# On Integrability of Evolutionary Equations in the Restricted Three-Body Problem with Variable Masses

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**Abstract.** The satellite version of the restricted three-body problem formulated on the basis of classical Gylden–Meshcherskii problem is considered. Motion of the point $P_2$ of infinitesimal mass about the point $P_0$ is described in the first approximation in terms of the osculating elements of the aperiodic quasi-conical motion, and an influence of the point $P_1$ gravity on this motion is analyzed. Long-term evolution of the orbital elements is determined by the differential equations written in the Hill approximation and averaged over the mean anomalies of points $P_1$ and $P_2$. Integrability of the evolutionary equations is analyzed, and the laws of mass variation have been found for which the evolutionary equations are integrable. All relevant symbolic calculations and visualizations are done with the computer algebra system Mathematica.

## 1 Introduction

The restricted three-body problem is a well-known model of celestial mechanics, having a lot of applications (see [9]). In the simplest case, it is assumed that two massive points $P_0$, $P_1$ move in the Keplerian orbits about their common center of mass, while the third point $P_2$ of negligible mass does not influence on their motion and moves in the gravitational field generated by $P_0$, $P_1$. This problem is not integrable, and so the perturbation theory is usually applied to the analysis of the point $P_2$ motion, and quite cumbersome symbolic calculations are involved. As a general solution of the two-body problem is known, one can consider in the first approximation that the point $P_2$ moves around the point $P_0$, for example, as a satellite and its Keplerian orbit is disturbed by the gravity of point $P_1$. Such a model has been used successfully in the study of satellite motion in the

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system Earth–Moon or Sun–planet [4,5]. It was shown that doubly averaged equations of motion determining the evolution of satellite orbit may become integrable. The corresponding general solution may be found in analytic form, and it enables investigation of main qualitative features of the orbit parameters (see, for example, [10]).

If masses of points $P_0$ and $P_1$ vary with time as it takes place in case of a binary star, losing the mass due to the corpuscular and photon radiation, the problem becomes much more complicated because a general solution of the corresponding two-body problem cannot be found in an analytical form (see [2,1,6]). Actually, using the relative coordinate system with origin at point $P_0$, one can write equation of the point $P_1$ motion in the form

$$\frac{d^2\boldsymbol{R}_1}{dt^2} = -G(m_0(t) + m_1(t))\frac{\boldsymbol{R}_1}{R_1^3} \ , \tag{1}$$

where $\boldsymbol{R}_1$ is a radius-vector of point $P_1$, $R_1 = |\boldsymbol{R}_1|$, and $G$ is the constant of gravitation. Equation (1) is known as the classical Gylden–Meshcherskii problem (see [2]), and its general solution can be found in symbolic form only for special cases. In the present paper, we assume that the masses $m_0(t)$ and $m_1(t)$ of points $P_0$ and $P_1$, respectively, vary isotropically with different rates, but their total mass reduces according to the joint Meshcherskii law

$$\frac{m_{00} + m_{10}}{m_0(t) + m_1(t)} = \sqrt{At^2 + 2Bt + C} \equiv v(t), \tag{2}$$

where $m_{00} = m_0(t_0)$, $m_{10} = m_1(t_0)$, $t_0$ is an initial instant of time, and parameters $A, B, C$ are chosen in a way to satisfy the condition $v(t_0) = 1$ and $v(t)$ to be an increasing function for $t > t_0$. Then equation (1) is reduced to ordinary equation of Keplerian motion for constant masses by means of variables transformation (see [2])

$$\boldsymbol{R}_1(t) = v(t)\boldsymbol{R}(\tau) \ , \ \frac{dt}{v^2(t)} = d\tau \ , \tag{3}$$

where $\boldsymbol{R}(\tau) = (X, Y, Z)$ is a new radius-vector, and $\tau$ is a new independent variable (new "time"). A uniform motion in a circle of radius $a_1$ situated in the coordinate plane $XOY$ is a particular case of Keplerian motion and is given by

$$X(\tau) = a_1 \cos M_1(\tau), \ Y(\tau) = a_1 \sin M_1(\tau), \ Z(\tau) = 0, \tag{4}$$

where $M_1(\tau) = \omega_1\tau$, and angular velocity $\omega_1$ is

$$\omega_1 = \left(AC - B^2 + \frac{K}{a_1^3}\right)^{1/2} , \ K = G(m_{00} + m_{10}) \ .$$

Assuming that motion of point $P_1$ is determined by equations (3)-(4), we consider here the satellite version of the restricted three-body problem when the point $P_2$ moves around point $P_0$, being perturbed by the gravity of point $P_1$.

We use the Hill approximation [3], when a distance between points $P_0$ and $P_1$ is considered to be much greater than distance between $P_0$ and $P_2$. The main purpose of this paper is to find a class of functions $m_0(t)$, $m_1(t)$, for which the evolutionary equations, describing the secular perturbations of point $P_2$ trajectory, become integrable, and to obtain the corresponding solutions in analytic form. The relevant cumbersome symbolic calculation and visualization of the results are done with the computer algebra system Mathematica [11].

