Chapter 4
Frege's *Grundgesetze* and a Reassessment of Predicativity

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It is well known that Frege's *Grundgesetze der Arithmetik* is inconsistent. The inconsistency is due to the coexistence of two assumptions within Frege's formal system, namely, the impredicative second-order comprehension axiom and unrestricted Basic Law V.[1] Still, it is also known that there are consistent fragments of *Grundgesetze*. In the 1980s, Peter Schroeder-Heister and Terence Parsons provided, respectively, a syntactic and a semantic proof of consistency for the first-order fragment of Frege's *Grundgesetze*. At the time, Parsons conjectured that any extension of the first-order fragment to some second-order system of *Grundgesetze* would result in an inconsistent set of axioms. Nevertheless, in 1996, Richard Heck proved that the *predicative* second-order fragment of *Grundgesetze* has a model. A few years later, Kai Wehmeier proved the consistency of the Δ^1_1-fragment of *Grundgesetze*.[2] This article will concern mainly Heck (1996). Heck's result shows that the fragment of *Grundgesetze* resulting from predicatively restricting the comprehension axiom,[3] while maintaining Basic Law V unrestricted, is consistent. Though Heck (1996) focuses on achieving a technical goal, one may wonder what the possible foundational applications of his result are. In particular, one may be

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[1] In *Grundgesetze*, one finds the so-called substitution rule, which is nevertheless equivalent to the usual second-order unrestricted comprehension axiom from second-order logic. For a matter of perspicuity, I will discuss the unrestricted comprehension axiom. Basic Law V is the renowned Frege's axiom according to which extensions α and β are identical if, and only if, their corresponding concepts F and G are coextensive.

[2] See also Burgess (2005) and Ferreira and Wehmeier (2002).

[3] By this restriction, no bound second-order variables are allowed on the right-hand side of the axiom's biconditional.

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interested in whether such a consistent fragment could provide a formal core for revising Frege's foundational programme.

Nevertheless, in order to be feasible at all, any such revision of Frege's logicism has to deal with two important issues: first, the issue of what the mathematical strength of such a revision is as compared to Frege's original programme of a logicist foundation of arithmetic; secondly, the issue of whether such a revision implies some radical modifications of Frege's philosophical assumptions.

As far as mathematical strength is concerned, the predicative fragment of *Grundgesetze* is known to be quite weak, since it is equi-interpretable with Robinson arithmetic Q.[4] Provided that Frege's logicism really is the claim that arithmetic is derivable from purely logical basis, such a system should be strengthened so that it recovers, in the best case scenario, full second-order Peano arithmetic. Nevertheless, my main interest in what follows will concern the issue of the possible revisions of Frege's *philosophical* assumptions in a predicative setting. In particular, since the predicative restriction on the comprehension axiom affects, first and foremost, Frege's view on *concepts*, I will investigate whether there is some possible interpretation of predicativity that is compatible with Frege's philosophical view on them.[5] This will be achieved through a general reassessment of the notion of *predicativity*, as it is first motivated by Gödel (1944). Gödel's objection to the use of predicativity in mathematics relies on ontological considerations. The reassessment of predicativity I propose detaches the acceptability of a predicative approach from ontological preferences and connects it to logico-mathematical reasoning. On these grounds, I will finally investigate where things stand as for Frege's philosophical view on concepts in a predicative setting, and I will conclude that such a view may be at least partially retained.

4.1 Predicativity and Predicativism: Russell's VCP

Predicativity is just a syntactic means by which the definition of a mathematical entity by quantification over a totality it belongs to is disallowed and some large portions of mathematics can or cannot be recovered. Predicativity in mathematics has to be distinguished from predicativism, which is the cluster of philosophical

[4]See Burgess (2005), Ferreira and Wehmeier (2002), Ganea (2007), and Heck (1996).

[5]Ferreira and Wehmeier (2002) shows that the Δ^1_1-comprehension axiom augmented by unrestricted Basic Law V is consistent. Δ^1_1-comprehension allows only for second-order existential formulæ that are provably equivalent in the system to second-order universal formulæ to appear on the right-hand side of the biconditional. Still, in what follows I will focus on Heck (1996). Δ^1_1-comprehension with Basic Law V, in fact, though very interesting mathematically because of its consistency proof, is still mathematically quite weak, since it is taken to interpret just Robinson arithmetic Q, like Heck (1996). So, if we take the recovery of portions of mathematics larger than Q as one of the two important issues any revisions of Frege's logicism should tackle, then Δ^1_1-comprehension with Basic Law V falls short of being an alternative to Heck (1996) as for broader foundational purposes.

claims motivating predicativity.[6] These claims are usually taken to be captured by Russell's *vicious circle principle* (VCP). Gödel (1944) is possibly the most classical article where a detailed discussion about VCP takes place. It has also been possibly the most influential view ever since on the divide between predicativity and impredicativity, and their philosophical implications. It is just fair, then, that VCP and Gödel's criticism are among the main topics of this article. In particular, I will defend a form of VCP against Gödel's criticisms as they are presented in Gödel (1944).

