices of the odd order (M-matrices). Two-, three-, and many-levels M-matrices are discussed in Sect. 10.4.5.

10.4.1 Hadamard Matrices

The Hadamard matrices are widely used in the theory of coding (codes correcting errors), theory of planning multifactor experiments (orthogonal block-diagrams), and other fields of mathematics. Below the definition of these matrices and description of their main properties are given [3, 21–25].

Definition 1 A Hadamard matrix of order n is such $n \times n$ matrix $\mathbf{A}$ with entries +1 or −1, for which $\mathbf{AA}^{\mathrm{T}} = n\mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

It is evident that the Hadamard matrix is a non-singular matrix and its rows in pairs are orthogonal. The transposition of rows or columns and multiplication them by −1 again yield a Hadamard matrix. These operations allow any matrix to be transformed into one of a "normalized" form, when in the first column and row all elements are equal to +1. Dividing the Hadamard matrix by $\sqrt{n}$, an orthogonal matrix $\mathbf{A}_0 = \mathbf{A}/\sqrt{n}$, is obtained that meets the condition $\mathbf{A}_0\mathbf{A}_0^{\mathrm{T}} = \mathbf{I}$. The simplest Hadamard matrix has the form

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

It is orthogonal: $\mathbf{A}^T\mathbf{A} = 2\mathbf{I}$ and symmetrical.

After dividing this matrix by $\sqrt{2}$ it becomes orthonormal

$$\mathbf{A}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

It is easy to make sure that if $\mathbf{M}$ and $\mathbf{N}$ are the Hadamard matrices of orders m and n, respectively, then their Kronecker product, i.e. the matrix $\mathbf{M} \otimes \mathbf{N}$, is the Hadamard matrix of order $m \cdot n$. For example, if $\mathbf{A}$ is the Hadamard matrix of the second order, then as a result of the Kronecker product $\mathbf{A} \otimes \mathbf{A}$ the Hadamard matrix of the 4th order is obtained

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

It is known that there are no Hadamard matrices of the odd order [26]. To provide the existence of Hadamard matrices of the even order $n > 2$, it is necessary to have n divisible by 4. It should be noted that thereby nothing but the required condition has been proved. From this condition it does not follow that at n divisible by 4 the Hadamard matrix has to exist. The hypothesis, according to which this condition is sufficient, also has not yet proved. In the geometry language the question concerning the existence of the Hadamard matrix of order $n = 4k$ is equivalent to the question concerning the possibility to inscribe a regular hyper-symplex into a $(4k - 1)$-dimensional cube.

To obtain the Hadamard matrices in practice, it is possible to use the command *hadamard* of the MATLAB packet. It allows the Hadamard matrices to be built for the cases, when n, $n/12$ or $n/20$ are powers of 2. Unfortunately, such n as 28, 36, 44, 52, 56, 60, and others, which are divisible by 4, do not refer to these cases, though for them the Hadamard matrices have been found long ago. A list of all known Hadamard matrices, which has been composed by Sloan, can be found at the site http://neilsloane.com/hadamard/. In the Sloan's library there are given all Hadamard matrices for $n = 28$ and at least by one matrix for all n values divisible by 4, right up to $n = 256$. They have names of the type: had.1.txt, had.2.txt, had.4.txt, had.8.txt, …, 256.syl.txt, and are arranged in the form of text files containing arrays of signs + and −, corresponding to positive and negative entries of the Hadamard matrices. Contents of several files of such a type are given in Table 10.1.

The system of notation is clear from the first column, where both versions of a system for recording the Hadamard matrix of order 4, are shown.

Let us notice that the Hadamard matrices of order 2, 4, 8, and 12 are single (accurate to the isomorphism). At $n = 16$, there are some various Hadamard matrices. In the Sloan's library they are denoted as: had.16.0, had.16.hed, had.16.syl, had.16.twin, had.16.1, had.16.2, had.16.3, had.16.4. Three non-equivalent Hadamard matrices for $n = 20$ are denoted as had.20.pal, had.20.will, had.20.toncheviv. Further, in the library there are given 60 matrices of order 24 and 487 matrices of order 28, as well as the examples of Hadamard matrices for number to 256 inclusive for each n divisible by 4.

In the process of designing them, there were used methods proposed by Paley, Placket-Burman, Sylvester, Tourin, and Williamson. Certain information about these methods can be found in the digest [27], the authors of which constructed the

Table 10.1 Examples of Hadamard matrices txt-files

+ + + +	++++++++	+ −− − − − −−−−−−	++++++++++++++++
+ − + −	+− +− +−+−	++−+−−−+++−+	+ −+−+ − + − + −+−+−+−
+ + − −	++ −− ++ −−	+++−+−−−+++−	++ − − ++ −−++−−++ − −
+ − − +	+−−++ − − +	+−++−+−−−+++	+ − ++ − − ++ − − ++− +
1 1 1 1	++++− − − −	++−++−+−−−++	++++− − − ++++ − - − −
1−1 1−1	+−+ −− +− +	+++−++−+−−−+	+− + −+− ++ − + − − + − +
1 1−1−1	++− − − ++	++++−++−+−−−	++ − − − ++++ − − − − ++
1−1−1 1	+−− +− ++ −	+−+++−++−+−	+− + −++ − + − − + − ++ −
		+−+++−++−+−	++++++++ − − − − − − −−
		+−−−+++−++−+	+ − + − + − +− − + −+− + −+
		++−−−+++−++−	++ − − ++− − − − ++− − ++
		+−+−−−+++−++	+− − ++ − − + −++ − − ++ −
			++++− − − − − − - − ++++
			+− + − − + −+− + − ++ − + −
			++ − − − − ++− − ++++ − −
			+−− + −++ − − ++ − + − − +

Hadamard matrix of order 428. The greatest order, for which Hadamard matrix is presently known, is 668.

Not all Hadamard matrices represented in Table 10.1 are symmetrical. In Table 10.2 there are given versions of the matrices, which are symmetrical relative to the main or side diagonals and in a number of cases are more convenient for being used in the strip-method. The Hadamard matrix is named regular, if every row and every column contain the same number of "1". Such matrices have the maximum number of "1" entries (among all possible Hadamard matrices of a given order). For example, the 1st row of Table 10.1 contains a regular Hadamard matrix of order 4.

10.4.2 Shortened Hadamard Matrices

With a permutation of rows, columns and a multiplication of them by –1, it is possible to provide their symmetrical form with positive entries in the first row and the first column. Discarding this row and column, a shortened (reduced) matrix of order n–1 will be obtained. This matrix will no longer be orthogonal but becomes the circular one. All its rows are obtained with a cyclic shift of the first. This property is

Table 10.2 Symmetrical Hadamard matrices

$n = 4$	$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$
$n = 8$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \end{bmatrix}$
$n = 12$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 \end{bmatrix}$

useful in processing signals with the strip-method since it provides "smoothness" of the signal transmitted [3].

Let some properties of the shortened Hadamard matrices be analyzed. At $n = 4$, taking as a basis the matrix from the first column of Table 10.1, the following Hadamard matrix of the third order will be obtained

$$\bar{A}_3 = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \quad \bar{A}_3^{-1} = -\frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

In the case given the inverse matrix, also one can find itself circulante.

Let us consider the eigenvalues λ_i and eigenvectors $\mathbf{H}_i$ of the matrix $\bar{\mathbf{A}}_3$

$$\lambda_1 = -1 \quad \lambda_2 = -2 \quad \lambda_3 = 2$$

$$\mathbf{H}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{H}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{H}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

The first vector corresponds to two-multiple noise that in filtration with the strip-method remains unchanged; however other two-multiple noises can increase.

For $n = 8$ we obtain the following shortened Hadamard matrix of the seventh order and the one inverse to it.

The eigenvalues of the matrix $\bar{\mathbf{A}}_7$ have the form −1, −2, 2. In general case, the eigenvalues of the shortened Hadamard matrix obtained from the Hadamard matrix of order n are divided into three groups: one of them is always equal to −1, a half of the rest ones is equal to $\sqrt{n}$, and another half is equal to $-\sqrt{n}$.

$\bar{\mathbf{A}}_7$	$\bar{\mathbf{A}}_7^{-1}$
$\begin{bmatrix} -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & -1 \end{bmatrix}$	$\frac{1}{4}\begin{bmatrix} -1 & -1 & 0 & 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & -1 \\ -1 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$

10.4.3 Conference Matrices

Definition 2 The name Conference-matrix (C-matrix) is given to any matrix C of order n with zero on the main diagonal and +1 and −1 on the rest places satisfying the condition $C^{\mathrm{T}}C = (n - 1)I$.

Thus, rows (and columns) of C-matrices are orthogonal in pairs. The simplest C-matrices have the form as Eq. 10.11.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{bmatrix} \tag{10.11}$$

The first and third of them are symmetrical, the second and fourth are skew-symmetrical. The skew-symmetric C-matrices as well as the Hadamard matrices exist only at $n = 2$ and n, divisible by 4. From the point of view of the strip-method, they in all respects are inferior to the Hadamard matrices, and therefore will not be considered below.

The symmetrical C-matrices of order n can exist only in the case, when $n - 2$ is divisible by 4, and $n - 1$ can be presented in the form of a sum of squares of two integer numbers. For example, at $n = 2, 6, 10, 14, 18$ they exist and for $n = 22$ do not, since number 21 is not presented by a sum of two squares. For $n = 26, 30$ the C-matrices exist since equalities $25 = 3^2 + 4^2$, $29 = 2^2 + 5^2$ have a place. For $n = 34$, as well as for $n = 22$, a negative answer is obtained. For $n = 38, 42, 46$ the answer will also be negative.

