A Qualitative Spatio-Temporal Framework Based on Point Algebra

Michael Sioutis, Jean-François Condotta, Yakoub Salhi, and Bertrand Mazure

Université Lille-Nord de France, Artois
CRIL-CNRS UMR 8188
Lens, France
{sioutis,condotta,salhi,mazure}@cril.fr

Abstract. Knowledge Representation and Reasoning has been quite successfull in dealing with the concepts of *time* and *space* separately. However, not much has been done in designing qualitative spatiotemporal representation formalisms, let alone reasoning systems for that formalisms. We introduce a qualitative constraint-based spatiotemporal framework using Point Algebra (PA), that allows for defining formalisms based on several qualitative spatial constraint languages, such as RCC-8, Cardinal Direction Algebra (CDA), and Rectangle Algebra (RA). We define the notion of a qualitative spatiotemporal constraint network (QSTCN) to capture such formalisms, where pairs of spatial networks are associated to every base relation of the underlying network of PA. Finally, we analyse the computational properties of our framework and provide algorithms for reasoning with the derived formalisms.

Keywords: point algebra, qualitative spatiotemporal reasoning, qualitative spatiotemporal framework, satisfiability, minimality, algorithm.

1 Introduction

Qualitative Reasoning is based on qualitative abstractions of aspects of the common-sense background knowledge, such as *space* and *time*, on which our human perspective on the physical reality is based. Spatiotemporal reasoning has become a significant field of research in Qualitative Reasoning, and, more generally, in Knowledge Representation and Reasoning. This field is essential for a plethora of areas and domains that include dynamic GIS, cognitive robotics, spatiotemporal design, and planning [5, 13].

The Point Algebra (PA) [2, 3, 21] is one of the dominant Artificial Intelligence approaches for representing and reasoning about qualitative temporal relations, and forms the basis of several richer temporal languages, such as Interval Algebra (IA) [1]. In particular, PA encodes temporal relations between two points in the timeline. Likewise, a fragment of the Region Connection Calculus [16], namely, RCC-8, Cardinal Direction Algebra (CDA) [8], and Rectangle Algebra (RA) [12], are among the dominant Artificial Intelligence approaches for representing and reasoning about qualitative spatial relations. In particular, RCC-8 encodes topological relations between two regions that are non-empty regular subsets of some

G. Agre et al. (Eds.): AIMSA 2014, LNAI 8722, pp. 117–128, 2014.

topological space, CDA encodes direction relations between spatial objects, and RA encodes relative position relations between multi-dimensional objects. All these qualitative constraint languages have been extensively studied *separately*, but there has not been a framework so far that allows for combining them in unique formalisms in order to reason about both time and space effectively.

The spectrum of spatiotemporal formalisms has mainly focused on adopting the propositional temporal logic (PTL), and combining it with RCC-8 [22] or even richer fragments than RCC-8, such as the modal logic $\mathsf{S4}_u$ [9] interpreted over topological spaces. A study of such formalisms along with their computational properties can be found in [23]. Most of the PTL-based formalisms are very elegant and expressive, but deciding their satisfiability is PSPACE-complete at best [9]. Delving deeper into modal logics, there have been multimodal logic approaches to qualitative spatiotemporal reasoning studied in the works of Burrieza et al. [6, 7], Muñoz-Velasco et al. [15], and Golinska-Pilarek et al. [11].

Unfortunately, constraint-based formalisms have not been paid the attention they deserve except in the work of Gerevini et al. [10] where the Interval Algebra (IA) [1] is combined with RCC-8 in a unique spatiotemporal formalism called STCC.

We take a similar approach to that of Gerevini et al. [10] by creating a framework that allows for combining a qualitative temporal constraint language with a spatial one, namely, PA with any spatial language such as RCC-8, CDA, and RA, and make the following contributions: (*i*) we define our framework in detail and describe the notion of a qualitative spatiotemporal constraint network (QSTCN), (*ii*) we analyse the computational properties of our framework and provide algorithms for reasoning with the derived formalisms.

Our approach is different to that of Gerevini et al. [10] in that we associate pairs of spatial networks to every base relation of a network of PA, while Gerevini et al. associate a spatial network to every variable of a network of Interval Algebra (IA) [10]. Thus, our approach is more flexible and richer. In particular, Gerevini et al. associate a static spatial configuration to a temporal interval (a variable of IA) which leads to a very rigid framework; every time two temporal intervals overlap in any way, it is clear that the associated spatial configurations must be identical for both intervals, as we can only have a unique spatial configuration within a period of time. This leads to $\mathcal{NP}$-completeness even in trivial cases where one only uses base relations and the two universal[1] relations of the qualitative constraint languages considered (Theorem 2 in [10]). On the other hand, and as we will explore later in the paper, our framework allows for many tractability cases that include large fragments of the relations of the qualitative constraint languages considered. Further, Gerevini et al. handle a maximum of $O(n)$ spatial configurations for a IA network of n variables. Since we associate spatial configurations to every base relation of a network of PA, and every such network of n variables can have $O(n^2)$ relations, we consider in total $O(n^2)$

[1] The universal relation of a qualitative constraint language is the non-restrictive relation that contains all base relations. It signifies the lack of knowledge between two entities in a qualitative constraint network.

pairs of spatial configurations. Moreover, taking into account the semantics of the base relations $\{<, =, >\}$ of PA, that is, their natural interpretation over time points in $\mathbb{Q}$, a pair allows us to capture both the past and the future of a spatial configuration with a particular base relation. For example, we can think of base relation $<$ as a relation that associates the past of a spatial configuration with its future. Therefore, we have the ability to define general laws about qualitative change, which the formalization proposed by Gerevini et al. lacks [10].

