Chapter 2
Plane and Solid Geometry: A Note on Purity of Methods

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2.1 Introduction

Traditional geometry concerns itself with planimetric and stereometric considerations, which are at the root of the division between plane and solid geometry. When one raises the problem of the relationship between these two areas, one encounters epistemological, ontological, semantical, and methodological problems. In addition, other issues related to psychology and pedagogy of mathematics emerge naturally. In this note (based on Arana and Mancosu 2012), we will focus on a methodological aspect: purity of methods (see Detlefsen 2008 and Detlefsen and Arana 2011). After a few historical remarks concerning the role played by solid geometry in the development of plane geometry, we will move on to the analysis of a specific case, Desargues' theorem on the plane (which we will call "Desargues' plane theorem"). This theorem was proved by Desargues by making use of metric notions (congruence principles) that were key to a theorem that played a central role in the demonstration, namely, Menelaus' theorem. However, the development of geometry in the nineteenth century led to the analysis of the foundations of projective geometry and to the attempt to eliminate as much as possible from this discipline non-projective notions such as congruence or measure. Desargues' theorem played a crucial role in this type of investigation. A purely projective proof of this theorem had already been given in 1822 by Poncelet. Poncelet had shown how a version of Desargues' theorem in space (we will call it "Desargues' solid theorem") provided, as a simple corollary,

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a projective demonstration of Desargues' plane theorem. The appeal to congruence in Desargues' original proof for the plane theorem was thus eliminated through the introduction of spatial notions. One can however ask whether this appeal to space is legitimate and necessary. The legitimacy question originates from considerations related to purity of methods. The issue about necessity is tied to logical considerations. One had to wait until the works of Peano and Hilbert to obtain an (affirmative) answer to the latter question. Moreover, these results are at the basis of a more articulate discussion of the legitimacy problem (namely, the purity problem). These considerations will be developed in the final part of the note.

2.2 Historical Notes on the Relationship Between Plane and Solid Geometry

In ancient geometry, we encounter few interesting applications of solid geometry to plane geometry (of course, solid geometry requires plane geometry). Euclid's *Elements* present us with a sharp separation between plane geometry and solid geometry (with the latter relegated to the last books of the *Elements*), a division that will have a lasting impact on the presentation of elementary geometry until the end of the nineteenth century. There are however, already in Greek times, some advanced directions of research in which techniques of solid geometry are applied to the study of problems in plane geometry. We can mention, for instance, the quadrature of the circle provided by (Pappus 1876–1878) which is obtained generating the quadratrix curve on the plane through a projection of the cylindrical helix. It is also important to note that the distinction between plane, solid, and linear problems given by Pappus is orthogonal to that between plane and solid geometry. Pappus' taxonomy concerns the types of curves required for the solution of problems (line and circle for plane problems, conic sections for solid problems, and "more complex" curves for the linear problems). Euclid's solid geometry ends up classified as "plane" in Pappus' taxonomy, and conversely, problems stated in plane geometry, such as the trisection of an arbitrary angle, are classified as "solid." While Pappus criticizes the use of curves that do not correspond to the nature of the problem (such as the use of conic sections for solving "plane" problems), we are not aware that any Greek mathematician (or philosopher) ever raised objections to the use of solid geometry in investigations of problems of plane geometry.

In the seventeenth century, one notices a lively interest for the application of solid geometry in the solution of problems in plane geometry. Consider for instance the example of Evangelista Torricelli. In his treatise, *De quadratura parabolae* (1644; see Torricelli 1919–1944), Torricelli presented 20 different proofs of the quadrature of the parabola (a theorem of plane geometry proved for the first time by Archimedes) and classified them according as to whether they were proved by "classical" demonstrations (using techniques by reductio ad absurdum) or with demonstrations obtained through the geometry of indivisibles of Cavalierian inspiration. What is striking in this treatise is Torricelli's attention vis-à-vis the use of solid geometry in the proofs of theorems of plane geometry. All the most important results of Euclidean and Archimedean stereometry are appealed to and Torricelli showed how one can obtain the quadrature of the parabola from each one of these stereometric theorems, either using exhaustion techniques (reductio ad absurdum) or indivisibilist arguments. Of course, not a single one of these stereometric results can be considered necessary for the proof of this plane theorem, for Archimedes had already given a proof that only appeals to concepts and results of plane geometry. Torricelli does not raise any methodological problems concerning the use of solid geometry in investigating problems of plane geometry.

