Chapter 13
Introduction to Coordinated Linear Systems

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13.1 Introduction

The purpose of this chapter is to introduce and motivate the concept of coordinated linear systems, a special class of hierarchical systems. Coordinated linear systems are structured linear systems consisting of one coordinator system and two or more subsystems, each with their own input and output. The coordinator state and input may influence the subsystem states, inputs, and outputs. The state and input of each subsystem, on the other hand, have no influence on the coordinator state, input, or output, and neither can they influence the state, input, or output of the other subsystem. This structure is illustrated in Fig. 13.1.

The concept of a coordinated linear system was first introduced in [1]. In [2], the construction of coordinated linear systems from unstructured linear systems is described, and the concept of a minimal coordinator is introduced. These results are summarized in Chap. 14. The controllability and observability properties of coordinated linear systems are discussed in [3].

Coordinated linear systems can be useful in the study of many applications with an inherent hierarchical structure, but also of applications without a predefined structure, which permit a hierarchical approach. The main motivation for the study of coordinated linear systems is that we expect the structure imposed on the systems to simplify control synthesis: Since the subsystem states and inputs have no influence on the states or outputs of any other part of the system, local control synthesis can be done for each subsystem independently after the control law for the coordinator has been fixed.

An application of coordinated linear systems involving the inherently hierarchical traffic network is described in Chap. 12. Another application, where a hierarchical

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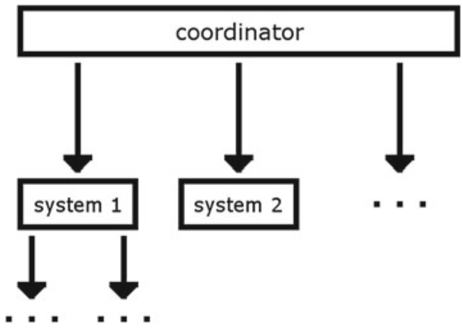

Fig. 13.1 Scheme of a coordinated system

structure is imposed on a group of autonomous underwater vehicles in order to meet the control requirements, can be found in [4].

The principle of a coordinated linear system, with several subsystems and a hierarchical top-to-bottom information structure, is in no way restricted to linear systems. Linear systems are merely one of several classes of systems to be considered; the corresponding problem of coordination control for discrete-event systems is studied in [5].

In the following, coordinated linear systems will be defined, and some of their basic properties will be discussed. For notational simplicity, we restrict attention to two subsystems. Coordinated linear systems with more than two subsystems, and their extension to hierarchical systems with more than two layers, are discussed in [6].

13.2 Definition

In accordance with the geometric framework for linear systems developed in [7], we define coordinated linear systems with inputs and outputs in terms of independence and invariance properties of the state, input, and output spaces:

Definition 13.1 Let a continuous-time, time-invariant linear system with inputs and outputs of the form

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t)$$

be given. Moreover, let the state space of the system be decomposed as $X = X_1 \dot{+} X_2 \dot{+} X_c$, and let the input and output spaces be given by $U = U_1 \dot{+} U_2 \dot{+} U_c$ and $Y = Y_1 \dot{+} Y_2 \dot{+} Y_c$. Then, we call the system a **coordinated linear system** if we have that

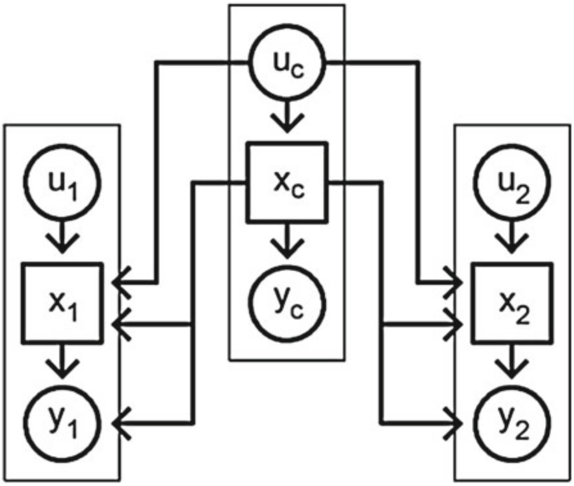

Fig. 13.2 A coordinated linear system with inputs and outputs

1. X_1 and X_2 are A-invariant,
2. $BU_1 \subseteq X_1$ and $BU_2 \subseteq X_2$,
3. and $CX_1 \subseteq Y_1$ and $CX_2 \subseteq Y_2$.

Conditions 1, 2, and 3 in Definition 13.1 imply that the state and input of each subsystem have no influence on the states or the outputs of the coordinator or the other subsystem.

