Chapter 12
Defending Maddy's Mathematical Naturalism from Roland's Criticisms: The Role of Mathematical Depth

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12.1 Maddy's Mathematical Naturalism

A naturalistic approach generally rejects the possibility of *a priori* philosophical inquiries. In Quine's words, naturalism is "the recognition that it is within science itself, and not in some prior philosophy, that reality is to be identified and described" (Quine 1981, p. 21).

Following Quine, Maddy's naturalism[1] does not extend to posing philosophical questions "from some special vantage point outside of science, but as an active participant, entirely from within" (Maddy 2011, p. 39). Maddy's naturalism "takes the correctness of successful scientific practice as a datum for philosophical theorizing rather than something susceptible to philosophical challenge" (Linnebo 2012, p. 134).

From this perspective, philosophical positions defined as naturalistic must state their theories not only as a matter of a simple deference to authoritative scientific statements but also for internal scientific reasons, which means grounding them on experiment and well-confirmed scientific theories (Maddy 2011, p. 39).

According to Maddy's naturalism, the appropriate method of investigating a particular domain of reality is by means of the science which specifically addresses

[1]Penelope Maddy's approach to the philosophy of mathematics has evolved from early cognitive realism (Maddy 1997) to her present mathematical naturalism. In this chapter, I focus on her present naturalistic account as presented in (Maddy 2007a, 2007b, 2011).

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this domain. With regard to mathematics, first and foremost it is worth highlighting that Maddy understands mathematics "as a human practice" (Maddy 2007b, p. 361). From this practical perspective, Maddy affirms the autonomy of mathematics from both philosophical and scientific considerations:

> My naturalist [. . .] begins, as Quine's does, within empirical science, and eventually turns, as Quine's does, to the scientific study of that science. She is struck by two phenomena: first, most of her best theories involve at least some mathematics, and many of her most prized and effective theories can only be stated in highly mathematical language; second, mathematics, as practice, uses methods different from those she's turned up in her study of empirical science. She could, like the Quinean, ignore those distinctive methods and hold mathematics to the same standards as natural science, but this seems to her misguided. The methods responsible for the existence of the mathematics she now sees before her are distinctively mathematical methods; she feels her responsibility is to examine, understand, and evaluate those methods on their own terms; to investigate how the resulting mathematics does (and doesn't) work in its empirical applications; and to understand how and why it is that a body of statements generated in this way can (and can't) be applied as they are. (Maddy 2007a, p. 448)

Maddy's suggestion is that mathematical objects and practice should be investigated with methods derived from mathematical practice itself.

In the same manner, the ontological questions concerning mathematical objects and statements must be answered within mathematics itself. Indeed, mathematical objects should not receive the same epistemological treatment reserved for physical objects. While for physical objects we need higher introduction and confirmation standards (e.g., identification through empirical instruments), standards for introducing mathematical objects are different: their role in successful mathematical theories is the only element we ought to use in confirming their existence.

Due to this epistemic disanalogy between mathematical and scientific objects, and due to the constitutive autonomy of mathematics with respect to philosophical concerns, Maddy does not think that a naturalistic investigation into the foundations of mathematics necessarily leads to a realist ontological position:

> [. . .] for my naturalist natural science is the final arbiter of what there is, and it doesn't seem to support its mathematical ontology [. . .]. Mathematics itself offers no ontological guidance beyond the minimal "mathematical things exist" [. . .]. In fact, I suspect that a decision on these matters will have more to do with the theory of truth than with the methodological or naturalized philosophical facts about mathematics or natural science. (Maddy 2007a, pp. 456–457)

It is for this reason that the second philosophy proposed by Maddy (2007b) ceases to give prominence to the defense of a particular position in the ontology of mathematics:

> Does mathematics have a subject matter like physics, chemistry, or astronomy? Are mathematical claims true or false in the same sense? If so, by what means do we come to know these things? What makes our methods reliable indicators of truth? The answers to these questions will not come from mathematics itself - which presents a wonderfully rich picture of mathematical things and their relations, but tells us nothing about the nature of their existence [. . .]. (Maddy 2007b, p. 361)

In considering mathematical objects and theories by looking at their role in mathematics as practiced by working mathematicians, her methodological naturalist focuses more on epistemological issues (how do we build mathematical theories? How do we account for them? "How can we properly determine if a new sort of entity is acceptable or a new method of proof reliable?" (Maddy 2011, p. 31)), than on ontological and semantical ones (do mathematical objects exist? Are mathematical theories true?). Maddy's only proviso is that, no matter which ontological position we endorse, it should not contradict second-philosophical methods of inquiry.