It is well known that Russell offers several formulations of VCP.[7] Gödel (1944) points out that these formulations boil down to three VCPs, respectively, formulated in terms of *definability*, *presupposition*, and *involvement*:

Definability VCP If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total.[8]

Presupposition VCP Given any set of objects such that, if we suppose the set to have a total, it will contain members which presuppose this total, then such a set cannot have a total.[9]

Involvement VCP Whatever involves all of a collection must not be one of the collection.[10]

Famously, Gödel (1944) focuses mainly on a critical appraisal of Definability VCP. Nonetheless, Jung (1999) shows that Presupposition VCP is the most basic formulation of the principle,[11] and it is quite plausible that Russell had this formulation in mind all along. Also, Gödel (1944) points out that Presupposition VCP (as well as Involvement VCP) is a more plausible principle than Definability VCP. In the reminder of this section, then, I will focus on Presupposition VCP, and in closing I will provide further motivation for viewing Definability VCP just as a formulation of it.

In order to spell out Presupposition VCP, the notion of presupposition has to be tackled. Nevertheless, such a notion is admittedly rather vague. In the literature, different notions of presupposition may be found.[12] Each of them gives rise to a

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[6]See, for instance, Hellman (2004).

[7]See, for instance, Jung (1999) for a detailed survey on them.

[8]Russell (1908, 63).

[9]Russell B. and Whitehead A., *Principia Mathematica*, vol. 1, p. 37.

[10]Russell B. and Whitehead A., *Principia Mathematica*, vol. 1, p. 63.

[11]In fact, Jung (1999, 69–74) shows that both Definability and Involvement VCPs follow from Presupposition VCP.

[12]See, for instance, Fine (1995) and Correia (2008). See also Linnebo (forthcoming). Hellman (2004) claims that there is also an epistemic justification for predicativism, namely, that rational beliefs in mathematics extend only to predicatively definable objects. Epistemic predicativism is indeed a possible interpretation of Russell's VCP. Nevertheless, I will not investigate it in this article, though it is worth mentioning that epistemic VCP may be interesting to anyone working on

formulation of Presupposition VCP. These notions of presupposition, and the related formulations of VCP, may be formulated as follows:

Presupposing for existence: An entity **A** existentially presupposes an entity **B** just in case **A** cannot exist unless **B** does. Consider, for instance, sets. A non-empty set x exists only if its members exist.[13]

Ontological VCP: No entity can presuppose for its existence a totality it belongs to.

Presupposing for essence: An entity **A** essentially presupposes an entity **B** just in case **A** cannot be what it is unless **B** is. For instance, what set x is presupposes what members it contains.[14]

Metaphysical VCP: No entity can essentially presuppose a totality it belongs to.

Presupposing for specification: An entity **A** presupposes an entity **B** for its specification just in case **A** cannot be specified unless **B** is. For instance, whether we can specify a set x presupposes that we are able to specify its members.[15]

Specifiability VCP: No entity can presuppose for its specification a totality it belongs to.

Both essential presupposition and presupposition for specification are tightly connected with identity (or at least equivalence). It may be argued that, after all, they come down to the same notion. Nevertheless, even though they both indeed involve some requirements for identity, I believe they should be carefully separated. We may

some Platonist response to Benacerraf's dilemma. Benacerraf's dilemma claims that the Platonist has to face severe epistemic problems as for the accessibility of the entities she takes to exist mind-independently. To this extent, epistemic predicativism seems to support Benacerraf's view.

[13]On existential presupposition, see Fine (1995) and Correia (2008).

[14]On essential presupposition, see Fine (1995) and Correia (2008).

[15]See, for instance, Linnebo (forthcoming), which is nevertheless focused on investigating *first*-order impredicativity in abstraction principles. An abstraction principle has the form $\S F = \S G \leftrightarrow R_E(F, G)$, where § is an abstraction operator mapping a given collection of entities into a collection of entities of different sort and R_E is an equivalence relation. Well-known examples are the so-called Hume's Principle and Basic Law V. The impredicativity Linnebo investigates concerns the fact that the entities introduced on the left-hand side of the biconditional can be among the values of the first-order variables appearing on the right-hand side (consider, for instance, Basic Law V: $\{x : Fx\} = \{x : Gx\} \leftrightarrow \forall x(Fx \leftrightarrow Gx)$). Thus, if abstraction principles serve the purpose of *individuating* or *specifying* the entities introduced on the left-hand side by the equivalence relation on the right-hand side, their impredicativity would imply that the entities the terms on the left-hand side refer to are individuated or specified on the basis of a totality they belong to. Nevertheless, I am here analysing the impredicativity underlying second-order logic, which originates from the comprehension axiom and concerns the specification of the second-order entities the left-hand side of the biconditional refers to.

in fact argue that there are examples of essential presupposition that do not apply to specifiability. Consider, for instance, the power set of ω. We may argue that such a set is the set it is, exactly because of the members it contains. Nevertheless, not all subsets of ω are specifiable. To this extent, should the power set of ω be specifiable at all, it would not be so on the basis of the presupposition of the specifiability of its members, even though it would still be the set it is because of the members it contains.

There is a further notion of presupposition I would like to mention, namely, referential presupposition. It is worth mentioning that referential presupposition, and its related VCP, will be the main focus of the paper. It may be surprising that in this section I set it aside right after I state it, but I will profusely come back to it in Sects. 4.4 and 4.5. All that is in between is propaedeutic to a motivation for referential presupposition and its related VCP, and their relation to the issue of second-order predicativity in a Fregean setting.

Presupposing for reference: The possibility of referring to an entity **A** presupposes the possibility of referring to an entity **B** just in case the possibility of referring to **A** makes ineliminable use of the possibility of referring to **B**. Let me mention one example to illustrate what I have in mind. Consider tropes. If I say 'Your smile is like no other', I am apparently referring to your smile. Now, could I refer to it, in case I were not able to refer to *you* in the first place? Hardly so. To this extent, the possibility of referring to your smile presupposes the possibility of referring to you.

