Chapter 9
Quasistatic Problems

This chapter deals with the study of quasistatic contact problems with a nonlocal Coulomb friction law. We first consider that the unilateral contact is modeled by the Signorini conditions. In this case, a variational formulation (see [7]) involves two inequalities with the simultaneous presence of the displacement field and of the velocity field. More precisely, the friction law generates an inequality with the velocity field as test function while the Signorini conditions lead to an inequality with the displacement field as test function. Applying Theorem 4.19 (p. 77), a known existence result (see [7]) is provided. We then prove, following the work [5], convergence results for a space finite element approximation and an implicit time discretization scheme of this problem. The last section is devoted, as in the work [6], to the study of a boundary control problem related to a quasistatic bilateral contact problem with nonlocal Coulomb friction.

Concerning the study of quasistatic contact problems in elasticity, we mention the existence and/or uniqueness results obtained, in the case of a normal compliance law, by Andersson [3] and Klarbring et al. [9], and, in the case of a local or nonlocal Coulomb law with unilateral contact, by Cocu et al. [7], Andersson [4], Cocou and Roca [8], Rocca [14]. For the study of quasistatic bilateral contact problems involving viscoelastic or viscoplastic materials, we refer to Shillor and Sofonea [15], Shillor et al. [16] and Amassad [1].

9.1 Classical and Variational Formulations

The quasistatic evolutionary of an elastic body in unilateral contact with a rigid foundation is considered. We suppose that the volume forces $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}, t)$ and the surface tractions $\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{x}, t)$ are applied so slowly that the inertial forces may be neglected.

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With the notation adopted in Sect. 8.1, the classical formulation of the quasistatic problem is obtained, as in the static case, by considering the equilibrium equations, the constitutive equation, the kinematic relation, the boundary conditions, and the initial condition.

Problem ($\mathscr{Q}$): Find a displacement field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t) : \Omega \times [0, T] \to \mathbb{R}^d$ such that

$$-\operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in } \Omega \times (0, T)\,, \tag{9.1}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u}) = \mathscr{A}\boldsymbol{\epsilon}\,, \qquad \boldsymbol{\epsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) \quad \text{in } \Omega \times (0, T)\,, \tag{9.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_0 \times (0, T)\,, \tag{9.3}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{\nu} = \boldsymbol{g} \quad \text{on } \Gamma_1 \times (0, T)\,, \tag{9.4}$$

$$u_\nu \le 0\,, \quad \sigma_\nu \le 0\,, \quad u_\nu \sigma_\nu = 0 \quad \text{on } \Gamma_2 \times (0, T)\,, \tag{9.5}$$

$$|\boldsymbol{\sigma}_\tau| \le \mu|\sigma_\nu| \ \text{and} \begin{cases} |\boldsymbol{\sigma}_\tau| < \mu|\mathscr{R}\sigma_\nu| \ \Rightarrow \ \dot{\boldsymbol{u}}_\tau = 0 \\ |\boldsymbol{\sigma}_\tau| = \mu|\mathscr{R}\sigma_\nu| \ \Rightarrow \ \exists\lambda \ge 0\,,\ \dot{\boldsymbol{u}}_\tau = -\lambda\boldsymbol{\sigma}_\tau \end{cases} \text{on } \Gamma_2 \times (0, T)\,, \tag{9.6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega\,. \tag{9.7}$$

where $\mathscr{A} = (a_{ijkh})$ is the fourth order tensor of elasticity with the elasticity coefficients satisfying the symmetry and ellipticity conditions:

$$a_{ijkh} = a_{jihk} = a_{khij}\,,\ \forall 1 \le i, j, k, h \le d\,,$$
$$\exists\,\alpha > 0 \text{ tel que } a_{ijkh}\xi_{ij}\xi_{kh} \ge \alpha|\boldsymbol{\xi}|^2\,,\ \forall\boldsymbol{\xi} = (\xi_{ij}) \in \mathbb{R}^{d^2}\,. \tag{9.8}$$

In order to derive a variational formulation of the problem (9.1)–(9.7), we suppose that

$$\boldsymbol{f} \in W^{1,2}(0, T; (L^2(\Omega))^d)\,,$$
$$\boldsymbol{g} \in W^{1,2}(0, T; (L^2(\Gamma_1))^d)\,,$$
$$a_{ijkl} \in L^\infty(\Omega),\ i, j, k, l = 1, \ldots, d,$$
$$\mu \in L^\infty(\Gamma_2),\ \mu \ge 0 \text{ a.e. on } \Gamma_2$$
$$\mathscr{R} : H^{-1/2}(\Gamma_2) \to L^2(\Gamma_2) \text{ is a linear continuous operator.} \tag{9.9}$$

We shall use the notation

$$\boldsymbol{V} = \{\boldsymbol{v} \in (H^1(\Omega))^d\,;\ \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_0\}\,,$$
$$\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{V}\,;\ v_\nu \le 0 \text{ a.e. on } \Gamma_2\}\,,$$
$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\epsilon}(\boldsymbol{v})\,\mathrm{d}x \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}\,. \tag{9.10}$$

Let $\boldsymbol{F} \in W^{1,2}(0,T;\boldsymbol{V})$ where, for all $t \in [0,T]$, $\boldsymbol{F}(t)$ is the element of $\boldsymbol{V}$ defined by

$$(\boldsymbol{F}(t),\boldsymbol{v}) = \int_{\Omega} \boldsymbol{f}(t)\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}(t)\cdot\boldsymbol{v}\,\mathrm{d}s \quad \forall \boldsymbol{v}\in\boldsymbol{V}\,, \tag{9.11}$$

where we have denoted by $(\cdot,\cdot)$ the inner product over the space $\boldsymbol{V}$.

We also put

$$\boldsymbol{W} = \{\boldsymbol{w}\in\boldsymbol{V};\ \operatorname{div}\boldsymbol{\sigma}(\boldsymbol{w})\in(L^2(\Omega))^d\}\,. \tag{9.12}$$

For simplicity, we denote by $\langle\cdot,\cdot\rangle$ the duality pairing between $(H^{-1/2}(\Gamma_2))^d$ and $(H^{1/2}(\Gamma_2))^d$ or between $H^{-1/2}(\Gamma_2)$ and $H^{1/2}(\Gamma_2)$. Then, as we have precise in Sect. 8.1, we have

$$\langle\boldsymbol{\sigma}(\boldsymbol{w})\cdot\boldsymbol{\nu},\boldsymbol{v}\rangle = \int_{\Omega}\boldsymbol{\sigma}(\boldsymbol{w})\boldsymbol{\epsilon}(\bar{\boldsymbol{v}})\,\mathrm{d}x + \int_{\Omega}\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{w})\bar{\boldsymbol{v}}\,\mathrm{d}x \quad \forall\boldsymbol{w}\in\boldsymbol{W}\,,\ \forall\boldsymbol{v}\in(H^{1/2}(\Gamma_2))^d$$

where $\bar{\boldsymbol{v}}\in(H^1(\Omega))^d$ satisfies $\bar{\boldsymbol{v}}=\boldsymbol{v}$ almost everywhere on Γ_2.

