Fractional Calculus in Economic Growth Modelling: The Spanish Case

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Abstract. A variety of fractional order models have been proposed in the literature to account for the behaviour of financial processes from different points of view. The objective of this work is to model the growth of national economies, namely, their gross domestic products (GDPs), by means of a fractional order approach. The particular case of Spain is addressed, and results show that fractional models have a better performance than the other alternatives considered and proposed in the literature.

Keywords: Fractional Calculus, Dynamic models, Gross Domestic Product, Economic Growth.

1 Introduction

Fractional derivatives, which are a generalisation of usual, integer-order derivatives, are non-local, and thus suitable for constructing dynamic models for long series where a memory effect can be found — more so than models using integer derivatives and integrals alone [25]. This is the reason why fractional differential equations possess large advantage in describing economic phenomena over large time periods.

A variety of fractional order models have been proposed in the literature to account for the behaviour of financial processes from different points of view. For example, as diffusion or stochastic processes by means of Lévy models [1,4,5,6,17] or continuous time random walks for the movement of log-prices (e.g. [10,16,18,19,22,23]), respectively. A modified fractality concept was applied in [12] to describe the stochastic dynamics of the stock and currency markets. Likewise, a macroeconomic state space model was proposed in [24] for national economies consisting of a group of fractional differential equations. A similar form was used in [27] but with variable orders.

The objective of this work is to model the growth of national economies, namely, their gross domestic products (GDPs), using a dynamic model of fractional order. In the literature many models for the evolution of the GDP have

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been published, among which the classical papers [3,21]. Yet, to the best of our knowledge, no fractional model of GDP as a function of a vector of inputs has yet been found.

In this paper, the GDP of a national economy was modelled as function of a vector with nine variables. The particular case of the economy of Spain along the last five decades is studied. The paper is organised as follows. Section 2 introduces fractional derivatives for reference purposes. Section 3 briefly describes the considered model to account for the behaviour of national economies. In Section 4, the obtained results after fitting are given. Finally, Section 5 draws the concluding remarks and future works.

2 Fractional Calculus

Let us define differential operator D as ${}_cD_t^n f(t) = \frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}$ and ${}_cD_t^{-n} f(t) = \underbrace{\int_c^t \cdots \int_c^t f(\tau)\,\mathrm{d}\tau \cdots \mathrm{d}\tau}_{|n| \text{ integrations}}$. It can be shown by mathematical induction that

$$ {}_cD_t^n f(t) = \lim_{h\to 0} \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t-kh)}{h^n}, \; n \in \mathbb{N} \tag{1}$$

where combinations of a things, b at a time are given by $\binom{a}{b} = \frac{a!}{b!(a-b)!}$. This can be generalised using the Gamma function, which verifies $\Gamma(n) = (n-1)!, n \in \mathbb{N}$ and is defined in $\mathbb{C}\backslash\mathbb{Z}^-$, as

$$ \binom{a}{b} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}, \text{ if } a, b, a-b \notin \mathbb{Z}^- \\ \frac{(-1)^b \Gamma(b-a)}{\Gamma(b+1)\Gamma(-a)}, \text{ if } a \in \mathbb{Z}^- \wedge b \in \mathbb{Z}_0^+ \\ 0, \text{ if } [(b \in \mathbb{Z}^- \vee b-a \in \mathbb{N}) \wedge a \notin \mathbb{Z}^-] \vee (a, b \in \mathbb{Z}^- \wedge |a| > |b|) \end{cases} \tag{2}$$

Using (2), it is reasonable to generalise (1) for non-integer orders as

$$ {}_cD_t^\alpha f(t) = \lim_{h\to 0^+} \frac{\sum_{k=0}^{\lfloor \frac{t-c}{h} \rfloor} (-1)^k \binom{\alpha}{k} f(t-kh)}{h^\alpha} \tag{3}$$

Values c and t are called terminals. The upper limit of the summation in (3) is diverging to $+\infty$. When $\alpha \in \mathbb{N}$, all terms with $k > \alpha$ will be zero; thus (3) reduces to (1) when $h > 0$. This is the only case in which the summation has a finite number of terms and the result does not depend on terminal c.

