The Stability Analysis of a Macroscopic Traffic Flow Model with Two-Classes of Drivers

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Abstract One of the most important objectives in the development of traffic theories is the improvement of traffic conditions. To achieve this goal, it is important a good understanding of multistyle and/or multilane traffic. In this work, we summarize the traffic model presented in Mendez and Velasco (FTC J Phys A Math Theor 46(46):462001, 2013) and additionally include the stability analysis of the same. The presented traffic model considers different driving styles, different vehicle types or both, for a two-classes of vehicles in which a model for the average desired speed is introduced (the aggressive drivers model) (Mendez and Velasco, Transp Res Part B 42:782–797, 2008; Velasco and Marques, Phys Rev E 72:046102, 2005). The kinetic model was solved for the steady and homogeneous state and also we obtained the local distribution function from an information entropy maximization procedure. The macroscopic traffic model is constructed by the usual methods in kinetic theory and a method akin with the Maxwellian iterative procedure is accomplished in order to close the macroscopic model for the mixture, where only the densities are present as relevant quantities. The linear stability analysis is carried out in order to have an insight of the unstable traffic regions of the model, which is very helpful in the numerical solution.

1 Introduction

Kinetic theory methods have been largely used in the study of the traffic flow phenomena [1–3, 5, 10, 12]. This kind of methods links the microscopic and macroscopic modeling giving a theoretical support to phenomenological models.

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Empirical observations have highlighted the wide variety of phenomena observed in traffic flow [6–9,14,15], and recently there have merged many efforts to describe and explain the wealth of traffic flow phenomena. In this work we present the stability analysis of a model that considers different driving styles, different vehicle types or both. The point of view we will follow to study such a problem is based on the kinetic theory of traffic flow. In Sect. 2 we summarize (see [12]) some details of the kinetic equation describing the evolution of each class of vehicles for aggressive drivers which has been also addressed before for a single class of driver [11, 16]. The kinetic model is solved for the steady and homogeneous state and also we have obtained a local distribution function from an information entropy maximization procedure. In Sect. 3 the macroscopic traffic model is constructed by means of a general transport equation obtained by the usual methods in kinetic theory. Then, a method akin with the Maxwellian iterative procedure is achieved in order to close the macroscopic model for the mixture where only the species densities are considered as relevant quantities. The linear stability analysis is presented and discussed in Sect. 4.

2 The Kinetic Model with Two-Classes of Vehicles

For a two-classes of drivers with punctual vehicles and diluted traffic, the equations describing the evolution of the distribution function $f_i = f_i(x, v_i, t)$ for individual vehicles of class $i = a, b$ is

$$\frac{\partial f_i}{\partial t} + v_i \frac{\partial f_i}{\partial x} + \frac{\partial}{\partial v_i}\left[\frac{W_i(x, v_i, t) - v_i}{\tau_i} f_i\right] = \sum_{j=a,b} Q(f_i f_j), \qquad (1)$$

where

$$Q(f_i f_j) = (1-p)\left[\int_v^{\infty} dw_j\, f_i(v_i) f_j(w_j)(w_j - v_i) - \int_0^{v} dw_j\, f_i(v_i) f_j(w_j)(v_i - w_j)\right]. \qquad (2)$$

for self-interaction we have terms $i = j$ and cross-interaction are represented by $i \neq j$.

In Eqs. (1) and (2) p is the overtaking probability and W_i is the mean desired velocity. We propose the following prescription for the former velocity

$$W_i(x, v_i, t) = \omega_i v_i, \qquad (3)$$

where $\omega_i \gtrsim 1$ is a constant called the aggressivity parameter of i−class vehicles. Relation (3) indicates that drivers desired velocity increases as their actual velocity increases, i.e., drivers want to drive faster than they do [11, 16].