Therefore, we define the normal component of the stress tensor $\sigma_\nu(\boldsymbol{w}) \in H^{-1/2}(\Gamma_2)$ by

$$\langle\sigma_\nu(\boldsymbol{w}),v\rangle = \int_{\Omega}\boldsymbol{\sigma}(\boldsymbol{w})\boldsymbol{\epsilon}(\bar{\boldsymbol{v}})\,\mathrm{d}x + \int_{\Omega}\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{w})\bar{\boldsymbol{v}}\,\mathrm{d}x \quad \forall\boldsymbol{w}\in\boldsymbol{W}\,, \quad \forall v\in H^{1/2}(\Gamma_2)$$

where $\bar{\boldsymbol{v}}\in(H^1(\Omega))^d$ satisfies $\bar{\boldsymbol{v}}_\tau=\boldsymbol{0}$ and $\bar{v}_\nu = v$ a.e. on Γ_2.

It is easy to verify that, for any $\boldsymbol{w}\in\boldsymbol{W}$, the above definitions of $\boldsymbol{\sigma}(\boldsymbol{w})\cdot\boldsymbol{\nu}$ and $\sigma_\nu(\boldsymbol{w})$ are independent on the choice of $\bar{\boldsymbol{v}}$.

For all $\boldsymbol{\Theta}\in\boldsymbol{V}$, we introduce the functional $\tilde{j}_{\boldsymbol{\Theta}}:\boldsymbol{K}(\boldsymbol{\Theta})\times\boldsymbol{V}\to\mathbb{R}$ defined by

$$\tilde{j}_{\boldsymbol{\Theta}}(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_2}\mu|\mathscr{R}\sigma_\nu(\boldsymbol{u})|\,|\boldsymbol{v}_t|\,\mathrm{d}s \quad \forall\boldsymbol{u}\in\boldsymbol{K}(\boldsymbol{\Theta}) \quad \forall\boldsymbol{v}\in\boldsymbol{V}\,, \tag{9.13}$$

where

$$\boldsymbol{K}(\boldsymbol{\Theta}) = \{\boldsymbol{w}\in\boldsymbol{K}\,;\ a(\boldsymbol{w},\boldsymbol{\psi}) = (\boldsymbol{\Theta},\boldsymbol{\psi})\,,\ \forall\boldsymbol{\psi}\in\boldsymbol{V} \text{ such that } \boldsymbol{\psi}=0 \text{ a.e. on } \Gamma_2\}\,.$$

A variational formulation of this problem (see [7]) involves two inequalities and the simultaneous presence of the displacement field and of the velocity field. More precisely, the friction law generates an inequality with the velocity field as test function while the Signorini conditions lead to an inequality with the displacement field as test function. So, we shall consider the following weak formulation of Problem $(\mathscr{Q})$.

Problem (Q): Find $\boldsymbol{u} \in W^{1,2}(0,T;\boldsymbol{V})$ such that

$$\begin{cases} \boldsymbol{u}(0) = \boldsymbol{u}_0\,, \quad \boldsymbol{u}(t) \in \boldsymbol{K} \quad \forall t \in [0,T] \\ a(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + \tilde{j}_{\boldsymbol{F}(t)}(\boldsymbol{u}(t), \boldsymbol{v}) - \tilde{j}_{\boldsymbol{F}(t)}(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \\ \geq (\boldsymbol{F}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + \langle \sigma_\nu(\boldsymbol{u}(t)), v_\nu - \dot{u}_\nu(t) \rangle \quad \forall \boldsymbol{v} \in \boldsymbol{V} \quad \text{a.e. } t \in (0,T) \\ \langle \sigma_\nu(\boldsymbol{u}(t)), z_\nu - u_\nu(t) \rangle \geq 0 \quad \forall \boldsymbol{z} \in \boldsymbol{K}\,,\ \forall t \in (0,T)\,. \end{cases} \tag{9.14}$$

Remark 9.1. If $\boldsymbol{u}$ verifies the first inequality of Problem **(Q)**, then $\boldsymbol{u}(t) \in \boldsymbol{K}(\boldsymbol{F}(t))$, $\forall t \in [0,T]$.

We suppose that the initial displacement $\boldsymbol{u}_0 \in \boldsymbol{K}$ satisfies the following compatibility condition

$$a(\boldsymbol{u}_0, \boldsymbol{v}) + \tilde{j}_{\boldsymbol{F}(0)}(\boldsymbol{u}_0, \boldsymbol{v}) \geq (\boldsymbol{F}(0), \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{K}\,. \tag{9.15}$$

In order to show that the classical formulation ($\mathscr{Q}$) and the variational formulation **(Q)** are equivalent, we first prove the following result.

Lemma 9.1. *Let $\tilde{\boldsymbol{u}} \in \boldsymbol{K} \cap \boldsymbol{W}$ be a regular function. Then, the following two conditions are equivalent:*

$$\tilde{u}_\nu \leq 0\,,\ \sigma_\nu(\tilde{\boldsymbol{u}}) \leq 0\,,\ \tilde{u}_\nu\,\sigma_\nu(\tilde{\boldsymbol{u}}) = 0 \quad \textit{on } \Gamma_2 \tag{9.16}$$

$$\langle \sigma_\nu(\tilde{\boldsymbol{u}}), z_\nu - \tilde{u}_\nu \rangle \geq 0 \quad \forall \mathbf{z} \in \boldsymbol{K}\,. \tag{9.17}$$

Proof. If the unilateral contact conditions (9.16) hold, then we have

$$\langle \sigma_\nu(\tilde{\boldsymbol{u}}), z_\nu - \tilde{u}_\nu \rangle = \langle \sigma_\nu(\tilde{\boldsymbol{u}}), z_\nu \rangle - \langle \sigma_\nu(\tilde{\boldsymbol{u}}), \tilde{u}_\nu \rangle = \langle \sigma_\nu(\tilde{\boldsymbol{u}}), z_\nu \rangle \geq 0 \quad \forall \mathbf{z} \in \boldsymbol{K}\,.$$

Conversely, if (9.17) is satisfied, then, by taking $\mathbf{z} = \mathbf{0}$ and $\mathbf{z} = 2\boldsymbol{u}$, we obtain

$$\langle \sigma_\nu(\tilde{\boldsymbol{u}}), \tilde{u}_\nu \rangle = 0\,, \tag{9.18}$$

and hence, by the inequality (9.17), we get

$$\langle \sigma_\nu(\tilde{\boldsymbol{u}}), z_\nu \rangle \geq 0 \quad \forall \mathbf{z} \in \boldsymbol{K}\,. \tag{9.19}$$

Finally, from the relations (9.18), (9.19) and the definition of $\boldsymbol{K}$, we conclude the proof. □

Following the standard procedure, we derive the next result.