Referential VCP: No entity can presuppose for its reference to a totality it belongs to.

In the opening of this section, I mentioned the three formulations of VCP available in Russell's writings, and in passing I mentioned Gödel's article on Russell's mathematical logic. In that article, Gödel claims that it is Definability VCP that is of interest in mathematics, since it disallows impredicative definitions and thus undermines the derivation of most of classical mathematics[16]:

> (...) the vicious circle principle (...) applies only if the entities involved are constructed by us. In this case, there clearly must exist a definition (namely the description of a construction) which does not refer to a totality to which the object defined belongs, because the construction of a thing can certainly not be based on a totality of things to which the thing to be constructed itself belongs. If, however, it is a question of objects that exist independently of our constructions, there is nothing in the least absurd in the existence of totalities containing members, which can be described (i.e., uniquely characterized) only by reference to this totality.[17]

A clear example of this is the definition of the set of the natural numbers, which is provided in terms of all inductive sets. If impredicative definitions are claimed to

[16]See Gödel (1944, 455–459).

[17]Gödel (1944, 456).

be impermissible, such a definition cannot be recovered. Nevertheless, according to Quine (1969, 242–3, emphasis added):

> No question of legitimacy can arise in connection with definition, so long as a mechanical procedure is provided for expanding the new notation in all cases uniquely into old notation. Now what Poincaré criticized is not the definition of some special symbol as short for '$\{x : x \notin x\}$', but rather the very assumption of the existence of a class y fulfilling '$(x)(x \in y \leftrightarrow x \notin x)$'. We shall do better to speak not of impredicative definitions but of *impredicative specification* of classes, and, what is the crux of the matter, impredicative assumptions of class existence.

Here, definability plays no role at all, since, independently of the possible underlying ontological assumptions concerning mathematical entities, definitions are just a matter of introducing a new notation that is always eliminable in terms of the old one. The best way to see this is to consider comprehension axioms, which are actually what Quine seems to have in mind in the quotation above. Comprehension axioms are not definitions, but they indeed provide a means to specify entities through predicative or impredicative formulæ. But if definitions are unproblematic, independently of the underlying ontology, how are we to make sense of Gödel's objection to Definability VCP? One possible way out is the one envisaged by Quine in the previous quotation, namely, to leave definability out of the picture and rather consider *specifiability* and thus rephrase Definability VCP as Specifiability VCP.[18] To this extent, specifiability may be considered as the rather general notion of 'singling out' an entity from other entities. A further formulation of VCP can be provided in terms of *individuation*. Interestingly, Quine (1985, 166–7) disallows impredicative individuation. In connection with individuation of events, he says:

> For my own, I welcome impredicative definitions. I have remarked that there is nothing wrong with identifying the most typical Yale man by averaging measurements and tests of all Yale men including him. But now we observe that impredicative definitions are no good for individuation. Here a difference between the impredicative and the predicative emerges which is significant quite apart from any constructivist proclivities. We can define impredicatively but we cannot individuate impredicatively.

This quotation calls for a distinction between impredicative individuation as impermissible, as opposed to impredicative specification as permissible. What would be Quine's reasons for allowing impredicative uses of the latter, while banning impredicative uses of the former? I take it that in Quine's view individuation goes through some identity criterion. Impredicative individuation has to be disallowed since if individuation presupposes identity, then the individuation of an object must not be performed by any identity statements concerning that very object. On the other hand, impredicative specification might be allowed since it is a weaker notion than individuation. In particular, it does not require an identity criterion: for instance, in the case of impredicative second-order comprehension for concepts, the singling out of a concept by a condition does not require, at least *prima facie*, an identity criterion for concepts. Though I acknowledge the different import of

[18] See also Jung (1999, 59) on this point.

impredicative individuation, on the one hand, and impredicative specification, on the other, I believe that none of them is permissible, at least as far as concepts are concerned. In Sect. 4.3, I provide reasons for equating specification of concepts to reference to them, on the grounds that, because of their intensional nature, their specification can only be performed by language. To this extent, specifiability of a concept and reference to it are equivalent notions. By bearing on this equivalence, in Sects. 4.4 and 4.5 I will provide motivations for banning, *pace* Quine, impredicative specification of concepts on the basis of Referential VCP.

4.2 Gödel's Criticism of VCP

VCP was subjected to some radical criticisms, especially by Gödel (1944) but also by Quine and Ramsey. So, it is quite important to go through these criticisms, and try to provide some counter-objection, to the aim of making the use of predicativity compatible with Frege's view on concepts. In this section, I will sum up Gödel's (or some broadly Gödelian) criticism of VCP.

Gödel's main objection to Russell's VCP is based on some philosophical considerations he puts forward: if a Platonist view of mathematical entities holds, then one is entitled to impredicativity, since VCP in a Platonist perspective is either easily rejected or trivial; on the other hand, if VCP holds and by this one has solid motivation for predicativity, one is also committed to some form of mathematical Constructivism. Thus, in Gödel's view, the divide between predicativity and impredicativity, and the acceptance or the rejection of VCP, ultimately hinges on ontology.

