Why Does Traffic Jam Acts Universally?

Tsumugi Ishida and Yūki Sugiyama

Abstract The theoretical reason for the existence of universal features of traffic jams forming in various places and with different number of cars on highways is provided using exact solutions of a mathematical model for traffic flow, the Optimal Velocity (OV) model. The OV model well reproduces real traffic data of jams in several aspects, such as the critical density of jam formation, the velocity of a jam cluster, etc. where each value is almost universal. The OV model with Heaviside step function as OV-function has essentially the same properties as a realistic OV model. Recently, we have obtained exact solutions of jam flow in the model for an arbitrary number of cars $N > 3$ and car-density. In the solutions, the dependence on N and the density are exponentially reduced for $N \to \infty$. This means that the properties of jams with more than about ten cars are almost the same as that of an infinite jam. This result is originated in the fact that the model is built mathematically based on the concept that traffic flow is a non-equilibrium dissipative system. This explains the universality of jam flow.

1 Introduction and OV Model

Traffic jams appear on a highway in several places, with different numbers of cars and under different conditions of the road. However, the properties of traffic jams, such as the critical car-density of emergence of a jam, the fundamental diagram (the relation between car-density and flow), the velocity of a cluster of jam, etc. These properties are very common independent of such conditions. We naturally wonder why traffic jams act highly universal. The question seems naive, but the answer is not trivial at all. In this paper we provide a physical and mathematical answer using the exact solution of a traffic flow model, the Optimal Velocity model.

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The OV model is a minimal 1-dimensional system of particles with nonlinear asymmetric interactions and a dissipative (viscous) term, which was first introduced as a traffic flow model in [1, 2]. The equation of motion is formulated as

$$\frac{d^2 x_n}{dt^2} = a\left(V(\Delta x_n) - \frac{dx_n}{dt}\right), \tag{1}$$

where x_n is the position of the nth car, and $\Delta x_n = x_{n+1} - x_n$ is the headway distance. a is a control parameter, which dimension is the inverse of time. The OV-function $V(\Delta x_n)$ determines the interaction with a car moving ahead. $V(\Delta x_n)$ should be a sigmoidal function. The model well describes the emergence of a jam in traffic flow and clearly explains its physical mechanism.

As for the simple case choosing the Heaviside step-function as OV-function, such that $V(\Delta x) = 0$ for $\Delta x < d$, and $V(\Delta x) = v_{max}$ for $\Delta x \geq d$, the emergence of a traffic jam is observed as in Fig. 1.

The jam flow solution in this model can be obtained analytically based on some hypothesis, which is justified for the case of large N [3]. Moreover, the N-body problem for the jam flow solution is exactly solved for $N = 2, 3, 4, \cdots$, in the special case of car-density $L/N = d$ [4]. Recently, we have obtained the most general jam flow solution for arbitrary N and L.

The profile of a jam flow is described by a limit cycle solution in the phase space $(\Delta x_n, \frac{dx_n}{dt})$, which expresses the two regions in a jam cluster and that cars move smoothly [2]. Profiles of jam flow solutions for several N, L, and car-densities are presented in Fig. 2.

We notice that solutions with the same density L/N but with different number of cars N, are different. However, these differences seem to be reduced for large N.

In this paper, using the exact solution for an arbitrary number of cars N and length of circuit L, we discuss the universality of traffic jams.