The paper is organized as follows. Section 2 introduces the notion of qualitative constraint languages. In Section 3 we define our framework and the concept of a QSTCN, analyse its computational properties, and present algorithms for reasoning with derived formalisms. Finally, in Section 4 we conclude and discuss future work.

2 Preliminaries

A (binary) qualitative temporal or spatial constraint language [18] is based on a finite set B of *jointly exhaustive and pairwise disjoint* (JEPD) relations defined on a domain D, called the set of base relations. The set of base relations B of a particular qualitative constraint language can be used to represent definite knowledge between any two entities with respect to the given level of granularity. B contains the identity relation Id, and is closed under the converse operation ($^{-1}$). Indefinite knowledge can be specified by unions of possible base relations, and is represented by the set containing them. Hence, 2^{B} will represent the set of relations. 2^{B} is equipped with the usual set-theoretic operations (union and intersection), the converse operation, and the weak composition operation. The converse of a relation is the union of the converses of its base relations. The weak composition $\diamond$ of two relations s and t for a set of base relations B is defined as the strongest relation $r \in 2^{\mathsf{B}}$ which contains $s \circ t$, or formally, $s \diamond t = \{b \in \mathsf{B} \mid b \cap (s \circ t) \neq \emptyset\}$, where $s \circ t = \{(x, y) \mid \exists z : (x, z) \in s \wedge (z, y) \in t\}$ is the relational composition.

The qualitative temporal constraint language PA [2, 3, 21] consists of the set of base relations $\{<, =, >\}$, where the relation symbols display the natural interpretation over time points in $\mathbb{Q}$. We denote the set of base relations of PA by B_{PA}. Thus, $2^{\mathsf{B}_{\mathsf{PA}}}$ represents the set of relations $\{\emptyset, <, =, >, \leq, \geq, \neq, = \vee \neq\}$, with $=$ being the identity relation. (Note that $\neq$ is an abbreviation for $> \vee <$, $\geq$ an abbreviation for $> \vee =$, and $\leq$ an abbreviation for $< \vee =$.) Likewise, qualitative spatial constraint languages RCC-8 [16], CDA [8], and RA [12] have their own set of base relations. As an example, RCC-8 consists of the set of base relations $\mathsf{B}_{\mathsf{RCC8}} = \{DC$ (disconnected), EC (externally connected), PO (partially overlaps), TPP (tangential proper part), $NTPP$ (non-tangential proper part), $TPPi$ (tangential proper part inverse), $NTPPi$ (non-tangential proper part inverse), EQ (equals)$\}$, with EQ being the identity relation, and $2^{\mathsf{B}_{\mathsf{RCC8}}}$ enumerates a total of 256 relations.

Qualitative temporal or spatial constraint languages can be formulated as qualitative constraint networks (QCNs) as follows:

Definition 1. *A* QCN *is a pair* $\mathcal{N} = (V, C)$ *where* V *is a finite set of variables and* C *a mapping associating a relation* $C(v, v') \in 2^{\mathsf{B}}$*, to each pair* (v, v') *of* $V \times V$*.* C *is such that* $C(v, v) \subseteq \mathsf{Id}$ *and* $C(v, v') = (C(v', v))^{-1}$ *for every* $v, v' \in V$*.*

Note that we always regard a QCN as a complete network. Given two QCNs $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V, C')$, $\mathcal{N} \cap \mathcal{N}'$ denotes the QCN $\mathcal{N}'' = (V, C'')$ where $C'' = C(v, v') \cap C'(v, v')$ for every $v, v' \in V$.

Definition 2. *A* solution *of a* QCN $\mathcal{N} = (V, C)$ *is a mapping* σ *defined from* V *to the domain* D*, such that* $\forall (v, v') \in V \times V$*,* $(\sigma(v), \sigma(v'))$ *satisfies* $C(v, v')$*, i.e., the base relation* b *defined by* $(\sigma(v), \sigma(v'))$ *exists in* $C(v, v')$*.* $\mathcal{N}$ *is* consistent *or* satisfiable *iff it admits a solution. A sub-*QCN $\mathcal{N}'$ *of* $\mathcal{N}$ *is a* QCN (V, C') *such that* $C'(v, v') \subseteq C(v, v')$ *for every* $v, v' \in V$*. An atomic* QCN *is a* QCN *where each constraint is defined by a base relation. A* scenario *of* $\mathcal{N}$ *is an atomic consistent sub-*QCN *of* $\mathcal{N}$*.* $\mathcal{N}$ *admits a solution iff it admits a scenario. Given a* QCN $\mathcal{N} = (V, C)$*, base relation* r *is feasible iff there exists a scenario* $\mathcal{N}_{\mathsf{atomic}} = (V, C_{\mathsf{atomic}})$ *of* $\mathcal{N}$ *such that* $C_{\mathsf{atomic}}(v, v') = \{r\}$*. A* QCN $\mathcal{N}$ *is minimal iff it comprises only feasible relations.*

Checking the consistency of a QCN of PA can be done in polynomial time, $O(n^3)$, using a path consistency algorithm [2].[2] It follows that the whole set of relations of PA, viz., $2^{\mathsf{B}_{\mathsf{PA}}}$, is tractable. On the other hand, QCNs of RCC-8, CDA, and RA are intractable in the general case. However, there exist large maximal tractable subclasses of their relations, for which the satisfiability problem is tractable. As an example, checking the consistency of a QCN of RCC-8 is $\mathcal{NP}$-complete in general [19], but there exist the maximal tractable subclasses $\hat{\mathcal{H}}_8, \mathcal{C}_8$, and $\mathcal{Q}_8$ [17] for which the satisfiability problem is tractable. Checking the consistency of a QCN of RCC-8, CDA, or RA comprising only relations from maximal tractable subclasses can be done in polynomial time, $O(n^3)$ in particular, using a path consistency algorithm.