Let us consider two problems, where we meet the C-matrices.

Conference arrangement problem. Let us suppose that n directors of some company have decided to arrange a conference by telephone in such a way as to provide any director with the possibility to speak to every one of his colleagues and the rest ones could listen to their discussion. The construction of such a conference-communications is equivalent to construction of a C-matrix.

Problem of weighing. What is the best scheme of weighing, if it needs to weigh n objects at n procedures of weighing?

The strategy of weighing is described by the C-matrix given by its entries c_{ij}:

$c_{ij} = 1$, if in weighing i the object j is located on the left pan;
$c_{ij} = -1$, if in weighing i the object j is on the right pan;
$c_{ij} = 0$, if in weighing i the object j does not take part.

For n divisible by 4, the best scheme of weighing is given with the Hadamard matrix and for even n, which are not divided by 4, is provided by the symmetrical C-matrix.

The normalized matrices, the order of which differs from the Hadamard ones on 2, are of the extreme quality similar to that the Hadamard matrices possess: their entry maximal in absolute value is minimal (for the class of orthogonal matrices). Further we will denote the entry maximal in absolute value as α. The value of this entry for the C-matrices equals $a = 1/\sqrt{n-1}$, i.e. it is only a little inferior to the

Hadamard matrices which have $a = 1/\sqrt{n}$. For example, for n = 6 the difference is less than 10 %.

These formulae taken together describe an accurate bottom boundary of the entry, maximal in absolute value, of the orthogonal matrices of the even order: the first one for n, which are not divisible by 4, in particular for 6, 10, 14, 18, 26; the second one for n divisible by 4, in particular for 4, 8, 12, 16, 20. In Table 10.3, the C-matrices for n = 10, 14, 18; cases for n = 2, 6, had been considered above (in Table 10.2) are shown.

The matrix $\mathbf{C}_{18}$ (as $\mathbf{C}_{14}$ too) have the symmetrical form with a zero diagonal. Moreover, there is an analogue matrix $\mathbf{X}_{18}$, having two zero diagonals disposed cross-wise.

Table 10.3 C-matrices for n = 10, 14, 18

$\mathbf{C}_{10}$	$\mathbf{C}_{14}$
0 -1 1 1 -1 1 -1 -1 -1 -1	0 1 1 1 1 1 1 1 1 1 1 1 1 1
-1 0 -1 1 1 -1 1 -1 -1 -1	1 0 -1 -1 -1 1 1 -1 1 -1 1 -1 1 1
1 -1 0 -1 1 -1 -1 1 -1 -1	1 -1 0 1 -1 -1 -1 1 -1 1 1 -1 1 1
1 1 -1 0 -1 -1 -1 -1 1 -1	1 -1 1 0 1 -1 1 -1 -1 -1 1 1 -1 1
-1 1 1 -1 0 -1 -1 -1 -1 1	1 -1 -1 1 0 -1 1 1 1 -1 -1 1 1 -1
1 -1 -1 -1 -1 0 1 -1 -1 1	1 1 -1 -1 -1 0 1 1 -1 1 1 1 -1 -1
-1 1 -1 -1 -1 1 0 1 -1 -1	1 1 -1 1 1 1 0 1 -1 -1 -1 -1 -1 1
-1 -1 1 -1 -1 -1 1 0 1 -1	1 -1 1 -1 1 1 1 0 -1 1 -1 -1 1 -1
-1 -1 -1 1 -1 -1 -1 1 0 1	1 1 -1 -1 1 -1 -1 -1 0 1 -1 1 1 1
-1 -1 -1 -1 1 1 -1 -1 1 0	1 -1 1 -1 -1 1 -1 1 1 0 -1 1 -1 1
	1 1 1 1 -1 1 -1 -1 -1 -1 0 1 1 -1
	1 -1 -1 1 1 1 -1 -1 1 1 1 0 -1 -1
	1 1 1 -1 1 -1 -1 1 1 -1 1 -1 0 -1
	1 1 1 1 -1 -1 1 -1 1 1 -1 -1 -1 0

$\mathbf{C}_{18}$	$\mathbf{X}_{18}$
0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0
1 0 1 1 1 -1 1 -1 1 -1 1 1 -1 1 -1 -1 -1 -1	1 0 1 1 -1 1 -1 1 -1 -1 1 -1 -1 -1 1 1 0 1
1 1 0 1 1 1 -1 1 -1 -1 1 -1 -1 -1 -1 1 1 -1	1 1 0 -1 -1 1 1 -1 -1 -1 -1 1 1 -1 1 0 1 -1
1 1 1 0 -1 1 1 1 -1 -1 -1 -1 -1 1 1 -1 -1 1	1 1 1 0 -1 -1 1 1 -1 1 -1 -1 1 1 0 -1 -1 1
1 1 1 -1 0 1 1 -1 1 1 -1 -1 1 -1 -1 1 -1 -1	1 1 1 -1 0 -1 -1 -1 1 -1 1 1 1 0 -1 1 -1 1
1 -1 1 1 1 0 -1 -1 1 -1 -1 -1 1 -1 1 -1 1 1	1 -1 1 1 -1 0 -1 -1 -1 1 1 1 0 1 -1 -1 1 -1
1 1 -1 1 1 -1 0 1 -1 1 -1 1 1 -1 -1 -1 -1 1	1 1 -1 1 -1 -1 0 1 1 -1 -1 0 -1 1 -1 1 1 -1
1 -1 1 1 -1 -1 1 0 -1 1 -1 1 -1 -1 1 1 1 -1	1 1 -1 1 1 1 -1 -1 0 0 -1 1 -1 1 1 -1 -1 1
1 1 -1 -1 1 1 -1 -1 0 1 -1 1 -1 1 1 -1 1 -1	1 -1 -1 -1 -1 1 -1 0 1 1 0 -1 1 1 1 1 -1 -1
1 -1 -1 -1 1 -1 1 1 1 0 -1 -1 -1 1 -1 1 1 1	1 -1 1 -1 1 1 1 -1 0 0 -1 -1 -1 1 -1 1 1 1
1 1 1 -1 -1 -1 -1 -1 -1 -1 0 1 1 1 -1 1 1 1	1 1 1 -1 1 -1 -1 0 1 1 0 -1 -1 -1 1 -1 1 -1
1 1 -1 -1 -1 -1 1 1 1 -1 1 0 1 -1 1 -1 1 -1	1 -1 1 1 1 1 0 1 1 -1 -1 0 1 -1 -1 -1 -1 -1
1 -1 -1 -1 1 1 1 -1 -1 -1 1 1 0 -1 1 1 -1 1	1 1 -1 1 1 0 1 -1 -1 1 1 -1 0 -1 -1 1 -1 -1
1 1 -1 1 -1 -1 -1 -1 1 1 1 -1 -1 0 1 1 -1 1	1 -1 -1 1 0 -1 1 -1 1 -1 1 -1 1 0 1 -1 1 1
1 -1 -1 1 -1 1 -1 1 1 -1 -1 1 1 1 0 1 -1 -1	1 -1 -1 0 1 -1 -1 1 -1 1 -1 1 1 -1 0 1 1 1
1 -1 1 -1 1 -1 -1 1 -1 1 1 -1 1 1 1 0 -1 -1	1 -1 0 -1 1 -1 1 1 -1 -1 1 1 -1 1 1 0 -1 -1
1 -1 1 -1 -1 1 -1 1 1 1 1 1 -1 -1 -1 -1 0 1	1 0 -1 -1 -1 1 1 1 1 1 1 1 -1 -1 -1 -1 0 1
1 -1 -1 1 -1 1 1 -1 -1 1 1 -1 1 1 -1 -1 1 0	0 1 -1 -1 1 1 -1 1 -1 -1 1 -1 1 1 -1 -1 1 0

It is quite close to the optimal one. The value of its maximum element after normalization is equal to $\alpha = 1/\sqrt{n-2} = 0.25$ (for the matrix C_{18}, $\alpha = 1/\sqrt{n-1} = 0.2425$).

The Hadamard matrices and C-matrices are closely connected. In particular, it is possible to construct Hadamard matrices from C-matrices [28].

Suppose $\mathbf{C}$ is a symmetric C-matrix of order m. Then the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{C} + \mathbf{I}_m & \mathbf{C} - \mathbf{I}_m \\ \mathbf{C} - \mathbf{I}_m & -\mathbf{C} - \mathbf{I}_m \end{bmatrix}$$

is a Hadamard matrix of order $2m$.

This matrix is rather close to the optimal one; after normalization the value of its maximal entry is equal to $\alpha = 1/\sqrt{n-2} = 0.25$ (for the matrix $\mathbf{C}_{18}$, $\alpha = 1/\sqrt{n-1} = 0.2425$). Moreover, if $\mathbf{C}$ is antisymmetric C-matrix, then $\mathbf{I} + \mathbf{C}$ is a Hadamard matrix of order m.

In the aggregate the Hadamard matrices and C-matrices give the solution of the orthogonal Procrustean problem (the problem to find orthogonal matrices with an entry minimal in absolute value) almost for all even n, with the exception of several values such as $n = 22$ and $n = 34$.

The situation for odd n is too much worse. Here only a few optimal matrices for small values of n are known. Information about them is given below.