The paper is organized as follows. In section 2, we obtain the equations of point $P_2$ motion in the framework of the Hill approximation, considering an aperiodic motion on quasi-conical section as the unperturbed motion. Doubly averaging the equations of motion, we obtain the differential equations determining the long-term evolution of the orbital parameters. Then we look in section 3 for the solutions of the evolutionary equations in analytic form and analyze the conditions under which such solutions exist and describe a quasi-elliptic motion of point $P_2$. Finally, in section 4 we determine the mass variation laws for which a general solution of the evolutionary equations can be found in analytical form. And we conclude in section 5.

## 2 Evolutionary Equations

Assume that position of point $P_2$ in the relative coordinate system with origin at point $P_0$ is given by the radius-vector $\boldsymbol{R}_2$. Then equations of its motion are given by (see [6])

$$\frac{d^2\boldsymbol{R}_2}{dt^2} = -Gm_0(t)\frac{\boldsymbol{R}_2}{R_2^3} - Gm_1(t)\frac{\boldsymbol{R}_1}{R_1^3} + Gm_1(t)\frac{\boldsymbol{R}_1 - \boldsymbol{R}_2}{R_{12}^3} \ , \tag{5}$$

where $R_2 = |\boldsymbol{R}_2|$, $R_{12} = |\boldsymbol{R}_1 - \boldsymbol{R}_2|$. Applying the scale transformation of spatial coordinates and time defined in (3), we reduce equation (5) to the form

$$\frac{d^2\boldsymbol{r}}{d\tau^2} = -(AC - B^2)\boldsymbol{r} - Gm_0(t)v(t)\frac{\boldsymbol{r}}{r^3} + Gm_1(t)v(t)\left(\frac{\boldsymbol{R} - \boldsymbol{r}}{\Delta_{12}^3} - \frac{\boldsymbol{R}}{R^3}\right) \ , \tag{6}$$

where $\boldsymbol{r}(\tau) = \boldsymbol{R}_2(t)/v(t)$ is a new radius-vector of point $P_2$, and $\Delta_{12} = |\boldsymbol{R} - \boldsymbol{r}|$.

Note that masses $m_0(t)$ and $m_1(t)$ in (6) are arbitrary non-increasing functions satisfying the condition (2). It is convenient to represent them in the form

$$m_j(t) = \frac{m_{j0}}{v(t)\gamma_j(\tau)} \ , \quad (j = 0, 1) \ , \tag{7}$$

where the functions $\gamma_j(\tau)$ are constrained by the condition

$$\frac{m_{00}}{\gamma_0(\tau)} + \frac{m_{10}}{\gamma_1(\tau)} = m_{00} + m_{10} \ , \tag{8}$$

that follows from (2). Then equation (6) takes the form

$$\frac{d^2\boldsymbol{r}}{d\tau^2} = -(AC - B^2)\boldsymbol{r} - \frac{Gm_{00}}{\gamma_0(\tau)}\frac{\boldsymbol{r}}{r^3} + \frac{Gm_{10}}{\gamma_1(\tau)}\left(\frac{\boldsymbol{R} - \boldsymbol{r}}{\Delta_{12}^3} - \frac{\boldsymbol{R}}{R^3}\right) \ . \tag{9}$$

In case of $\gamma_0 = 1$, when each of the masses $m_0$ and $m_1$ decreases with time according to the joint Meshcherskii law (see (2), (7)–(8)), equation (9) reduces to the restricted three-body problem with constant masses. Note that appearance of a linear term $(AC - B^2)\boldsymbol{r}$ in the right-hand side of (9) does not destroy its integrability for $m_1 = 0$, although it can be integrated only in quadratures. So it is convenient to analyse the corresponding evolutionary equations under an assumption that in the first approximation, point $P_2$ moves around point $P_0$ on Keplerian orbit but its orbital parameters are disturbed by the gravity of point $P_1$ and by additional force being a linear function of $\boldsymbol{r}$. One can show that the differential equations determining evolution of the orbital parameters can then be integrated in analytic form.

To analyse a general case and to find other functions $\gamma_0(\tau)$ for which the evolutionary equations are integrable, one can apply similar approach, exploiting integrability of the differential equation

$$\frac{d^2\boldsymbol{r}}{d\tau^2} = \frac{\ddot{\gamma}_0}{\gamma_0}\boldsymbol{r} - \frac{Gm_{00}}{\gamma_0(\tau)}\frac{\boldsymbol{r}}{r^3} , \tag{10}$$

where $\ddot{\gamma}_0 \equiv d^2\gamma_0/d\tau^2$. Note that $\gamma_0(\tau)$ in (10) is an arbitrary twice continuously differentiable function and this equation determines an aperiodic motion of a point on quasi-conical section (see [6,8]). The corresponding solution $\boldsymbol{r} = (x, y, z)$ can be represented in the form