3 A Spatio-Temporal Framework Based on Point Algebra

We obtain a spatiotemporal framework by defining the concept of a qualitative spatiotemporal constraint network (QSTCN) that builds on PA and allows plugging in any spatial constraint language, such as RCC-8, CDA, and RA. In particular, in a QSTCN we assign a pair of spatial QCNs to every base relation of the underlying QCN of PA. We formally define a QSTCN as follows.

Definition 3. *A* QSTCN *is a tuple* $\mathcal{N} = (V_{\mathsf{T}}, V_{\mathsf{S}}, C, \alpha)$*, where* V_{T} *is a finite set of temporal variables,* V_{S} *is a finite set of spatial variables,* C *a mapping associating a relation* $C(v, v') \in 2^{\mathsf{B}_{\mathsf{PA}}}$ *to each pair* $(v, v') \in V_{\mathsf{T}} \times V_{\mathsf{T}}$*, and* α *a mapping associating a pair of spatial* QCN*s* $(\mathcal{N}_v^{r(v,v')}, \mathcal{N}_{v'}^{r(v,v')})$ *to each base*

[2] Actually, in [2, chap. 3] there exists an even faster, but very particular algorithm, that checks the consistency of a QCN of PA in $O(n^2)$ time. The path consistency algorithm is a more general approach that applies to most qualitative calculi.

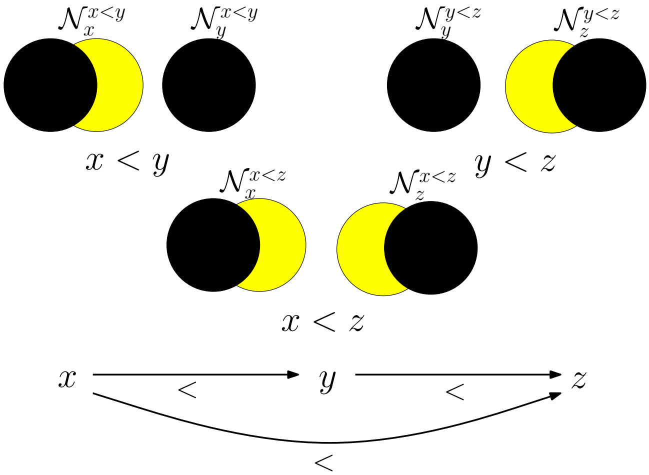

Fig. 1. Example of a QSTCN

relation $r(v, v') \in C(v, v')$*, i.e.,* $\alpha(r(v, v')) = (\mathcal{N}_v^{r(v,v')}, \mathcal{N}_{v'}^{r(v,v')})$*. By overloading notation* $\alpha(C(v, v')) = \{\alpha(r(v, v')) \mid r(v, v') \in C(v, v')\}$*.* C *is such that* $C(v, v) \subseteq \{=\}$ *and* $C(v, v') = (C(v', v))^{-1}\ \forall v, v' \in V_{\mathsf{T}}$*, and* α *is such that all spatial* QCN*s* $\mathcal{N}_v$*, where* $v \in V_{\mathsf{T}}$*, share the same set of spatial variables* V_{S}*,* $\mathcal{N}_v^{v=v'} \equiv \mathcal{N}_{v'}^{v=v'}$*, and* $\alpha(C(v, v')) = \{($*second entry of* 2*-tuple* t*, first entry of* 2*-tuple* $t) \mid t \in \alpha(C(v', v))\}$ *for all* $v, v' \in V_{\mathsf{T}}$*.*

Example 1. An example of a QSTCN $\mathcal{N}$ is presented in Figure 1. To begin with, $\mathcal{N}$ builds upon a QCN of PA that comprises the set of temporal variables $\{x, y, z\}$ and the set of constraints $\{x < y, y < z, x < z\}$ (using infix notation). In a sense, a scenario of a QCN of PA yields a totally ordered set of points in a finite timeline ranging over values in $\mathbb{Q}$. Thus, it allows us to a acquire a description of how a spatial configuration evolves over time. For every base relation between every pair of variables of the QCN of PA, a pair of spatial configurations is attached. For the sake of our example, we can view these configurations as QCNs of RCC-8. All QCNs of RCC-8 share the same set of spatial variables V_{S}, which in our case comprises the image of the moon and the sun. In fact, our example describes an eclipse. The pair of QCNs of RCC-8 attached to the base relation $<$ between temporal variables x and y is the pair $(\mathcal{N}_x^{x<y}, \mathcal{N}_y^{x<y})$. $\mathcal{N}_x^{x<y}$ comprises the set of constraints $\{PO(\mathsf{moon}, \mathsf{sun})\}$, and $\mathcal{N}_y^{x<y}$ comprises the set of constraints $\{EQ(\mathsf{moon}, \mathsf{sun})\}$. Upon instatiation of the base relation $<$ between variables x and y, variable x acquires QCN $\mathcal{N}_x^{x<y}$ and variable y acquires QCN $\mathcal{N}_y^{x<y}$.

In the above example, we can think of relation $<$ as a relation that associates the past of a spatial configuration with its future. Likewise, relation $>$ associates the future of a spatial configuration with its past, and relation $=$ describes only one unique spatial configuration at a point of time, i.e., $\mathcal{N}_v^{v=v'} \equiv \mathcal{N}_{v'}^{v=v'}$ for all $v, v' \in V_{\mathsf{T}}$. Another way to think of a base relation of a QCN of PA is like the command c of a Hoare triple $\{p\}\ c\ \{q\}$, where p and q are its assertions, p is named the precondition and q the postcondition. When the precondition is met, executing the command establishes the postcondition. In our example, $\mathcal{N}_y^{x<y}$ is identical to $\mathcal{N}_y^{y<z}$. However, in general we do not require spatial QCNs associated to a variable $v \in V_\mathsf{T}$ from different sources to be identical to one another.