This is why, regarding the ontology of mathematics, Maddy "does not address alternative theories of ontology she does not find in that practice" (McLarty 2013, p. 390), that is, the day-to-day practice of mathematics, and adopts quite indifferently what she calls *Thin Realism* (the thesis that mathematical objects exist, but they only have the properties ascribed to them by mathematical theories, any other question about their nature being irrelevant) or *Arealism* (the thesis that mathematical objects do not exist).[2] These two ontological positions are taken to be "equally accurate, second-philosophical descriptions of the nature of pure mathematics" (Maddy 2011, p. 112).

12.2 Roland's Objections

Before Maddy (2011) appeared, Jeffrey W. Roland (2007) charged Maddy's account of mathematics with failing to be naturalistic: in his opinion, Maddy would be unable to explain the reliability of mathematical beliefs without breaking one of the main principles of naturalism.

As I have explained above, Maddy's latest position with respect to mathematics is compatible with "there being no fact of the matter regarding the truth and falsity of mathematical claims" (Roland 2007, p. 425). But, in Roland's opinion, "if there is no fact of the matter with respect to truth and falsity in mathematics, that undermines the project of giving an epistemology of mathematics" (Roland 2007, p. 425).

Roland writes:

> Epistemology is centrally concerned with systematic connections between justification and truth. If there is no fact of the matter as to whether claims concerning Fs are true or false, then there simply is no question of systematic connections between what justifies our F-beliefs and the truth about Fs. (Roland 2007, p. 425)

In Roland (2009), arguing for the impossibility of naturalizing any epistemology of mathematics, Roland explains in the following manner what, in his opinion, is essential for the possibility of an epistemology of any discipline, including mathematics, that is, the "truth-conduciveness" of beliefs:

> Suppose we have an epistemology E that ratifies our acceptance of pure mathematics as justified. [. . .] The notion of justification endorsed by E must be truth directed; i.e. it must be such that beliefs justified according to that notion tend to be true. [. . .] What makes a

[2]Arealism is taken as different from nominalism: Maddy states the difference in Maddy (2011, pp. 96–98), claiming that "[. . .] if Arealism is to be considered a version of nominalism, it certainly isn't the 'stereotypical' variety" (Maddy 2011, pp. 97–98).

conception C of justification a conception of *epistemic* justification is at least in large part that beliefs which are justified according to C tend to be true, i.e. that there is some sort of systematic connection between beliefs justified according to C and what is actually the case. Moreover, endorsing the truth-directedness of epistemic justification [. . .] is to recognize a widely accepted conviction that an epistemic notion of justification must be systematically connected to truth, i.e. truth-conducive. (Roland 2009, p. 71)

Roland claims that a naturalistic account is able to answer the epistemological question of "how we are justified in believing what we (justifiably) do about the world?" (Roland 2007, p. 430) because of the two positions it entails: *disciplinary holism* and ontological realism.

What Roland calls *disciplinary holism* is the "cross-discipline criticism and support allowed for" by naturalism (Roland 2007, p. 431):

[. . .] the family of disciplines that fall under the heading 'science' is large enough and varied enough that meaningful criticism of one discipline can be mustered in another while remaining within science (broadly construed to include natural and social sciences plus the mathematics and logic applied in the practice of these sciences). [. . .] While science as a whole is insulated from outside criticism on the naturalist's view, individual branches of science [. . .] are not insulated from each other. (Roland 2007, p. 430)

As to the naturalist's ontological realism, Roland explains why he thinks that the epistemology of naturalism is linked to realism in the following terms:

Naturalism has it that our inductive practices are underwritten by our appreciation, conscious or not, of natural kinds. Successful inductions are those done on projectible[3] properties of (predicates applied to singular terms denoting) objects, and the naturalist, following Quine, holds that 'a projectible predicate is one that is true of all and only the things of a natural kind' (Quine 1969, p. 116). Thus, our ability to successfully engage in induction is linked to our ability to tell projectible predicates from nonprojectible ones, which is in turn linked to our ability to track general features of the world [. . .]. So since naturalists are generally realists about natural kinds, naturalism, in its account of our inductive practices, takes a realist stance toward the general *prima facie* subject matter of the sciences. (Roland 2007, p. 431)

In particular, Roland claims that a causal form of realism is essential to a naturalistic epistemology:

[. . .] An account of the reliability of perception [in the case of natural science] must bridge theory and the world. This bridge is provided by a causal theory of detection [. . .]. (Roland 2007, p. 433)

A causal theory of perception is "the ground level of detection" (Roland 2007, p. 433), and, due to disciplinary holism, all sciences are rooted in this ground level:

This is the sense in which it is reasonable to say that physics and physiology, in addition to biology, chemistry, psychology, neuroscience - even sociology and economics - ultimately depend on perception. The experience on which empirical science depends is perceptual

[3] According to Roland, projectibility is a property of predicates that measures the degree to which past instances can be taken as guides to future ones.

> experience, broadly construed to include detection (indirect perception) by instruments, but empirical science only fulfills its primary mission, i.e. to tell us about the world, if that experience is causally connected to the world. (Roland 2007, p. 433)

So, the naturalist's account of the reliability of scientific beliefs requires the commitment to a realist conception of causation.