$$\begin{aligned} x &= \gamma_0 a((\cos E - e)(\cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i) - \\ &-\sqrt{1-e^2}(\sin\omega\cos\Omega\sin E + \cos\omega\sin\Omega\sin E\cos i)) , \\ y &= \gamma_0 a((\cos E - e)(\cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i) - \\ &-\sqrt{1-e^2}(\sin\omega\sin\Omega\sin E - \cos\omega\cos\Omega\sin E\cos i)) , \\ z &= \gamma_0 a((\cos E - e)\sin\omega\sin i + \sqrt{1-e^2}\cos\omega\sin E\sin i) , \end{aligned} \tag{11}$$

where the constants $a$, $e$, $i$, $\Omega$, $\omega$ are the analogues of orbital elements known from the classical two-body problem with constant masses (see, for example, [7]), and the eccentric anomaly $E$ is determined by the equation

$$E - e\sin E = M = \frac{\sqrt{K_0}}{a^{3/2}}(\Phi(\tau) - \Phi(\tau_0)) . \tag{12}$$

The function $\Phi(\tau)$ in (12) is given by

$$\Phi(\tau) = \int_{\tau_0}^{\tau}\frac{d\tau}{\gamma_0^2(\tau)}, \; K_0 = Gm_{00} .$$

In case of $\gamma_0 = 1$ equation (10) reduces to a pure Keplerian problem with constant masses when the variable $M$ becomes a linear function of time known as the mean anomaly and the constant $\tau_0$ is the time of perihelion passage (see [6,7]). Note that orbital parameters $a$, $e$, $i$, $\Omega$, $\omega$, and $\tau_0$ are determined from the initial conditions of motion, and expressions (11), (12) determine an exact solution of the two-body problem (10) for any given function $\gamma_0(\tau)$ satisfying the conditions above.

As equation (9) does not contain a linear term being proportional to the second derivative of the function $\gamma_0$, one can add and subtract the corresponding term and rewrite the equation in the form

$$\frac{d^2\boldsymbol{r}}{d\tau^2} = \frac{\ddot{\gamma}_0}{\gamma_0}\boldsymbol{r} - \frac{Gm_{00}}{\gamma_0(\tau)}\frac{\boldsymbol{r}}{r^3} - \left[(AC - B^2)\boldsymbol{r} + \frac{\ddot{\gamma}_0}{\gamma_0}\boldsymbol{r} - \frac{Gm_{10}}{\gamma_1(\tau)}\left(\frac{\boldsymbol{R}-\boldsymbol{r}}{\Delta_{12}^3} - \frac{\boldsymbol{R}}{R^3}\right)\right] . \quad (13)$$

Then its solution can be sought in the form (11) under the condition that the orbital parameters are functions of time. Such approach is known as a method of variation of constants and is widely used in the theory of differential equations.

To derive the differential equations determining the time evolution of orbital parameters in the simplest form, it is convenient to rewrite equation (13) in the Hamiltonian form and to change to the special set of canonical variables known as Delaunay's variables (see [6,8]). Three pairs of the corresponding canonical conjugate coordinates and momenta $(l, L)$, $(g, G)$ and $(h, H)$ are related to the analogues of the Keplerian orbital elements as

$$l = M,\ L = \sqrt{K_0 a},\ g = \omega,\ G = L\sqrt{1-e^2},\ h = \Omega,\ H = G\cos i . \quad (14)$$

The Hamiltonian function in the Delaunay variables may be written in the form

$$\mathcal{H} = -\frac{K_0^2}{2\gamma_0^2 L^2} + \frac{1}{2}\left(AC - B^2 + \frac{\ddot{\gamma}_0}{\gamma_0}\right)(x^2 + y^2 + z^2) - V, \quad (15)$$

where the function $V$ is given by

$$V = \frac{K_1}{\gamma_1(\tau)}\left(\frac{1}{\Delta_{12}} - \frac{xX + yY + zZ}{R^3}\right) ,\ K_1 = Gm_{10} ,$$

and components of vectors $\boldsymbol{R}$ and $\boldsymbol{r}$ are given by (4) and (11), respectively.

Assuming further the ratio of the distances $r$ and $R$ to be small ($r/R << 1$), one can expand the function $V$ into a power series in terms of $r/R$ and keep only the main term of the expansion in the Hamiltonian (15). It means that we consider the problem in the Hill approximation [3]. Then the Hamiltonian takes the form

$$\mathcal{H} = -\frac{K_0^2}{2\gamma_0^2 L^2} + \frac{1}{2}\left(AC - B^2 + \frac{\ddot{\gamma}_0}{\gamma_0} + \frac{K_1}{\gamma_1 a_1^3}\right)(x^2 + y^2 + z^2) - \\ - \frac{3K_1}{2\gamma_1 a_1^3}\left(x^2\cos^2 M_1 + y^2\sin^2 M_1 + xy\sin(2M_1)\right) . \quad (16)$$