The upper limit $\lfloor \frac{t-c}{h} \rfloor$ was set so that, if $\alpha = -1, -2, -3, \ldots$, (3) becomes a Riemann integral (calculated from c to t).

For more details on operator D, properties, alternative definitions, and Laplace transforms, see [25,26].

3 Economic Growth Model

Consider a simple model of a national economy in the following form:

$$y(t) = f(x_1, x_2 \ldots) \tag{4}$$

where the output model y is the GDP (in 2012 euros) and the x_k are the variables on which the output depends. The inputs considered and their rationale are the following:

- natural resources are represented by x_1 (land area, km^2), and their quality by x_2 (arable land, km^2);
- human resources are represented by x_3 (population), and their quality by x_4 (average years of school attendance);
- manufactured resources are represented by x_5 (gross capital formation (GCF), in 2012 euros);
- external impacts in the economy are represented by x_6 (exports of goods and services, in 2012 euros);
- internal impacts in the economy are represented as follows: budgetary impacts by x_7 (general government final consumption expenditure (GGFCE), in 2012 euros), monetary impacts by x_8 (money and quasi money (M2), in 2012 euros), and investment by x_5. Rather than having x_5 play two roles, we will rather use a ninth variable $x_9 \equiv x_5$ to represent the impact of investment in the economy.

This choice of variables joins those traditionally considered in growth accounting [8,13,14] to those acknowledged by Keynesian models having short-term inputs related to impacts in the economy.

Thus, the following integer and fractional order models were considered:

$$\begin{aligned} y(t) =& C_1 x_1(t) + C_2 x_2(t) + C_3 x_3(t) + C_4 x_4(t) + C_5 \int_{t_0}^{t} x_5(t)\mathrm{d}t + \\ &+ C_6 x_6(t) + C_7 x_7(t) + C_8 \frac{\mathrm{d}x_8(t)}{\mathrm{d}t} + C_9 \frac{\mathrm{d}x_9(t)}{\mathrm{d}t}, \end{aligned} \tag{5}$$

$$y(t) = \sum_{k=1}^{9} C_k \, D^{\alpha_k} x_k(t), \tag{6}$$

where C_k and α_k are constant weights and the differentiation orders for each of the variables, respectively, and t_0 is the first year considered. Notice that in the integer model the accumulated gross capital formation ($\int_{t_0}^{t} x_5(t)\mathrm{d}t$) is used as a measure of manufactured resources; the variation of M2 ($\frac{\mathrm{d}x_8(t)}{\mathrm{d}t}$) is used as a measure of the monetary impacts in the economy; and the variation of the gross capital formation ($\frac{\mathrm{d}x_9(t)}{\mathrm{d}t} = \frac{\mathrm{d}x_5(t)}{\mathrm{d}t}$) is used as a measure of the impact of investment in the economy.

4 Results for the Spanish Economy

Using the models above, the economy of Spain was modelled in the period between 1960 to 2012. This period was considered not only because it is the one for which reliable data can be easily obtained, but also because this is the period where modern economic growth consistently took hold of the Spain economy. (See data in the Appendix.) The goal of the fitting is to calculate parameters α_k and C_k of the dynamic models (5) and (6) for this particular national economy. The fitting procedure was implemented in MATLAB, using Nelder-Mead's simplex search method as implemented in function *fminsearch*, by minimising the mean square error (MSE). To evaluate the goodness-of-fit of the obtained models, the following performance indices were also calculated: (i) mean absolute deviation (MAD); (ii) coefficient of determination (R^2); (iii) $t-$ and $p-$values for each variable.