> Reliability has to do with truth [. . .]. An adequate explanation of the reliability of certain types of interactions in terms of causation (i.e. causal powers, processes, or structure) must give us reason to think that beliefs formed as a result of the right types of interactions are true in a robust sense. A realist conception of causation can do this [. . .]. (Roland 2007, p. 435)

These two features of naturalism, that is, disciplinary holism and ontological realism, enable us to explain the reliability of scientific beliefs and also to account for the accuracy of the naturalist's epistemic norms and standards[4] (Roland 2007, p. 431).

With regard to Maddy's mathematical naturalism, Roland points out that, according to Maddy, mathematics should be regarded as being detached from natural sciences (and from philosophy), which fails to meet his requirement for disciplinary holism, and that she does not think that a naturalistic investigation of mathematics necessarily leads to ontological realism, which in turn disregards his requirement for causal realism. Judging Maddy's epistemology for mathematics in light of his own conception of scientific naturalism, Roland puts forth two criticisms of Maddy.

The first is what I call ''reliability criticism''. Roland claims that Maddy's accordance of autonomy to mathematics is the equivalent of a rejection of disciplinary holism (Roland 2007, p. 436). But in Roland's view disciplinary holism, as we have seen, is essential to providing naturalism with an epistemology for science which is able to guarantee the accuracy of its epistemic norms and standards. The same goes for mathematics: in Roland's view, disciplinary holism is necessary in order to provide an epistemology for mathematics which is able to guarantee the accuracy of its epistemological standards. So, by rejecting disciplinary holism, Maddy's mathematical naturalism disqualifies itself from the possibility of being considered as a genuine naturalistic position:

> [. . .] An account of the reliability of the method of mathematical naturalism analogous to the account of the reliability of scientific practice available to the naturalist is out of reach for the mathematical naturalist. (Roland 2007, p. 437)

The second objection, strictly connected to the first, is what I call ''ontological criticism''. We have seen that, in Roland's view, in order to provide an adequate epistemology for both science and mathematics it is essential to rely on causal realism. But, as we have seen, Maddy's mathematical naturalism leaves the question of the existence of mathematical objects and of the truth of mathematical statements substantially open, considering it to be an extra-mathematical question, and as such without interest for her naturalist. In order for a naturalistic epistemology for

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[4]In Roland's words, this means providing a "dissident epistemology" for science (Roland 2007, p. 432).

mathematics to be adequate, Roland requires the identification of "truth-makers" for mathematics, "in virtue of which mathematical beliefs (statements, etc.) have the truth values they do" (Roland 2009, p. 72). Given her ontological agnosticism, Maddy cannot rely on existing mathematical objects as truth-makers of this kind:

> [. . .] The mathematical naturalist can countenance nothing to play a role in the epistemology of mathematics analogous to that of the causal order in naturalistic epistemology. (Roland 2007, p. 439).

Therefore, in Roland's view, since she ultimately refuses to ground her position on ontological realism, Maddy should not define her second philosophy as a form of naturalism with regard to mathematics.

It is worth noting that Roland's objections to Maddy only hold if we also accept Roland's conception of naturalism for mathematics, that is, if we think about mathematical naturalism as modeled on his conception of scientific naturalism, described in Roland (2007) as a position committed to disciplinary holism and to causal realism. A way to challenge Roland's criticisms would thus be to show that there are other conceptions of naturalism available[5] and that, in particular, Maddy's view of naturalism is different from Roland's.

But even if for the sake of argument we accept Roland's epistemic requirement of reliability for mathematical beliefs, defined as the need for a link of truth-conduciveness between mathematical beliefs and some objective facts ("what is actually the case," in Roland's words (Roland 2009, p. 71)), I argue that a concept introduced by Maddy in 2011 could be used to provide an answer to Roland's criticisms: namely, the concept of "mathematical depth."

In the following I shall present this notion and try to show how, within Maddy's (Maddy 2011) framework, mathematical depth could deliver the sort of reliability of mathematical beliefs that Roland demands.

12.3 Maddy's Mathematical Depth

Maddy (2011) uses the term "mathematical depth" to refer to the capacity for fruitfulness of mathematics. Mathematical notions, theories, and statements are ultimately fruitful both internally, in mathematics itself (e.g., the foundational role of set theory), and externally in the applications of mathematical concepts to empirical sciences (e.g., Maxwell's equations which established the foundation of classical electrodynamics).[6]

Indeed, I shall distinguish the depth of mathematics from its fruitfulness *tout court*. To this purpose, it could be useful to consider Godfrey Harold Hardy's

[5]As an example, consider that the requirement of a causal link between the world's facts and beliefs, in the case of mathematical knowledge, is certainly not common among naturalists.