10.4.4 Optimal Orthogonal Matrices of the Odd Order (M-Matrices)

Let us name the matrices providing a solution of the orthogonal Procrustean problem for odd n minimax, or simply M-matrices. Their main property is the minimality of the value α, i.e. the values of the entry maximal in absolute value on the class of all orthogonal matrices of a given dimension. Here it is possible to indicate three problems [29]:

Problem 1 Search of particular M-matrices for various numbers n.

Problem 2 Determination of an accurate bottom boundary α^* for the value of maximal entries of M-matrices α depending on n: $\alpha \geq \alpha^* = f(n)$.

Problem 3 Determination of the number k of entry levels in the M-matrix for different n.

Therefore, the Hadamard matrices can be called one-level since all their entries are equal in absolute value. The C-matrices are two-level, modulus of their entries is equal to 0 or 1. For an odd n, the M-matrices appear to be the k-level ones; k depending on n [30, 31].

It should be expected that the solution of all three problems set will depend on what remainder is, when the odd number n is divided by 4 (1 or 3). Correspondingly, a set of M-matrices breaks up into two subsets that differ in bottom boundaries, number of levels k and type of matrices.

Let us move to description of particular M-matrices for n = 3, 5, 7, 9, 11. Searching for these matrices is performed by numerical and symbolic modelling in the MATLAB and MAPLE packets with the help of specially developed software. As a result we have managed to determine an analytic type of entries of the optimal matrices M_3, M_5, M_7, M_9, as well as to find the matrix M_{11} in the numerical form, having preliminary obtained a system of non-linear algebraic equations for determining its entries. A more detailed procedure of searching is explained below by an example of the matrix M_{11}.

For the case n = 3, the optimal matrix providing the solution of the orthogonal Procrustean problem, is provided by Eq. 10.12.

$$\mathbf{M}_3 = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \tag{10.12}$$

This matrix is orthogonal and symmetrical, the value of its maximal entry is equal to α = 2/3. The matrix contains entries of two types, i.e. it has two levels. For n = 5 the optimal matrix occurs to be of three levels (Eq. 10.13)

$$\mathbf{M}_3 = \frac{1}{11}\begin{bmatrix} -2 & 3 & 6 & 6 & 6 \\ 3 & 6 & -6 & 6 & -2 \\ 6 & -6 & -3 & 2 & 6 \\ 6 & 6 & 2 & -6 & 3 \\ 6 & -2 & 6 & 3 & -6 \end{bmatrix}. \tag{10.13}$$

It is also orthogonal and symmetrical, the value of its maximal entry α = 6/11. Distribution of the absolute value of its entries by levels is shown in Fig. 10.6.

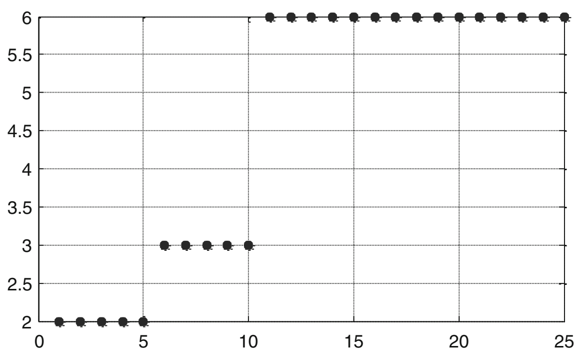


Fig. 10.6 Distribution of the matrix $\mathbf{M}_5$ entries by levels

From its 25 entries, 15 ones are on the upper level, the rest ones by 5 are on the remaining two levels. Thus, the entries of the upper levels amounts to 60 % of the total number (67 % for the matrix M_3 and 100 % for the Hadamard matrices).

In investigating the case $n = 7$, there were found two matrices: the five-level matrix M_7 of the value $\alpha = \frac{5+7\sqrt{7}}{53} \approx 0.444$ and two-level matrix N_7 of the value $\alpha = \frac{2+3\sqrt{2}}{14} \approx 0.446$. The structures of these matrices are the following:

$$\mathbf{M}_7 \quad = \quad \begin{matrix} [a, & -d, & c, & a, & -a, & -a, & -a] \\ [-d, & c, & a, & a, & a, & a, & -a] \\ [c, & a, & -d, & a, & -a & a, & a] \\ [a, & a, & a, & -c & b, & -b, & b] \\ [-a, & a, & -a, & b, & c, & -a, & -d] \\ [-a, & a, & a, & -b, & -a, & -d, & -e] \\ [-a, & -a, & a, & b, & -d, & -e, & a], \end{matrix}$$

$$\mathbf{N}_7 \quad = \quad \begin{matrix} [a, & a, & a, & a, & b, & b, & -b] \\ [a, & -b, & -b, & a, & -a, & b, & a] \\ [a, & -b, & a, & -b, & b, & -a, & a] \\ [a, & a, & -b, & -b, & -a, & -a, & -b] \\ [b, & -a, & b, & -a, & -b, & a, & -a] \\ [b, & b, & -a, & -a, & -b, & a, & b] \\ [-b, & a, & a, & -b & -a, & b, & a]. \end{matrix}$$

Unlike the preceding cases, the entries of these matrices are irrational.

For the matrix $\mathbf{M}_7$ they contain $\sqrt{7}$: $a = 3 + 3\sqrt{7}$, $b = 9$, $c = 5 - \sqrt{7}, d = -6 + 3\sqrt{7},\ e = 4 + \sqrt{7}$.

In normalizing all of them should be divided by $22 + \sqrt{7}$. Entries of the matrix N_7 contain $\sqrt{2}$: $a = 2 + \sqrt{2},\ \ b = 2$. In normalizing all of them should be divided by $2 + 4\sqrt{2}$. Let us show both of these matrices in detailed writing (without any normalization) (Eq. 10.14).

$$M_7 = \begin{bmatrix} 3+3\sqrt{7}, & 6-3\sqrt{7}, & 5-\sqrt{7}, & 3+3\sqrt{7}, & -3-3\sqrt{7}, & -3-3\sqrt{7}, & -3-3\sqrt{7} \\ 6-3\sqrt{7}, & 5-\sqrt{7}, & 3+3\sqrt{7}, & 3+3\sqrt{7}, & 3+3\sqrt{7}, & 3+3\sqrt{7}, & -3-3\sqrt{7} \\ 5-\sqrt{7}, & 3+3\sqrt{7}, & 6-3\sqrt{7}, & 3+3\sqrt{7}, & -3-3\sqrt{7}, & 3+3\sqrt{7}, & 3+3\sqrt{7}, \\ 3+3\sqrt{7}, & 3+3\sqrt{7}, & 3+3\sqrt{7}, & -5+\sqrt{7}, & 9, & -9, & 9 \\ -3-3\sqrt{7}, & 3+3\sqrt{7}, & -3-3\sqrt{7}, & -9, & 4+\sqrt{7}, & -3-3\sqrt{7}, & 6-3\sqrt{7} \\ -3-3\sqrt{7}, & 3+3\sqrt{7}, & 3+3\sqrt{7}, & -9, & -3-3\sqrt{7}, & 6-3\sqrt{7}, & -4-\sqrt{7} \\ -3-3\sqrt{7}, & -3-3\sqrt{7}, & 3+3\sqrt{7}, & 9, & 6-3\sqrt{7}, & -4-\sqrt{7}, & 3+3\sqrt{7} \end{bmatrix} \tag{10.14}$$

$$N_7 = \begin{bmatrix} 2+\sqrt{2} & 2+\sqrt{2} & 2+\sqrt{2} & 2+\sqrt{2} & 2 & 2 & -2 \\ 2+\sqrt{2} & -2 & -2 & 2+\sqrt{2} & -2-\sqrt{2} & 2 & 2+\sqrt{2} \\ 2+\sqrt{2} & -2 & 2+\sqrt{2} & -2 & 2 & -2-\sqrt{2} & 2+\sqrt{2} \\ 2+\sqrt{2} & 2+\sqrt{2} & -2 & -2 & -2-\sqrt{2} & -2-\sqrt{2} & -2 \\ 2 & -2-\sqrt{2} & 2 & -2-\sqrt{2} & -2 & 2+\sqrt{2} & -2-\sqrt{2} \\ 2 & 2 & -2-\sqrt{2} & -2-\sqrt{2} & 2+\sqrt{2} & 2+\sqrt{2} & 2 \\ -2 & 2+\sqrt{2} & 2+\sqrt{2} & -2 & -2-\sqrt{2} & 2 & 2+\sqrt{2} \end{bmatrix}$$

Distribution of the entry modulus for the normalized matrix $\mathbf{M}_7$ level by level, which has been obtained in MATLAB with the help of the command “plot(sort(abs (M7(:))),‘*’)”, is shown in Fig. 10.7.

From this figure, it is seen that the bottom level contains 6 entries. The next ones contain 4, 3, and 6 entries, respectively. The most numerous upper level contains 30 entries, which amounts to about 61 % (approximately as much as in the case with the matrix M_5).