As we are interested in the secular evolution of the point $P_2$ orbit under an influence of massive point $P_1$, one may disregard the short-period perturbations of orbital elements by means of averaging of the Hamiltonian (16) over the mean anomalies of points $P_1$ and $P_2$ (see [5]). The averaged Hamiltonian is determined as (see (12))

$$\bar{\mathcal{H}} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\mathcal{H}dMdM_1 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\mathcal{H}(1 - e\cos E)dEdM_1,$$

and is given by

$$\bar{\mathcal{H}} = -\frac{K_0^2}{2\gamma_0^2 L^2} + \frac{\gamma_0^2 L^4}{4K_0^2}\left(AC - B^2 + \frac{\ddot{\gamma}_0}{\gamma_0} + \frac{K_1}{\gamma_1 a_1^3}\right)\left(5 - \frac{3G^2}{L^2}\right) - \tag{17}$$

$$-\frac{3K_1\gamma_0^2 L^4}{16\gamma_1 a_1^3 K_0^2}\left(2 + \frac{2H^2}{G^2} + \left(1 - \frac{G^2}{L^2}\right)\left(3 + \frac{3H^2}{G^2} + 5\cos(2g)\left(1 - \frac{H^2}{G^2}\right)\right)\right) ,$$

where the relationships (14) have been taken into account.

Obviously, the averaged Hamiltonian (17) does not depend on the mean anomaly $M \equiv l$ and so its canonical conjugate variable $L$ is constant. The first term in the right-hand side of expression (17), depending only on $L$, influences on the time evolution of mean anomaly $M$ but doesn't influence on other orbital parameters of point $P_2$. Therefore, if the rest terms of the Hamiltonian (17) contained the same multiplier $\gamma_0^2/\gamma_1$, depending on time, it would be possible to reduce the differential equations, determining the secular evolution of orbital parameters $g$, $h$, $G$, $H$, to the autonomous case by means of the scale transformation of time. We shall show later that such autonomous differential equations may be integrated. So let us consider a class of functions $\gamma_0(\tau)$ satisfying the condition

$$\frac{\ddot{\gamma}_0}{\gamma_0} + AC - B^2 = -\alpha\frac{K_1}{\gamma_1 a_1^3} , \tag{18}$$

where $\alpha$ is a parameter. Then the Hamiltonian (17) can be rewritten as

$$\bar{\mathcal{H}} = -\frac{K_0^2}{2\gamma_0^2 L^2} + \frac{K_1\gamma_0^2 L^4}{4\gamma_1 a_1^3 K_0^2}\left[(1-\alpha)\left(5 - \frac{3G^2}{L^2}\right) - \frac{3}{2}\left(1 + \frac{H^2}{G^2}\right) - \right.$$

$$\left. -\frac{3}{4}\left(1 - \frac{G^2}{L^2}\right)\left(3 + \frac{3H^2}{G^2} + 5\cos(2g)\left(1 - \frac{H^2}{G^2}\right)\right)\right] . \tag{19}$$

Differential equations for orbital parameters $g$, $h$, $G$, $H$ are obtained in the standard Hamiltonian form as

$$\frac{dg}{d\tau} = \frac{\partial\bar{\mathcal{H}}}{\partial G} , \ \frac{dG}{d\tau} = -\frac{\partial\bar{\mathcal{H}}}{\partial g} , \ \frac{dh}{d\tau} = \frac{\partial\bar{\mathcal{H}}}{\partial H} , \ \frac{dH}{d\tau} = -\frac{\partial\bar{\mathcal{H}}}{\partial h} . \tag{20}$$

Substituting expression (19) into (20) and taking into account (14), after quite standard symbolic calculations we obtain differential equations determining the secular evolution of the orbital parameters in the form

$$\frac{dz}{dn} = 20z\sqrt{1-z}\sin^2 i\sin(2\omega) , \tag{21}$$

$$\frac{di}{dn} = -\frac{10z}{\sqrt{1-z}}\sin i\cos i\sin(2\omega) , \tag{22}$$

$$\frac{d\omega}{dn} = \frac{4}{\sqrt{1-z}} \left(5\cos^2 i \sin^2\omega + (1-z)(2\alpha + 2 - 5\sin^2\omega)\right) , \tag{23}$$

$$\frac{d\Omega}{dn} = -\frac{4\cos i}{\sqrt{1-z}} \left(1 - z + 5z\sin^2\omega\right) , \tag{24}$$

where $z = e^2$, and $n$ is a new dimensionless independent variable determined by the equation

$$dn = \frac{3K_1\gamma_0^2(\tau)a^2}{16\gamma_1(\tau)a_1^3\sqrt{K_0 a}}\, d\tau . \tag{25}$$

Note that the system of differential equations (21)-(24) looks similarly to the corresponding equations describing evolution of satellites of Uranus (see [10]). But due to dependence of the points masses on time equation (23) contains additional term $2\alpha(1-z)$ in the parentheses in the right-hand side and additional parameter $\alpha$. Therefore, the system behaviour and its analysis should be more complicated, although it can be investigated in a similar way as in [10].