The formalization that we propose does not only permit us to describe a spatial configuration that changes over time, but also to state general laws of *how* the spatial configuration changes by describing the transition from its past to its future within a pair. The formalization proposed by Gerevini et al. [10], lacks this ability to define general laws about qualitative change. Depending on the qualitative spatial constraint language considered, one can use the pairs of spatial configurations to restrict the movement, the direction, the position, the topology, or any other sort of property of a spatial configuration between two different points in time. This makes our approach very favorable for applications in many fields that deal with qualitative change, such as dynamic GIS, cognitive robotics, spatiotemporal design, and spatiotemporal planning [5, 13].

Definition 4. *A* solution *of a* QSTCN $\mathcal{N} = (V_\mathsf{T}, V_\mathsf{S}, C, \alpha)$ *is a solution of the underlying* QCN *of* PA*, that is compatible with a solution of every spatial* QCN *associated to every* $v \in V_\mathsf{T}$*.* $\mathcal{N}$ *is* consistent *or* satisfiable *iff it admits a solution. A sub-*QSTCN $\mathcal{N}'$ *of* $\mathcal{N}$ *is a* QSTCN *where the underlying* QCN *of* PA *is a sub-*QCN *of the underlying* QCN *of* PA *of* $\mathcal{N}$ *and all associated spatial* QCN*s are sub-*QCN*s of the corresponding spatial* QCN*s of* $\mathcal{N}$*. An atomic* QSTCN *is a* QSTCN *where the underlying* QCN *of* PA *is atomic and all associated spatial* QCN*s are atomic. A* scenario *of* $\mathcal{N}$ *is an atomic consistent sub-*QSTCN *of* $\mathcal{N}$*.* $\mathcal{N}$ *admits a solution iff it admits a scenario.*

Theorem 1. *Checking the consistency of a given* QSTCN $\mathcal{N}$ *is* $\mathcal{NP}$*-complete.*

Proof. Suppose that you are provided with a candidate scenario of a given QSTCN $\mathcal{N} = (V_\mathsf{T}, V_\mathsf{S}, C, \alpha)$. To check whether the candidate scenario is indeed a scenario of the given QSTCN, we must first use the path consistency algorithm to check if the temporal scenario is consistent, which takes $O(n^3)$ time, where $n = |V_\mathsf{T}|$. Then, we can check the consistency of all the linear in the number n of temporal variables spatial QCNs in $O(n \cdot m^3)$ time, where $m = |V_\mathsf{S}|$, again with the path consistency algorithm. $\mathcal{NP}$-hardness follows from the fact that checking the consistency of a single spatial QCN is $\mathcal{NP}$-complete (intractable). Hence, checking the consistency of a given QSTCN $\mathcal{N}$ is $\mathcal{NP}$-complete. □

The rest of this section is devoted to characterizing cases for which we have tractability.

We first define a classical constraint satisfaction problem (CSP [14]) $\mathcal{N}_\mathsf{S}$ that corresponds to the spatial aspect of a QSTCN $\mathcal{N}$ where the spatial QCNs represent definite knowledge between entities, i.e., they are atomic. Inconsistent atomic QCNs are filtered out with a path consistency algorithm. An atomic QCN can be seen as a constant value, as it yields a unique minimal signature of itself.

Definition 5. *Given a* QSTCN $\mathcal{N} = (V_{\mathsf{T}}, V_{\mathsf{S}}, C, \alpha)$ *where the associated spatial* QCN*s are atomic,* $\mathcal{N}_{\mathsf{S}} = \langle X, D, R \rangle$ *will correspond to the constraint satisfaction problem where:*

- *the set of variables* $X = \{x_1, \ldots, x_n\}$ *with* $n = |V_{\mathsf{T}}|$,
- *the set of domains* $D = \{d_1, \ldots, d_n\}$ *where* $d_i = \{\bigcap(\{\textit{first entry of } 2\textit{-tuple } t \mid t \in \alpha(C(v_i, v_j))\} \textit{ or } \{\textit{second entry of } 2\textit{-tuple } t \mid t \in \alpha(C(v_j, v_i))\}) \mid v_j \in V_{\mathsf{T}}\}$ *for all* i *with* $0 \leq i \leq n$,
- *and the set of binary constraints* $R = \{r(x_i, x_j) \mid 0 \leq i, j \leq n\}$ *where each* $r(x_i, x_j) = \{\textit{first entry of } 2\textit{-tuple } t \in d_i \textit{ and second entry of } 2\textit{-tuple } t \in d_j \mid t \in \alpha(C(v_i, v_j))\}$.

The observant reader will note that constructing the set of variables X, the set of domains D, and the set of constraints R for $\mathcal{N}_{\mathsf{S}}$ in that particular order, as defined, will result in a node and arc consistent network.

Example 2. Let us consider the QSTCN $\mathcal{N} = (V_{\mathsf{T}}, V_{\mathsf{S}}, C, \alpha)$, where $V_{\mathsf{T}} = \{x, y, z\}$, $C(x, y) \subseteq \{<\}$, $C(y, z) \subseteq \{>\}$, $C(x, z) \subseteq \{<, =\}$, $\alpha(C(x, y)) = \{(a_x^{x<y},\ b_y^{x<y})\}$, $\alpha(C(y, z)) = \{(b_y^{y>z},\ c_z^{y>z})\}$, and $\alpha(C(x, z)) = \{(a_x^{x<z},\ c_z^{x<z}),(b_x^{x=z},\ b_z^{x=z})\}$. Note that a, b, and c correspond to different atomic spatial QCNs over a set of variables V_{S}, i.e., we can consider them as constant domain values. We can view the CSP $\mathcal{N}_{\mathsf{S}}$ with a set of variables $\{x_s, y_s, z_s\}$, value domain $\{a\}$ for variable x_s, value domain $\{b\}$ for variable y_s, value domain $\{c\}$ for variable z_s, and constraints $r(x_s, y_s) = \{(a_x^{x<y},\ b_y^{x<y})\}$, $r(y_s, z_s) = \{(b_y^{y>z},\ c_z^{y>z})\}$, and $r(x_s, z_s) = \{(a_x^{x<z},\ c_z^{x<z})\}$. (Note that $r(x_s, z_s) \subset \alpha(C(x, z))$.)