As usual in kinetic theory, the local variables such as the traffic density and the mean velocity of each vehicle class i are defined through the first two moments of the distribution function as follows

$$\int f_i dv_i = \rho_i \qquad \int f_i v_i dv_i = \rho_i V_i, \qquad \sum_i \int f_i v_i dv_i = \rho V. \tag{4}$$

2.1 Equilibrium Solution and the Information Entropy

For the homogeneous and steady state, Eq. (1) can be solved analytically assuming relation (3) for the mean desired velocity. The details of this calculation are analogous to the single class case and can be consulted in [16], the result is

$$f_{ie}(v_i) = \frac{\rho_{ie}\alpha_i}{\Gamma(\alpha_i) V_e} \left(\frac{\alpha_i v_i}{V_e}\right)^{\alpha_i - 1} \exp\left[-\frac{\alpha_i v_i}{V_e}\right], \qquad \alpha_i = \frac{\tau_i}{\omega_i - 1}(1-p)\rho_e V_e. \tag{5}$$

α_i is a constant that contains information on the equilibrium state and the model parameters τ_i, ω_i and p and $\Gamma(\alpha_i)$ is the gamma function with argument α_i. In this case the e-subindex means that the quantity corresponds to the steady and homogeneous case, usually called as the equilibrium state.

The local zeroth-order approximation for the one-vehicle distribution function follows through a maximization procedure of the information entropy referred to the equilibrium state. First we write the information entropy as

$$S\left[f_i(x, v_i, t)\right] = \sum_{i=a,b} \int_0^\infty f_i^{(0)}(x, v_i, t) \ln\left(\frac{f_i^{(0)}(x, v_i, t)}{f_{ie}(v_i)}\right) dv_i, \tag{6}$$

and consider relations (4) as restrictions for the optimization procedure. With this information the corresponding Lagrangian function, $\mathscr{F}$, is constructed and the distribution function which satisfies the optimality condition $\delta\mathscr{F}/\delta f_i^{(0)} = 0$ is given by

$$f_i(x, v_i, t) = \frac{\rho_i(x,t)\,\alpha_i}{\Gamma[\alpha_i]\, V(x,t)} \left(\frac{\alpha_i v_i}{V(x,t)}\right)^{\alpha_i - 1} \exp\left[-\frac{\alpha_i v_i}{V(x,t)}\right]. \tag{7}$$

At this point it seems important to emphasize that the distribution function (7) has the same functional structures that (5), both the equilibrium values are replaced by the local variables.

2.2 Model Equation

The mathematical complexity of the non-linear interaction operator Q_{ij} can be avoided if we replace the right hand side of Eq. (1) by a simple relaxation-time term of the form [4]:

$$\frac{\partial f_i}{\partial t} + v_i \frac{\partial f_i}{\partial x} + \frac{\partial}{\partial v_i}\left[\left(\frac{W_i(x, v_i, t) - v_i}{\tau_i}\right) f_i\right] = -\sum_{j=a,b} \sigma_{ij}\left(f_i - f_{ij}\right), \tag{8}$$

where f_{ij} is a reference distribution function to be determined. Our aim is to derive a model equation for a two-classes of vehicles characterized by the ρ_i, V_i. We propose the following form for the reference distribution function f_{ij}:

$$f_{ij} = f_i^{(0)}\left(1 + A^{ij} + B^{ij} C_i\right) \quad \text{where} \quad C_i = (v_i - V) \tag{9}$$

$f_i^{(0)}$ given by (7) and A^{ij} and B^{ij} are undetermined coefficients. To specify these coefficients we assume

$$f_i = f_i^{(0)}(1 + \phi_i) \tag{10}$$

where the deviation ϕ_i is a linear function of the spatial gradients and must satisfy compatibility conditions. Inserting (9) and (10) in (8)

$$\phi_i = \frac{1}{\sigma_i}\left[1 + \frac{2}{V_i}(v_i - V_i) - \frac{\alpha_i}{V_i^2}(v_i - V_i)^2\right]\left(\frac{\partial V_i}{\partial x}\right). \tag{11}$$

where restrictions (4) have been used. It is worth noticing that the deviation ϕ to the distribution function depends on the velocity gradients.