## 3 Integration of the Evolutionary Equations

Direct symbolic calculation shows that the system of three equations (21)-(23) has two independent integrals of motion

$$(1-z)\cos^2 i = c_1 = const, \tag{26}$$

$$z\left(\frac{2}{5}N - \sin^2 i \sin^2\omega)\right) = c_2 = const, \tag{27}$$

where $N = 1+\alpha$ is a new parameter. This enables us to eliminate two variables in the system (21)-(23) and to reduce it to an ordinary differential equation with respect to the function $z(n)$ that can be integrated. As determination of the function $\Omega(n)$ reduces then to simple integrating the right-hand side of equation (24) we focus here on analysis of system (21), (26), (27) and will discuss solving the equations (22), (23) only if the corresponding solutions cannot be obtained from the integrals (26), (27).

Note that in case of quasi-elliptic motion of point $P_2$ eccentricity of its orbit should be less than 1 or $0 \le z < 1$. Hence, the first integral $c_1$ must belong to the interval $0 \le c_1 \le 1$. Consequently, for given $c_1$, expression (26) restricts possible values of $z$ to the interval $0 \le z \le 1 - c_1$. Eliminating the variable $i$ in the system (26)-(27), we obtain

$$\sin^2\omega = \frac{(1-z)(2Nz - 5c_2)}{5z(1-z-c_1)} . \tag{28}$$

Then the condition $0 \le \sin^2\omega \le 1$ gives two inequalities

$$2Nz - 5c_2 \ge 0 , \quad (5-2N)z^2 - z(5 - 2N - 5c_1 - 5c_2) - 5c_2 \le 0 . \tag{29}$$

Applying the Mathematica built-in function *Reduce* to the system (29) combined with inequalities $0 \le z \le 1 - c_1$, $0 \le c_1 \le 1$ and separating the results with the function *LogicalExpand*, one can get a long list of different solutions, determining possible values of the integral $c_2$ and the variable $z$, corresponding to quasi-elliptic motion of point $P_2$. Depending on the value of parameter $N$, one can separate three different cases which are considered below.

### 3.1 Case $N = \frac{5}{2}$

In this case, the system (29) reduces to the following inequalities

$$0 \le c_2 \le z \le \frac{c_2}{c_1 + c_2} \ , \quad c_1 + c_2 \le 1 \ , \quad 0 \le c_1 \le 1 \ . \tag{30}$$

Therefore, the domain of possible values of the integrals $c_1$, $c_2$ in the plane $Oc_1c_2$ is a triangle bounded by the lines $c_1 = 0$, $c_2 = 0$, $c_1 + c_2 = 1$.

Using expressions (26), (28), we eliminate the variables $i$ and $\omega$ in (21) and obtain the following differential equation

$$\frac{dz}{dn} = 40\mathrm{sgn}(\sin(2\omega_0))\sqrt{(z - c_2)(c_2 - z(c_1 + c_2))} \ , \tag{31}$$

where the function $\mathrm{sgn}(x)$ determines a sign of $\sin(2\omega_0)$ at the initial instant of time ($\omega_0 = \omega(t_0)$). This equation is easily integrated, and its solution is given by

$$z = c_2 + \frac{c_2(1 - c_1 - c_2)}{c_1 + c_2} \sin^2\left(20\mathrm{sgn}(\sin(2\omega_0))\sqrt{c_1 + c_2}n + \varphi_0\right) \ , \tag{32}$$

where

$$\varphi_0 = \arcsin\sqrt{\frac{(z_0 - c_2)(c_1 + c_2)}{c_2(1 - c_1 - c_2)}} \ , \quad z_0 = z(0) \ .$$

Expression (32) shows that $z(n)$ is an oscillating function, and its values belong to the interval (30). The function $i(n) = \arccos\sqrt{c_1/(1 - z)}$ also oscillates, and an interval of its values is determined by inequality

$$\frac{c_1}{1 - c_2} \le \cos^2 i \le c_1 + c_2 \ .$$

One can readily check that the function $\omega(n)$ increases with time because its derivative (see (23)) is positive under the conditions (30), while its values are determined by the expression (28).

One should note that in case of $c_2 = 0$ there exists additional stationary solution of equation (21) that cannot be obtained as a limit case of (32). Actually, an equality $c_2 = 0$ takes place either in case of $z = 0$ or in case of $\sin^2 i = 1$ and $\sin^2\omega = 1$ (see (27)). The second case implies $c_1 = 0$, and the corresponding solution is given by

$$0 \le z = const < 1 \ , \quad i = \frac{\pi}{2} \ , \quad \omega = \frac{\pi}{2} \ \text{ or } \ \omega = \frac{3\pi}{2} \ . \tag{33}$$

Solution (33) describes motion of point $P_2$ on elliptic orbit in a plane that is perpendicular to the orbital plane of point $P_1$. Note that in case of constant masses, such motion always results in collision of points $P_0$ and $P_2$ (see [4]).