Let us see how Gödel's argument may be reconstructed. Consider a Platonist attitude towards the existence of the entities of a certain domain. To this extent, these entities exist mind-independently, and none of them presupposes the totality it belongs to for its existence, because they are just there from the start. Consequently, in a Platonist framework, Ontological VCP is trivial:

> Such a state of affairs[19] would not even contradict (...) the third form[20] if 'presuppose' means 'presuppose for the existence'.[21]

Let us consider now Specifiability VCP, which, under the revision of Gödel's objections to Definability VCP, is the real target of a Gödelian objection. From a Platonist perspective, Specifiability VCP is false, since 'there is nothing in the least absurd in the existence of totalities containing members, which can be described

[19]That is, Platonism.

[20]That is, Presupposition VCP.

[21]Gödel (1944, 456). The same argument goes as for Metaphysical VCP. Consider, for instance, sets. A set x is the set it is because of the members it contains, not because of the totality it belongs to. Thus, we may argue against Metaphysical VCP that it is trivial as much as Ontological VCP, under a Platonist stance of the universe of sets. In such a view, in fact, Metaphysical VCP would hold by default. From now on, then, I will not consider Metaphysical VCP anymore.

(i.e., uniquely characterized) only by reference to this totality'.[22] And in fact Quine's quotation from above continues:

> And what now of the vicious circle? (. . .) [I]mpredicative specification of classes (. . .) is hardly a procedure to look askance at, except as one is pressed by the paradoxes to look askance at something or other. For we are not to view classes literally as created through being specified - hence as dated one by one, and as increasing in number with the passage of time. (. . .) The doctrine of classes is rather that they are there from the start. This being so, there is no evident fallacy in impredicative specification. It is reasonable to single out a desidered class by citing any trait of it, even though we chance thereby to quantify over it along with everything else in the universe. Impredicative specification is not visibly more vicious than singling out an individual as the most typical Yale man on the basis of averages of Yale scores including himself.[23]

Finally, if the Platonist view on the entities of a given domain holds, also Referential VCP is false. Gödel, in fact, claims that

> [i]t is demonstrable that the formalism of classical mathematics does not satisfy the vicious circle principle in its first form,[24] since the axioms imply the existence of real numbers definable in this formalism only by reference to all real numbers. Since classical mathematics can be built up on the basis of *Principia* (including the axiom of reducibility), it follows that even *Principia* (. . .) does not satisfy the vicious circle principle in its first form (. . .).
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> I would consider this rather as a proof that the vicious circle principle is false than that classical mathematics is false, and this is indeed plausible on its own account. For, first of all one may, on good grounds, deny that reference to a totality necessarily implies reference to all single elements of it or, in other words, that 'all' means the same as an infinite logical conjunction.[25]

If Platonism is the best ontological account of classical mathematics on the market and we hold it, then we may indeed quantify over all the entities of a totality without presupposing reference to each and every one of them. Consider, for instance, real numbers. If we help ourselves to a Platonist view of the reals, we think of their collection as a totality of some sort. But quantifying over the totality of the real numbers does not imply that we have as many individual constants as the reals (and in fact we have not) and that we can attach referents to them.

Even more so, a further criticism of Referential VCP may target the principle on the basis of the relation between reference, on the one hand, and existence and specifiability, on the other. In fact, without existence, we may not specify, let alone refer. But even if we grant existence with no specification, can we actually refer to an entity without being able to single it out by some means? According to a common intuition, in order to refer by a term, the referent of that term must exist and it has to be, at least, specifiable in some way. In particular, Referential VCP may be criticised from a further angle, which is connected with its relation to Ontological

[22] Gödel (1944, 456).

[23] Quine (1969, 242–3).

[24] That is, Definability VCP.

[25] Gödel (1944, 454–455).

and Specifiability VCPs. It may be argued, in fact, that Referential VCP is hardly news as compared to Ontological and Specifiability VCPs. These objections will be addressed in a few sections.

4.3 Predicativity and Presupposition Versus Frege's Platonist Logicism About Concepts

If Gödel's argument holds, *prima facie* none of the aforementioned formulations of VCP is compatible with Frege's Platonist view on concepts. Accordingly, if we try to motivate a predicative restriction on second-order comprehension in a Fregean setting and we take predicativity to be motivated by some notion of presupposition, the consistent predicative fragments of *Grundgesetze* do *not* seem to provide a philosophical basis to revive Frege's foundational programme.

Previously, I claimed that specifiability and reference may be seen as distinct though connected notions. As for concepts, however, the situation is different. Whether the underlying ontological assumption for either of those principles is Platonism or Constructivism, Specifiability VCP and Referential VCP boil down to the very same principle. Specification means to single out a concept from the other concepts. But then again, how can we specify an intensional entity such as a concept? Given their intensionality, a rather natural way to specify a concept is via some linguistic expression expressing it, like a *predicate*. To this extent, the specification of a concept comes down to providing a way to refer to it via an appropriate linguistic means. To this, it may be objected that a possible alternative way of specifying a concept is via thought. For instance, in order to specify the concept 'being red', we can picture red things or maybe just think 'red'. In this case, specifiability presupposition would not come down to referential presupposition. Nevertheless, consider that I am interested in second-order languages in a Fregean setting and in Frege's view on concepts. In Frege's perspective, mental representations are to be avoided in the philosophical investigation on mathematics and logic. The Fregean, thus, would not be troubled with the above objection to the equivalence between specifiability presupposition and referential presupposition for concepts. In a Fregean perspective, it is quite natural to view concepts as specified by language.[26] But then again, to this extent, specifiability presupposition and referential presupposition turn out to be equivalent notions. If then a Platonist view of Fregean concepts holds, Specifiability and Referential VCPs will be both rejected, on the grounds that on that view Referential VCP is false.