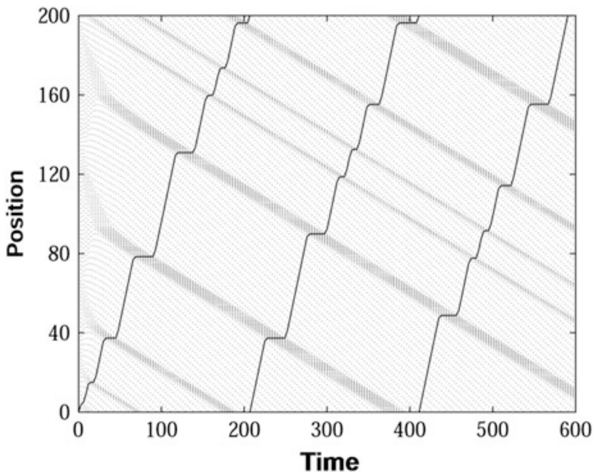

Fig. 1 Plot of all cars, $N = 100$, for the process of jam formation on a circuit in the OV model with Heaviside step function as OV-function. The initial condition is set as homogeneous flow. The orbit for a car is represented as an example

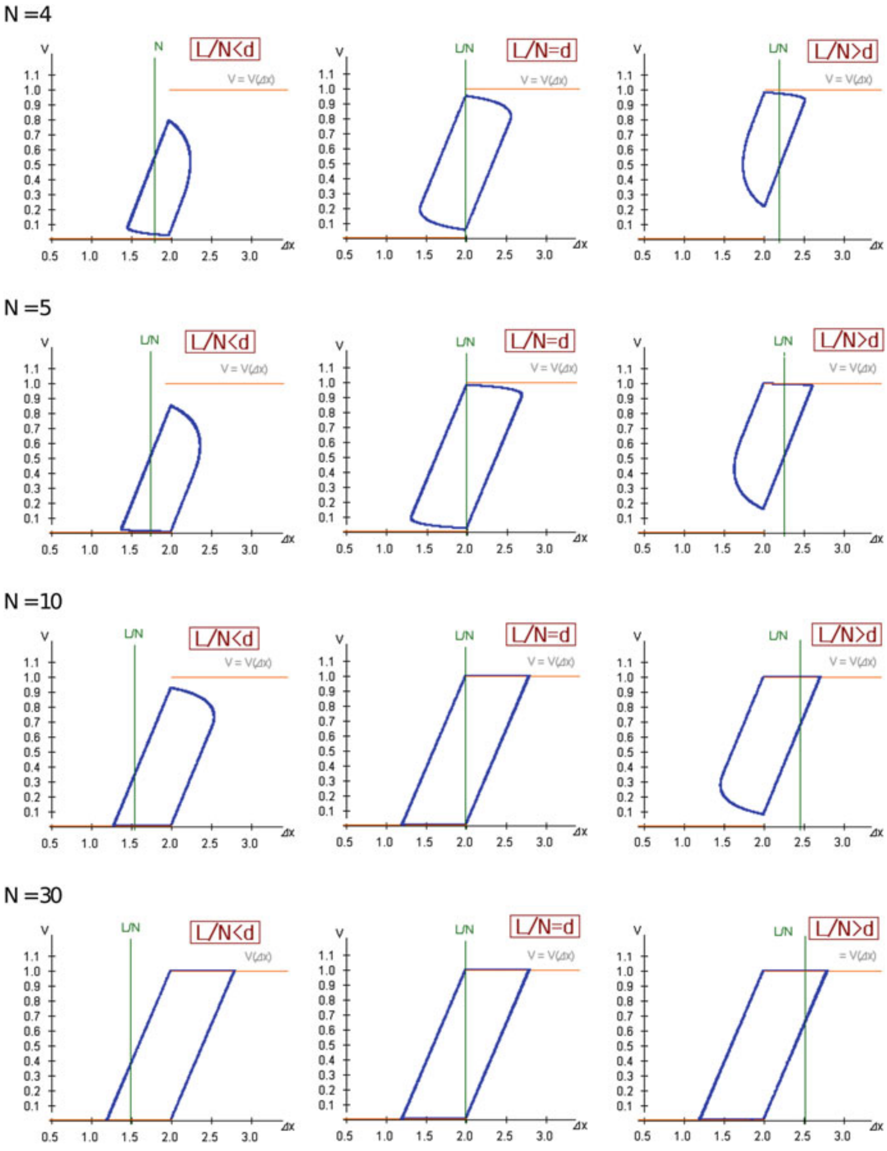

Fig. 2 Profiles of jam flow solutions (limit cycles) for $N = 4, 5, 10, 30$ with several densities, $L/N = d \pm \epsilon$. The horizontal axis is the headway distance Δx, and the vertical axis is the velocity $\frac{dx}{dt}$