### 3.2 Case $N > \frac{5}{2}$

Analysis of inequalities (29) shows that the domain of possible values of the integrals $c_1$ and $c_2$ in the plane $Oc_1c_2$ is a triangle determined by inequalities

$$c_1 \geq 0 \ , \quad c_2 \geq 0 \ , \quad c_1 \leq 1 - \frac{5c_2}{2N} \ . \tag{34}$$

On its boundary $c_2 = 0$, equations (21), (22) have only a stationary solution $z = 0$, $\cos^2 i = c_1$, while equation (23) takes the form

$$\frac{d\omega}{dn} = 4\left(2N - 5(1 - c_1)\sin^2\omega\right) \ ,$$

and is integrated in terms of elementary functions, the result is easily found with the Mathematica built-in function *DSolve*, for example.

On the other border $c_1 = 1 - 5c_2/(2N)$, inequalities (29) can be written in the form

$$z - \frac{5c_2}{2N} \geq 0 \ , \quad 2N - z(2N - 5) \leq 0 \ .$$

One can readily see that inside of the interval $z \in [0, 1]$ there is only one point $z = 5c_2/(2N)$ satisfying these inequalities. Therefore, equations (21) and (22) have only a stationary solution $z = 5c_2/(2N) = 1 - c_1$, $\cos^2 i = 1$, and equation (23), taking the form

$$\frac{d\omega}{dn} = \frac{4}{\sqrt{c_1}}\left(2Nc_1 + 5(1 - c_1)\sin^2\omega)\right) \ ,$$

is again integrable in terms of elementary functions.

On the third border $c_1 = 0$, when $0 \leq c_2 < 2N/5$ and $\cos^2 i = 0$, inequalities (29) can be written in the form

$$z - \frac{5c_2}{2N} \geq 0 \ , \quad (1 - z)\,(5c_2 - z(2N - 5)) \geq 0 \ . \tag{35}$$

Therefore, the variable $z$ belongs to the interval

$$\frac{5c_2}{2N} \leq z < \frac{5c_2}{2N - 5} < 1 \ ,$$

for $0 \leq c_2 < 2N/5 - 1$, and

$$\frac{5c_2}{2N} \leq z \leq 1 \ ,$$

for $2N/5 - 1 \leq c_2 < 2N/5$. In this case, solution of equation (21) becomes more complicated, although it may be integrated in terms of elliptic functions. As the method applied is similar to the case when values of the integrals $c_1$, $c_2$ belong to the domain inside of the triangle (34) in the plane $Oc_1c_2$, let us consider such general case.

Using expressions (26), (28) and eliminating the variables $i$ and $\omega$ in equation (21), one can rewrite it in the form

$$\frac{dz}{dn} = 8\mathrm{sgn}(\sin(2\omega_0))\sqrt{Q(z)} \ , \tag{36}$$

where the third-degree polynomial $Q(z)$ is given by

$$Q(z) = (2Nz - 5c_2)(5c_2 + z(5 - 2N - 5c_1 - 5c_2) - z^2(5 - 2N)) \ , \tag{37}$$

and it is assumed that the variable $z$ takes only such values, for which $Q(z) \geq 0$. Solving the equation $Q(z) = 0$, we obtain in general three different roots

$$z_{1,2} = \frac{1}{2}\left[1 + \frac{5(c_1 + c_2)}{2N - 5} \pm \left(\left(1 + \frac{5(c_1 + c_2)}{2N - 5}\right)^2 - \frac{20c_2}{2N - 5}\right)^{1/2}\right] \ ,$$

$$z_3 = \frac{5c_2}{2N} \ . \tag{38}$$

Analysis of expressions (37), (38) shows that inside of the domain (34) two roots $z_2$, $z_3$ of polynomial $Q(z)$ belong to the interval $[0, 1]$, and $Q(z) \geq 0$ if $z_3 \leq z \leq z_2 < 1$, while the third root $z_1 \geq 1$. Then $Q(z)$ may be represented in the form

$$Q(z) = 2N(2N - 5)(z_1 - z)(z_2 - z)(z - z_3)$$

and equation (36) may be integrated in the elliptic quadrature. Its solution is

$$8 \ \mathrm{sgn}(\sin(2\omega_0))\sqrt{2N(2N - 5)} \ n = \int_{z_0}^{z} \frac{dz}{\sqrt{(z_1 - z)(z_2 - z)(z - z_3)}} \ . \tag{39}$$

An integral in the right-hand side of (39) is calculated in terms of the elliptic functions and the solution may be represented as

$$z(u) = z_3 + (z_2 - z_3)\mathrm{sn}^2 u \ , \tag{40}$$

where

$$u = 4\mathrm{sgn}(\sin(2\omega_0))\sqrt{2N(2N - 5)(z_1 - z_3)}n + u_0 \ , \quad u_0 = F(\varphi_0, \kappa^2) \ ,$$

$$\kappa^2 = \frac{z_2 - z_3}{z_1 - z_3} < 1 \ , \quad \sin^2\varphi_0 = \frac{z_0 - z_3}{z_2 - z_3} \ , \quad z_0 = z(0) \ .$$

Here $\mathrm{sn}u$ and $F(\varphi_0, \kappa^2)$ are the Jacobi elliptic sine and the incomplete elliptic integral of the first kind, respectively.