With respect to the decompositions $X = X_1 \dot{+} X_2 \dot{+} X_c$, $U = U_1 \dot{+} U_2 \dot{+} U_c$, and $Y = Y_1 \dot{+} Y_2 \dot{+} Y_c$, the system is then of the form

$$\begin{aligned} \dot{\begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix}}(t) &= \begin{bmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ 0 & 0 & A_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix}(t) + \begin{bmatrix} B_{11} & 0 & B_{1c} \\ 0 & B_{22} & B_{2c} \\ 0 & 0 & B_{cc} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_c \end{bmatrix}(t), \\ \begin{bmatrix} y_1 \\ y_2 \\ y_c \end{bmatrix}(t) &= \begin{bmatrix} C_{11} & 0 & C_{1c} \\ 0 & C_{22} & C_{2c} \\ 0 & 0 & C_{cc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix}(t). \end{aligned} \tag{13.1}$$

The structure of the system matrices in (13.1) follows directly from conditions 1, 2, and 3 in Definition 13.1.

The interconnections between the different variables of a coordinated linear system are illustrated in Fig. 13.2.

13.3 Basic Properties

The set of coordination-structured matrices

$$\mathbb{R}_{\text{CLS}} = \left\{ \begin{bmatrix} M_{11} & 0 & M_{1c} \\ 0 & M_{22} & M_{2c} \\ 0 & 0 & M_{cc} \end{bmatrix},\ M_{jj} \in \mathbb{R}^{m_j \times n_j},\ j = 1, 2, c \right\}$$

forms an algebraic ring (i.e., it is closed with respect to addition and multiplication). In particular, e^M is of the form

$$\exp\left(\begin{bmatrix} M_{11} & 0 & M_{1c} \\ 0 & M_{22} & M_{2c} \\ 0 & 0 & M_{cc} \end{bmatrix} \right) = \begin{bmatrix} \mathrm{e}^{M_{11}} & 0 & \star_{1c} \\ 0 & \mathrm{e}^{M_{22}} & \star_{2c} \\ 0 & 0 & \mathrm{e}^{M_{cc}} \end{bmatrix},$$

where the entries denoted by $\star$ are not specified further. Hence, the information structure imposed by the invariance properties of Definition 13.1 is left unchanged over time by the system dynamics. This means that the conditional independence imposed on subsystems 1 and 2, given the coordinator state and input, is preserved under the system dynamics: No subsystem is ever influenced by the state or input of the other subsystem, not even indirectly via the coordinator.

If $A, B, C, D \in \mathbb{R}_{\text{CLS}}$ are of the appropriate sizes, then the transfer function of the system is given by

$$\hat{G}(z) = D + C(zI - A)^{-1}B = \begin{bmatrix} G_{11}(z) & 0 & \star_{1c} \\ 0 & G_{22}(z) & \star_{2c} \\ 0 & 0 & G_{cc}(z) \end{bmatrix},$$

where $G_{jj}(z) = D_{jj} + C_{jj}(zI - A_{jj})^{-1}B_{jj}$ corresponds to the transfer function of subsystem j for $j = 1, 2, c$ when disregarding the rest of the system, and

$$\star_{ic} = D_{ic} + C_{ii}(zI - A_{ii})^{-1}B_{ic} + (C_{ic} - C_{ii}(zI - A_{ii})^{-1}A_{ic})(zI - A_{cc})^{-1}B_{cc}.$$

Note that the diagonal entries of the linear combination, product, and inverse of matrices in $\mathbb{R}_{\text{CLS}}$ are just the linear combination, product, and inverse of the corresponding diagonal entries of the original matrices, respectively. This means that these operations also preserve the structure of matrices corresponding to more nested hierarchies: If $A \in \mathbb{R}_{\text{CLS}}$ with a diagonal entry $A_{ii} \in \mathbb{R}_{\text{CLS}}$, then operations as above will yield matrices in $\mathbb{R}_{\text{CLS}}$ with the ii-th entry again in $\mathbb{R}_{\text{CLS}}$. Hence, coordinated linear systems can act as building blocks for constructing linear systems with a more complex hierarchical structure: An extension to an arbitrary number of subsystems is straightforward, and nested hierarchies can be modeled by using another coordinated linear system as one of the subsystems of a coordinated linear system. Hierarchical systems that are modeled by such a combination of coordinated linear systems can again be shown to have an information structure that is invariant with respect to the system dynamics.

This invariance property has important consequences for control synthesis: In Chap. 14, it is shown that the problem of stabilizing the overall system via a static state feedback reduces to local stabilization problems for the different parts of the system, and hence can be solved in a fully decentralized manner. LQ optimal control problems for coordinated linear systems decouple partly, allowing for some subproblems to be solved in a decentralized manner (see Chap. 15).

13.4 Concluding Remarks and Further Reading

In this chapter, coordinated linear systems were introduced, and related concepts were summarized. The concept was defined mathematically, and some basic properties were described. An in-depth analysis of coordinated linear systems and related concepts can be found in [6]. Many of the results developed in [1] and [6] are summarized in the following chapters of this part.

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