3 The Macroscopic Equations and the Iterative Procedure

Once we have the distribution function f_i, through (10) and (11), it is possible to obtain the macroscopic equations and close them. The procedure is the standard in kinetic theory, the resulting balance equations are

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial \rho_i V_i}{\partial x} = (1 - p)\rho\rho_i(V - V_i), \tag{12}$$

$$\frac{\partial V_i}{\partial t} - \frac{V_i}{\rho_i}\frac{\partial \rho_i V_i}{\partial x} + \frac{1}{\rho_i}\frac{\partial p_i}{\partial x} - \gamma_i V_i = -(1 - p)\rho\theta_i. \tag{13}$$

where $\rho_i \theta_i = \int f(x, v_i, t)(v_i - V_i)^2 dv_i, \quad i = a, b$ are the speed variances. To be practical, we now apply a method akin to the Maxwell iterative procedure [13], to obtain a first order model for each different class of vehicle. We assume that $V_i(x,t) = V_{ie}^{(0)}(\rho_i(x,t)) + \hat{V}_i(x,t)$. In this case we obtain the closure relation

$$\hat{V}_i = -\frac{\rho [V_{ie}^{(0)}]^2 \Gamma_i}{\rho_e V_e \left[1 - 2\frac{\rho V_{ie}^{(0)}}{\rho_e V_e}\right]} \frac{\partial V_{ie}^{(0)}}{\partial x}, \quad \text{with} \quad \Gamma_i = \frac{2(\alpha_i + 1)}{\sigma_i \alpha_i}. \tag{14}$$

By introducing (14) in (12) we get the model

$$\frac{\partial \rho_i}{\partial t} + b(\rho_i, \rho_j)\rho_i \frac{\partial \rho_i}{\partial x} - D(\rho_i, \rho_j)\frac{\partial^2 \rho_i}{\partial x^2} = 0 \qquad i, j = a, b \quad j \neq i \tag{15}$$

where $D(\rho_i, \rho_j) = \left[\left(\Gamma_i \rho V_{ie}^2 V_e'\right) / (\rho_e V_e - 2\rho V_{ie})\right] \rho_i$, we have considered $V_e(\rho_e) = V_e$, $V_i^{(0)} = V_e(\rho_i) = V_{ie}$ and

$$b(\rho_i, \rho_j) = \frac{V_i}{\rho_i} - V_e' + (V_i - V_{ei})\left[\frac{1}{\rho} + 2\frac{V_{ei}'}{V_{ei}} + \frac{V_{ei}''}{V_{ei}'} + 2\frac{\left(V_{ei} + \rho V_{ei}'\right)}{(\rho_e V_e - 2\rho V_{ei})}\right].$$

4 Linear Stability Analysis and Concluding Remarks

In order to determine if our model equation produces unstable traffic regions, a linear stability analysis of Eqs. (15) will be carried out. We write the perturbed densities as

$$\rho_i(x,t) = \rho_{ie} + \overline{\rho}_i \exp(-\gamma t + ikx) = \rho_{ie} + \tilde{\rho}_i(x,t) \tag{16}$$

for $i = a, b$ where k is the wave number and γ is the growth parameter. The procedure consists in inserting perturbations (16) into Eqs. (15) and neglect nonlinear contributions. The following linear system is obtained

$$\begin{pmatrix} \mathscr{A}(k,\gamma) & (1-p)\rho_{ae}\rho_{be}V_e'(\beta_{be}ik - 1) \\ (1-p)\rho_{ae}\rho_{be}V_e'(\beta_{ae}ik - 1) & \mathscr{B}(k,\gamma) \end{pmatrix} \begin{pmatrix} \tilde{\rho}_a \\ \tilde{\rho}_b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{17}$$