There seems to be a further difficulty with predicativity in a Fregean perspective. In Frege's view, concepts are logical entities whose existence is guaranteed by the laws of thought alone, i.e. the laws of the True. According to Frege, these laws are

[26]Consider, for instance, the Fregean view that predicates, i.e. the linguistic items standing for concepts, are obtained by extrapolation of singular terms from sentences.

rigorously formalised in classical logic as it is presented in the *Begriffsschrift* as a higher-order logical system. Frege is not just a Platonist about concepts; he is a *logicist* Platonist: if we grant Frege that the very laws of thought are exhaustively captured by higher-order logic, then in his view higher-order logic is *the* logic of the laws of thought. In a Fregean perspective, conceptual Platonism is not just an ontological stance on concepts independent of his logicism; it is instead strongly motivated by, or at the very least deeply connected with, his logicism. In this view, unrestricted comprehension for concepts is both motivated by Frege's Platonist view on concepts, but even more so it entails, given his logicism, that unrestricted comprehension is a fundamental tool to capture the laws of the True.

In what follows, I will argue that there is a motivated way to make the consistent predicative fragment of *Grundgesetze* compatible with Frege's view. This motivation hinges on a revision of Gödel's dichotomy between predicativity and impredicativity and their philosophical implications. To this aim, in the following sections, I will propose to detach the acceptance of predicativity from ontological considerations and to connect it to considerations concerning logical and mathematical reasoning. On these grounds, I will finally claim that, in the light of these considerations, it is possible to make Frege's Platonist logicism about concepts compatible with a predicative revision of his foundational programme. A word of caution is here needed, though. I will argue that Frege's conceptual Platonism is indeed compatible with second-order predicativity. As for Frege's logicism, it will turn out that Frege's claim that arithmetic can be reduced to logic alone has to be revised. Nevertheless, this revision will not necessarily make Frege's original view a complete nonstarter. It will turn out that a predicative restriction on second-order comprehension will indeed provide a starting point for a revision of Frege's logicism. Before that, though, I will have to take a slight detour through the notion of *arbitrary reference*.

4.4 TAR

Let us recall the formulation of referential presupposition from above: the possibility of referring to an entity **A** presupposes the possibility of referring to an entity **B** just in case the possibility of referring to **A** makes ineliminable use of the possibility of referring to **B**. First of all, in order to make sense of it and provide some counter-objections to Gödel's arguments against Referential VCP, the notion of *possibility of reference* needs to be clarified.

According to Martino (2001, 2004), the possibility of directly referring, at least ideally, to any entity of a universe of discourse is presupposed both by logical and mathematical reasoning, even when non-denumerable domains are concerned. Martino (2001) labels this claim the *Thesis of Arbitrary Reference* (TAR).[27] Such

[27]More suggestively, Martino (2004) calls this claim the *Thesis of Ideal Reference*. In what follows, it will become clear why.

a possibility of direct reference is very well expressed by the crucial role *arbitrary reference* plays both in formal and informal reasoning. Its cruciality lies in that arbitrary reference exhibits two different logical features that make it essential for performing proofs, i.e. *arbitrariness* and *determinacy*. Through arbitrary reference, we may consider *any* object a of a universe of discourse. Consequently, the arguments about a retain their general validity. At the same time, though, within the arguments about it, 'a' is required to denote a determinate object, distinct from all the other objects in the domain it belongs to. Typically, when a derivation is completed, we may detach 'a' from the individual we attached it to and reuse it in a different derivation. But within a derivation on a, 'a' has to refer to a determinate individual. The same argument holds, *mutatis mutandis*, also for reference to concepts via second-order free variables. In fact, every argument in favour of the genuine referentiality of arbitrary reference put forward in this section applies both to first- and second-order arbitrary reference.

In order to motivate TAR, an account of the *genuine* referentiality of arbitrary reference and its *directness* has to be provided. It may be argued, in fact, that arbitrary reference is not genuinely referential, since parameters and free variables do not refer at all.[28] Evidence in favour of the genuine referentiality of arbitrary reference may be found in Breckenridge and Magidor (2012) and Martino (2001, 2004). In this section, I will provide a more general argument to this aim. My claim is that the soundness of arguments in mathematical and logical reasoning requires the underlying assumption of the genuine referentiality of arbitrary reference. The relation between soundness and referentiality will be accounted for in terms of sameness and determinacy of reference. Though my argument will be spelled out in terms of reference to individuals, nothing will prevent its application to reference to concepts as well.

Usually, a parameter 'a' is used to refer to the *same* individual a within a derivation on a. A crucial reason for this is to be found in the requirement of soundness we want to impose on some valid argument schemas. If sameness of reference were not a basic ingredient of derivations, soundness would be in jeopardy.[29] Consider the rule of existential elimination in natural deduction. When we pass from a premise of the form $\exists x\phi x$ to the auxiliary assumption $\phi(a)$, 'a' has to be an unused parameter, or at least it has not to appear in any of the assumptions which $\exists x\phi x$ depends upon. Consider now the following (invalid) deduction:

(1) $\exists x Hx$ $\quad \mathscr{A}$
(2) $\exists x \neg Hx$ $\quad \mathscr{A}$
(3) Ha $\quad \mathscr{A}$
(4) $\neg Ha$ $\quad \mathscr{A}$
(5) $Ha \wedge \neg Ha$ $\quad$ 3,4 intr. $\wedge$

[28] See, for instance, Pettigrew (2008).