With the development of projective geometry in the nineteenth century, the use of spatial techniques in the study of problems of plane projective geometry begins to show its fruitfulness. Monge's school, in particular, made extensive use of the interaction between planar and spatial notions. In his famous 1837 *Aperçu*, Chasles described the school of Monge by means of its propensity towards the use of three dimensions in the proof of plane theorems.

We conclude these brief historical remarks by recalling that within elementary geometry, the separation between plane and solid geometry was challenged seriously for the first time in the work of the Italian geometer Riccardo de Paolis in *Elementi di geometria* (1884). In this book, de Paolis emphasized the importance of analogies between plane and solid geometry (angles-diedra, polygons-polyhedra, etc.) as well as the importance of using space for the understanding and the simplification of theorems of plane geometry. This "fusionist" position, namely, the request that plane and solid geometry be developed together, was at a source of the debate known as "fusionism" which saw the involvement of Italian, French, and German geometers. The debate between those who advocated "fusionism" and their opponents led to discussions concerning the legitimacy as well as the necessity of using space in proofs of theorems in plane geometry. But in order to seriously tackle such issues, one had to wait for the foundational works of Peano and Hilbert, which we will discuss below.

2.3 The Foundations of Projective Geometry

In the early nineteenth century, geometers set out to develop the foundations of projective geometry, independently of Euclidean geometry. Some, for instance, Möbius and Plücker, sought to develop an analytic projective geometry, analogous to Cartesian analytic geometry for Euclidean geometry. Others, for instance, Steiner, sought a coordinate-free development of projective geometry that had the same power as the new analytic projective geometry. In these research programs, these geometers freely used metric considerations. They appealed either to the Euclidean distance metric or to principles of proportionality or congruence. However, these are not projectively invariant.

Beginning with his *Geometrie der Lage* (1847), von Staudt sought to eliminate these metric considerations from projective geometry, on purity grounds:

> I have tried in this work to make the geometry of position into an independent science that does not require measurement.

Though there were gaps concerning continuity that were later filled by others, von Staudt's work yielded a means of defining projective coordinates by purely projective means. The key to his accomplishment was a particular construction ("quadrilateral construction"), which provides a way, given any three collinear points, to find uniquely a fourth point on that line with a certain relation to the other three points; it is then said that the four points together form a "harmonic range" (a notion that we do not need to define here). The uniqueness of the fourth harmonic point can be shown by metric considerations. Following his aim of purifying projective geometry of metric considerations, von Staudt proved the uniqueness of the fourth harmonic point by purely projective means, in particular by appealing to Desargues' theorem whose statement does not involve any metric notion.

Desargues' Plane Theorem If two triangles lying in the same plane are such that the lines connecting their corresponding vertices intersect at a point, then the intersections of their corresponding sides are collinear.

As we have already mentioned, Desargues' original proof appeals to metric notions, since it appealed to congruence by way of Menelaus' theorem. Von Staudt's aim was to purify projective geometry of metric considerations and, in particular, to define projective coordinates by purely projective means. The key to doing so was Desargues' theorem.

His aim would only have been satisfied if he had a nonmetrical proof of Desargues' theorem. However, Desargues had also stated a solid version of the result.

Desargues' Solid Theorem If two triangles lying in different planes are such that the lines connecting their corresponding vertices intersect at a point, then the intersections of their corresponding sides are collinear.

The planar Desargues' theorem can be proved by "projecting" the solid version into the plane, as Poncelet showed in his *Traité des propriétés projectives des figures* (1822). This proof is purely projective, avoiding metrical considerations (all one needs to observe is that two planes intersect at a line and that the lines that connect the vertices of the triangles lying on different planes can only meet in the line of intersection of the two planes). Hence, von Staudt was able to achieve his aim by using this proof. However, this proof draws on considerations from solid geometry, despite the fact that Desargues' theorem concerns just triangles in the plane.

We have thus reached the point where the fusionist debate began. That fusionism had to be a necessity in the foundations of projective geometry was also the conclusion reached by Felix Klein in his article *Über die sogenannte Nicht-Euklidische Geometrie* (1873).

In an influential 1891 lecture (Wiener 1892) remarked, without proof, that Desargues' theorem cannot be proved by purely planar projective considerations, observing that "this area of geometry is not self-contained." Peano and Hilbert took up this metamathematical question shortly thereafter.