Theorem 9.1. *The mechanical problem* ($\mathscr{Q}$) *is formally equivalent to the weak formulation* **(Q)** *in the following sense:*

(i) If $\boldsymbol{u}$ is a sufficiently smooth function which verifies the mechanical problem (9.1)–(9.7), then $\boldsymbol{u}$ is a solution of the variational problem (9.14).

(ii) If $\boldsymbol{u}$ *is a regular solution of the variational problem (9.14), then* $\boldsymbol{u}$ *verifies (9.1)–(9.7) in the distributional sense.*

Proof. For simplicity, we shall omit the variable t.

(i) Multiplying Eq. (9.1) by $\boldsymbol{v} - \dot{\boldsymbol{u}}$ with $\boldsymbol{v} \in \boldsymbol{V}$ and integrating by parts over Ω, we obtain

$$a(\boldsymbol{u}, \boldsymbol{v} - \dot{\boldsymbol{u}}) - \int_{\Gamma} \boldsymbol{\sigma} \cdot \boldsymbol{\nu}(\boldsymbol{v} - \dot{\boldsymbol{u}})\, \mathrm{d}s = \int_{\Omega} \boldsymbol{f}(\boldsymbol{v} - \dot{\boldsymbol{u}})\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V},$$

and so, by using (9.3) and (9.4), we get

$$a(\boldsymbol{u}, \boldsymbol{v} - \dot{\boldsymbol{u}}) - \int_{\Gamma_2} (\sigma_\nu (v_\nu - \dot{u}_\nu) + \boldsymbol{\sigma}_\tau (\boldsymbol{v}_\tau - \dot{\boldsymbol{u}}_\tau))\, \mathrm{d}s = (\boldsymbol{F}, \boldsymbol{v} - \dot{\boldsymbol{u}}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{9.20}$$

Hence, for $\boldsymbol{v} = \boldsymbol{\psi} + \dot{\boldsymbol{u}}$ with $\boldsymbol{\psi} \in \boldsymbol{V}$ such that $\boldsymbol{\psi} = 0$ a.e. on Γ_2, we deduce that $\boldsymbol{u} \in \boldsymbol{K}(\boldsymbol{F})$.

On the other hand, the Coulomb friction law (9.6) implies

$$\tilde{j}_{\boldsymbol{F}}(\boldsymbol{u}, \boldsymbol{v}) - \tilde{j}_{\boldsymbol{F}}(\boldsymbol{u}, \dot{\boldsymbol{u}}) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau (\boldsymbol{v}_\tau - \dot{\boldsymbol{u}}_\tau)\, \mathrm{d}s \geq 0 \quad \forall \boldsymbol{v} \text{ smooth function}. \tag{9.21}$$

Indeed, let us denote $E = \mu|\mathscr{R}\sigma_\nu|(|\boldsymbol{v}_\tau| - |\dot{\boldsymbol{u}}_\tau|) + \boldsymbol{\sigma}_\tau(\boldsymbol{v}_\tau - \dot{\boldsymbol{u}}_\tau)$.
If $|\boldsymbol{\sigma}_\tau| < \mu|\mathscr{R}\sigma_\nu|$, then $\dot{\boldsymbol{u}}_\tau = \boldsymbol{0}$, and hence

$$E \geq -|\boldsymbol{\sigma}_\tau|\,|\boldsymbol{v}_\tau| + \mu|\mathscr{R}\sigma_\nu|\,|\boldsymbol{v}_\tau| \geq 0.$$

If $|\boldsymbol{\sigma}_\tau| = \mu|\mathscr{R}\sigma_\nu|$, then we have $\dot{\boldsymbol{u}}_\tau = -\lambda\boldsymbol{\sigma}_\tau$, and so

$$E = \boldsymbol{\sigma}_\tau \boldsymbol{v}_\tau + |\boldsymbol{\sigma}_\tau|\,|\boldsymbol{v}_\tau| \geq 0.$$

Combining (9.20) and (9.21), we deduce that $\boldsymbol{u}$ verifies the first inequality of (9.14).

The second inequality of (9.14) is obtained from (9.5) and Lemma 9.1 for $\tilde{\boldsymbol{u}} = \boldsymbol{u}$.

(ii) If we take $\boldsymbol{v} = \dot{\boldsymbol{u}} \pm \boldsymbol{\varphi}$ in the first inequality of Problem (**Q**), with $\boldsymbol{\varphi} \in (\mathscr{D}(\Omega))^d$ and we apply Green's formula (8.7), then we obtain (9.1) in the distributional sense.

It is immediate, from Lemma 9.1 and the second inequality of (9.14), that the Signorini contact conditions (9.5) are satisfied.

In order to obtain (9.4), we multiply the relation (9.1) by $\boldsymbol{v} - \dot{\boldsymbol{u}}$ with $\boldsymbol{v} \in \boldsymbol{V}$, and so, by integrating by parts and using the first inequality of (9.14), we obtain

$$\begin{aligned} &\tilde{j}_{\boldsymbol{F}}(\boldsymbol{u}, \boldsymbol{v}) - \tilde{j}_{\boldsymbol{F}}(\boldsymbol{u}, \dot{\boldsymbol{u}}) + \int_{\Gamma} (\boldsymbol{\sigma} \cdot \boldsymbol{\nu})(\boldsymbol{v} - \dot{\boldsymbol{u}})\, \mathrm{d}s - \int_{\Gamma_1} \boldsymbol{g}(\boldsymbol{v} - \dot{\boldsymbol{u}})\, \mathrm{d}s \\ &\geq \langle \sigma_\nu(\boldsymbol{u}), v_\nu - \dot{u}_\nu \rangle \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \end{aligned} \tag{9.22}$$

By choosing $\boldsymbol{v} = \dot{\boldsymbol{u}} \pm \boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in (C^\infty(\Omega))^d$ and supp $\boldsymbol{\varphi} \subset \Gamma_1$, we deduce

$$\int_{\Gamma_1} ((\boldsymbol{\sigma} \cdot \boldsymbol{\nu}) - \boldsymbol{g}) \cdot \boldsymbol{\varphi} \, \mathrm{d}s = 0,$$

that is the relation (9.4). Thus, the relation (9.22) becomes

$$\tilde{j}_{\boldsymbol{F}}(\boldsymbol{u}, \boldsymbol{v}) - \tilde{j}_{\boldsymbol{F}}(\boldsymbol{u}, \dot{\boldsymbol{u}}) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau (\boldsymbol{v}_\tau - \dot{\boldsymbol{u}}_\tau) \, \mathrm{d}s \geq 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,. \tag{9.23}$$