[29] A further argument to this aim, from the uniformity of substitution of predicate and individual letters in argument schemas, may be found in Boccuni (2010).

Invalidity stems out from that, in eliminating the existential quantifiers, respectively, from (1) and (2), we use the very same parameter in (3) and (4).[30] Say that H is the property of being even and x varies over the natural numbers: (1) and (2) say, respectively, that there is at least a number which is even and there is at least a number which is not. Both these sentences are true in the standard model of Peano arithmetic. Nevertheless, if we use the same parameter to perform existential elimination in the derivation above, in (3) and (4) we, respectively, say that a number is even and that *the very same number* is not, from which the contradiction in (5) arises. For this reason, using an already used parameter in (4) cannot be allowed.

In order to explain the invalidity of the derivation (1)–(5), 'a' must be referring to the same, though arbitrary, individual both in lines (3) and (4). Thus, in order to achieve soundness in the previous example, in line (4) we have to use a different parameter than 'a', because we need to express that a *different* individual than a is $\neg H$ within the same derivation, in accordance with the restrictions imposed on universal introduction and existential elimination. But then again, in order to distinguish between a and any other arbitrary individual that is $\neg H$, we have to assume that a is a *determinate*, though arbitrary, individual of the domain. A similar argument can be provided as for second-order existential elimination. Say, for instance, that you have two second-order existential assumptions such as $\exists F \phi F$ and $\exists F \neg \phi F$, which are both true in second-order arithmetic, for ϕ being the formula 'is finite'. Indeed, there are finite concepts in second-order arithmetic as there are infinite ones, so the two existential assumptions are true. But now if we use the same parameter to eliminate both of them, we incur in the contradictory conclusion that the same concept satisfies a formula and its negation.

The motivation for this requirement is very nicely explained by Suppes:

> (...) ambiguous names,[31] like all names, cannot be used indiscriminately. The person who calls a loved one by the name of a *former* loved one is quickly made aware of this. (...) Such a happy-go-lucky naming process is bound to lead to error, just as we could infer a false conclusion from true facts about two individuals named 'Fred Smith' if we did not somehow devise a notational device for distinguishing which Fred Smith was being referred to in any given statement. The restriction which we impose to stop such invalid arguments is to require that when we introduce by existential specification an ambiguous name in a derivation, that name has not previously been used in the derivation.[32]

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[30] See Suppes (1999, 82) for this example.

[31] That is, parameters like 'a'.

[32] Suppes (1999, 82). Of course, it is not always the case that using the same parameter leads to invalidity nor that different parameters always have to refer to different entities. For instance, consider using 'a' for eliminating the quantifiers both from $\forall x F x$ and $\forall x G x$ in the same derivation, where x varies over the natural numbers and both formulæ have a model in Peano arithmetic. Or consider using 'a' and 'b' for eliminating, respectively, the first quantifier and the second, where a and b may be the same individual. In none of these examples, sameness of reference seems to lead to invalidity, but such an innocuousness does not by itself speak against the genuine referentiality of 'a' or the importance of sameness of reference to derivations. It rather testifies that there are contexts in which the co-referentiality of all the occurrences of 'a' (or of 'a' and 'b', for that matter) is not problematic.

The reasons for restricting the rules of introduction and elimination of quantifiers in natural deduction are semantic: in derivations, we perform a semantic reasoning that we want to be captured by deductive rules and restrictions on them. Such a reasoning is crucially based on sameness and determinacy of reference of parameters. But then again, in order to make sense of sameness and determinacy and consequently of the requirements we impose on deductive rules for the sake of soundness, we have to assume the genuine referentiality of parameters. Genuine referentiality is a necessary condition for soundness. This relation can be highlighted by investigating the role that sameness and determinacy of arbitrary reference have in derivations. In fact, if 'a' were not referential at all, how could we account for a being the same individual throughout an argument? Those who support the non-referentiality of arbitrary reference should provide some argument for explaining how formal and informal reasoning functions in the way it does (for instance, by certain constraints on introduction and elimination of quantifiers).[33]

A further issue concerns the connection of arbitrary reference with existence. In fact, arbitrary reference may be used within derivations in which one of the assumptions leads to contradiction. Consider the example 'Let r be the set of all the sets that are not members of themselves'. It is well known that a contradiction is derived from the assumption of the existence of the Russell set. We readily conclude that r does not exist. But then again 'r', under the assumption of the genuine referentiality of arbitrary reference, has to genuinely refer to an (arbitrary) individual throughout the derivation on it. Is then 'r' genuinely referential after all, i.e. in the derivation of Russell's paradox does it refer to an individual that does not exist? Reference in arguments by *reductio ad absurdum* may be explained as follows. Throughout the derivation leading to the contradiction, we temporarily assign an individual as the value of 'r'. But then we find out that the individual we picked does not, and in fact cannot, satisfy the condition 'set of all the non-self-membered sets'. Since the individual chosen is an arbitrary individual, that the Russell condition is not satisfied holds of all the individuals of the domain. But in the process, 'r' has indeed been referential: it referred to an arbitrary individual that does not satisfy the Russell condition. The derivation of the contradiction does not just say that there is nothing 'r' refers to, but that for every assignment of a referent to 'r', r does not satisfy the defining formula.