For $n = 9$ the best from found matrices has four levels and the value $\alpha = \frac{3+\sqrt{3}}{12} = 0.3943$.

$$\boldsymbol{M}_9 = \begin{matrix} [d, & b, & b, & b, & b, & b, & b, & b, & b] \\ [b, & a, & a, & a, & -a, & -a, & -c, & -c, & -c] \\ [b, & a, & -c, & -a, & -c, & a, & a, & -c, & -a] \\ [b, & a, & -a, & -c, & a, & -c, & -a, & -c, & a] \\ [b, & -a, & -c, & a, & a, & -c, & a, & -a, & -c] \\ [b, & -a, & a, & -c, & -c, & a, & -c, & -a, & a] \\ [b, & -c, & a, & -a, & a, & -c, & -c, & a, & -a] \\ [b, & -c, & -c, & -c, & -a, & -a, & a, & a, & a] \\ [b, & -c, & -a, & a, & -c, & a, & -a, & a, & -c], \end{matrix}$$

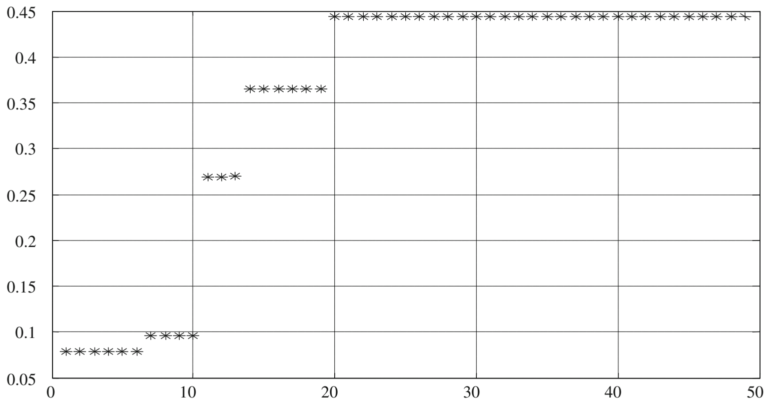


Fig. 10.7 Distribution of the matrix $\mathbf{M}_7$ entries by levels

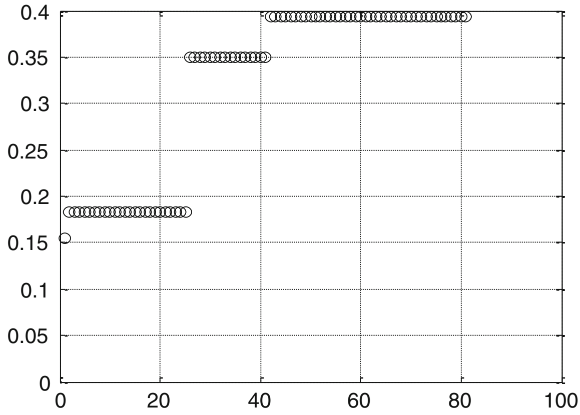


Fig. 10.8 Distribution of the matrix $\mathbf{M}_9$ entries by levels

$$\begin{aligned} 12a &= 3+\sqrt{3}, & a &= 0.3943,\\ 6b &= \sqrt{6\sqrt{3}-6}, & b &= 0.3493,\\ 4c &= \sqrt{3}-1, & c &= 0.1830,\\ 3d &= 2\sqrt{3}-3, & d &= 0.1547, \end{aligned}$$

$$\text{Maximal entry } \frac{3+\sqrt{3}}{12} = 0.394337.$$

Its structure and entries are the following. Here we deal with an irrationality of the type "a root from a root" arising from the solution of a biquadratic equation. Distribution of modulus of the matrix $\mathbf{M}_9$ entries on levels is shown in Fig. 10.8.

From Fig. 10.8, it is seen that on the bottom level there is one entry, on the next two levels there are 34 and 16 entries, respectively. On the upper level, there are 40 entries, which amounts to 49 % of their total number. Unfortunately, $n = 9$ is the final case, when it has been managed to get explicit expressions for entries of the M-matrix.

For $n = 11$ the best orthogonal matrix founded in MATLAB, has a six-level structure

$$\mathbf{M}_{11} = \begin{bmatrix} -b & a & f & a & a & d & c & e & a & -a & -a\\ -d & f & a & -a & e & -a & b & c & -a & -a & a\\ -a & -e & -c & a & d & -a & a & -a & f & a & b\\ a & -d & a & b & a & a & -f & -a & -e & -c & a\\ a & a & e & a & -b & -a & a & -d & -a & -f & -c\\ a & -a & a & -d & a & -e & a & f & c & b & -a\\ -f & b & d & -c & -a & a & a & -a & a & e & a\\ e & a & a & a & f & -c & -a & a & b & d & a\\ a & a & -a & -f & c & a & d & b & -a & a & e\\ a & -c & -b & e & -a & -f & a & a & a & -a & d\\ -c & -a & a & a & -a & b & e & a & -d & a & -f \end{bmatrix}.$$

The numerical values of its entries are as follows: $a = 0.34295283$, $b = 0.33572291$, $c = 0.30893818$, $d = 0.2439851$, $e = 0.15671878$, $f = 0.045364966$. Their distribution over the levels is shown in Fig. 10.9.

The index $\alpha = 0.3429$ is equal to the value of the entry a. Let us notice that a percentage of entries maximal in absolute value amounts to 6/11 ≈ 54.5 %, which accurately coincides with the value of the index α for the matrix $\mathbf{M}_5$.

Equally with the search of optimal matrices of the odd order, there is a similar task with regard to those matrices of the even order, for which there are no C-matrices. First of all this refers to the orders $n = 22$, $n = 34$, and $n = 66$. Let us give the best result obtained for $n = 22$. The two level matrix $\mathbf{M}_{22}$ has the following form:

$\mathbf{M}_{22} =$

0	1	1	−1	−1	1	−1	1	−1	1	1	0	1	1	−1	1	1	1	1	1	−1	−1
1	0	1	1	−1	−1	1	−1	1	−1	1	−1	0	1	1	−1	1	1	1	1	1	−1
1	1	0	1	1	−1	−1	1	−1	1	−1	−1	−1	0	1	1	−1	1	1	1	1	1
−1	1	1	0	1	1	−1	−1	1	−1	1	1	−1	−1	0	1	1	−1	1	1	1	1
1	−1	1	1	0	1	1	−1	−1	1	−1	1	1	−1	−1	0	1	1	−1	1	1	1
−1	1	−1	1	1	0	1	1	−1	−1	1	1	1	1	−1	−1	0	1	1	−1	1	1
1	−1	1	−1	1	1	0	1	1	−1	−1	1	1	1	1	−1	−1	0	1	1	−1	1
−1	1	−1	1	−1	1	1	0	1	1	−1	1	1	1	1	1	−1	−1	0	1	1	−1
−1	−1	1	−1	1	−1	1	1	0	1	1	−1	1	1	1	1	1	−1	−1	0	1	1
1	−1	−1	1	−1	1	−1	1	1	0	1	1	−1	1	1	1	1	1	−1	−1	0	1
1	1	−1	−1	1	−1	1	−1	1	1	0	1	1	−1	1	1	1	1	1	−1	−1	0
0	−1	−1	1	1	1	1	1	−1	1	1	0	−1	−1	1	−1	1	−1	1	1	−1	−1
1	0	−1	−1	1	1	1	1	1	−1	1	−1	0	−1	−1	1	−1	1	−1	1	1	−1
1	1	0	−1	−1	1	1	1	1	1	−1	−1	−1	0	−1	−1	1	−1	1	−1	1	1
−1	1	1	0	−1	−1	1	1	1	1	1	1	−1	−1	0	−1	−1	1	−1	1	−1	1
1	−1	1	1	0	−1	−1	1	1	1	1	1	1	−1	−1	0	−1	−1	1	−1	1	−1
1	1	−1	1	1	0	−1	−1	1	1	1	−1	1	1	−1	−1	0	−1	−1	1	−1	1
1	1	1	−1	1	1	0	−1	−1	1	1	1	−1	1	1	−1	−1	0	−1	−1	1	−1
1	1	1	1	−1	1	1	0	−1	−1	1	−1	1	−1	1	1	−1	−1	0	−1	−1	1
1	1	1	1	1	−1	1	1	0	−1	−1	1	−1	1	−1	1	1	−1	−1	0	−1	−1
−1	1	1	1	1	1	−1	1	1	0	−1	−1	1	−1	1	−1	1	1	−1	−1	0	−1
−1	−1	1	1	1	1	1	−1	1	1	0	−1	−1	1	−1		−1	1	1	−1	−1	0

The distribution of modules of its elements over the levels is shown in Fig. 10.10.

The index of this matrix $\alpha = 0.2236$, which is worse than estimate 0.2182 for the non-existent C-matrix only by 0.0054. This index is a little worse, i.e. only by 0.0033, than the index $\alpha = 0.2269$ for six level matrix M_{22}, obtained in [32, 33]. Similar two level matrices exist and for cases $n = 34$ and $n = 66$.