We now take $\boldsymbol{v} \in \boldsymbol{V}$ such that $\boldsymbol{v}_\tau = \pm\delta\boldsymbol{\varphi}$ with $\delta \in \mathbb{R}_+$, $\boldsymbol{\varphi} \in (C^\infty(\Omega))^d$ and supp $\boldsymbol{\varphi} \subset \Gamma_2$. As $\boldsymbol{\sigma}_\tau \boldsymbol{v}_\tau = \pm\delta\boldsymbol{\sigma}_\tau\boldsymbol{\varphi}_\tau = \pm\delta\boldsymbol{\sigma}_\tau\boldsymbol{\varphi}$, we obtain

$$\delta \int_{\Gamma_2} (\mu|\mathscr{R}\sigma_\nu|\,|\boldsymbol{\varphi}| \pm \boldsymbol{\sigma}_\tau\boldsymbol{\varphi}) \, \mathrm{d}s - \int_{\Gamma_2} (\mu|\mathscr{R}\sigma_\nu|\,|\dot{\boldsymbol{u}}_\tau| + \boldsymbol{\sigma}_\tau\dot{\boldsymbol{u}}_\tau) \, \mathrm{d}s \geq 0 \quad \forall \delta \geq 0$$

which gives

$$\begin{cases} \displaystyle\int_{\Gamma_2} (\pm\boldsymbol{\sigma}_\tau\boldsymbol{\varphi} + \mu|\mathscr{R}\sigma_\nu|\,|\boldsymbol{\varphi}|) \, \mathrm{d}s \geq 0 \\ \displaystyle\int_{\Gamma_2} (\boldsymbol{\sigma}_\tau\dot{\boldsymbol{u}}_\tau + \mu|\mathscr{R}\sigma_\nu|\,|\dot{\boldsymbol{u}}_\tau|) \, \mathrm{d}s \leq 0 \end{cases}$$

or, equivalently to

$$|\boldsymbol{\sigma}_\tau| \leq \mu|\mathscr{R}\sigma_\nu| \tag{9.24}$$

and

$$\boldsymbol{\sigma}_\tau\dot{\boldsymbol{u}}_\tau + \mu|\mathscr{R}\sigma_\nu|\,|\dot{\boldsymbol{u}}_\tau| \leq 0\,. \tag{9.25}$$

It is easy to see that the relations (9.25) and (9.24) give

$$\boldsymbol{\sigma}_\tau\dot{\boldsymbol{u}}_\tau + \mu|\mathscr{R}\sigma_\nu|\,|\dot{\boldsymbol{u}}_\tau| = 0\,. \tag{9.26}$$

Indeed, if $|\boldsymbol{\sigma}_\tau| < \mu|\mathscr{R}\sigma_\nu|$, then, supposing that $\dot{\boldsymbol{u}}_\tau \neq 0$, it follows that $0 > \boldsymbol{\sigma}_\tau\dot{\boldsymbol{u}}_\tau + |\boldsymbol{\sigma}_\tau|\,|\dot{\boldsymbol{u}}_\tau| \geq 0$, which is a contradiction. It follows that $\dot{\boldsymbol{u}}_\tau = \boldsymbol{0}$.

If $|\boldsymbol{\sigma}_\tau| = \mu|\mathscr{R}\sigma_\nu|$, then it follows that $0 = \boldsymbol{\sigma}_\tau\dot{\boldsymbol{u}}_\tau + |\boldsymbol{\sigma}_\tau|\,|\dot{\boldsymbol{u}}_\tau|$, and so, there exists $\lambda > 0$ such that $\dot{\boldsymbol{u}}_\tau = -\lambda\boldsymbol{\sigma}_\tau$. Therefore, the friction conditions (9.6) are satisfied and, by taking into account that $\boldsymbol{u}(0) = \boldsymbol{u}_0$ et $\boldsymbol{u}(t) \in \boldsymbol{K}$ for all $t \in [0, T]$, we conclude the proof. □

Using an implicit time discretization scheme (as in Sect. 4.3, p. 69), we obtain the following sequence $\{(\mathbf{Q})_{\mathbf{n}}^{\mathbf{i}}\}_{i=0,1,\ldots,n-1}$ of incremental formulations.

Problem $(\mathbf{Q})_\mathbf{n}^\mathbf{i}$: Find $\boldsymbol{u}^{i+1} \in \boldsymbol{K}^{i+1}$ such that

$$\left\{ \begin{array}{l} a(\boldsymbol{u}^{i+1}, \boldsymbol{v} - \partial \boldsymbol{u}^i) + \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{v}) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \partial \boldsymbol{u}^i) \\ \quad \geq (\boldsymbol{F}^{i+1}, \boldsymbol{v} - \partial \boldsymbol{u}^i) + \langle \sigma_\nu(\boldsymbol{u}^{i+1}), v_\nu - \partial u_\nu^i \rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \\ \langle \sigma_\nu(\boldsymbol{u}^{i+1}), z_\nu - u_\nu^{i+1} \rangle \geq 0 \quad \forall\, \boldsymbol{z} \in \boldsymbol{K} \end{array} \right. \tag{9.27}$$

where $\boldsymbol{K}^{i+1} = \boldsymbol{K}(\boldsymbol{F}^{i+1})$ and $\boldsymbol{u}^0 = \boldsymbol{u}_0$. By setting $\boldsymbol{w} = \boldsymbol{v}\Delta t + \boldsymbol{u}^i$, we deduce that the problem $(\mathbf{Q})_\mathbf{n}^\mathbf{i}$ is equivalent to the following problem $(\tilde{\mathbf{Q}})_\mathbf{n}^\mathbf{i}$.

Problem $(\tilde{\mathbf{Q}})_\mathbf{n}^\mathbf{i}$: Find $\boldsymbol{u}^{i+1} \in \boldsymbol{K}^{i+1}$ such that

$$\left\{ \begin{array}{l} a(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i+1}) + \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i) \\ \quad \geq (\boldsymbol{F}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i+1}) + \langle \sigma_\nu(\boldsymbol{u}^{i+1}), w_\nu - u_\nu^{i+1} \rangle \quad \forall\, \boldsymbol{w} \in \boldsymbol{V}, \\ \langle \sigma_\nu(\boldsymbol{u}^{i+1}), z_\nu - u_\nu^{i+1} \rangle \geq 0 \quad \forall\, \boldsymbol{z} \in \boldsymbol{K}. \end{array} \right. \tag{9.28}$$

In order to obtain an existence result for the problem **(Q)** (by applying Theorem 4.19), we first prove the following equivalence result which states that the hypothesis (4.105) of Theorem 4.19 is satisfied.