Proposition 1. *Given a* QSTCN $\mathcal{N} = (V_{\mathsf{T}}, V_{\mathsf{S}}, C, \alpha)$ *where the associated spatial* QCN*s are atomic, we have that for* $\mathcal{N}_{\mathsf{S}} = \langle X, D, R \rangle$ *the size of domain* d_i *for each variable* $x_i \in X$ *is at most* 3*, and each constraint* $r \in R$ *contains at most three* 2*-tuples.*

Proof. It is easy to see that each $v \in V_{\mathsf{T}}$ can be associated with at most three different atomic QCNs, as there can be at most three PA base relations in any constraint $C(v, v')$, where $v' \in V_{\mathsf{T}}$, and each PA base relation contributes a single atomic spatial QCN to v (and a single atomic spatial QCN to v'). Thus, the possible atomic spatial QCNs for each $v \in V_{\mathsf{T}}$ will be the intersection of all atomic spatial QCNs contributed from each constraint $C(v, v')$, which can be at most three. Further, each PA base relation can contribute a single unique 2-tuple, thus, each constraint $C(v, v')$ can contribute at most three. □

We note that in a CSP a binary constraint $r(x_i, x_j)$ between variables x_i and x_j can be represented as a $(0, 1)$-matrix with $|d_i|$ rows and $|d_j|$ columns by imposing an ordering on the domains of the variables [14]. The entries that correspond to the 2-tuples of a binary constraint have value 1, and all others have value 0.

Definition 6 ([4]). *A binary relation* $r(x_i, x_j)$ *represented as a* $(0, 1)$*-matrix is row convex iff in each row all of the* 1*s are consecutive; that is, no two* 1*s within a single row are separated by a* 0 *in that same row.*

Based on the definition of the notion of row convexity we can obtain the definition of the weaker notion of directional row convexity which is sufficient for the results in our paper.

Definition 7 ([4]). *Given a binary* CSP $\mathcal{N} = \langle X, D, R \rangle$ *and an ordering* $x_i \ldots x_n$ *of its variables, network* $\mathcal{N}$ *is directionally row convex if each of the binary relations* $r(x_i, x_j)$ *represented as a* $(0,1)$*-matrix, where* x_i *occurs before variable* x_j *in the ordering, is row convex.*

Then we have the following result from literature:

Theorem 2 ([4]). *Let* $\mathcal{N} = \langle X, D, R \rangle$ *be a path consistent binary* CSP*. If there exists an ordering of the domains* $d_i, \ldots, d_n$ *of* D *such that the constraints of* R *are directionally row convex then a solution for* $\mathcal{N}$ *can be found without backtracking.*

By Proposition 1 we can deduce that for a CSP $\mathcal{N}_{\mathsf{S}} = \langle X, D, R \rangle$ that corresponds to a QSTCN $\mathcal{N} = (V_{\mathsf{T}}, V_{\mathsf{S}}, C, \alpha)$ where the associated spatial QCNs are atomic, all binary constraints in the set of constraints R can be represented by a $i \times j$ $(0,1)$-matrix, with $i, j \leq 3$, with at most three entries of 1s. By exhaustive enumeration of all the possible constraints of a CSP $\mathcal{N}_{\mathsf{S}}$ it can be found that such a network always is, or can be made, directionally row convex. Therefore, by Theorem 2 we can have the following result:

Theorem 3. *Given a* QSTCN $\mathcal{N} = (V_{\mathsf{T}}, V_{\mathsf{S}}, C, \alpha)$ *where the associated spatial* QCN*s are atomic, applying path consistency on the derived* CSP $\mathcal{N}_{\mathsf{S}} = \langle X, D, R \rangle$ *is sufficient to guarantee a backtrack-free solution.*

Proof. As noted earlier, given a CSP $\mathcal{N}_{\mathsf{S}} = \langle X, D, R \rangle$ that corresponds to a QSTCN $\mathcal{N} = (V_{\mathsf{T}}, V_{\mathsf{S}}, C, \alpha)$ where the associated spatial QCNs are atomic, all binary constraints in the set of constraints R can be represented by a $i \times j$ $(0,1)$-matrix, with $i, j \leq 3$, with at most three entries of 1s. This property will obviously hold even after path consistency is applied on $\mathcal{N}_{\mathsf{S}}$ as the size of the domains can only decrease. Thus, we will obtain a path consistent CSP $\overline{\mathcal{N}_{\mathsf{S}}}$ where the maximum domain size will be at most 3. Then, we can sort the variables of $\overline{\mathcal{N}_{\mathsf{S}}}$ according to their domain size in decreasing order and we will obtain $i \times j$ $(0,1)$-matrices, with $i, j \leq 3$ and $i \geq j$. All 3×3 $(0,1)$-matrices are row convex since they can have at most three entries of 1s and, thus, only a single 1 can exist at each row and column, otherwise it would be a $i \times j$ $(0,1)$-matrix with $i < 3$ or $j < 3$. The rest of the matrices of $\overline{\mathcal{N}_{\mathsf{S}}}$, always assuming the aforementioned ordering, will be $i \times j$ $(0,1)$-matrices with $i \leq 3$ and $j \leq 2$ and $j \leq i$. It is clear that for a number of columns less than or equal to 2 ($j \leq 2$) the corresponding matrix is row convex, as we need at least three colummns for a 0 to exist between two 1s in a single row. Thus, there exists an ordering for which $\overline{\mathcal{N}_{\mathsf{S}}}$ is directionally row convex. The result follows directly from the implication of Theorem 2. □