10.4.5 *Two-, Three-, and Many-Levels M-Matrices*

The Hadamard matrices have many remarkable properties marking them out on a set of orthogonal matrices. Unfortunately, at $n > 2$ there exist no Hadamard matrices, if n is odd or becomes odd after dividing by 2. In such cases there is a

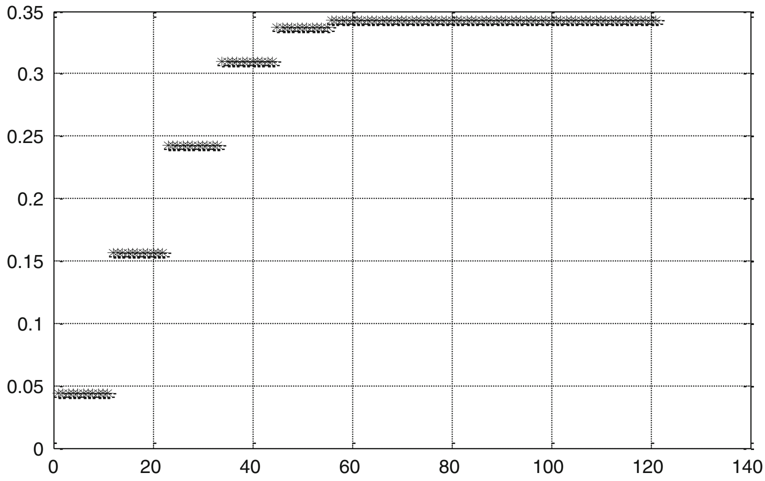
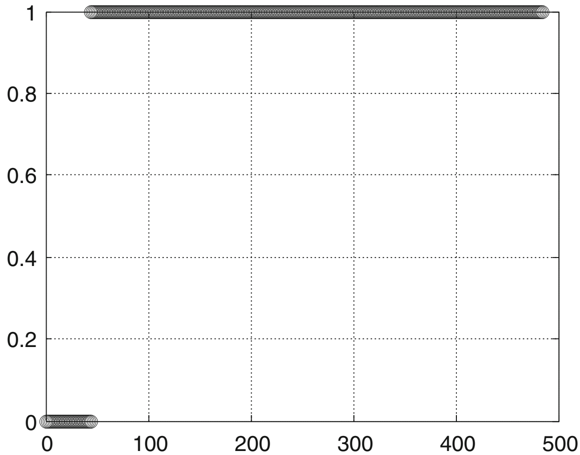


Fig. 10.9 Distribution of the matrix $\mathbf{M}_{11}$ entries by levels

Fig. 10.10 Distribution of the matrix $\mathbf{M}_{22}$ entries by levels



problem of searching some orthogonal matrices that due to their properties are close to Hadamard matrices. On mathematical statement of this problem it is necessary to indicate, what properties of Hadamard matrices should be saved in particular. According to the author's opinion three versions of setting the problem are the most natural ones:

1. An orthogonal matrix of a given order n, for which the highest possible (maximal) absolute value element is minimal (the minimax problem) has to be found.
2. An orthogonal matrix of a given order n, for which the minimum absolute value element is maximal (the maximin problem) has to be found.

3. An orthogonal matrix of a given order n, for which the difference between the maximal module and minimal module elements is minimal (the problem concerning the matrices with a minimal swing of elements).

In all versions, the cases connected with logical design of arraying indicated matrices, finding specific matrices with properties given, and analyzing the asymptote at a large n originate. It should be noted that from the theoretical point of view all three problems are equally substantial. At the same time the first problem seems to be of the greatest practical interest since it well agrees with the standard criterion signal-to-noise ratio traditionally used in the communication theory. It is interesting that in all three cases the optimal orthogonal matrices have a property, due to which their elements are able to group by a value, i.e., their elements are divided into a small number of groups (levels) with equal absolute values. At the same time, for optimal minimax matrices (problem 1) it is typical that the group of elements, which are maximal by their absolute values, is the most numerous one. On the contrary, for optimal maximin matrices (problem 2) the group of elements that are minimal by their absolute values is the most numerous one. In problem 3, it is possible to expect a more symmetrical pattern of the level-wise element distribution.

Let the orthogonal matrices be called the r-level ones, if the absolute values of their elements possess precisely r values. For example, the Hadamard matrices are single-level ones, the unitary matrix and permutation matrices are two-level ones.

From the point of view of tasks of processing images and signals, encoding, masking, constructing noise combating codes, the integer-valued two-level orthogonal matrices and ones obtained from them by the way of multiplication by a constant, are of a particular interest.

It is possible to outline a number of classes of such matrices.

The C-matrices. Such matrices are orthogonal with elements ±1 and zero main diagonal.

The D-matrices. Matrices of such a type are orthogonal of the following form:

$$\mathbf{D}_n = \frac{1}{n}\begin{bmatrix} 2-n & 2 & 2 & \cdots & 2 \\ 2 & 2-n & 2 & \cdots & 2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2 & 2 & 2 & \cdots & 2-n \end{bmatrix}.$$

In particular, at n = 3, 4, 5 they look like as mentioned below:

$$\mathbf{D}_3 = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \mathbf{M}_3, \quad \mathbf{D}_4 = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix},$$

$$\mathbf{D}_5 = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 & 2 & 2 \\ 2 & -3 & 2 & 2 & 2 \\ 2 & 2 & -3 & 2 & 2 \\ 2 & 2 & 2 & -3 & 2 \\ 2 & 2 & 2 & 2 & -3 \end{bmatrix}.$$

Let us note that the matrix $\mathbf{D}_3$ coincides with the optimal matrix $\mathbf{M}_3$, and the matrix $\mathbf{D}_4$ with the Hadamard matrix $\mathbf{A}_4$.

At n = 6, 8 the D-matrices have a view:

$$\mathbf{D}_6 = \frac{1}{3}\begin{bmatrix} -2 & 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 & 1 & 1 \\ 1 & 1 & -2 & 1 & 1 & 1 \\ 1 & 1 & 1 & -2 & 1 & 1 \\ 1 & 1 & 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & 1 & 1 & -2 \end{bmatrix}$$

$$\mathbf{D}_8 = \frac{1}{4}\begin{bmatrix} -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 \end{bmatrix}.$$

The using of the D-matrices of higher orders is not efficient, since by a value of the maximum element $a = \frac{2}{n} - 1$ they significantly worse than the optimal M-matrices. At the same time the above indicated D-matrices can be used as "building blocks" for constructing other two-level and three-level matrices.

Let us note that the matrix $\mathbf{D}_4$ is a particular case of the family of two-level matrices having the form

$$\begin{bmatrix} a & -b & -b & b \\ b & a & b & b \\ b & -b & a & -b \\ -b & -b & b & a \end{bmatrix},$$

which are orthogonal at any a, b. Particularly the versions [a, b] = [1 2], [1 3], [2 3] are possible.

The composed orthogonal matrices. One of the methods applied for constructing the two-level orthogonal matrices is based on application of the Kronecker product of single-level and two-level matrices. For example, when analyzing the Kronecker product of the matrix $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ and Hadamard matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, one get a family of two-level matrices of the 4th order.

At $n = 6$, multiplying the matrix $\mathbf{D}_3$ on the Hadamard matrix $\mathbf{A}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and on unitary matrix $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, one get two two-level matrices of the 6th order:

$$\mathbf{D}_3 \otimes \mathbf{A}_2 = \begin{bmatrix} \mathbf{D}_3 & \mathbf{D}_3 \\ \mathbf{D}_3 & -\mathbf{D}_3 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 2 & -1 & 2 & 2 \\ 2 & -1 & 2 & 2 & -1 & 2 \\ 2 & 2 & -1 & 2 & 2 & -1 \\ -1 & 2 & 2 & 1 & -2 & -2 \\ 2 & -1 & 2 & -2 & 1 & -2 \\ 2 & 2 & -1 & -2 & -2 & 1 \end{bmatrix},$$

$$\alpha = \sqrt{3}/2 = 0.4714,$$

$$\mathbf{D}_3 \otimes \mathbf{I} = \begin{bmatrix} \mathbf{D}_3 & 0 \\ 0 & \mathbf{D}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 0 & 0 & 0 \\ 2 & 1 & -2 & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & -2 & 2 & 1 \end{bmatrix},$$

$$\alpha = 2/3 = 0.6667.$$

Let us note that at the Kronecker multiplication of orthogonal matrices $\mathbf{A} = \mathbf{A}_1 \otimes \mathbf{A}_2$ their indices are multiplied: $\alpha = \alpha_1 \alpha_2$ The unitary matrix $\mathbf{I}$ has the index $\alpha = 1$, therefore at $\mathbf{A}_2 = \mathbf{I}$ we get $\alpha = \alpha_1$. Provided $\mathbf{A}_2$ is the Hadamard matrix then the result index is equal to $\alpha = \alpha_1/\sqrt{n}$.

The Kronecker product of the matrix D_5 and Hadamard matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ gives the two-level matrix of the 10th order with the index $\alpha = 0.3\sqrt{2} = 0.4242$ (the matrix C_{10} has the similar index $\alpha = 1/3$).

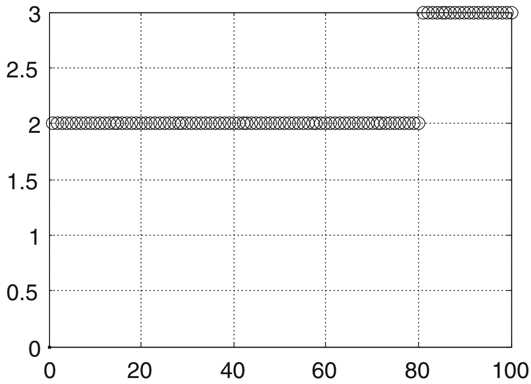


Fig. 10.11 Distribution of the matrix $\mathbf{C}_{10}$ entries by levels

$$\mathbf{C}_{10} = \begin{matrix} 2 & -3 & 2 & -2 & -2 & 2 & 2 & 2 & 3 & -2 \\ 2 & 2 & -2 & 3 & 2 & 2 & 2 & -2 & 2 & -3 \\ 2 & 2 & -2 & -2 & 2 & 2 & -3 & 3 & 2 & 2 \\ -2 & -2 & -2 & 2 & 2 & -2 & 3 & 3 & 2 & 2 \\ 3 & -2 & 3 & 2 & 2 & -2 & -2 & -2 & 2 & 2 \\ -2 & -2 & 2 & 2 & 3 & 3 & -2 & 2 & -2 & -2 \\ 3 & -2 & -3 & 2 & -2 & -2 & -2 & 2 & -2 & -2 \\ -2 & -2 & -2 & 2 & -3 & 3 & -2 & -2 & 2 & 2 \\ 2 & 2 & 2 & 3 & -2 & 2 & 2 & 2 & -2 & 3 \\ -2 & 3 & 2 & 2 & -2 & -2 & -2 & 2 & 3 & -2. \end{matrix}$$

The distribution of the matrix $\mathbf{C}_{10}$ entries over the levels is shown in Fig. 10.11 (n = 10, 2 levels). Each of its line has 8 twins and 2 triplets.