The obtained models are shown in Fig. 1, with the values of the orders α and the coefficients C given in Table 1. It can be seen that the differentiation orders obtained for x_1, x_2, x_3, x_6 and x_7 of fractional order model (6) are zero (or almost zero), which leads us to consider a simpler model, in which only variables x_4, x_5, x_8 and x_9 are assumed to have fractional order influence, as follows:

$$y(t) = \sum_{k=1,2,3,6,7} C_k x_k(t) + \sum_{k=4,5,8,9} C_k\, D^{\alpha_k} x_k(t). \tag{7}$$

What this means is that not all economic indicators have the same influence over time on the GDP: for some (those with $\alpha = 0$) only the current value matters. The results related to this model were also included in Fig. 1 and Table 1.

The performance indices calculated for models (5), (6) and (7) are summarized in Table 2, where $t-$values corresponding to variables which are necessary for the model, assuming a 5% significance level, are in bold. As observed, the population (x_3) and the variation of GCF (x_9) have a considerable effect on the integer model, whereas the remaining variables have low influence. In contrast, for the fractional model, it is clear that the arable land (x_2) and the GGFCE (x_7) are variables without much influence in the GDP. Likewise, the fractional model has a clearly better performance, at the expense of needing more variables (7 against 2 for the integer model).

Taking into account the low influence of variables x_2 and x_7 in the model, let us consider a simpler model with only 7 inputs, with an integer form given by

$$\begin{aligned} y(t) =& C_1 x_1(t) + C_3 x_3(t) + C_4 x_4(t) + C_5 \int_{t_0}^{t} x_5(t)\mathrm{d}t + \\ &+ C_6 x_6(t) + C_8 \frac{\mathrm{d}x_8(t)}{\mathrm{d}t} + C_9 \frac{\mathrm{d}x_9(t)}{\mathrm{d}t}, \end{aligned} \tag{8}$$

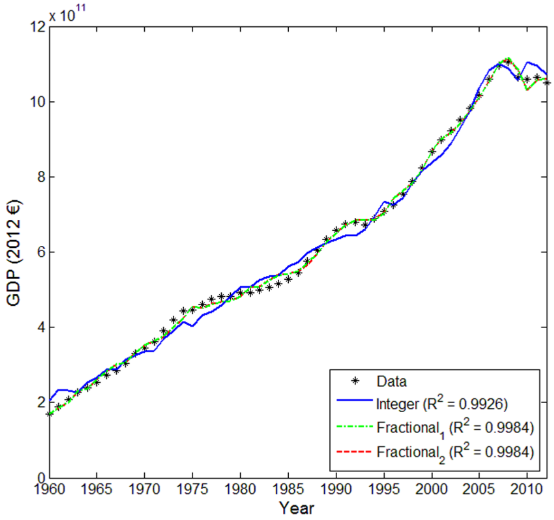

Fig. 1. Fitting results for the Spanish case (models with 9 variables): the integer model is given by (5), the fractional$_1$ model by (6), and the fractional$_2$ model by (7)

Table 1. Fitting results: orders of the fractional operator and coefficients

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
Integer (5)	0	0	0	0	-1	0	0	1	1
Fractional (6)	0	0	0	0.068	0.860	0	1.250×10^{-4}	-1.020	-0.834
Fractional (7)	0	0	0	0.066	0.855	0	0	-1.016	-0.822
Integer (8)	0	—	0	0	-1	0	—	1	1
Fractional (9)	0	—	0	0.198	-0.809	0	—	-0.995	0.988

	C_1 ($\times 10^5$)	C_2 ($\times 10^6$)	C_3 ($\times 10^4$)	C_4 ($\times 10^{10}$)	C_5 ($\times 10^{-2}$)	C_6 ($\times 10^{-2}$)	C_7 ($\times 10^{-1}$)	C_8 ($\times 10^{-2}$)	C_9 ($\times 10^{-1}$)
Integer (5)	-3.209	-3.137	2.365	2.974	1.385	-4.541	8.841	14.711	10.315
Fractional (6)	8.760	1.616	-1.767	-3.431	645.925	46.860	-5.935	-2.310	4.788
Fractional (7)	8.789	1.579	-1.735	-3.742	647.620	49.818	-3.536	-2.393	4.917
Integer (8)	-12.09	—	1.841	5.111	4.067	13.23	—	204.30	7.070
Fractional (9)	10.31	—	-1.153	-3.991	41.643	38.34	—	-20.98	829.228

and a fractional form as

$$y(t) = \sum_{k=\ 1,3,6} C_k x_k(t) + \sum_{k=4,5,8,9} C_k \mathrm{D}^{\alpha_k} x_k(t). \tag{9}$$

The obtained models consisting of 7 variables are shown in Fig. 2. The values of the orders α and the coefficients C are also given in Table 1, and the performance indices in Table 2.