It is time to introduce our path consistency algorithm, that operates both on the temporal and the spatial aspect of a QSTCN $\mathcal{N}$. We note that the composition of two constraints for the corresponding CSP $\mathcal{N}_{\mathsf{S}}$ is the standard relational

Algorithm 1. $\mathsf{stPC}(\mathcal{N}, \mathcal{N}_{\mathsf{S}})$

in : A QSTCN $\mathcal{N} = (V_{\mathsf{T}}, V_{\mathsf{S}}, C, \alpha)$, and CSP $\mathcal{N}_{\mathsf{S}} = \langle X, D, R \rangle$.

output : False if network $\mathcal{N}$ results in a trivial inconsistency (contains the empty relation), True if the modified network $\mathcal{N}$ is path consistent.

1 **begin**

2 $\quad Q \leftarrow \{(i, j) \mid (i, j) \in V_{\mathsf{T}} \times V_{\mathsf{T}}\}$;

3 $\quad$ **while** $Q \neq \emptyset$ **do**

4 $\quad\quad (i, j) \leftarrow Q.pop()$;

5 $\quad\quad$ **foreach** $k \leftarrow 1$ ***to*** V_{T}, $(i \neq k \neq j)$ **do**

6 $\quad\quad\quad t \leftarrow C(i, k) \cap ((C(i, j) \diamond C(j, k)) \cap \alpha^{-1}(r(i_s, j_s) \circ r(j_s, k_s)))$;

7 $\quad\quad\quad$ **if** $t \neq C(i, k)$ **then**

8 $\quad\quad\quad\quad$ **if** $t = \emptyset$ **then return** False;

9 $\quad\quad\quad\quad C(i, k) \leftarrow t$; $C(k, i) \leftarrow t^{-1}$;

10 $\quad\quad\quad\quad Q \leftarrow Q \cup \{(i, k)\}$;

11 $\quad\quad\quad t \leftarrow C(k, j) \cap ((C(k, i) \diamond C(i, j)) \cap \alpha^{-1}(r(k_s, i_s) \circ r(i_s, j_s)))$;

12 $\quad\quad\quad$ **if** $t \neq C(k, j)$ **then**

13 $\quad\quad\quad\quad$ **if** $t = \emptyset$ **then return** False;

14 $\quad\quad\quad\quad C(k, j) \leftarrow t$; $C(j, k) \leftarrow t^{-1}$;

15 $\quad\quad\quad\quad Q \leftarrow Q \cup \{(k, j)\}$;

16 $\quad$ **return** True;

composition. Given for example two 2-tuples of atomic QCNs (a, b) and (b, c), $(a, b) \circ (b, c)$ yields 2-tuple (a, c).

Algorithm stPC presented in Algorithm 1 receives as input a QSTCN $\mathcal{N}$ and its spatial CSP $\mathcal{N}_S$, and performs path consistency on both the underlying PA network and network $\mathcal{N}_{\mathsf{S}}$. This is achieved by iteratively performing a composition operation both on the temporal and the spatial aspect of network $\mathcal{N}$ in lines 6 and 11. Let us refer to Example 2. We have a QSTCN $\mathcal{N} = (V_{\mathsf{T}}, V_{\mathsf{S}}, C, \alpha)$, where $V_{\mathsf{T}} = \{x, y, z\}$, $C(x, y) \subseteq \{<\}$, $C(y, z) \subseteq \{>\}$, $C(x, z) \subseteq \{<, =\}$, $\alpha(C(x, y)) = \{(a_x^{x<y}, b_y^{x<y})\}$, $\alpha(C(y, z)) = \{(b_y^{y>z}, c_z^{y>z})\}$, and $\alpha(C(x, z)) = \{(a_x^{x<z}, c_z^{x<z}), (b_x^{x=z}, b_z^{x=z})\}$, and CSP $\mathcal{N}_{\mathsf{S}}$ with a set of variables $\{x_s, y_s, z_s\}$, value domain $\{a\}$ for variable x_s, value domain $\{b\}$ for variable y_s, value domain $\{c\}$ for variable z_s, and constraints $r(x_s, y_s) = \{(a_x^{x<y}, b_y^{x<y})\}$, $r(y_s, z_s) = \{(b_y^{y>z}, c_z^{y>z})\}$, and $r(x_s, z_s) = \{(a_x^{x<z}, c_z^{x<z})\}$. The composition $C(x, y) \diamond C(y, z)$ regarding PA, yields the set of relations $\{<, =, >\}$ which is the universal relation B_{PA}. Since $C(x, z) \subseteq \{<, =\} \subset \mathsf{B}_{\mathsf{PA}}$, the underlying PA network of QSTCN $\mathcal{N}$ is path consistent. However, the composition $r(x_s, y_s) \circ r(y_s, z_s)$ yields the set $\{(a_x^{x<z}, c_z^{x<z})\}$. Therefore, since $\alpha^{-1}(r(x_s, y_s) \circ r(y_s, z_s)) = \{<\}$, we must intersect $\{<\}$ with $\{<, =\}$ to acquire the set of relations $\{<\}$ for $C(x, z)$. As a result, network $\mathcal{N}$ will be path consistent for both its temporal and spatial aspect.

We can assert the following proposition for the case of an atomic QSTCN:

Proposition 2. *Given an atomic* QSTCN $\mathcal{N} = (V_{\mathsf{T}}, V_{\mathsf{S}}, C, \alpha)$ *algorithm* stPC *enforces path consistency on* $\mathcal{N}$ *and is able to correctly decide its consistency in* $O(n^3)$ *time, where* $n = |V_{\mathsf{T}}|$.