2.4 Peano and Hilbert

The key to Klein's observation on the necessity of appealing to space in the foundations of projective geometry is Beltrami's theorem (1865), which says that a "smooth" (i.e., Riemannian) surface has constant curvature if and only if it can be mapped to a plane so that the geodesics of that surface are mapped to straight lines in that plane. The result applies to the Euclidean plane and even to the projective plane. Klein understood Beltrami's theorem as asserting that a Riemannian surface of nonconstant curvature cannot be represented on a plane so that the geodesics of that surface "behave" like straight lines in the plane.

In 1894 Peano developed this suggestion, sketching a proof that his axioms of planar geometry have models in which Desargues' theorem fails by appealing to Riemannian surfaces of nonconstant curvature. Hence, his planar axioms are provably insufficient for proving Desargues' theorem. Once solid axioms are added, Peano's axioms prove Desargues' theorem as expected.

In lectures delivered in 1898–1899, Hilbert developed his own axiomatization for geometry, dividing his axioms into classes I (incidence), II (order), III (parallel), IV (congruence), and V (continuity). He observed that Desargues' theorem is provable in this system using spatial axioms, or alternately using axioms of congruence. He then showed that Desargues' theorem cannot be proved in plane geometry (in fact, from axioms I 1–2, II, III, IV 1–5, and V), by presenting explicitly a model in which these axioms are satisfied but Desargues' theorem is not. Hence, it follows that his planar axioms (I 1–2) are provably insufficient for proving Desargues' theorem.

In his lectures of 1898–1899 (Lectures on Euclidean Geometry), Hilbert commented upon the result by emphasizing the importance for the issue of purity of methods:

> This theorem gives us an opportunity now to discuss an important issue. The content [Inhalt] of Desargues' theorem belongs completely to planar geometry; for its proof we needed to use space. Therefore we are for the first time in a position to put into practice a critique of means of proof. In modern mathematics such criticism is raised very often, where the aim is to preserve the purity of method [die Reinheit der Methode], i.e. to prove theorems if possible using means that are suggested by [nahe gelegt] the content of the theorem. (Hallett and Majer 2004, pp. 315–316)

What is critical for a proof's being pure or not, then, is whether the means it draws upon are "suggested by the content of the theorem" being proved. Since the "content of Desargues' theorem belongs completely to planar geometry," solid considerations would not appear to be "suggested by the content of the theorem," and therefore, it would seem that Hilbert judged solid proofs of Desargues' theorem impure. Hilbert also showed that if a planar geometry satisfies axioms I 1–2 (the planar incidence axioms), II (the order axioms), and III (the parallel axiom), then Desargues' theorem is necessary and sufficient for that planar geometry to be an element of a spatial geometry satisfying all the incidence axioms I in addition to the axioms of II and III. That is, a plane satisfying axioms I 1–2, II, and III, and also satisfying Desargues' theorem, will also satisfy the spatial incidence axioms I 3–7. Hilbert proved this by showing, firstly, how, in a planar geometry satisfying axioms I 1–2, II, III, and Desargues' theorem, to construct an algebra of segments that is an ordered division ring and, secondly, how this ordered division ring can be used to construct a model of axioms of I, II, and III, that is, a model of spatial geometry. (Order is inessential here.)

Here is how Hilbert summarized the situation in his 1898–1899 lectures:

> Then the Desargues Theorem would be the very condition which guarantees that the plane itself is distinguished in space, and we could say that everything which is provable in space is already provable in the plane from Desargues. (Hallett and Majer 2004, p. 240)

In other words, Desargues' theorem can be used as a replacement for Hilbert's solid axioms: it has the same provable consequences as those axioms in Hilbert's axiomatic system (see Hilbert 1899, 1971).