Let us consider some other examples of two-level matrices. At n = 13, there is a matrix consisting of zeros and units, which has the index α = 1/3 and contains 4 zeros in each of its lines

$$
\mathbf{C}_{13} =
\begin{array}{ccccccccccccc}
0 & -1 & 1 & 1 & -1 & 0 & -1 & -1 & -1 & -1 & 0 & -1 & 0 \\
-1 & -1 & 0 & 0 & 1 & -1 & 0 & -1 & -1 & 1 & 1 & 1 & 0 \\
-1 & 0 & 1 & -1 & -1 & 1 & -1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 1 & -1 & 1 & -1 & 1 \\
0 & -1 & -1 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\
-1 & 0 & -1 & 0 & 1 & 1 & -1 & 1 & -1 & -1 & 0 & 0 & 1 \\
-1 & 1 & 0 & -1 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & -1 & -1 \\
-1 & -1 & 1 & -1 & 0 & -1 & 1 & 1 & 0 & -1 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & -1 & -1 & -1 & 0 & -1 & 1 & -1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 1 \\
1 & 0 & 0 & -1 & 1 & 0 & -1 & -1 & 0 & -1 & -1 & 1 & -1.
\end{array}
$$

The distribution of the matrix $\mathbf{C}_{13}$ entries over the levels is shown in Fig. 10.12 ($n = 13$, 2 levels, $\alpha = 1/3$).

At $n = 15$ there is a matrix, each line of which contains 7 twins and 8 triplets.

$$
\mathbf{C}_{15} =
\begin{array}{ccccccccccccccc}
3 & 2 & 2 & -3 & 3 & -3 & -2 & 2 & 2 & -2 & 3 & -3 & -3 & -3 & 2 \\
-2 & -3 & -3 & -3 & 3 & 2 & 3 & 2 & 2 & 3 & -2 & -3 & -3 & 2 & 2 \\
2 & -2 & 3 & -2 & -3 & -2 & -3 & 3 & -2 & 2 & -3 & -2 & 3 & 3 & 3 \\
2 & 3 & 3 & 3 & 2 & -2 & 2 & -2 & 3 & 2 & -3 & 3 & -2 & 3 & 3 \\
2 & -2 & 3 & -2 & 2 & 3 & 2 & 3 & 3 & -3 & 2 & 3 & 3 & 3 & -2 \\
3 & 2 & 2 & -3 & -2 & 2 & 3 & -3 & 2 & 3 & 2 & -3 & 2 & -3 & -3 \\
-3 & 3 & -2 & -2 & 2 & -2 & -3 & -2 & 3 & -3 & -3 & -2 & 3 & 3 & -2 \\
2 & 3 & -2 & 3 & 2 & 3 & 2 & 3 & -2 & -3 & -3 & -2 & 3 & -2 & 3 \\
3 & -3 & -3 & 2 & 3 & -3 & -2 & 2 & 2 & 3 & -2 & 2 & 2 & -3 & -3 \\
-3 & 3 & 3 & -2 & 2 & 3 & -3 & 3 & -2 & 2 & -3 & 3 & -2 & -2 & -2 \\
3 & 2 & -3 & -3 & -2 & -3 & 3 & 2 & -3 & -2 & -2 & 2 & -3 & 2 & -3 \\
-3 & 3 & -2 & -2 & -3 & -2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & -2 & 3 \\
3 & 2 & -3 & -3 & 3 & 2 & -2 & -3 & -3 & 3 & 3 & 2 & 2 & 2 & 2 \\
-2 & 2 & 2 & 2 & 3 & -3 & 3 & 2 & -3 & 3 & 3 & -3 & 2 & 2 & -3 \\
2 & 3 & -2 & 3 & -3 & 3 & -3 & 3 & 3 & 2 & 2 & -2 & -2 & 3 & -2.
\end{array}
$$

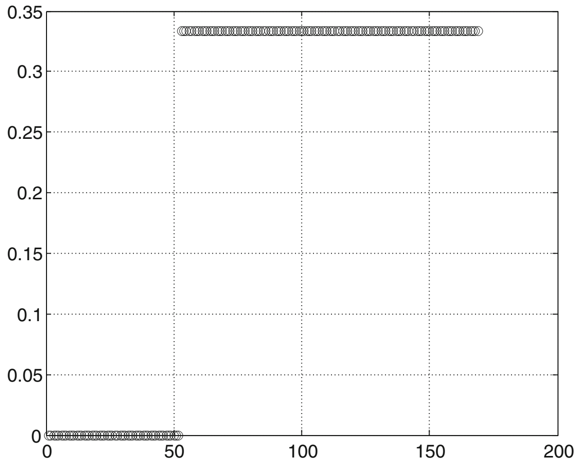


Fig. 10.12 Distribution of the matrix $\mathbf{C}_{13}$ entries by levels

The distribution of the matrix $\mathbf{C}_{15}$ entries over the levels is shown in Fig. 10.13 (n = 15, 2 levels, α = 0.3).

Moreover, at $n = 15$ there is the matrix $\mathbf{C}_{15}$ (the variant) that contains 7 twins and 8 units in each of its lines:

$\mathbf{C}_{15}. =$

2	2	2	−2	1	2	−1	1	−1	−1	−1	−1	2	−1	2
1	2	−1	−2	−2	2	2	1	−1	2	−1	2	−1	−1	−1
1	−1	2	1	−2	2	−1	−2	2	−1	−1	2	2	−1	−1
−1	−2	1	−1	−1	1	1	−1	−2	−2	−2	−2	−2	1	−2
1	−1	2	−2	1	−1	2	−2	2	2	−1	−1	−1	−1	2
−2	2	2	1	−2	−1	2	1	2	−1	2	−1	−1	−1	−1
−2	−1	−1	1	1	2	2	−2	−1	2	2	−1	2	−1	−1
2	1	1	2	2	1	1	2	1	1	−2	−2	1	1	−2
−1	−2	1	−1	2	1	−2	2	1	1	1	1	−2	−2	−2
−2	−1	−1	−2	1	−1	2	1	2	−1	−1	2	2	2	−1
2	−2	−2	−1	−1	1	1	2	1	−2	1	−2	1	−2	1
−1	1	−2	−1	−1	−2	−2	−1	1	1	−2	−2	1	−2	−2
−1	−2	1	2	−1	−2	1	2	−2	1	−2	1	1	−2	1
−2	−1	−1	1	−2	2	−1	1	2	2	−1	−1	−1	2	2
1	−1	2	−2	−2	−1	−1	1	−1	2	2	−1	2	2	−1

The distribution of the matrix $\mathbf{C}_{15}$ (the variant) entries over the levels as it is shown in Fig. 10.14 (n = 15, 2 levels, α = 1/3).

Similar matrices with elements consisting of two adjacent integer numbers exist at $n = 2^{2k} - 1$, i.e. at n = 3, 15, 63, 255, … At a given k there exist two matrices, one

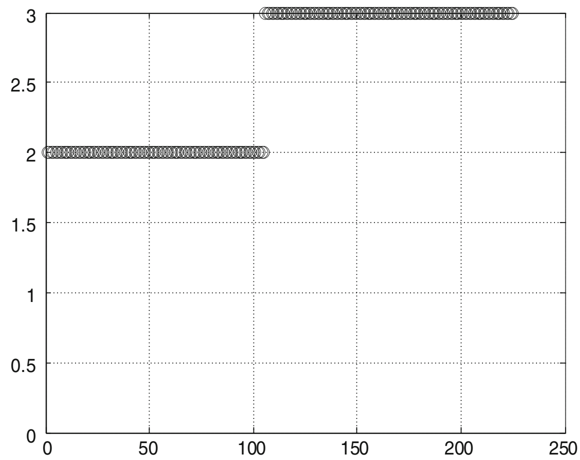


Fig. 10.13 Distribution of the matrix $\mathbf{C}_{15}$ entries by levels

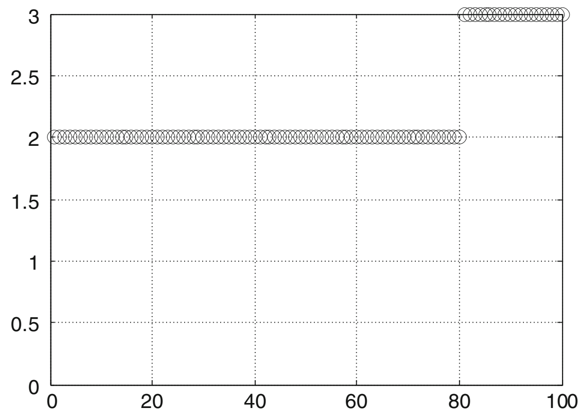


Fig. 10.14 Distribution of the matrix $\boldsymbol{C}_{15}$ (the variant) entries by levels

with elements $k - 1$ and k, the other with elements k and $k + 1$. In particular, at $n = 63$ ($k = 3$) such matrices will have elements which have modules equal to 3 and 4; 4 and 5.