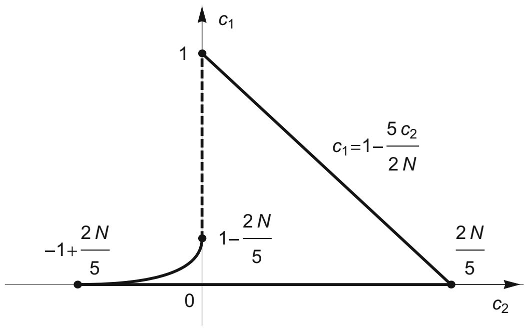


**Fig. 1.** Domain of possible values of the integrals $c_1$, $c_2$ for $0 \leq N < 5/2$

### 3.3 Case $N < \frac{5}{2}$

In case of $N \geq 0$ possible values of integrals $c_1$, $c_2$ must belong to the domain shown in Fig. 1, which are bounded by the lines

$$c_1 = 0 \ , \quad c_2 = 0 \ , \quad c_1 = 1 - \frac{5c_2}{2N} \ , \tag{41}$$

and the curve

$$c_1 = 1 - c_2 - \frac{2N}{5} - 2\sqrt{(-c_2)\left(1 - \frac{2N}{5}\right)} \ . \tag{42}$$

On the line $c_2 = 0$, polynomial $Q(z)$ has three roots, namely, $z_1 = 1 - c_1/(1 - 2N/5)$ and $z_{2,3} = 0$. The root $z_1$ is negative for $1 - 2N/5 < c_1 \leq 1$ (the corresponding points are shown in Fig. 1 as a dashed bold line), and equation (21) has the only solution $z = 0$. But for $0 \leq c_1 < 1 - 2N/5$, the root $z_1$ becomes smaller than 1, and the polynomial (37), taking a form

$$Q(z) = 2Nz^2(5 - 2N - 5c_1 - z(5 - 2N)) \ , \tag{43}$$

is non-negative for $0 \leq z \leq z_1$.

Substituting the polynomial (43) into equation (36), one can readily see that the differential equation is integrated in terms of elementary functions and its solution is determined by the equation

$$\ln \frac{\sqrt{a} - \sqrt{a - bz}}{\sqrt{a} + \sqrt{a - bz}} = 8\sqrt{10Na}\ \mathrm{sgn}(\sin(2\omega_0))n + B_0 \ , \tag{44}$$

where

$$B_0 = \ln \frac{\sqrt{a} - \sqrt{a - bz_0}}{\sqrt{a} + \sqrt{a - bz_0}} \ , \quad a = 1 - c_1 - 2N/5 \ , \quad b = 1 - 2N/5 \ .$$

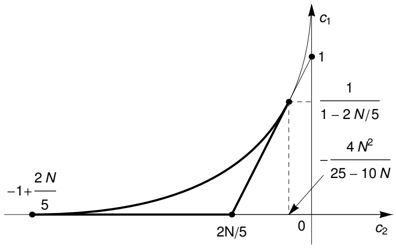


**Fig. 2.** Domain of possible values of the integrals $c_1$, $c_2$ for $N \leq 0$

On the line $c_1 = 1 - 5c_2/(2N)$, polynomial $Q(z)$ takes the form

$$Q(z) = (2Nz - 5c_2)^2 \left( z \left( 1 - \frac{5}{2N} \right) - 1 \right) ,$$

and has three zeros

$$z_1 = \frac{1}{1 - \frac{5}{2N}}, \ z_{2,3} = \frac{5c_2}{2N}.$$

For $N \geq 0$ we have $c_2 > 0$ (see Fig. 1), and the root $z_1$ is negative, so equation (21) has only a stationary solution $z = z_{2,3} = 1 - c_1$. But for $N < 0$, when the line $c_1 = 1 - 5c_2/(2N)$ touches the curve (42) (see Fig. 2) and parameters $c_1$, $c_2$ must satisfy the conditions $0 \leq c_1 \leq 1/(1-2N/5)$, $2N/5 \leq c_2 \leq -4N^2/(25-10N)$, we obtain $0 < z_1 \leq z_{2,3} \leq 1$ and polynomial (37) is non-negative for $z \in [z_1, z_{2,3}]$. Then equation (36) is integrated in terms of elementary functions similar to the previous case (see (43), (44)).