Proof. It is clear that algorithm stPC enforces path consistency on the underlying QCN of PA and the corresponding CSP $\mathcal{N}_{\mathsf{S}}$. Suppose though that the path consistency of the temporal aspect is not interdependent to the path consistency of the spatial aspect, and vice versa. Then, there should exist a triple of variables i, j, and k for which we have that $(C(i,j) \diamond C(j,k)) \cap \alpha^{-1}(r(i_s, j_s \circ C(j_s, k_s)) = \emptyset$. Because of line 6 in the algorithm this is a contradiction, as stPC would have returned False if this was the case. Since path consistency decides the consistency of an atomic QCN of PA, and it also decides the consistency of CSP $\mathcal{N}_{\mathsf{S}}$ by Theorem 3, it holds that it is able to decide the consistency of QSTCN $\mathcal{N}$. Algorithm stPC is a standard path consistency algorithm as the one described in [20] for qualitative spatial reasoning which runs in $O(n^3)$ time. In our case we only extend the usual composition operation with an additional check on the spatial aspect of a given QSTCN $\mathcal{N}$ which can be done in constant time. □

Let us now consider the more complicated case, where a QSTCN $\mathcal{N} = (V_{\mathsf{T}}, V_{\mathsf{S}}, C, \alpha)$ comprises atomic spatial QCNs and an underlying QCN of PA with relations from the convex class of relations $\{\emptyset, <, =, >, \leq, \geq, = \vee \neq\}$, i.e., relation $\neq$ is not premitted [2]. Then we have the following result from literature:

Theorem 4 ([2]). *Let $\mathcal{N}$ be a path consistent* QCN *of* PA*. If $\mathcal{N}$ comprises relations from the convex class of relations $\{\emptyset, <, =, >, \leq, \geq, = \vee \neq\}$ then $\mathcal{N}$ is* minimal *and* globally consistent *and a solution is found with no backtracking.*

By Proposition 2, Theorem 3, and Theorem 4, we have the following result:

Theorem 5. *Given a* QSTCN *$\mathcal{N} = (V_{\mathsf{T}}, V_{\mathsf{S}}, C, \alpha)$ that comprises atomic spatial* QCN*s and an underlying* QCN *of* PA *with relations from the convex class of relations $\{\emptyset, <, =, >, \leq, \geq, = \vee \neq\}$, algorithm* stPC *enforces path consistency on $\mathcal{N}$ and is able to correctly decide its consistency in $O(n^3)$ time, where $n = |V_{\mathsf{T}}|$.*

Proof. Enforcing path consistency with stPC on QSTCN $\mathcal{N}$ will result in a globally consistent underlying QCN of PA by Theorem 4, denoted by $\mathcal{N}_{\mathsf{PA}}$, and a path consistent corresponding spatial CSP $\mathcal{N}_{\mathsf{S}}$, in a total of $O(n^3)$ time, where $n = |V_{\mathsf{T}}|$. All the scenarios (path consistent atomic networks) that exist for $\mathcal{N}_{\mathsf{PA}}$ are interdependent to respective scenarios of $\mathcal{N}_{\mathsf{S}}$ due to Proposition 2, and vice versa. Thus, all scenarios of $\mathcal{N}$, are both scenarios of $\mathcal{N}_{\mathsf{PA}}$ and $\mathcal{N}_{\mathsf{S}}$. Due to global consistency for $\mathcal{N}_{\mathsf{PA}}$ and the implication of Theorem 3 for $\mathcal{N}_{\mathsf{S}}$, a solution of $\mathcal{N}$ can be obtained by instantiating a single base relation of $\mathcal{N}$, and consistently extending it to a scenario of $\mathcal{N}$ in a backtrack-free manner. □

Up to this point, and as long as tractability was the issue, we have been concerned with a QSTCN $\mathcal{N} = (V_{\mathsf{T}}, V_{\mathsf{S}}, C, \alpha)$ that comprises atomic spatial QCNs. If the associated spatial QCNs are not atomic, it is not possible to construct the corresponing spatial CSP $\mathcal{N}_{\mathsf{S}}$ as provided by Definition 5. This is mainly because there is no way to know the possible values of the spatial QCNs that we will obtain in a scenario of QSTCN $\mathcal{N}$, i.e., it is no longer the case that spatial QCNs yield unique constant values of themselves. Two non-atomic QCNs can intersect and yield a different value, not just the empty relation $\emptyset$. A possible approach would be to enumerate all the scenarios for each non-atomic QCN and use stPC in the way that we described so far. However, for a QSTCN that comprises an atomic underlying QCN of PA we can have the following result and a simple algorithm sketched in its proof.

Theorem 6. *Checking the consistency of a* QSTCN $\mathcal{N} = (V_\mathsf{T}, V_\mathsf{S}, C, \alpha)$ *that comprises an atomic underlying* QCN *of* PA*, has the same complexity with checking the consistency of the associated spatial* QCN*s.*

Proof. We can check the consistency of the underlying QCN of PA in $O(n^3)$ time, where $n = |V_\mathsf{T}|$, with a path consistency algorithm. We then have to obtain the set of spatial QCNs that correspond to each $v \in V_\mathsf{T}$. This would be the set $S = \{\mathcal{N}_1, \ldots, \mathcal{N}_{|V_\mathsf{T}|}\}$ where $\mathcal{N}_i = \{\bigcap(\{\text{first entry of 2-tuple } t \mid t \in \alpha(C(v_i, v_j))\}$ or $\{\text{second entry of 2-tuple } t \mid t \in \alpha(C(v_j, v_i))\}\}) \mid v_j \in V_\mathsf{T}\}$ for all i with $0 \leq i \leq |V_\mathsf{T}|$. Set S can be constructed in $O(n^2 \cdot m^2)$ time, where $m = |V_\mathsf{S}|$. In the case of atomic QCNs we could create constant values out of them, hash values, and compare them in constant time. In this case, we need to go over the $O(m^2)$ constraints for each spatial QCN and intersect them with the constraints of another QCN. After that, checking the consistency of the spatial QCNs is in $\mathcal{P}$ if they are tractable (in $O(n \cdot m^3)$ time with a path consistency algorithm that will go over n spatial QCNs), and in $\mathcal{NP}$ if the they are not tractable. □