With the help of the Kronecker product it is possible to construct three-level matrices too. When performing the multiplication, it is possible to use the following orthogonal matrices as the basic ones:

$$\mathbf{A}_2 = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \quad \mathbf{A}_3 = \begin{bmatrix} a & b & c \\ -c & a & -b \\ -b & c & a \end{bmatrix} \quad \mathbf{A}_4 = \begin{bmatrix} a & -b & -c & d \\ b & a & d & c \\ c & -d & a & -b \\ -d & -c & b & a \end{bmatrix}.$$

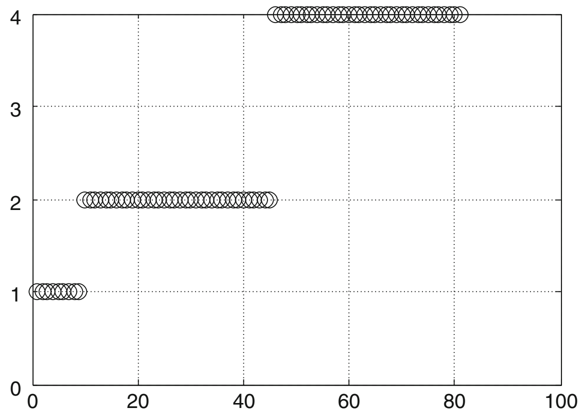


Fig. 10.15 Distribution of the Hadamard-Mersenne matrix $\mathbf{M}_9$ entries by levels

At the same time a part of elements in these matrices can be made equal to each other. Below the three-level matrix of the 9th order is shown, which is based on the optimal matrix $\mathbf{M}_3$, the distribution of modules of its elements is illustrated too.

The distribution of the Hadamard-Mersenne matrix $\mathbf{M}_9$ entries over the levels is shown in Fig. 10.15 ($n = 9$, $\alpha = 4/9$).

The integer-valued three-level matrix of the 25th order with a unit diagonal can be obtained with the help of cyclic shift of line:

$$1\ {-4}\ {-4}\quad 6\ {-4}\quad 6\quad 6\quad 6\quad 6\quad 6\quad 6\quad 6\ {-4}\ {-4}\quad 6\ {-4}\ {-4}\quad 6\ {-4}\quad 6\ {-4}\quad 6\ {-4}\ {-4}\ {-4}.$$

The Hadamard-Mersenne matrices. The minimax matrices of orthogonal bases (i.e. *M*-matrices) with a minimal number of levels depending on a division remainder r by 4, can be divided into 4 cases:

- The matrices with $r = 0$: Hadamard matrices (*H*) [3, 21, 29, 34], containing matrices of the Sylvester chain.
- The matrices with $r = 1$: Hadamard-Fermat matrices (*F*) [35], including orders from the Fermat chain.
- The matrices with $r = 2$: Hadamard-Euler matrices (*E*) [36] (and *C*-matrices [28], with exceptions based on Euler criterion).
- The matrices with $r = 3$: Hadamard-Mersenne matrices (*M*) [37], including orders from a chain of Mersenne numbers.

The Hadamard-Mersenne matrices represent a class of two-level matrices of the odd order, which are close to the Hadamard matrices. The dimensionality of these matrices is equal to Mersenne numbers $2^k - 1$, and their elements tend to values $\{1, -1\}$, as the values of a integer-valued argument k increases, as it takes place with the Hadamard matrices [38].

The classical method of constructing Hadamard matrices of the order $n = 2$ is based on using the iteration formula, where the iteration process begins from the matrix $\mathbf{A}_1 = 1$

$$\mathbf{A}_{2n} = \begin{bmatrix} \mathbf{A}_n & \mathbf{A}_n \\ \mathbf{A}_n & -\mathbf{A}_n \end{bmatrix}.$$

On the analogy, at the start of constructing the Hadamard-Mersenne matrices we will use a modified Eq. 10.15, where the M_n-matrix contains elements of the form $\pm a$ and $\pm b$ (without any limitation of the commonness, let us consider that $a = 1$), and the matrix M_n^* was formed with the help of permutation of the levels a and $-b$.

$$\mathbf{M}_n^* = \mathbf{S}_{2n} = \begin{bmatrix} \mathbf{H}_n & \mathbf{H}_n \\ \mathbf{H}_n & \mathbf{H}_n^* \end{bmatrix} \quad (10.15)$$

The matrix S_{2n} obtained with this formula is symmetrical; its order is even and less than the order of the next Hadamard-Mersenne matrix M_{2n+1} by 1. At the second step the matrix S_{2n} is "bordered" by the way of adding a line and column, where λ and e are the proper number and eigenvector of the matrix S_{2n}

$$\mathbf{M}_{2n+1} = \mathbf{H}_{2n+1} = \begin{bmatrix} -\lambda & e' \\ e & \mathbf{S}_{2n} \end{bmatrix}.$$

If the iteration process is started from

$$\mathbf{H}_3 = \begin{bmatrix} a & -b & a \\ -b & a & a \\ a & a & -b \end{bmatrix},$$

then the matrix obtained by such a manner will be symmetrical and orthogonal.

The eigenvalue of the matrix $\mathbf{S}_{2n}$ will be equal to $\lambda = -a$. At the same time half of the components of the eigenvector consists of $-b$, and the remaining half consists of a. It takes place for next values of generating pair: $b = a/2$ at $n = 3$ and $b = \frac{p \pm \sqrt{4p}}{p-4} a$, $p = n + 1$ on the contrary.

The structure of the Hadamard-Mersenne matrix of the 15th order is shown in Fig. 10.16, where the white fields are the matrix elements with the value $a = 1$, and the black fields are the elements of the matrix with the value b.

Hadamard-Fermat matrices. These matrices represent a class of three-level matrices of the odd order, close by their properties to the properties of the Hadamard matrices. The size of these matrices is equal to numbers $2^{2k} + 1$ and as the values of integer-valued argument k increase, their elements tend to values $\{1, -1\}$, as it takes place with the Hadamard matrices.

Let the Hadamard-Fermat matrix $\mathbf{F}_n$ be of the order n and designate by $\mathbf{S}_{n-1}$ a symmetrical matrix obtained from the matrix $\boldsymbol{F}_n$ by means of deletion of its first line

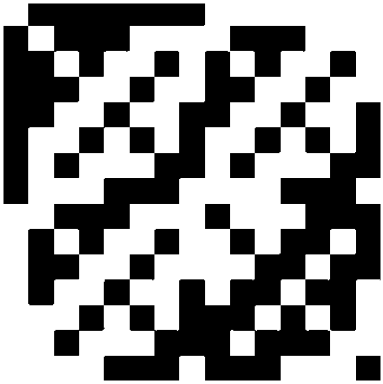


Fig. 10.16 The structure of the Hadamard-Mersenne matrix $\mathbf{M}_{15}$

and column. After that, let Eq. 10.15 be modified by replacing it with quadruplicating its order according to rule [39] in Eq. 10.16, where the matrix $\mathbf{S}^*_{n-1}$ was formed by replacing the values of levels a with $-b$ and vice versa.

$$\mathbf{S}_{4n-4} = \begin{pmatrix} \mathbf{S}^*_{n-1} & \mathbf{S}_{n-1} & \mathbf{S}_{n-1} & \mathbf{S}_{n-1} \\ \mathbf{S}_{n-1} & \mathbf{S}^*_{n-1} & \mathbf{S}_{n-1} & \mathbf{S}_{n-1} \\ \mathbf{S}_{n-1} & \mathbf{S}_{n-1} & \mathbf{S}^*_{n-1} & \mathbf{S}_{n-1} \\ \mathbf{S}_{n-1} & \mathbf{S}_{n-1} & \mathbf{S}_{n-1} & \mathbf{S}^*_{n-1} \end{pmatrix} \tag{10.16}$$

The matrix $\mathbf{S}_{4n-4}$ obtained according to Eq. 10.16 is symmetrical. Its order is even and less than the order of the next Hadamard-Fermat matrix $\mathbf{F}_{4n-3}$ by a unit. To complete the recursive process an additional bordering (addition of a line and column) is needed. The most important requirement is the orthogonality of the matrix obtained due to edging.

To find the location of the orthogonal edging, let the method based on the properties of proper numbers and eigenvectors of block matrices be applied. Matrix $\mathbf{F}_{4n-3}$ is formed by edging the matrix $\mathbf{S}_{4n-4}$ (Eq. 10.16) in such a way, where λ and e are the proper number and eigenvector of the matrix $\mathbf{S}_{4n-4}$, respectively,

$$\mathbf{F}_{4n-3} = \begin{pmatrix} -\lambda & e' \\ e & \mathbf{S}_{4n-4} \end{pmatrix}. \tag{10.17}$$

The matrix obtained in such a way will be symmetrical and orthogonal, if the iteration process is started from the matrix $\mathbf{S}_{4n-4}$

$$\mathbf{F}_5 = \begin{pmatrix} a & s & s & s & s \\ s & a & -b & -b & -b \\ s & -b & a & -b & -b \\ s & -b & -b & a & -b \\ s & -b & -b & -b & a \end{pmatrix}.$$

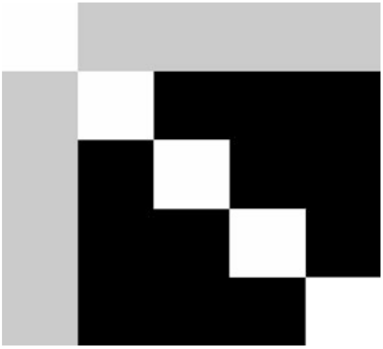


Fig. 10.17 The structure of the Hadamard-Fermat matrix $\mathbf{F}_5$

Fig. 10.18 The structure of the Hadamard-Fermat matrix $\mathbf{F}_{17}$

The matrix $\mathbf{S}_4$ is obtained by deleting its first line and first column ("edging"). Here $a = -\lambda$ is the proper number of the matrix $\mathbf{S}_4$, taken with an inverse sign; s are the elements of the corresponding eigenvector, at that $b < s < a$.