[6]Maxwell's equations are the usual example given by Maddy (1997, p. 114, 2007b, p. 332, 2011, p. 19), but we could mention any other successful case of application.

attempt to define the notion of depth. After claiming that "there are two things at any rate which seem essential [to make a mathematical idea significant], a certain generality and a certain depth; but neither quality is easy to define at all precisely" (Hardy 2005, p. 24), Hardy attempts to characterize depth as follows:

> It has *something* to do with *difficulty*; the 'deeper' ideas are usually the harder to grasp: but it is not at all the same. The ideas underlying Pythagoras's theorem and its generalization are quite deep, but no mathematicians now would find them difficult. [. . .] It seems that mathematical ideas are arranged somehow in strata, the ideas in each stratum being linked by a complex of relations both among themselves and with those above and below. The lower the stratum, the deeper (and in general more difficult) the idea. Thus the idea of an 'irrational' is deeper than that of an integer; and Pythagoras's theorem is, for that reason, deeper than Euclid's. (Hardy 2005, pp. 27–28)

On the other hand, Hardy clearly separates the idea of depth from that of fruitfulness, since he does not think that mathematics must be judged in terms of its utility (Hardy 2005, pp. 32–33).

Nevertheless, for the purpose of the present work, I shall focus on Maddy's proposal and take the viewpoint that despite the depth of a mathematical notion, statement or theory does not have to be identified with its fruitfulness; it could undeniably be seen as the encoding of a set of virtues (to be further specified) which constitute a fundamental part of its fruitfulness and then a condition for it.

In Maddy (2011), Maddy suggests that mathematical practice is grounded in the phenomenon of mathematical depth:

> [. . .] What guides our [mathematical] concepts formation, beyond the logical requirement of consistency, is the way some logically possible concepts track deep mathematical strains that the others miss. (Maddy 2011, p. 79)

Maddy continues by saying:

> [. . .] Judgments of mathematical depth are not subjective [. . .]. [. . .] mathematical fruitfulness isn't defined as 'that which allows us to meet our [mathematical] goals', irrespective of what these might be; rather, our mathematical goals are only proper insofar as satisfying them furthers our grasp of the underlying strains of mathematical fruitfulness. [. . .] there is a well-documented objective reality underlying Thin Realism [or Arealism], what I've been loosely calling the facts of mathematical depth. The fundamental nature of sets (and perhaps all mathematical objects) is to serve as means for tapping into that well. (Maddy 2011, pp. 81–83)

It is in light of this notion of mathematical depth that Thin Realism and Arealism are ultimately equivalent positions:

> [Thin Realism and Arealism] are equally well-supported by precisely the same objective reality: those facts of mathematical depth. [. . .] They are alternative ways of expressing the very same account of the objective facts that underlie mathematical practice. (Maddy 2011, p. 112)

Maddy provides the reader with some examples of mathematical depth. Her examples refer to concept formation in set theory and group theory and to the different formulations and applications of the axiom of choice (Maddy 2011, pp. 78–81). Unfortunately, these examples do not seem to be clear enough to shed light on the concept we are seeking. Let me briefly show why.

Let us begin with the example of the axiom of choice: the axiom states that for every set of nonempty sets there is a choice function which selects one element within each set. Maddy's description of the fruitfulness of this axiom is not as complete as one might have wished: she references a few applications of the axiom of choice but only makes explicit reference to its application in geometry, connected to the Banach-Tarski paradox (Maddy 2011, pp. 34–35). She does not cite the axiom's other applications, such as in algebra (the existence of bases in vector spaces); topology (Tychonoff's theorem about the product of compact topological spaces); or analysis (Hahn-Banach theorem and the existence of non-Lebesgue measurable sets of reals). Maddy restricts herself to reminding us, in a note, of the "internal mathematical considerations in favor of the axiom" (Maddy 2011, pp. 35–36, note 74) she described in Maddy (1997, pp. 54–57).

In her presentation, which I have sketched out above, Maddy does not provide any real insight into the fruitfulness of the axiom nor does she with regard to the connection between the axiom and the discovery of mathematical depth.

The example of group theory appears more promising. The concept of group turns out to be essential in several mathematical domains: originally used to study permutations and the solvability of algebraic quintic equations, group theory went on to be recognized as the appropriate tool to study the concept of symmetry. Today, group theory is an indispensable tool in mathematics: it essentially occurs in model constructions within different scientific contexts.

Faced with this variety of use, Maddy states that group theory's fruitfulness lies in its capacity to unify different structures which share several properties (Maddy 2011, p. 79) (e.g., a mathematical structure and a physical one), by representing them with the same model. For this reason, the example of the concept of group seems particularly well suited to demonstrating what mathematical depth is through the study of its applications.

Despite all of this, Maddy (2011) does not elaborate about group theory and its applications, leaving the reader without a clear explanation of the connection between its fruitfulness and the phenomenon of mathematical depth we are examining.