2.5 The Problem of Content

In a recent article (Hallett 2008) and in his introductions to the Hilbert's lectures on geometry published in the first volume of the Hilbert Editions (Hallett and Majer 2004), Michael Hallett has drawn some interesting consequences, which in our opinion are questionable, on the notion of the content of Desargues' theorem and on the issue of purity of methods. Hallett writes:

> What this shows is that the Planar Desargues's Theorem is a sufficient condition for the orderly incidence of lines and planes, in the sense that it can be used to generate a space. We thus have an explanation for why the Planar Desargues's Theorem cannot be proved from planar axioms alone: the Planar Desargues's Theorem appears to have spatial content. (Hallett 2008, p. 229)

Moreover, in his introduction to Hilbert's 1898–1899 lectures, Hallett writes that Hilbert's work "reveals that Desargues' planar Theorem has hidden spatial content, perhaps showing that the spatial proof of the Planar Theorem does not violate 'Reinheit' after all" (pp. 227–228). Thus, Hallett believes that Hilbert's work should cause us to revise our judgments of what counts as a pure proof of Desargues' theorem. While solid considerations would seem "at first sight" to be impure for proving Desargues' theorem, Hallett infers from Hilbert's work reveals that they are not, for Desargues' theorem is in fact a theorem with (hidden) solid content.

This position defended by Hallett appeals to the notion of "hidden higher-order content" developed by Dan Isaacson in the context of some articles aimed at providing an interpretation of Gödel's incompleteness results for Peano arithmetic (Isaacson 1987). In our paper (Arana and Mancosu 2012), we develop a detailed analysis both of Isaacson's notion of "hidden higher-order content" as well as the consequences drawn from it by Hallett with respect to the issue of purity of methods.

The central aspect of the issue is that the notion of content proposed by Hallett, on the basis of the Hilbertian analysis of Desargues' theorem, is based on the deductive role played by this theorem with an axiomatic context. This notion is very close to that of content as deductive equivalence (within an axiomatic system) that had been proposed by Carnap. Hallett sees in Desargues' plane theorem a statement with (hidden) solid content exactly because, within a certain axiomatic theory, Desargues' theorem plays the same inferential role as the space incidence axioms.

Our criticism to Hallett's position is based on the following five objections, which we simply state here without giving any arguments (for which we refer to Arana and Mancosu 2012):

(a) If the content of Desargues' theorem were spatial, it would seem to follow that an investigator with no beliefs or commitments concerning space (such as a character of Flatland) could not understand Desargues' theorem, which seems implausible.
(b) To claim that Desargues' plane theorem has a solid content on account of the inferential role it plays in Hilbert's axiomatic system requires a deep metatheoretical analysis such as the one carried out by Hilbert. But what to say then of statements that have not yet been subjected to such a deep metatheoretical analysis or, worse, for which we don't know whether they are true or false (such as the twin prime conjecture)? Intuitively, we understand the content of the twin prime conjecture even though we have no metatheoretical analysis of it.
(c) Hallett's view implies a radical contextualism regarding the content of statements like Desargues' theorem. The inferential role of Desargues theorem within metrical geometries is quite different than its inferential role within projective geometry; in the former, spatial considerations are unnecessary, while in the latter they are necessary. If Hallett were correct, the content of Desargues' theorem would change dramatically depending on which axiomatic context we use it in, without its formulation changing at all.
(d) Hallett infers that the spatial content "revealed" by Hilbert's work belongs specifically to Desargues' theorem, when Hilbert's work shows only that Desargues' theorem added to the planar axioms of classes I, II, and III has the same spatial consequences as the spatial axioms of those classes. Even if it is reasonable to maintain that the planar axioms plus Desargues' theorem have tacit spatial content on account of their shared inferential role (which we have contested), it is illicit to single out that content as belonging to Desargues' theorem. For those spatial consequences belong only to the axiomatic system as a whole, not to Desargues' theorem alone. While it is true that without Desargues' theorem these spatial consequences are not ensured, it is also true that Desargues' theorem alone does not ensure them. Hence it would be more accurate to say that these spatial consequences are partly the result of the planar

axioms and partly the result of Desargues' theorem. Indeed, Hallett's argument would just as well establish that one of the planar axioms, say I.1, has tacit spatial content.

(e) From the analysis of the notion of content defended by Hallett, it follows that every theorem has a pure proof. It seems to us implausible that this can be true *a priori*, simply as a consequence of the analysis of the notion of content. Purity would end up being trivialized.

We conclude that the notion of content offered by Hallett can be of interest for other theoretical goals but not for the clarification of the ascription of purity that are often found in mathematical practice. The notion of content that in our opinion is useful for clarifying judgments about purity of proofs must be tied to the understanding of the meaning of the statement of a theorem and not to its inferential role within an axiomatic system. Moreover, our position on Desargues' theorem seems to us to be identical to the one defended by Hilbert: Desargues' plane theorem does not have a pure proof in a projective context.

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