On the curve (42), the polynomial $Q(z)$ takes the form

$$Q(z) = 2N(5 - 2N) \left( z - \frac{5c_2}{2N} \right) \left( z - \sqrt{\frac{(-c_2)}{1 - 2N/5}} \right)^2 ,$$

where we have taken into account that $c_2 < 0$ and $N < 5/2$. The corresponding roots are given by

$$z_{1,2} = \sqrt{\frac{(-c_2)}{1 - 2N/5}} , \quad z_3 = \frac{5c_2}{2N} .$$

One can readily check that for $0 \leq N < 5/2$, we have $0 \leq z_{1,2} \leq 1$, $z_3 < 0$ and, hence, equation (21) has only a stationary solution $z = z_{1,2}$. This solution remains also for $N < 0$ and $-1 + 2N/5 \leq c_2 < 2N/5$ when the root $z_3$ becomes greater than 1. But for $N < 0$ and $2N/5 \leq c_2 < -4N^2/(25 - 10N)$ we obtain

$$\frac{1}{1 - 5/(2N)} < z_{1,2} < z_3 \leq 1 .$$

Polynomial $Q(z)$ is non-negative for $z \in [z_{1,2}, z_3]$, and equation (21) is integrated in terms of elementary functions similarly to the cases above.

On the last boundary $c_1 = 0$, we have $\cos^2 i = 0$, and polynomial $Q(z)$ taking the form

$$Q(z) = (2Nz - 5c_2)(1 - z)(5c_2 + z(5 - 2N)) \ ,$$

has three different roots

$$z_1 = 1, \ z_2 = \frac{5c_2}{2N - 5}, \ z_3 = \frac{5c_2}{2N} \ .$$

If $0 \le N < 5/2$ and $c_2 \ge 0$ then these roots satisfy the inequalities

$$z_2 \le 0 \le z_3 \le z_1 = 1 \ ,$$

and the polynomial $Q(z) \ge 0$ for $z \in [z_3, z_1]$. If $c_2$ becomes negative then we obtain

$$z_3 \le 0 \le z_2 \le z_1 = 1 \ ,$$

and $Q(z) \ge 0$ for $z \in [z_2, z_1]$. Finally, for $N < 0$ and $(-1 + 2N/5) \le c_2 \le 2N/5$ the corresponding inequalities become

$$0 \le z_2 \le z_1 = 1 \le z_3 \ ,$$

and again $Q(z) \ge 0$ for $z \in [z_2, z_1]$. In all three cases one of the roots is outside the interval $[0, 1]$ and two other roots $z_j$, $z_k$ are inside it, while $Q(z) \ge 0$ for $z \in [z_j, z_k]$. Then equation (36) is integrated in elliptic quadratures and its solution looks similarly to the expression (39).

Analysis of expressions (38) shows that at the internal points of the domains shown in Fig. 1, 2, we have similar situation, when three roots of the polynomial $Q(z)$ are different and only two of them belong to the interval $[0, 1]$. In all such cases, equation (21) is reduced to the form (36), and the result of its integration is expressed in terms of the elliptic functions with some permutation of the roots $z_1$, $z_2$, $z_3$.

## 4 Mass Variations

As we have seen above, the evolutionary equations are integrable in terms of elementary and elliptic functions if the functions $\gamma_0(\tau)$, $\gamma_1(\tau)$ satisfy equation (18). Taking into account condition (8), we can rewrite (18) in the form

$$\frac{d^2\gamma_0}{d\tau^2} + \left( AC - B^2 + \alpha \frac{K}{a_1^3} \right) \gamma_0(\tau) = \alpha \frac{K_0}{a_1^3} \ . \tag{45}$$

One can readily see that equation (45) is integrable, and its solution satisfying the condition $\gamma_0(0) = 1$ is given by

$$\gamma_0(\tau) = \alpha \frac{K_0}{\sigma^2 a_1^3} + \left( 1 - \alpha \frac{K_0}{\sigma^2 a_1^3} \right) \cos(\sigma\tau) + \Phi \sin(\sigma\tau) \ , \tag{46}$$

where $\Phi$ is an arbitrary constant, and

$$\sigma^2 = AC - B^2 + \alpha \frac{K}{a_1^3} \ .$$

Taking into account equation (8), one can represent differential equation (25), determining the variable $n$, in the form

$$\frac{dn}{d\tau} = \frac{3a^{3/2}}{16a_1^3 K_0^{1/2}} \left( K\gamma_0^2(\tau) - K_0\gamma_0(\tau) \right) \ ,$$

where the function $\gamma_0(\tau)$ is given by (46). Obviously, this equation is easily integrated, and an explicit expression for the function $n(\tau)$ together with different explicit and implicit solutions $z(n)$ found in previous section gives a complete solution of the evolutionary equations in the considered restricted three-body problem with variable masses.

## 5 Conclusion

We have considered the satellite version of the restricted three-body problem formulated on the basis of the classical Gylden–Meshcherskii problem. We have obtained the evolutionary equations of the massless point $P_2$, describing a long-term evolution of its orbital elements, in the Hill approximation, and investigated their integrability. It was shown that the evolutionary equations are integrable in terms of the elementary and elliptic functions if masses of points $P_0$, $P_1$ vary isotropically with different rates determined by the expressions (2), (8), (46). Solutions of these equations describe quasi-elliptic motion of the point $P_2$ if initial conditions of motion are chosen in such a way that two integrals of motion $c_1$, $c_2$ belong to the domains shown in Fig. 1, 2. All relevant symbolic calculations and visualizations are done with the computer algebra system Mathematica.

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