Theorem 9.2. *For all $i \in \{0, \ldots, n-1\}$, the problem $(\tilde{\mathbf{Q}})_\mathbf{n}^\mathbf{i}$ is equivalent to the problem $(\tilde{\mathbf{R}})_\mathbf{n}^\mathbf{i}$ defined below.*

Problem $(\tilde{\mathbf{R}})_\mathbf{n}^\mathbf{i}$: Find $\boldsymbol{u}^{i+1} \in \boldsymbol{K}^{i+1}$ such that

$$\begin{aligned} & a(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i+1}) + \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i) \\ & \quad \geq (\boldsymbol{F}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i+1}) \quad \forall\, \boldsymbol{w} \in \boldsymbol{K}\,. \end{aligned} \tag{9.29}$$

To help the reader acquire a better understanding of the proof of Theorem 9.2, we divide it into two steps, Propositions 9.1 and 9.2 below. For this reason we introduce the following mechanical problem.

Problem $(\mathscr{Q})_n^i$: Find a displacement field $\boldsymbol{u}^{i+1} : \Omega \to \mathbb{R}^d$ such that

$$-\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}^{i+1}) = \boldsymbol{f}^{i+1} \quad \text{in } \Omega\,, \tag{9.30}$$

$$\boldsymbol{u}^{i+1} = \boldsymbol{0} \quad \text{on } \Gamma_0\,, \tag{9.31}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}^{i+1}) \cdot \boldsymbol{\nu} = \boldsymbol{g}^{i+1} \quad \text{on } \Gamma_1\,, \tag{9.32}$$

$$u_\nu^{i+1} \leq 0\,, \quad \sigma_\nu(\boldsymbol{u}^{i+1}) \leq 0\,, \quad u_\nu \sigma_\nu(\boldsymbol{u}^{i+1}) = 0 \quad \text{on } \Gamma_2\,, \tag{9.33}$$

$$\left\{ \begin{array}{l} |\boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})| \leq \mu |\mathscr{R}\sigma_\nu(\boldsymbol{u}^{i+1})| \ \text{ and} \\ |\boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})| < \mu |\mathscr{R}\sigma_\nu(\boldsymbol{u}^{i+1})| \ \Rightarrow \ \boldsymbol{u}_\tau^{i+1} = \boldsymbol{u}_\tau^i \\ |\boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})| = \mu |\mathscr{R}\sigma_\nu(\boldsymbol{u}^{i+1})| \ \Rightarrow \ \exists \lambda \geq 0\,, \ \boldsymbol{u}_\tau^{i+1} - \boldsymbol{u}_\tau^i = -\lambda \boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1}) \end{array} \right. \quad \text{on } \Gamma_2\,. \tag{9.34}$$

Lemma 9.2. *Let* $\boldsymbol{\Theta} \in \boldsymbol{V}$ *and* $\boldsymbol{d} \in \boldsymbol{K}$ *be given and let* $\tilde{\boldsymbol{u}} \in \boldsymbol{K}(\boldsymbol{\Theta})$ *be a regular function such that*

$$\tilde{j}_{\boldsymbol{\Theta}}(\tilde{\boldsymbol{u}}, \boldsymbol{w} - \mathbf{d}) - \tilde{j}_{\boldsymbol{\Theta}}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}} - \boldsymbol{d}) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\boldsymbol{w}_\tau - \tilde{\boldsymbol{u}}_\tau)\, \mathrm{d}s \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{K}\,. \tag{9.35}$$

Then $\tilde{\boldsymbol{u}}$ *verifies (in the distributional sense)*

$$\begin{cases} |\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})| \leq \mu|\mathscr{R}\sigma_\nu(\tilde{\boldsymbol{u}})| \ \textit{and} \\ |\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})| < \mu|\mathscr{R}\sigma_\nu(\tilde{\boldsymbol{u}})| \ \Rightarrow \ \tilde{\boldsymbol{u}}_\tau = \boldsymbol{d}_\tau \\ |\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})| = \mu|\mathscr{R}\sigma_\nu(\tilde{\boldsymbol{u}})| \ \Rightarrow \ \exists\lambda \geq 0\,,\ \tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau = -\lambda\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}}) \end{cases} \quad \textit{on } \Gamma_2\,. \tag{9.36}$$

Proof. If we take $\boldsymbol{w} = \boldsymbol{d} + \delta\boldsymbol{\varphi}_\tau$ in (9.35), with $\boldsymbol{\varphi} \in (C^\infty(\Omega))^d$, $\operatorname{supp} \boldsymbol{\varphi} \subset \Gamma_2$ and $\delta > 0$, we obtain

$$\begin{aligned} &\int_{\Gamma_2} \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|(|\boldsymbol{w}_\tau - \boldsymbol{d}_\tau| - |\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau|) + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\boldsymbol{w}_\tau - \tilde{\boldsymbol{u}}_\tau)\, \mathrm{d}s \\ &= \delta \int_{\Gamma_2} (\mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\, |\boldsymbol{\varphi}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})\boldsymbol{\varphi})\ \mathrm{d}s \\ &- \int_{\Gamma_2} (\mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\, |\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau))\ \mathrm{d}s \geq 0 \quad \forall\delta > 0\,, \end{aligned}$$

which gives, as $|\boldsymbol{\varphi}| \geq |\boldsymbol{\varphi}_\tau|$,

$$\int_{\Gamma_2} (\mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\, |\boldsymbol{\varphi}| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})\boldsymbol{\varphi})\, \mathrm{d}s \geq 0 \quad \forall \boldsymbol{\varphi} \in (C^\infty(\Omega))^d\,,\ \ \operatorname{supp} \boldsymbol{\varphi} \subset \Gamma_2\,, \tag{9.37}$$

and

$$\int_{\Gamma_2} (\mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\, |\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau))\, \mathrm{d}s \leq 0\,. \tag{9.38}$$

Putting $\boldsymbol{\varphi} = \pm\boldsymbol{\varphi}$ in (9.37), it results

$$\int_{\Gamma_2} |\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})|\, |\boldsymbol{\varphi}|\, \mathrm{d}s \leq \int_{\Gamma_2} \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))||\boldsymbol{\varphi}|\, \mathrm{d}s \quad \forall \boldsymbol{\varphi} \in (C^\infty(\Omega))^d\,,\ \ \operatorname{supp} \boldsymbol{\varphi} \subset \Gamma_2\,,$$

i.e.

$$|\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})| \leq \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\,. \tag{9.39}$$

Therefore, (9.38) implies

$$0 \geq \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\,|\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau) \geq (\mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))| - |\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})|)\,|\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau| \geq 0$$

that is

$$\mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\,|\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau) = 0\,. \tag{9.40}$$