Maddy examines the case of set theory more closely. In fact, much of Maddy's work is devoted to set theory, in particular to answering the following questions (Maddy 2011, p. 37): what are the methods of set theory? And according to what criteria must we choose new axioms to adopt in order to increase the deductive and explanatory power of set theory?

Through analysis of the history of mathematics and the evolution of the connection between mathematics and the study of the empirical word (Maddy 2011, pp. 3–27), Maddy establishes that set theory is essential to the unification of mathematical structures and their languages. Indeed, according to Maddy's naturalism, it follows from the autonomy of mathematics that the unified model allowing us to study different mathematical structures and methods, if one exists, must come from mathematics itself. That unified model is now represented by set theory.

It is worth highlighting that set theory can be seen as providing us with a theoretic framework that could be used as a model in which it is possible to represent numbers and functions without being forced to make specific claims about the existence of those objects. In addition, given Gödel's incompleteness theorems, set theory does not even give what Maddy calls, quoting Saunders MacLane (1986, p. 406), a "parachute" against the risk of incompleteness (Maddy 2011, p. 133). However, in spite of these ontological remarks that do not have much importance for Maddy's naturalist, set theory has a unifying role within mathematics. Maddy then states that set theory provides us with a shared framework within which every single mathematical problem concerning consistency and proof may be treated:

> What set theory does is provide a generous, unified arena to which all local questions of coherence and proof can be referred. In this way, set theory furnishes us with a single tool that can give explicit meaning to questions of existence and coherence; make previously unclear concepts and structures precise; identify perfectly general fundamental assumptions that play out in many different guises in different fields; facilitate interconnections between disparate branches of mathematics now all uniformly presented; formulate and answer questions of provability and refutability; open the door to new strong hypotheses to settle old open questions; and so on. In this philosophically modest but mathematically rich sense, set theory can be said to found contemporary pure mathematics. (Maddy 2011, p. 34)

Maddy's explanation of the depth of the concept of set ends here. She confines herself to saying that, due to their foundational role, we are allowed to consider sets as "maximally effective trackers of mathematical depth" (Maddy 2011, p. 82). But the connection between the meaning of the concept of set and the emergence of the concept of mathematical depth is not analyzed in detail. The concept of mathematical depth is thus left rather unclear.

On one hand, and consistently with her peculiar form of naturalism, Maddy claims:

> [. . .] I doubt that an attempt to give a general account of what mathematical depth really is would be productive; it seems to me the phrase is best understood as a catch-all for the various kinds of special virtues we clearly perceive in our illustrative examples of concept-formation and axiom choice. (Maddy 2011, p. 81)

> This is why I spend so much time rehearsing these various cases, to give the reader a feel for what 'mathematical depth' looks like. (Maddy 2011, p. 81, note 39)

From the examples presented in Maddy (2011), the reader is therefore supposed to obtain a satisfying understanding of what mathematical depth is. Unfortunately, the examples are not discussed thoroughly enough in order to obtain the "feel for what mathematical depth looks like" that Maddy is seeking to impart.

As already highlighted by previous quotes, mathematical depth is presented by Maddy as something objective:

> [. . .] the topography of mathematical depth [. . .] stands over and above the merely logical connections between statements, and furthermore, it is entirely objective. (Maddy 2011, p. 80)

Maddy claims that the phenomenology of mathematical practice itself guarantees the objectivity of mathematical depth. In her opinion, anyone who does even a little mathematics can easily come to recognize this objectivity: in Maddy's words, the first sensation which strikes anyone who does mathematics is "the immediate recognition" that it is "an objective undertaking *par excellence*" (Maddy 2011, p. 114).

It is worth underlining that the form of the fruitfulness of mathematical depth is an extrinsic justification for mathematical theories and statements. The justification of a statement of a mathematical theory is intrinsic if the truth of the statements follows from the properties ascribed to its objects by the theory in question; on the other hand, an extrinsic justification of a statement is a justification in terms of its consequences, inside or outside the theory.[7] Maddy writes:

> We're out to explain what underlies the justificatory methods of set theory [...]. Part of the answer, for intrinsic justifications, may be that they spell out what's implicit in our "concept of set," but the bulk of the justifications that interest us are extrinsic. (Maddy 2011, pp. 78–79)

In Maddy's opinion, the use of a mathematical theory that has certain consequences on the improvement of our knowledge, inside or outside mathematics, is in itself a good justification to use the theory in question. In favor of this conception of fruitfulness as an extrinsic justification of theories, Maddy quotes a number of selected passages of Zermelo's defense of the axiom of choice (Zermelo 1967, pp. 187–189), specifically insisting on its fruitfulness:

> [This axiom] has frequently been used, and successfully at that, in the most diverse fields of mathematics, especially in set theory. (Maddy 2011, p. 46)

Moreover:

> So long as [...] the principle of choice cannot be definitely refuted, no one has the right to prevent the representatives of productive science from continuing to use this hypothesis. [...] Principles must be judged from the point of view of science, and not science from the point of view of principles fixed once and for all. (Maddy 2011, p. 47)

This is what allows Maddy to conclude in favor of her hypothesis of the importance of the capacity for fruitfulness in the evaluation of each mathematical notion, statement, or theory. Like Zermelo, Maddy's naturalist counts the fruitfulness of a mathematical statement as a point in its favor – indeed, as the most important point.