Table 2. Performance indices

Index / Statistic	Variable	Models with 9 variables Integer (5)	Fractional (6)	Fractional (7)	Models with 7 variables Integer (8)	Fractional (9)
MSE ($\times 10^{20}$)		5.610	1.228	1.241	6.084	1.320
R^2		0.9926	0.9984	0.9984	0.9920	0.9983
MAD ($\times 10^{10}$)		2.033	0.912	0.920	2.0820	0.9257
$t-$values	x_1	-0.425	**3.953**	**3.831**	-2.150	**5.190**
	x_2	-1.836	2.036	2.044	—	—
	x_3	**3.276**	**−4.117**	**−3.962**	**2.917**	**−3.634**
	x_4	0.724	**−8.277**	**−7.355**	1.879	**−8.121**
	x_5	0.385	**11.977**	**10.731**	1.339	**17.764**
	x_6	-0.113	**4.019**	**4.008**	0.474	**3.669**
	x_7	0.719	-1.489	-0.936	—	—
	x_8	2.237	**−16.560**	**−16.236**	**3.437**	**−15.678**
	x_9	**2.736**	**12.264**	**9.508**	2.093	**12.359**
$p-$values	x_1	0.673	2.762×10^{-4}	4.013×10^{-4}	3.682×10^{-2}	4.634×10^{-6}
	x_2	7.903×10^{-2}	4.777×10^{-2}	4.695×10^{-2}	—	—
	x_3	3.993×10^{-3}	1.662×10^{-4}	2.688×10^{-4}	5.441×10^{-3}	7.001×10^{-4}
	x_4	0.480	1.622×10^{-10}	3.465×10^{-9}	6.657×10^{-2}	1.964×10^{-10}
	x_5	0.720	2×10^{-15}	7.3×10^{-14}	0.187	0
	x_6	0.912	2.252×10^{-4}	2.330×10^{-4}	0.637	6.315×10^{-4}
	x_7	0.502	0.143	0.354	—	—
	x_8	3.334×10^{-2}	0	0	1.257×10^{-3}	0
	x_9	8.941×10^{-3}	1×10^{-15}	3.090×10^{-12}	4.190×10^{-2}	0

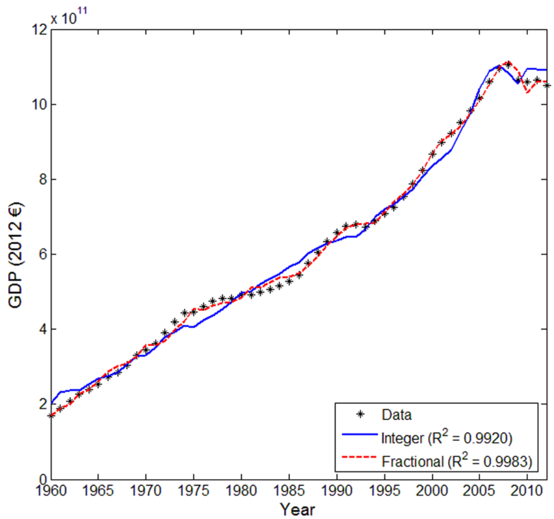

Fig. 2. Fitting results for the Spanish case (models with 7 variables): the integer model is given by (8), and the fractional model by (9)

There is, of course, a slight deterioration of performance, but the results obtained with fractional model (9) remain highly satisfactory. Again, this is achieved at the expense of the model needing more independent variables than its integer counterpart, as seen from the $t-$ and $p-$values.