where

$$\mathscr{A}(k,\gamma) = \left[-\gamma + ikc_{ae} + \beta_{ae}k^2 V_e' \rho_{ae} + (1-p)\rho_{ae}\rho_{be}V_e'(1 - \beta_{ae}ik)\right],$$

$$\mathscr{B}(k,\gamma) = \left[-\gamma + ikc_{be} + \beta_{be}k^2 V_e' \rho_{be} + (1-p)\rho_{be}\rho_{ae}V_e'(1 - \beta_{be}ik)\right],$$

$$\beta_{ae} = \frac{\alpha_a V_{ae}^2 \Gamma_a}{[V_e - 2V_{ae}]} \qquad \text{and} \qquad c_{ae} = \left(V_{ea} + V_e' \rho_{ae}\right).$$

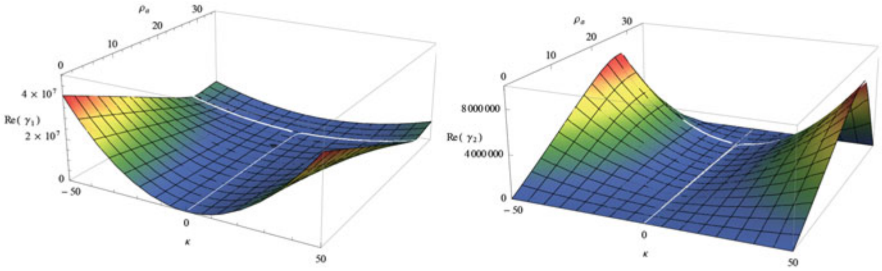

Fig. 1 The stable region for $\rho_e = 35$ veh/km

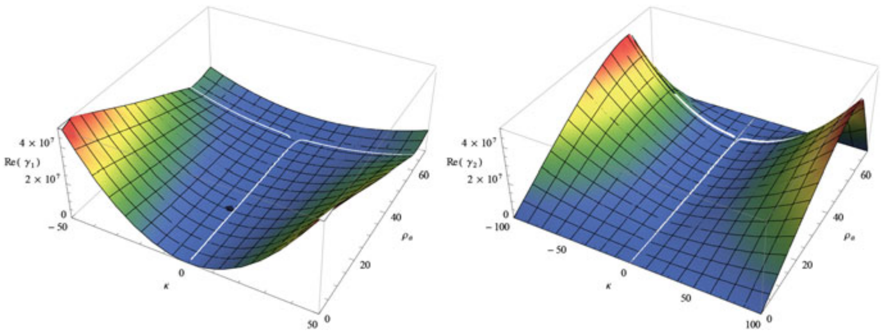

Fig. 2 The stable region for $\rho_e = 70$ veh/km

Table 1 Parameters for the graphics

Vehicle class		
Class a	$\alpha_a = 120$	$\Gamma_a = 0.012$
Class b	$\alpha_b = 100$	$\Gamma_b = 0.06$

The nontrivial solutions of system (17) are obtained when the determinant of the coefficient matrix vanishes, leading to a dispersion relation of the form $\gamma^2 + \gamma b(k) + c(k) = 0$.

This relation is very complex and is solved in Mathematica for some specific values. The results in Figs. 1 and 2 show the stable regions, corresponding to $\mathbb{R}[\gamma_{1,2}] > 0$, that is, when both γ_1 and γ_2 are positive. The parameters used in these figures are in Table 1 and we have used also the relations $V_e(\rho) = V_{max}\,(1 - \rho/\rho_{max})$ and $(1 - p) = \rho/\rho_{max}$ with $\rho_{max} = 140$ veh/km and $V_{max} = 120$ km/h.

Figures 1 and 2 show that we have stable regions. Worth be mentioned that a linear stability give us just a guide of the real stability regions, in order to have a better understanding a non-liner stability analysis should be done.

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