The directness of arbitrary reference may be appreciated by considering its relation with quantification. Consider once again the rule of existential elimination. As Martino (2004) points out, the possibility of passing from a purely existential assumption such as $\exists x \phi x$ to the consideration of an arbitrary object a such that ϕa

[33] A further issue concerns the semantics that better captures the genuine referentiality of arbitrary reference. To the best of my knowledge, there are two competing options on the market: Kit Fine's view according to which arbitrariness is a property of some special kind of objects, namely, those referred to by parameters, and an epistemic view, championed by Breckenridge and Magidor (2012) and Martino (2001, 2004), according to which arbitrariness is an epistemic feature of our reasoning – a is a determinate individual, and 'a' determinately refers to it, but we do not know *which* individual a is.

is guaranteed by the rule of elimination of the existential quantifier which allows to substitute the given existential assumption with the auxiliary assumption ϕa. If the rules of inference that govern the use of the logical constants in natural deduction are justified by the meaning of the constants themselves, the meaning of the existential quantifier presupposes the possibility of singularly referring, at least ideally, to any individual of a domain, and consequently existential quantification logically presupposes such a possibility of reference.[34] Thus, before we simultaneously consider several entities through quantification, we are required to be able to refer to each of them, at least ideally: quantification logically presupposes the ideal possibility of referring to each and every element of a domain, before we consider those elements through generalisation.[35] In this perspective, no individual can be referred to only on the basis of reference to a totality it belongs to, as an ideal way to directly refer to it is required in order not to violate TAR: Referential VCP is a corollary of TAR.[36]

4.5 Gödelian Criticism to Referential VCP

Let us recall that Referential VCP may be criticised with respect to two issues: (1) the fact that quantification does not imply an infinite conjunction or disjunction and (2) Referential VCP being hardly news over Ontological and Specifiability VCPs. In this section, I will tackle these two issues.

According to TAR, quantification presupposes arbitrary reference: if the meaning of the quantifiers is governed by their rules of introduction and elimination in natural deduction, then quantification presupposes the possibility of reference, at least in principle. In this respect, under certain restrictions, we may pass from the consideration of an arbitrary individual a such that ϕ to the generalisation that every x is such that ϕ.[37] In this regard, one is reminded of Gödel's objection to quantification as presupposing reference to each and every member of a domain. Nevertheless, Gödel's objection apparently does not take into consideration the cruciality of arbitrary reference, but is limited to considering that quantification

[34] Analogously as far as the rule of introduction for universal quantification is concerned. See Martino (2004, 110).

[35] For further justification and applications of arbitrary reference, see also Breckenridge and Magidor (2012).

[36] See Martino (2004, 119). Notice that Referential VCP follows from TAR also when non-denumerable domains are concerned. Even though a language cannot display non-denumerably many names, TAR still holds, as the ideal possibility of directly referring to each and every individual in a non-denumerable domain may be performed via arbitrary reference, as in the case of, e.g. 'let a be an arbitrary real number'. Also, Martino (2001, 2004) provides a special semantics, the acts of choice semantics, in order to make sense of how the directness of arbitrary reference should work.

[37] And analogously as for existential quantification.

does not imply infinite conjunctions or disjunctions of formulæ where the bound variables is substituted by a constant. Referential VCP as implied by TAR, though, does not necessarily concern reference through constants, either first- or second-order, but at the very least concerns arbitrary reference. The restriction to arbitrary reference is really all that is needed to make a case for predicativity concerning Fregean concepts, because the predicativity that is employed to that aim is motivated by a reflection on the relation between quantification and the ideal possibility of reference. To this extent, TAR *does* imply that quantification involves arbitrary reference, but it does *not* imply that quantification involves reference to each member of a totality by an infinite conjunction or disjunction.[38] According to TAR, quantification implies reference to *any* individual of that totality, by arbitrary reference: if we refer to a totality, then we must be able, at least ideally, to refer to each and every member of it through arbitrary reference prior to the consideration of the totality itself. In this respect, the first objection to Referential VCP is a nonstarter.

What now of the criticism that, given the relation between reference, on the one hand, and existence and specifiability on the other, Referential VCP is hardly news over Ontological and Specifiability VCP? I take this objection to come down to whether there are cases in which specifiability and ontological presupposition do not hold, whereas referential presupposition does. As for Specifiability VCP, arbitrary reference is such that specification and reference are detached. In order to fix the reference of 'a', we do not need to be able to specify a. In a sense, our ignorance of *which* individual a is justifies the possibility of introducing universal quantification in reasoning (given the appropriate restrictions on universal introduction). For instance, we are not required to specify an arbitrary real number r, in order to genuinely refer to it via an arbitrary name 'r'. Furthermore, there are also cases in which ontological presupposition does not hold, whereas referential presupposition does. Consider, for instance, the empty set $\{x : x \neq x\}$. Its existence does not depend upon the existence of non-self-identical individuals, as in fact there are none. Nevertheless, the possibility of referring to $\{x : x \neq x\}$ seems to depend upon the possibility of referring to the individuals of the domain through x in the formula $x \neq x$, if only to acknowledge that no individual is non-self-identical. This holds on the grounds of TAR and in particular on the grounds that parameters and free variables are genuinely referential. Consider, furthermore, a view on concepts