If $|\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})| < \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|$, then, supposing $\tilde{\boldsymbol{u}}_\tau \neq \boldsymbol{d}_\tau$, (9.40) gives

$$0 = \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\,|\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau) > |\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})|\,|\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau) \geq 0\,,$$

and so, we must have $\tilde{\boldsymbol{u}}_\tau = \boldsymbol{d}_\tau$

If $|\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})| = \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|$, then (9.40) implies

$$|\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})|\,|\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau) = 0$$

and thus, there exists $\lambda \geq 0$ such that $\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau = -\lambda\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})$. ☐

Lemma 9.3. *Let $\boldsymbol{\Theta} \in \boldsymbol{V}$ and $\boldsymbol{d} \in \boldsymbol{K}$ be given. Let $\tilde{\boldsymbol{u}} \in \boldsymbol{K}(\boldsymbol{\Theta})$ be a sufficiently smooth function which verifies (9.36). Then*

$$\tilde{j}_{\boldsymbol{\Theta}}(\tilde{\boldsymbol{u}}, \boldsymbol{w} - \mathbf{d}) - \tilde{j}_{\boldsymbol{\Theta}}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}} - \boldsymbol{d}) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\boldsymbol{w}_\tau - \tilde{\boldsymbol{u}}_\tau)\,\mathrm{d}s \geq 0 \;\; \forall \boldsymbol{w} \textit{ smooth function}\,. \tag{9.41}$$

Proof. Let $\boldsymbol{w}$ be a smooth function.

If $|\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})| < \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|$ and $\tilde{\boldsymbol{u}}_\tau = \boldsymbol{d}_\tau$, then one has

$$\begin{aligned}
&\tilde{j}_{\boldsymbol{\Theta}}(\tilde{\boldsymbol{u}}, \boldsymbol{w} - \mathbf{d}) - \tilde{j}_{\boldsymbol{\Theta}}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}} - \boldsymbol{d}) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\boldsymbol{w}_\tau - \boldsymbol{d}_\tau)\,\mathrm{d}s \\
&= \int_{\Gamma_2} (\mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\,|\boldsymbol{w}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\boldsymbol{w}_\tau - \boldsymbol{d}_\tau))\,\mathrm{d}s \\
&\geq \int_{\Gamma_2} (\mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))| - |\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})|)\,|\boldsymbol{w}_\tau - \boldsymbol{d}_\tau|\,\mathrm{d}s \geq 0
\end{aligned}$$

If $|\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})| = \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|$ and $\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau = -\lambda\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})$, then one gets

$$\begin{aligned}
&\tilde{j}_{\boldsymbol{\Theta}}(\tilde{\boldsymbol{u}}, \boldsymbol{w} - \mathbf{d}) - \tilde{j}_{\boldsymbol{\Theta}}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}} - \boldsymbol{d}) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\boldsymbol{w}_\tau - \tilde{\boldsymbol{u}}_\tau)\,\mathrm{d}s \\
&= \int_{\Gamma_2} |\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})|\,|\boldsymbol{w}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\boldsymbol{w}_\tau - \boldsymbol{d}_\tau)\,\mathrm{d}s \geq 0\,,
\end{aligned}$$

which completes the proof of Lemma. ☐

Proposition 9.1. *The problem* $(\tilde{\mathbf{Q}})^{\mathbf{i}}_{\mathbf{n}}$ *is formally equivalent (in the sense considered in Theorem 9.1) to the mechanical problem* $(\mathscr{Q})^i_n$.

Proof. Let $\boldsymbol{u}^{i+1}$ be a regular solution of $(\tilde{\mathbf{Q}})^{\mathbf{i}}_{\mathbf{n}}$. If we chose, in the first inequality of (9.28), $\boldsymbol{w} = \boldsymbol{u}^{i+1} \pm \boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in (\mathscr{D}(\Omega))^d$ and we apply the Green's formula, then we obtain (9.30).

From the second inequality of (9.28) and Lemma 9.1 for $\tilde{\boldsymbol{u}} = \boldsymbol{u}^{i+1}$, we deduce (9.33).

Multiplying (9.30) by $\boldsymbol{w} - \boldsymbol{u}^{i+1}$ for $\boldsymbol{w} \in \boldsymbol{V}$, integrating by parts and using against the Green's formula and the first inequality of (9.28), we get

$$\tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})(\boldsymbol{w}_\tau - \boldsymbol{u}^{i+1}_\tau)\, \mathrm{d}s$$
$$+ \int_{\Gamma_1} (\boldsymbol{\sigma}(\boldsymbol{u}^{i+1}) \cdot \boldsymbol{\nu} - \boldsymbol{g}^{i+1})(\boldsymbol{w} - \boldsymbol{u}^{i+1})\, \mathrm{d}s \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{V}, \tag{9.42}$$

and thus, by taking $\boldsymbol{w} = \boldsymbol{u}^{i+1} \pm \boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in (C^\infty(\Omega))^d$ and $\operatorname{supp} \boldsymbol{\varphi} \subset \Gamma_1$, one obtains (9.32). Therefore, the relation (9.42) implies

$$\begin{aligned} &\tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i) \\ &+ \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})(\boldsymbol{w}_\tau - \boldsymbol{u}^{i+1}_\tau)\, \mathrm{d}s \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{V}. \end{aligned} \tag{9.43}$$

Therefore, by Lemma 9.2 for $\boldsymbol{\Theta} = \boldsymbol{F}^{i+1}$, $\boldsymbol{d} = \boldsymbol{u}^i$ and $\tilde{\boldsymbol{u}} = \boldsymbol{u}^{i+1} \in \boldsymbol{K}(\boldsymbol{F}^{i+1})$, it follows that the conditions (9.34) are satisfied. As $\boldsymbol{u}^{i+1} \in \boldsymbol{K} \subset \boldsymbol{V}$, it yields the condition (9.31) holds which completes the proof.

Conversely, let $\boldsymbol{u}^{i+1}$ be a sufficiently smooth solution of the mechanical problem $(\mathscr{Q})^i_n$. Then, by applying Lemma 9.1 for $\tilde{\boldsymbol{u}} = \boldsymbol{u}^{i+1}$, it follows that $\boldsymbol{u}^{i+1}$ satisfies the second inequality of (9.28).