2 Procedure of Solving the OV Model

The equation of motion Eq. (1) consists of two parts according to $\Delta x < d$ or $\Delta x \geq d$. Each case is easily solved as follows. At the initial condition, $x_n(t_0)$, $\dot{x}_n(t_0)$ for $t = t_0$, in the case $\Delta x < d$,

$$x_n(t) = x_n(t_0) + \frac{\dot{x}_n(t_0)}{a}(1 - e^{-a(t-t_0)}), \tag{2}$$

and in the other case $\Delta x \geq d$,

$$x_n(t) = x_n(t_0) + v_{max}(t - t_0) - \frac{v_{max} - \dot{x}_n(t_0)}{a}(1 - e^{-a(t-t_0)}) \tag{3}$$

The motion of each particle is constructed by changing these two solutions depending on its headway distance Δx. The important point for obtaining a jam flow solution is how to connect the above two solutions in the appropriate condition.

If we obtain five unknown variables, we can determine the connection condition to build up a jam flow solution for N and L. They are the following: The velocity v_{RB} of a particle at the time when $\Delta x = d$ where the solution changes from Eqs. (2) to (3), and the velocity v_{BR} when the solution changes from Eqs. (3) to (2); the time delay τ; the period of the same relative position shifting one number of particle, satisfying $x_n(t + \tau) = x_{n+1} - v_c\tau$; the shift of time beyond τ, denoted by Δt, that is the difference between the period in $\Delta x < d$ and that in $\Delta x > d$; and v_c, the velocity of a cluster moving in the opposite direction of the car motion. We can write down five independent equations including the above five unknown variables and derive these variables from the solution.

3 Exact Solution of a Jam Flow

A cluster flow solution for a given arbitrary number of cars, N and circuit length, L, with a density $L/N = d + \epsilon$, $(\epsilon \geq 0)$ is expressed by using five unknown variables; v_{RB}, v_{BR}, v_c, τ, Δt.[1] They are obtained as

$$v_{RB} = \frac{v_{max}(1 - e^{\frac{N}{2}a(\tau-\Delta t)})}{(1 + e^{\frac{N}{2}a\tau})(1 - e^{\frac{N}{2}a\tau})}, \tag{4}$$

$$v_{BR} = \frac{v_{max}(1 - e^{-\frac{N}{2}a(\tau-\Delta t)})}{(1 + e^{-\frac{N}{2}a\tau})(1 - e^{-\frac{N}{2}a\tau})}, \tag{5}$$

[1] The solution with a density $L/N = d - \epsilon$ is also obtained, which has the symmetry of duality.

$$v_c = \frac{d - \frac{1}{2}v_{max}\tau}{\tau} + f(\Delta t; N), \tag{6}$$

$$\frac{a\tau(1 + e^{-\frac{N}{2}a\tau})(1 - e^{-\frac{N}{2}a\tau})}{1 - \frac{e^{-\frac{N}{2}a\tau}(e^{-\frac{N}{2}a\Delta t} + e^{\frac{N}{2}a\Delta t})}{2}} = 2(1 - e^{-a\tau}), \tag{7}$$

where $f(\Delta t; N)$ is given as

$$f(\Delta t; N) = \frac{v_{max}}{2a\tau} \frac{e^{\frac{N}{2}a\Delta t} - e^{-\frac{N}{2}a\Delta t}}{e^{\frac{N}{2}a\tau} - e^{-\frac{N}{2}a\tau}}(1 - e^{-a\tau}). \tag{8}$$