12.3.1 Problems with the Notion of Mathematical Depth

Maddy names her conception of the phenomenon of mathematical depth "post-metaphysical objectivism" (Maddy 2011, p. 116): with this term, Maddy refers to a form of objectivity which has nothing to do with the metaphysical and ontological level and which is constituted by the practice-oriented reality of the depth of certain mathematical theories.

[7] For references and discussion on the distinction between intrinsic and extrinsic mathematical explanation, see, for example, Mancosu (2008).

I have already quoted Maddy talking about the "facts" of mathematical depth (Maddy 2011, pp. 83, 112), stating that they represent a level of objectivity which, from an ontological point of view, could be acceptable by Thin Realism as well as by Arealism (Maddy 2011, pp. 102–112). Here one could legitimately ask what exactly these "facts of mathematical depth" are and more broadly what exactly "mathematical depth" is.

Indeed, Maddy's concept of mathematical depth is appealing to us because it provides an answer to Roland's objections, insofar as mathematical notions and the methods which are able to identify mathematical depth seem to represent in and of themselves the connection between objective facts (the facts of mathematical depth) and the corresponding mathematical beliefs. Nevertheless, we cannot ignore that Maddy's notion of mathematical depth has several problems, engendered by the lack of precision with which the notion is presented.

First of all, consider that Maddy (2011, p. 114) introduces the idea that in order to account for the phenomenon of mathematical depth one could appeal to some sort of intuition that would be shared by anyone who practices mathematics, though she does not clarify what exactly she means by this. The suggestion that the mathematical depth could best be explained in terms of the concept of intuition[8] is certainly intriguing and should be further explored. Let me stress only that we are not dealing here with a mathematical intuition conceived as a rational faculty, somehow *à la* Gödel; as mentioned above, the intuition of the depth of mathematics that Maddy is talking about is rather the psychological intuition that, when practicing mathematics, we enter a domain where our methods and conclusions are to a certain extent imposed or forced, not arbitrary.

However, accepting the favorable intuitions of mathematicians as a sufficient criterion to judge of the depth of a concept would seem to clash with the naturalistic principles sketched above, which require grounding concepts on experiment and on well-confirmed scientific theories, or, in the specific case of mathematics, on proofs and mathematical theories. Mathematicians could be wrong in their intuitions; this explains why, even if we could use the shared intuitions of mathematicians as a clue of the depth of a mathematical concept, one should hope that the depth of a mathematical notion, theory, or statement would count as an objective feature of it and not as a psychological sensation subjectively associated with it.

Moreover, Maddy frequently uses a metaphorical language, without clearly defining the words she employs; again, it would, for example, be legitimate to ask what exactly these "facts of mathematical depth" are. Furthermore, what does it mean exactly that sets and set-theoretic methods "track strains" of mathematical depth? And what is the exact definition of "post-metaphysical objectivism"?

Thirdly, and more generally, not only does Maddy not provide us with a sufficiently clear explanatory definition of what mathematical depth is, but she also presents examples which are not explored in enough detail. If it is not possible to

[8]The role of intuition in philosophy is a topic of debate: for references and discussion, see, for example, the essays in Gendler (2010).

formalize a strict definition of the facts of mathematical depth, consistent with the naturalistic attitude Maddy emphasizes, we should at least be given more clarifying examples.

Nevertheless, it is true that Maddy (2011) does not pretend to have provided a satisfactory definition of mathematical depth: her sole intention is to focus our attention on the challenge of understanding the phenomenon that is supposed to drive the practice of pure mathematics in her latest account.

Maddy uses the metaphor of the "black box" (Maddy 2011, p. 85) to describe the effectiveness of mathematics. This lexical choice provides an idea of something that contains all the information we need, but which we do not know how to read in order to have complete knowledge of the issue. Beyond the metaphor, at present the only thing that seems clear is that in Maddy's account (Maddy 2011) the goal of any epistemological inquiry concerning mathematics ends in those facts of mathematical depth.

In order to provide a satisfactory answer to Roland's challenge – and even in a general sense, to make Maddy's new account stronger – clarifications are needed regarding the concept of depth in mathematics.

12.3.2 A Possible Direction Toward Clarifying the Notion of Mathematical Depth

In light of the previous analysis of Maddy's notion of mathematical depth and the related problems thereof, I suggest that in order to clarify Maddy's account we should see those "facts" of mathematical depth not as mathematical theoretic facts, but rather as the historical facts of the fruitful use of particular notions, statements, and theories during the history of mathematical practice.