4 Conclusion and Future Work

In this paper, we defined a qualitative constraint-based spatiotemporal framework using Point Algebra (PA), that allows for defining formalisms based on several qualitative spatial constraint languages, such as RCC-8, Cardinal Direction Algebra (CDA), and Rectangle Algebra (RA). We formally defined the notion of a qualitative spatiotemporal constraint network (QSTCN), studied its computational properties for the consistency checking problem, and presented algorithms for reasoning with derived formalisms.

Future work consists of further exploring cases of tractability, especially for QSTCNs that comprise non-atomic spatial QCNs. Then, we would like to formally define algorithms for these general QSTCNs, explore heuristics, introduce random and real datasets, identify the phase transition region for such datasets, and create and experiment with a benchmark of QSTCN instances for evaluation. Further, we would like to extend our framework to pointisable IA [2].

Acknowledgments. This work was funded by a PhD grant from Université d'Artois and region Nord-Pas-de-Calais.

References

1. Allen, J.F.: Maintaining Knowledge about Temporal Intervals. Commun. ACM 26, 832–843 (1983)
2. van Beek, P.: Reasoning About Qualitative Temporal Information. Artif. Intell. 58, 297–326 (1992)

3. van Beek, P., Cohen, R.: Exact and approximate reasoning about temporal relations. Computational Intelligence 6, 132–144 (1990)
4. van Beek, P., Dechter, R.: On the Minimality and Decomposability of Row-Convex Constraint Networks. JACM 42, 543–561 (1995)
5. Bhatt, M., Guesgen, H., Wölfl, S., Hazarika, S.: Qualitative Spatial and Temporal Reasoning: Emerging Applications, Trends, and Directions. Spatial Cognition & Computation 11, 1–14 (2011)
6. Burrieza, A., Muñoz-Velasco, E., Ojeda-Aciego, M.: A PDL Approach for Qualitative Velocity. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 19(1), 11–26 (2011)
7. Burrieza, A., Ojeda-Aciego, M.: A Multimodal Logic Approach to Order of Magnitude Qualitative Reasoning with Comparability and Negligibility Relations. Fundam. Inform. 68, 21–46 (2005)
8. Frank, A.U.: Qualitative Spatial Reasoning with Cardinal Directions. In: ÖGAI (1991)
9. Gabelaia, D., Kontchakov, R., Kurucz, A., Wolter, F., Zakharyaschev, M.: On the Computational Complexity of Spatio-Temporal Logics. In: FLAIRS (2003)
10. Gerevini, A., Nebel, B.: Qualitative Spatio-Temporal Reasoning with RCC-8 and Allen's Interval Calculus: Computational Complexity. In: ECAI (2002)
11. Golińska-Pilarek, J., Muñoz-Velasco, E.: Reasoning with Qualitative Velocity: Towards a Hybrid Approach. In: Corchado, E., Snášel, V., Abraham, A., Woźniak, M., Graña, M., Cho, S.-B. (eds.) HAIS 2012, Part I. LNCS, vol. 7208, pp. 635–646. Springer, Heidelberg (2012)
12. Guesgen, H.W.: Spatial Reasoning Based on Allen's Temporal Logic. Tech. rep., International Computer Science Institute (1989)
13. Hazarika, S.: Qualitative Spatio-Temporal Representation and Reasoning: Trends and Future Directions. IGI Global (2012)
14. Montanari, U.: Networks of constraints: Fundamental properties and applications to picture processing. Inf. Sci. 7, 95–132 (1974)
15. Muñoz-Velasco, E., Burrieza, A., Ojeda-Aciego, M.: A logic framework for reasoning with movement based on fuzzy qualitative representation. Fuzzy Sets and Systems 242, 114–131 (2014)
16. Randell, D.A., Cui, Z., Cohn, A.: A Spatial Logic Based on Regions and Connection. In: KR (1992)
17. Renz, J.: Maximal Tractable Fragments of the Region Connection Calculus: A Complete Analysis. In: IJCAI (1999)
18. Renz, J., Ligozat, G.: Weak Composition for Qualitative Spatial and Temporal Reasoning. In: van Beek, P. (ed.) CP 2005. LNCS, vol. 3709, pp. 534–548. Springer, Heidelberg (2005)
19. Renz, J., Nebel, B.: On the Complexity of Qualitative Spatial Reasoning: A Maximal Tractable Fragment of the Region Connection Calculus. AI 108, 69–123 (1999)
20. Renz, J., Nebel, B.: Efficient Methods for Qualitative Spatial Reasoning. JAIR 15, 289–318 (2001)
21. Vilain, M., Kautz, H., van Beek, P.: Constraint Propagation Algorithms for Temporal Reasoning: A Revised Report. In: Readings in Qualitative Reasoning about Physical Systems, pp. 373–381. Morgan Kaufmann Publishers Inc. (1990)
22. Wolter, F., Zakharyaschev, M.: Spatio-temporal representation and reasoning based on RCC-8. In: KR (2000)
23. Wolter, F., Zakharyaschev, M.: Qualitative Spatiotemporal Representation and Reasoning: A Computational Perspective. In: Exploring Artificial Intelligence in the New Millennium. Morgan Kaufmann Publishers Inc. (2003)