The difference of density from the self-dual case $d = L/N$, denoted by ϵ, is expressed with Δt as

$$\epsilon = \frac{v_{max}\Delta t}{2} - f(\Delta t; N)\tau \tag{9}$$

4 Rapid Convergence to the Universal Solution for Large N

The property of convergence for large N in the formula of the obtained solution is determined by the largest of the two values $a\tau$ and $a\Delta t$. Figure 3 shows the relation

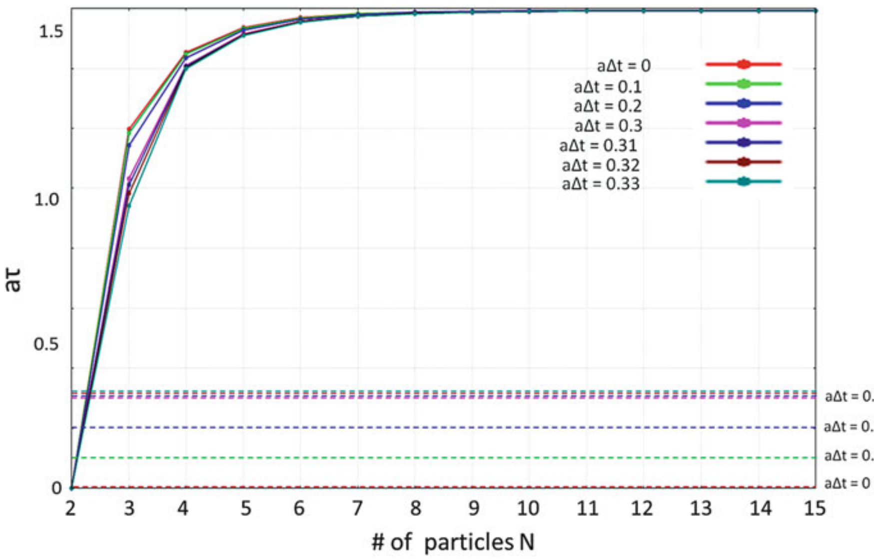

Fig. 3 N-dependence of $a\tau$ and $a\Delta t$

obtained by solving Eq. (7) numerically for given N, which indicates

$$a\tau > a\Delta t, \tag{10}$$

for $N \geq 3$. Thus, $f(\Delta t; N) \to 0$ for $N \to \infty$ in Eq. (8), and Eqs. (4)–(7) are reduced to

$$v_{RB} = 0, \tag{11}$$

$$v_{BR} = v_{max}, \tag{12}$$

$$v_c = \frac{d - \frac{1}{2} v_{max}\tau}{\tau}, \tag{13}$$

$$a\tau = 2(1 - e^{-a\tau}), \tag{14}$$

independent of Δt or ϵ, which means independent of the density L/N. The jam flow solution for $N \to \infty$ is already known 10 years before [3]. In addition, as for Eq. (9),

$$\epsilon = \frac{v_{max}\Delta t}{2}. \tag{15}$$

Jam flow solutions depending on N and L converge exponentially, as $e^{-\frac{N}{2}a\tau}$ and $e^{-\frac{N}{2}a\Delta t} \to 0$, for large N to the universal self-dual solution. They are independent of the car density $L/N = d \pm \epsilon$ and the number of cars N. Based on these results, we can explain analytically that the motion of every car in a jam flow built with more than several 10 cars shows the same universal profile of the limit cycle as in Fig. 2.

5 Answer of the Question

The property of the rapid convergence to the universal profile of the limit cycle solution for jam flow explains that any jam flow shows the same behavior with the common features independent of the condition and situation on real highways.

The characteristic factor of convergence as $e^{-\frac{N}{2}a\tau}$ is originated in the dissipative (viscous) term in the basic equation of motion in OV model. This is evidence that phenomena of traffic flow, such as jam flow, follow from the dissipative features in physics.

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