[38]What if we wanted to extend Referential VCP to reference via individual constants? In this case, the relation between reference as involved in Referential VCP and arbitrary reference in TAR should be further motivated. Brandom (1996) suggests a way to deal with this issue. While arbitrary reference – which he calls 'parametrical' – and reference via individual constants are both genuinely referential, we may explain their relation by saying that (i) either they convey different notions of reference (ii) or arbitrary reference embodies the only notion of reference there is, and either reference via individual constants is built up from it or it is reducible to it. I discuss option (i) in the main text. In the case of (ii), arbitrary reference would be primitive, so Referential VCP would concern it by default and would easily follow from TAR. I sincerely thank an anonymous reviewer for pressing me on this issue.

quite like Frege's. In such a view, the existence of concepts does not depend upon the existence of predicates expressing them. Nevertheless, since concepts can be specified and referred to only via language, the possibility of referring to a concept depends upon the possibility of referring to a predicate expressing it. At least a few cases, in which referential presupposition holds, are not covered by ontological presupposition. To this extent, referential presupposition and Referential VCP are indeed news.

4.5.1 Frege's Platonist Logicism About Concepts

I shall now turn to the issue of whether there is a way to make the use of predicativity compatible with Frege's view on concepts. I mentioned earlier that concepts are intensional entities and, as such, the only way to specify and refer to them is via language, i.e. through formulæ which express them. I also claimed that specification of and reference to concepts boil down to the very same notion. Consider the impredicative comprehension principle in second-order logic: $\exists F \forall x (Fx \leftrightarrow \phi)$. Considering the concept F means considering the formula ϕ, under the intended interpretation. But then again, the concept F cannot be referred to by quantification over the domain it belongs to, since quantification would require that we are able to refer to F prior to specifying it through a predicate. In ϕ, therefore, there cannot be any bound second-order variables, on pain of violating TAR and Referential VCP.

Now, what about ontology and in particular Frege's Platonism on concepts? The relation of logical presupposition of quantification from arbitrary reference provides a motivation for formulating Referential VCP *independently* of the underlying ontology. Such a relation in fact holds no matter what ontological preferences one has. It is just a matter of a relation between two crucial logical tools, i.e. (arbitrary) reference and quantification, and as such it is impervious to ontological considerations. One thing is the ontological assumptions underlying the consideration of a domain of entities; another matter is the consideration of how reference works. The underlying ontological assumptions, though inevitably connected with reference, are not the only considerations playing a role in the referential picture. TAR provides independent reasons for the dependence of quantification on reference, to which the underlying ontological preferences are irrelevant. TAR is a claim concerning how language works *logically*: ontology entering the picture is merely accidental. To this extent, holding a Platonist stance towards the existence of the entities of a certain domain does not affect Referential VCP. Given that Referential VCP is based on TAR and TAR just embodies a claim of logical dependence of quantification from arbitrary reference, a rejection of Referential VCP does not follow from a Platonist attitude towards existence. To this extent, the Fregean may still be a Platonist about concepts, even though she holds Referential VCP. Moreover, all concepts may exist: the only restriction TAR imposes is on the concepts which are expressible within a language. These latter may well exist, depending on the ontological preferences one has; as for the ones which are not expressible, they too may well exist, even though the language cannot talk about them.

What about logicism, finally? As for Frege's idea that impredicative higher-order logic captures the very laws of thought, it has to be drastically revised. Since TAR is a claim concerning logic and in particular the relation between arbitrary reference and quantification, I here hold the strong claim that logic as the body of the laws of thought should be inherently predicative. Frege's logicism has to be restricted to the extent that it finally takes into account the relation of logical presupposition of quantification from arbitrary reference. Is this bad news to the Fregean? Yes and no. It is bad news to the extent that a predicative second-order fragment of Frege's *Grundgesetze* like Heck (1996) cannot interpret more than Robinson arithmetic Q, which is a rather poor, though not trivial, fragment of mathematics, especially as compared to Frege's original attempt to found (full second-order) arithmetic on pure logic. In this respect, the objection from mathematical strength to a predicative revision of Frege's logicist programme hits the target.

The limitation that TAR imposes on the underlying logic, though, may still be good news to the Fregean. First of all, TAR provides some independent motivation, other than the mere search for consistency, for restricting the underlying second-order logic. Secondly, given its compatibility with a Platonist stance towards the existence of concepts, TAR provides some room to manœuvre for claiming that the predicative fragments of *Grundgesetze* may indeed provide a consistent formal core for reviving Frege's foundational programme. Most of all, the limitation may just imply that higher-order logic *as the Fregean might interpret it*, namely, as a theory of concepts, has to be revised, but there may be further extensions of Heck (1996) that may both count as logic and recover second-order Peano arithmetic.[39]

4.6 Concluding Remarks

In this paper, I tried to show that the consistent predicative fragments of *Grundgesetze*, though *prima facie* at odds with Frege's philosophical stance on concepts, may indeed provide some philosophical grounds for revising Frege's programme. I proceeded through an independent reassessment of the very notion of predicativity, not on ontological grounds as Gödel does, but on some logical basis. It turns out that this reassessment is compatible with Frege's Platonist view on the existence of concepts but requires a drastic revision of his logicist view. In spite of this pessimistic conclusion, I suggest to look on the bright side of the matter: once both consistency and Frege's conceptual Platonism are secured by TAR on independent philosophical grounds, the predicative fragments of *Grundgesetze* may provide a stepping-stone for further investigations towards a (possibly partial) revival of Frege's philosophical views on mathematics.

[39] See, for instance, Boccuni (2010).

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