At $n = 5$, in particular, we have $b = s = 2a/3$, where in the general case $b = \frac{n-q}{q}a$, $s = \frac{\sqrt{np-2\sqrt{p}}}{2q}a$, $q = \frac{p+\sqrt{p}}{2}$, $p = n - 1$.

The structure of the matrix $\mathbf{F}_5$ and matrix $\mathbf{F}_{17}$ constructed according to the iteration are shown in Figs. 10.17 and 10.18. The intermediate level of the second matrix corresponds to the elements of the marked vector. Here, the white field is the matrix element of the form $a = 1$, the black field is the element of the form $-b$, the grey field is the element of "edging" элемент $b < s < a$.

The Hadamard-Euler matrices. In [40] a class of two-level matrices is named as the Hadamard-Euler matrices. These matrices are represented by the square matrices $\mathbf{E}_n$ of the order n, consisting of the numbers $\pm a$ and $\pm b$. Such matrices are constructed on the basis of the formula, where $\mathbf{H}_{n/2}$ is the two-level Hadamard-Mersenne matrix of the half odd order, which consists of the numbers $\{a = 1, -b\}$ with a recalculation of their level in such a manner as to have $b = 1/2$ at $n = 6$, and in the remaining cases, $b = \frac{q-\sqrt{8q}}{q-8}$, $q = n + 2$

Fig. 10.19 The structure of the Hadamard-Euler matrix $\mathbf{E}_{30}$

$$\mathbf{E}_n = \begin{bmatrix} \mathbf{H}_{n/2} & \mathbf{H}_{n/2} \\ \mathbf{H}_{n/2} & -\mathbf{H}_{n/2} \end{bmatrix}.$$

The Hadamard-Mersenne matrix of the 3rd order has the form:

$$\mathbf{H}_3 = \begin{bmatrix} a & -b & a \\ -b & a & a \\ a & a & -b \end{bmatrix}.$$

The Hadamard-Euler matrix obtained on its basis will be of the 6th order, where $a = 1$, $b = 0.5$

$$\mathbf{E}_6 = \begin{bmatrix} a & -b & a & a & -b & a \\ -b & a & a & -b & a & a \\ a & a & -b & a & a & -b \\ a & -b & a & -a & b & -a \\ -b & a & a & b & -a & -a \\ a & a & -b & -a & -a & b \end{bmatrix}.$$

The Hadamard-Euler matrices can be used in a number of cases instead of C-matrices, e.g., when the last ones do not exist. An important property of the Hadamard-Mersenne matrices is kept for them, i.e. with an increase of n the modules of all elements tend to 1. As an example, in Fig. 10.19 the structure of the Hadamard-Euler of the 30th order is demonstrated.

Thus, in asymptotes (at large n) all three described classes of matrices (Hadamard-Mersenne, Hadamard-Fermat, and Hadamard-Euler) tend to a common limit of the form of orthogonal matrices with elements ±1.

10.5 Conclusion

This chapter contains the description and analysis of the matrix method of transforming images, which is based on a procedure of the strip-transformation that can be considered as a finite dimensional analogue of the holographic principle of image transformation. The main tasks on investigation and implementation of the strip-method are formulated for increase of immunity with regard to pulse noises present in communication channels, determination of requirements for strip-transformation operators, development of the strip-method for the case of transmitting, storage and processing two-dimensional images, and search of optimal matrices of two-dimensional strip-transformation. In solving these tasks the main requirements for an operator of transformation are considered. It is shown that the operator should be linear, isometric, or finite dimensional. This leads to use the matrices, which have the symmetry of a certain type and have entries equal in absolute value.

The possibility to apply two-dimensional strip-transformation for storage and noise immune transmission of images is considered. At the same time two-sided matrix transformations of an original image have been used, in the process of which image fragments are mixed and superimposed on each other. Great attention is paid to the implementation of the Hadamard matrices and matrices close to them. They include Hadamard-Mersenne, Hadamard-Fermat, and Hadamard-Euler matrices.

References

1. Mironovskii LA, Slaev VA (2006) The strip method of transforming signals containing redundancy. Meas Tech 49(7):631–638
2. Mironovskii LA, Slaev VA (2006) The strip method of noise-immune image transformation. Meas Tech 49(8):745–754
3. Mironowsky LA, Slaev VA (2011) Strip-Method for Image and Signal Transformation. De Gruyter, Berlin
4. Totty RE, Clark GC (1967) Reconstruction error in waveform transmission. IEEE Trans Inf Theory 13(2):333–338
5. Andrews HC (1970) Computer techniques in image processing. Academic Press, New York
6. Haar A (1955) Zur Theorie der Orthogonal Funktionen-System. Inaugural Dissertation. Math Annalen 5:17–31
7. Livak EN Algorithms Compression. http://mf.grsu.by/UchProc/livak/po/comprsite/theory_jpeg.html. Accessed 14 June 2014 (in Russian)
8. Mironowsky LA, Slayev VA (1975) Equalization of the variance of a nonstationary. Signal Telecom Radio Eng 29–30(5):65–72
9. Costas JP (1952) Coding with Linear Systems. Proc IRE 40(9):1101–1103

10. Lang GR (1963) Rotational transformation of signals. IEEE Trans Inf Theory 9(3):191–198
11. Leith EN, Upatnieks J (1962) Reconstructed wavefronts and communication theory. J Opt Soc America 52(10):1123–1133
12. Medianik AI (1997) Proper simplex inscribed in a cube and hadamard matrices of the half-circulante type. Math Phys Analis Geom 4(4):458–471
13. Mironovskii LA, Slaev VA (2002) Optimal Chebishev pre-emphasis and filtering. Meas Tech 45(2):126–136
14. Paley REAC (1933) On orthogonal matrices. J Math Phys 12:311–320
15. Pierce WH (1968) Linear-real codes and coders. Bell Syst Techn J 47(6):1067–1097
16. Rao KR, Narasimhan MA, Revuluri K (1975) Image data processing by hadamard-haar transforms. IEEE Trans Comput C-23(9):888–896
17. Votolin DS (1998) Algorithms of images compression. Moscow State University (in Russian)
18. Mironowsky LA, Slayev VA (2013) Double-sided noise-immune strip transformation and its root images. Meas Tech 55(10):1120–1127
19. Selyakov IS (2005) Analysis and computer imitation of images STRIP-transformation. Master's dissertation, Saint-Petersburg State University of Aerospace Instrumentation
20. Mironowsky LA, Slaev VA (2011) Root images of two-sided noise immune strip-transformation. International workshop on physics and mathematics IWPM 2011. Hangzhou, China
21. Hadamard J (1893) Resolution d'une Question Relative aux Determinants. Bull Sci Math ser 2 17(1):240–246
22. Ruben S (1990) Methods of cipher video compression. Comput Press 10:22–30 (in Russian)
23. Umnyashkin SV (2004) Mathematical Foundations of Signal Cipher Processing and Coding. National Research University of Electronic Technology (in Russian)
24. Williamson J (1944) Hadamard's determinant theorem and the sum of four squares. Duke J Math 11:65–81
25. Zalmanzon LA (1989) Transformations of Furier, Walsh, Haar and their implementation in control, communication and other areas. Science, Moscow (in Russian)
26. Shintyakov DV (2006) Algorithm for searching hadamard matrices of odd order. Techn Sci 2:207–211
27. Contemporary Design Theory: A Collection of Essays (1992) Orthogonal arrays. Wiley, New York
28. Belevitch V (1950) Theorem of 2n-networks with application to conference telephony. Electr Commun 26:231–244
29. Balonin NA, Mironovsky LA (2006) Hadamard matrices of the odd order. Inf Control Syst 22 (3):46–50 (in Russian)
30. Golova EA (2013) Properties investigation of M-matrices applied for filtration. Master's dissertation, Saint-Petersburg State University of Aerospace Instrumentation
31. Golub GH, van Loan CF (1989) Matrix computations, 3rd edn. John Hopkins University Press, Baltimore
32. Balonin NA, Sergeyev MB (2011) M-matrices. Inf Control Syst 50(1):14–21 (in Russian)
33. Balonin NA, Sergeyev MB (2011) M-matrix of the 22-th order. Inf Control Syst 54(5):87–90 (in Russian)
34. Hadamard matrices monitoring. http://mathworld.wolfram.com/HadamardMatrix.html. http://mathscinet.ru. Accessed 14 June 2014 (in Russian)
35. Balonin NA, Sergeyev MB, Mironovsky LA (2012) Calculation of Hadamard-Fermat matrices. Inf Control Syst 61(6):90–93 (in Russian)
36. Balonin NA, Sergeyev MB (2013) Two ways to construct Hadamard-Euler matrices. Inf Control Syst 62(1):7–10 (in Russian)
37. Balonin NA, Mironovsky LA, Sergeyev MB (2012) Computation of Hadamard-Mersenne matrices. Inf Control Syst 60(5):92–94 (in Russian)

38. Gantmacher F (1959) Matrix theory. Chelsea Publishing, New York
39. Gersho AB, Gray RM (1992) Vector quantization and signal compression. Kluwer, Boston
40. Golov AS (2011) Robustness Investigation of strip-method to channel noises. Bachelor's thesis, Saint-Petersburg State University of Aerospace Instrumentation