Furthermore, fractional orders of x_8 and x_9 appearing in Table 1 are nearly ± 1. It is worth mentioning that the sign of α_8 is different in the integer and fractional models (8) and (9). This is particularly significant since it shows that M2 has an effect over a long time (a derivative of order almost -1 is not a local operator). On the other hand, variables x_1, x_3, x_6 and x_9 turn out to have influence in the present only. Finally, we can note some similarity between the fractional models obtained and those of fractional diffusion processes [15]. We can thus hypothesize that such diffusion models (useful in areas such as bioengineering or soil dynamics) can also explain how these variables affect the economy; this hypothesis can only be checked when more countries are studied.

5 Conclusions

This paper investigated modelling of national economic growth, namely, the gross domestic product (GDP), using models from Fractional Calculus. Nine macroeconomic indicators, chosen according to the practice established in the literature, were used to account for the behaviour of this financial process. The particular case of Spain was studied for the period 1960–2012, and results show that fractional models have a better performance than the other alternatives considered and proposed in the literature. In the end, a simplified model with only seven inputs was obtained. External and internal impacts, manufactured resources, and the quality of the natural and human resources are seen to be important factors.

Our future efforts will focus on study other economies of the European zone, and verifying our results using other criteria, such as the AIC or the BIC [11].

Appendix

Sources for the economic data in Table 3 are as follows:

- x_1 is taken from [2]. The data concerns what is currently the territory of Spain only, and not what are now Equatorial Guinea and Western Sahara, which were always separate national economies. Slight variations in area, found in the database, which are spurious, since the territory of Spain did not change in the period considered, were discarded. This input is thus constant.
- x_2 and x_3 are taken from [2].
- x_4 is taken from [7]. As the data has a 5-year sampling time (starting in 1960), a third-order spline interpolation was used for intercalary years.
- x_5, x_6 and x_7 are taken from [2], in current euros. The price index mentioned below was used to convert values to 2012 euros.

Table 3. Spanish economic data for years 1960–2012. GDP, x_5, x_6, x_7 and x_8 in 2012 euros, x_1 in km^2, x_2 in % of x_1, x_3 in people and x_4 in years