Next, from (9.34), by Lemma 9.3 for $\boldsymbol{\Theta} = \boldsymbol{F}^{i+1}$, $\boldsymbol{d} = \boldsymbol{u}^i$ and $\tilde{\boldsymbol{u}} = \boldsymbol{u}^{i+1}$, we obtain

$$\begin{aligned} &\tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i) \\ &+ \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})(\boldsymbol{w}_\tau - \boldsymbol{u}^{i+1}_\tau)\, \mathrm{d}s \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{V}. \end{aligned} \tag{9.44}$$

On the other hand, multiplying (9.30) by $\boldsymbol{w} - \boldsymbol{u}^{i+1}$ with $\boldsymbol{w} \in \boldsymbol{V}$, integrating by parts and using (9.32), we deduce

$$a(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i+1}) = (\boldsymbol{F}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i+1}) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})(\boldsymbol{w}_\tau - \boldsymbol{u}^{i+1}_\tau)\, \mathrm{d}s$$
$$+ \langle \sigma_\nu(\boldsymbol{u}^{i+1}), w_\nu - u^{i+1}_\nu \rangle \quad \forall \boldsymbol{w} \in \boldsymbol{V}. \tag{9.45}$$

Combining (9.44) and (9.45), we obtain the first inequality of (9.28) which completes the proof. □

Proposition 9.2. *The problem* $(\tilde{\mathbf{R}})^{\mathbf{i}}_{\mathbf{n}}$ *is formally equivalent to the mechanical problem* $(\mathcal{Q})^i_n$.

Proof. If $\boldsymbol{u}^{i+1}$ is a regular solution of $(\tilde{\mathbf{R}})^{\mathbf{i}}_{\mathbf{n}}$, then, with a similar proof as for Proposition 9.1, one obtains (9.30). Therefore, from (9.30) and (9.29), one gets

$$\tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{w}-\boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}-\boldsymbol{u}^i) + \int_{\Gamma_2} \boldsymbol{\sigma}(\boldsymbol{u}^{i+1}) \cdot \boldsymbol{\nu}(\boldsymbol{w}-\boldsymbol{u}^{i+1})\, \mathrm{d}s$$
$$+ \int_{\Gamma_1} (\boldsymbol{\sigma}(\boldsymbol{u}^{i+1}) \cdot \boldsymbol{\nu} - \boldsymbol{g}^{i+1})(\boldsymbol{w}-\boldsymbol{u}^{i+1})\, \mathrm{d}s \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{K}, \tag{9.46}$$

from which, by taking $\boldsymbol{w} = \boldsymbol{u}^{i+1} \pm \boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in (C^\infty(\Omega))^d$ and supp $\boldsymbol{\varphi} \subset \Gamma_1$, one deduces (9.32). Thus, the relation (9.46) becomes

$$\tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{w}-\boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}-\boldsymbol{u}^i)$$
$$+ \int_{\Gamma_2} (\sigma_\nu(\boldsymbol{u}^{i+1})(w_\nu - u^{i+1}_\nu) + \boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})(\boldsymbol{w}_\tau - \boldsymbol{u}^{i+1}_\tau))\, \mathrm{d}s \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{K}. \tag{9.47}$$

By choosing $\boldsymbol{w} = \delta\varphi_\nu \boldsymbol{\nu} + \boldsymbol{u}^{i+1}_\tau$ with $\boldsymbol{\varphi} \in (C^\infty(\Omega))^d$, $\varphi_\nu \leq 0$ on Γ_2 and $\delta > 0$, it follows

$$\delta \int_{\Gamma_2} \sigma_\nu(\boldsymbol{u}^{i+1})\, \varphi_\nu\, \mathrm{d}s \geq \int_{\Gamma_2} \sigma_\nu(\boldsymbol{u}^{i+1})\, \boldsymbol{u}^{i+1}_\nu\, \mathrm{d}s \quad \forall \delta > 0$$

which gives

$$\begin{cases} \displaystyle\int_{\Gamma_2} \sigma_\nu(\boldsymbol{u}^{i+1})\, \varphi_\nu\, \mathrm{d}s \geq 0 \quad \forall \boldsymbol{\varphi} \in \boldsymbol{V}\,,\ \varphi_\nu \leq 0 \text{ on } \Gamma_2\,, \\ \displaystyle\int_{\Gamma_2} \sigma_\nu(\boldsymbol{u}^{i+1})\, \boldsymbol{u}^{i+1}_\nu\, \mathrm{d}s \leq 0\,, \end{cases} \tag{9.48}$$

and, as $\boldsymbol{u}^{i+1} \in \boldsymbol{K}$, we obtain (9.33).

Now, if we choose in (9.47), $\boldsymbol{w} = u^{i+1}_n \boldsymbol{\nu} + \boldsymbol{v}$ with $\boldsymbol{v} \in \boldsymbol{K}$ arbitrary, we obtain

$$\tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{v}-\boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}-\boldsymbol{u}^i) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})(\boldsymbol{v}_\tau - \boldsymbol{u}^{i+1}_\tau)\, \mathrm{d}s \geq 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K}$$

which gives, together with Lemma 9.2, for $\boldsymbol{\Theta} = \boldsymbol{F}^{i+1}$, $\boldsymbol{d} = \boldsymbol{u}^i$ and $\tilde{\boldsymbol{u}} = \boldsymbol{u}^{i+1}$, the conditions (9.34).

Conversely, if $\boldsymbol{u}^{i+1}$ is a sufficiently smooth function which verifies $(\mathscr{Q})_n^i$, then, from Lemmas 9.3 and 9.1, we obtain

$$\tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})(\boldsymbol{w}_\tau - \boldsymbol{u}_\tau^{i+1})\,\mathrm{d}s \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{K} \tag{9.49}$$

and

$$\int_{\Gamma_2} \boldsymbol{\sigma}_\nu(\boldsymbol{u}^{i+1})(\boldsymbol{w}_\nu - \boldsymbol{u}_\nu^{i+1})\,\mathrm{d}s \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{K}\,. \tag{9.50}$$

Next, by arguing as in the proof of Proposition 9.1, we conclude that $\boldsymbol{u}^{i+1}$ is a solution of $(\tilde{\mathbf{R}})_{\mathbf{n}}^{\mathbf{i}}$ which completes the proof. □

Proof of Theorem 9.2. Using Propositions 9.1 and 9.2, the assertion follows. However, we remark that if $\boldsymbol{u}^{i+1}$ is a solution of $(\tilde{\mathbf{Q}})_{\mathbf{n}}^{\mathbf{i}}$, then, obviously, $\boldsymbol{u}^{i+1}$ is a solution of $(\tilde{\mathbf{R}})_{\mathbf{n}}^{\mathbf{i}}$. Hence, in order to prove the condition (4.105), it would have been enough to prove that $(\tilde{\mathbf{R}})_{\mathbf{n}}^{\mathbf{i}} \Rightarrow (\mathscr{Q})_n^i \Rightarrow (\tilde{\mathbf{Q}})_{\mathbf{n}}^{\mathbf{i}}$. □

In the sequel we shall use the similar definitions to (4.118) (p. 72), i.e.