A simple reference to the history of mathematical practice would probably not be sufficient because not all the history of mathematics is a history of success casting light on the depth of the concepts involved. Moreover, we should distinguish fruitful developments from the unfruitful ones. Nevertheless, I suggest that we should think about the history of mathematics as a gradual process akin to a sort of natural selection that promotes the development of fruitful mathematical notions and makes the unfruitful ones short-lived. Although I will not develop this suggestion here, it is useful to bear it in mind in order to see the facts of mathematical depth as the occurrences of certain uses of mathematical notions, statements and theories which turn out to be fruitful when we survey the history of mathematical practice.[9]

Defined in this manner, the facts of mathematical depth are beyond a doubt empirical facts, being part of the history and the current practice of mathematics.

[9]Note that we are not denying our initial distinction between depth and fruitfulness, since we clearly stated that, despite this distinction, the depth of a mathematical notion, statement, or theory could be seen as constituting a condition for its fruitfulness.

Maddy's depth could be defined as the capacity for fruitfulness of mathematical notions, statements, and theories, inside and outside mathematics itself. Mathematical notions, statements, and theories could then be seen as "tracking strains" of mathematical depth when their use in mathematical practice produces a useful insight or progress in the practice itself.

Even if Maddy does not explicitly express things in this way, her examples, discussed above, of notions and statements that point to the phenomenon of mathematical depth do not seem to conflict with this practice-oriented direction of clarifying the concept of mathematical depth.

Since historical facts are empirical, it is no longer necessary to explain why these facts of mathematical depth are objective. Fruitful uses of certain notions, statements, and theories in the history of the practice of mathematics stand out *a posteriori* and independently from the subjective intuitions of mathematicians.[10] This constitutes another reason to not base the judgments concerning depth on intuitions: the facts of mathematical depth are best understood as what turns out to be fruitful in the history of mathematics, independently from any subjective beliefs.

The definition of Maddy's objectivism with respect to mathematical depth as "post-metaphysical" would thus become clearer: the objectivity of the facts of mathematical depth is not a theoretic objectivity, depending on the ontological existence of mathematical objects, but is grounded on the empirical reality of the practice of mathematics[11] and is objective in this empirical sense.

Essentially, what was missing in Maddy's presentation of mathematical depth was a clarification of the definition of the concept, the nature of the facts it relates to, and the reasons why we should take them as objective; my suggestion indicates a possible direction toward solving these issues.

At this point, I put forward that looking at the facts of mathematical depth in the manner I proposed, that is, as the empirical, historical facts of mathematical practice, may allow us to answer to Roland's objections concerning the reliability issue and his request to link our mathematical beliefs to objectively existing facts.

12.4 Answers to Roland's Objections

In order to accept an epistemology for mathematics that ratifies our acceptance of mathematical beliefs as justified, Roland demands the existence of a connection between justification and truth, ultimately stated as "some sort of systematic

[10] Maddy's description of the objectivity of the mathematical depth seems to be robustly consistent with this sense of objective: see, for example, Maddy (2011, pp. 80–81).

[11] As McLarty clearly explains: "Maddy calls the existence claim [about sets] mathematical, since mathematicians routinely affirm it. She calls claims about possible existence, which do not occur in mathematics and are prominent in metaphysical discussion, metaphysical. She never argues against pursuing metaphysics and even the metaphysics of mathematics. She argues that we can understand what mathematics is and how it is justified by looking at mathematics and other sciences which mathematicians routinely do address, and not metaphysics." (McLarty 2013, p. 386)

connection between beliefs justified [...] and what is actually the case" (Roland 2009, p. 71). Roland identifies this "what is actually the case" with "truth *simpliciter*" (Roland 2007, p. 435), endorsing ontological realism and thus denying the possibility of including Maddy's ontologically agnostic account in his definition of naturalistic epistemology.

But if, with the aim of providing an epistemology for mathematics in a naturalistic manner, we take the facts of mathematical depth as "what is actually the case," in the sense of what is empirically the case, instead of the theoretic notion of truth *simpliciter*, then I argue that we might be able to find a connection between our mathematical beliefs and an objective reality, as Roland demands, while staying within Maddy's account.

As I have proposed, the facts of mathematical depth could be seen as the empirical facts of mathematical practice, objectively existing in the history of mathematics and in its current practice. Considering these facts in this light allows us to state the existence of a connection between mathematical beliefs we take to be justified and "what is actually the case," that is, the objectively existing facts of the fruitful use of mathematical notions, statements, and theories. In Maddy's approach, the fact on the basis of which to judge the reliability of beliefs in mathematics are these facts of mathematical depth and not the alleged ontology of mathematical objects that is commonly posited as grounding the truth of mathematical statements.