Year	GDP ($\times 10^{11}$)	x_1	x_2	x_3	x_4	GCF ($\times 10^{10}$)	x_6 ($\times 10^{10}$)	x_7 ($\times 10^{10}$)	x_8 ($\times 10^{10}$)
1960	1.69	499780	32.51	30455000	4.7	13.5	1.41	1.52	9.91
1961	1.89	499780	32.51	30739250	4.74	15.2	1.50	1.66	11.33
1962	2.07	499780	32.61	31023366	4.77	1.58	1.72	1.80	12.34
1963	2.27	499780	32.42	31296651	4.79	14.5	1.75	2.04	13.42
1964	2.39	499780	31.85	31609195	4.81	13.7	2.11	2.10	15.28
1965	2.54	499780	31.95	31954292	4.82	13.3	2.08	2.29	16.46
1966	2.73	499780	31.03	32283194	4.83	12.4	2.43	2.55	17.27
1967	2.85	499780	31.49	32682947	4.85	10.8	2.43	2.87	18.34
1968	3.03	499780	31.40	33113134	4.87	10.2	3.21	2.96	20.48
1969	3.30	499780	32.18	33441054	4.91	10.6	3.74	3.24	23.18
1970	3.45	499780	31.39	33814531	4.95	9.51	4.29	3.49	25.37
1971	3.61	499780	32.69	34191678	5.01	9.15	4.81	3.72	29.16
1972	3.90	499780	32.59	34502705	5.07	10.4	5.34	3.98	32.99
1973	4.20	499780	32.12	34817071	5.15	11.7	5.74	4.28	36.82
1974	4.44	499780	31.85	35154338	5.22	13.8	6.01	4.71	38.07
1975	4.47	499780	31.66	35530725	5.3	13.1	5.67	4.50	38.78
1976	4.61	499780	31.34	35939437	5.37	12.8	5.95	5.58	39.61
1977	4.74	499780	31.29	36370050	5.44	12.2	6.45	5.84	38.18
1978	4.81	499780	31.31	36872506	5.51	11.3	6.85	6.14	37.81
1979	4.82	499780	31.18	37201123	5.58	11.0	6.77	6.40	38.33
1980	4.92	499780	31.15	37439035	5.66	11.7	7.22	6.88	39.55
1981	4.92	499780	31.17	37740556	5.75	10.9	8.21	7.34	41.16
1982	4.98	499780	31.16	37942805	5.85	10.9	8.67	7.52	42.40
1983	5.07	499780	31.22	38122429	5.95	10.7	9.92	7.89	43.74
1984	5.16	499780	31.34	38278575	6.06	10.3	11.27	7.90	45.36
1985	5.28	499780	31.16	38418817	6.17	10.7	11.29	8.28	47.26
1986	5.45	499780	31.16	38535617	6.28	11.5	10.17	8.38	48.38
1987	5.75	499780	31.20	38630820	6.38	13.0	10.45	9.14	52.48
1988	6.04	499780	31.19	38715849	6.49	14.9	10.72	9.51	56.20
1989	6.33	499780	31.06	38791473	6.61	16.4	10.78	10.30	60.39
1990	6.57	499780	30.70	38850435	6.73	17.2	10.60	10.97	62.88
1991	6.74	499780	30.55	38939049	6.86	17.1	10.89	11.71	65.46
1992	6.80	499780	30.44	39067745	6.7	15.9	11.29	12.44	64.49
1993	6.73	499780	29.99	39189400	7.14	14.1	12.23	12.68	67.90
1994	6.89	499780	29.64	39294967	7.28	14.5	14.36	12.57	69.98
1995	7.08	499780	28.12	39387017	7.42	1.55	15.86	12.81	72.81
1996	7.25	499780	28.93	39478186	7.56	15.7	17.14	13.05	75.56
1997	7.53	499780	28.60	39582413	7.69	16.6	19.82	13.17	77.00
1998	7.87	499780	27.40	39721108	7.83	18.5	20.99	13.63	75.94
1999	8.24	499780	26.96	39926268	7.97	20.7	21.99	14.16	79.58
2000	8.66	499780	26.85	40263216	8.13	22.8	25.17	14.85	84.71
2001	8.98	499780	26.20	40720484	8.29	23.7	25.63	15.29	89.47
2002	9.22	499780	25.87	41313973	8.47	24.6	25.20	15.82	92.14
2003	9.51	499780	26.07	42004522	8.64	26.1	25.02	16.46	100.5
2004	9.82	499780	26.09	42691689	8.81	27.8	25.46	17.44	115.3
2005	10.17	499780	25.87	43398143	8.97	30.0	26.10	18.27	143.3
2006	10.58	499780	25.49	44116441	9.11	32.7	27.83	19.02	175.1
2007	10.95	499780	25.22	44878945	9.23	33.9	29.46	20.07	202.2
2008	11.05	499780	25.04	45555716	9.32	32.2	29.28	21.54	214.6
2009	10.63	499780	25.05	45908594	9.37	2.55	25.43	22.69	223.2
2010	10.60	499780	25.12	46070971	9.39	2.42	28.82	22.69	224.0
2011	10.65	499780	25.08	46174601	9.47	2.29	32.22	22.30	214.6
2012	10.49	499780	25.04	46217961	9.56	2.06	33.80	21.14	199.5

- x_8 is taken from [9] in current euros in the 1999–2012 period. In the 1962–1968 period, it is taken from [2] also in current euros. These two series are clearly coherent. [20] has data for 1941–1970 in current pesetas; values for 1962–1970 are consistently 60% of those in [2]: and so for 1960–1961 we used the values of [20] converted to euros and divided by 0.6. The price index mentioned below was used to convert values to 2012 euros.
- The price index mentioned several times above is the one implicit in [2], that for several variables provides values in current euros and in constant euros.

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