In order to have an adequate explanation of the reliability of mathematical beliefs in Maddy's mathematical naturalism, we needn't "think that beliefs [...] are true in a robust sense" (Roland 2007, p. 435), as Roland believes, but only that they are linked in a robust sense to the objective, empirical facts of mathematical depth.

The "truth-makers" of mathematical statements (i.e., "that in virtue of which mathematical beliefs (statements, etc.) have the truth values they do" (Roland 2009, p. 72)), which Roland requires for Maddy's epistemology of mathematics to be considered naturalistic, could now be seen as corresponding to the empirical facts of the successful use of mathematical notions and statements in mathematical practice.

With the facts of mathematical depth in place of ontological truth, we have the "bridge" between "theory and the world" that Roland's conception of epistemology calls for (Roland 2007, p. 433) without being compelled to endorse a causal form of realism.[12]

We are now able to answer Roland's question about the reliability of mathematical beliefs without being forced to adopt a form of ontological realism, which means we can offer an answer to the two objections he raised against Maddy's epistemological account for mathematics. We find this answer to Roland's criticisms within Maddy's account itself, thanks to the introduction of the concept of

[12]Even the projectibility of predicates Roland (2009, p. 431) applied to terms denoting objects in mathematical statements could still be there, because in Maddy's account the successful use of a mathematical notion, statement, or theory may be taken as a guide to future uses of the same notion, statement, or theory in mathematical practice. In Maddy's view, successful mathematical practice relies on the knowledge of the history of mathematics and of the patterns of mathematical depth that we discover studying and practicing mathematics.

mathematical depth and through seeing its "facts" as the empirical, historical facts of mathematical practice. This is why Maddy's account can continue to be seen as guaranteeing the reliability of mathematical beliefs in a deeply naturalistic way.

Maddy's work shows that, in regard to mathematics, it is indeed possible to be a naturalist without being a realist. For in Maddy (1997) and completely in Maddy (2007b, 2011), Maddy applies to mathematics the radical naturalistic approach that Quine applied to science. This is why I think Roland's ontological criticism is misplaced. I agree with Rosen (1999, p. 407) that Maddy's naturalism rectifies Quinean asymmetry: while Quine expects science to be completely autonomous from any philosophical considerations, he still views mathematics as dependent on empirical sciences, considering that mathematical statements need empirical support to be proven. Maddy on the other hand extends Quinean naturalism to mathematics, bestowing upon it methodological autonomy and independence from any extra-mathematical considerations, be they philosophical or scientific.

I do not agree with Roland that a naturalistic approach, adopted within Quinean tradition, forces us toward ontological realism with respect to mathematics. With regard to the philosophy of mathematics, a naturalistic account surely commits us to certain methodological Quinean (Quine 1969) standards (e.g., rejection of *a priori* philosophical inquiries, a claim of continuity between philosophy and sciences, employment of proper methods of inquiry for different scientific subjects), but it does not seem to force us to choose a realist ontological position.

Maddy's account does not in fact consider the ontological issue as being essential to her approach to the philosophy of mathematics; what really matters is the methodological statement of inquiry. This chapter's attempt to clarify the concept of mathematical depth moves in this practice-oriented direction, consistent with Maddy's naturalistic approach.

With regard to the disciplinary holism that Roland demands for any form of naturalism, I should emphasize that the autonomy accorded to mathematics by Maddy, criticized as not naturalistic, does not prevent her from establishing a fruitful connection between mathematical work and the results of other scientific subjects. Her frequent references to studies in psychology and cognitive sciences[13] to support her theory of mathematical reasoning demonstrate this. This is why I think that Roland's concern about the separation between mathematics and other sciences in Maddy's view is simply not grounded.

Recall, moreover, that in Roland's argument disciplinary holism is essential to naturalism in order to justify the possibility of espousing a causal form of realism in any scientific domain. But now that we have argued for the possibility of a link between mathematical beliefs and an empirically objective reality which assures their reliability without being grounded on causal realism, disciplinary holism no longer seems to be essential.

[13] Maddy (1997) referred to cognitive studies made by Hebb, Piaget, Phillips, and Gelman (Maddy 1997, pp. 58–67). On the other hand, Maddy (2007b) refers to more recent neuroscientific works of Dehaene, Spelke, Wynn, and others (Maddy 2007b, pp. 264–269, 319–328).

12.5 Conclusions

I briefly recalled the main features of Maddy's mathematical naturalism in order to present Roland's reliability and ontological criticisms of her account.

Thanks to the introduction of the concept of mathematical depth and seeing its "facts" as the empirical, historical facts of mathematical practice, I proposed an answer to Roland's objections that does not force us to abandon naturalism, as Roland stressed. In light of this answer, Maddy's account can continue to be seen as guaranteeing the reliability of mathematical beliefs in a naturalistic way.

However, I submitted that the concept of mathematical depth needs some important clarifications. In this respect, I suggested a possible manner in which the notion could be developed further, also through future investigations.

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