$$\begin{cases} \boldsymbol{u}_n(0) = \hat{\boldsymbol{u}}_n(0) = \boldsymbol{u}^0, \\ \boldsymbol{F}_n(0) = \boldsymbol{F}(0) = \boldsymbol{F}^0\,, \\ \left.\begin{array}{l} \boldsymbol{u}_n(t) = \boldsymbol{u}^{i+1} \\ \hat{\boldsymbol{u}}_n(t) = \boldsymbol{u}^i + (t - t_i)\partial \boldsymbol{u}^i \\ \boldsymbol{F}_n(t) = \boldsymbol{F}^{i+1} \end{array}\right\} \forall\, i \in \{0, 1, \ldots, n-1\} \quad \forall\, t \in (t_i, t_{i+1}]\,, \end{cases} \tag{9.51}$$

Therefore, $\boldsymbol{u}_n \in L^2(0,T;\boldsymbol{V})$ and $\hat{\boldsymbol{u}}_n \in W^{1,2}(0,T;\boldsymbol{V})$ satisfy, for all $t \in [0,T]$, the following incremental problem.
Problem $(\mathbf{Q})_\mathbf{n}$: Find $\boldsymbol{u}_n \in K(\boldsymbol{F}_n(t))$ such that

$$\begin{cases} a\left(\boldsymbol{u}_n(t), \boldsymbol{v} - \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{u}}_n(t)\right) + \tilde{j}_{\boldsymbol{F}_n(t)}(\boldsymbol{u}_n(t), \boldsymbol{v}) \\ -\tilde{j}_{\boldsymbol{F}_n(t)}\left(\boldsymbol{u}_n(t), \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{u}}_n(t)\right) \geq \left(\boldsymbol{F}_n(t), \boldsymbol{v} - \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{u}}_n(t)\right) \\ +\left\langle \sigma_\nu(\boldsymbol{u}_n(t)), v_\nu - \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{u}_{n\nu}(t)\right\rangle \quad \forall\, \boldsymbol{v} \in \mathbf{V}\,, \\ \langle \sigma_\nu(\boldsymbol{u}_n(t)), z_\nu - u_{n\nu}(t)\rangle \geq 0 \quad \forall\, \boldsymbol{z} \in \boldsymbol{K}\,. \end{cases} \tag{9.52}$$

We have the following convergence and existence result.

Theorem 9.3. *Suppose the hypotheses (9.8) and (9.9) hold and that meas $\Gamma_0 > 0$. Then, there exists a constant $\mu_1 > 0$ such that for any $\mu \in L^\infty(\Gamma_2)$ with $\mu \geq 0$ a.e. on Γ_2 and $\|\mu\|_{L^\infty(\Gamma_2)} < \mu_1$, the problem* **(Q)** *has at least one solution. More precisely, there exists a subsequence $\{(\boldsymbol{u}_{n_k}, \bar{\boldsymbol{u}}_{n_k}\}_k$ such that*

$$\begin{aligned}
&\boldsymbol{u}_{n_k}(t) \to \boldsymbol{u}(t) \text{ strongly in } \boldsymbol{V} \quad \forall\, t \in [0, T]\,,\\
&\hat{\boldsymbol{u}}_{n_k} \to \boldsymbol{u} \text{ strongly in } L^2(0, T; \boldsymbol{V})\,,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{u}}_{n_k} \rightharpoonup \dot{\boldsymbol{u}} \text{ weakly in } L^2(0, T; \boldsymbol{V})
\end{aligned}$$

as $k \to \infty$, where $\boldsymbol{u}$ is a solution of the problem $(\mathbf{Q})$*.*

Proof. By putting

$$j(\boldsymbol{\Theta}, \boldsymbol{v}, \boldsymbol{w}) = \tilde{j}_{\boldsymbol{\Theta}}(\boldsymbol{v}, \boldsymbol{w}) - (\boldsymbol{\Theta}, \boldsymbol{w}) \quad \forall \boldsymbol{\Theta} \in \boldsymbol{V}\,,\ \forall \boldsymbol{v} \in \boldsymbol{K}(\boldsymbol{\Theta})\,,\ \forall \boldsymbol{w} \in \boldsymbol{V}\,, \tag{9.53}$$

it follows that

$$\begin{aligned}
&|j(\boldsymbol{\Theta}_1, \boldsymbol{v}_1, \boldsymbol{w}_2) + j(\boldsymbol{\Theta}_2, \boldsymbol{v}_2, \boldsymbol{w}_1) - j(\boldsymbol{\Theta}_1, \boldsymbol{v}_1, \boldsymbol{w}_1) - j(\boldsymbol{\Theta}_2, \boldsymbol{v}_2, \boldsymbol{w}_2)|\\
&= \left| \int_{\Gamma_2} \mu(|\mathscr{R}\sigma_\nu(\boldsymbol{v}_1)| - |\mathscr{R}\sigma_\nu(\boldsymbol{v}_2)|)(|\boldsymbol{w}_{1\tau}| - |\boldsymbol{w}_{2\tau}|)\,\mathrm{d}s + (\boldsymbol{\Theta}_1 - \boldsymbol{\Theta}_2, \boldsymbol{w}_1 - \boldsymbol{w}_2) \right|\\
&\le C_1 \|\mu\|_{L^\infty(\Gamma_2)} \int_{\Gamma_2} |\mathscr{R}\sigma_\nu(\boldsymbol{v}_1) - \mathscr{R}\sigma_\nu(\boldsymbol{v}_2)|\, |\boldsymbol{w}_1 - \boldsymbol{w}_2|\,\mathrm{d}s + \|\boldsymbol{\Theta}_1 - \boldsymbol{\Theta}_2\|\, \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|\\
&\le C_2 \|\mu\|_{L^\infty(\Gamma_2)} (\|\boldsymbol{\Theta}_1 - \boldsymbol{\Theta}_2\| + \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|) \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|\\
&\qquad\qquad \forall\, \boldsymbol{w}_i \in \boldsymbol{V}\,,\ \forall \boldsymbol{v}_i \in \boldsymbol{K}(\boldsymbol{\Theta}_i)\,,\ \forall \boldsymbol{w}_i \in \boldsymbol{V}\,,\ i = 1, 2\,,
\end{aligned} \tag{9.54}$$

where C_1, C_2 are positive constants and $\|\cdot\|$ denotes the norm over $\boldsymbol{V}$.

In order to apply Theorem 4.19, we put

$$b(\boldsymbol{\Theta}, \boldsymbol{v}, \boldsymbol{w}) = \langle \sigma_\nu(\boldsymbol{v}), w_\nu \rangle \quad \forall \boldsymbol{\Theta} \in \boldsymbol{V}\,,\ \forall \boldsymbol{v} \in \boldsymbol{K}(\boldsymbol{\Theta})\,,\ \forall \boldsymbol{w} \in \boldsymbol{V}\,, \tag{9.55}$$

$$\begin{aligned}
&H = L^2(\Gamma_2)\,,\\
&\beta(\boldsymbol{\Theta}, \boldsymbol{v}) = \mu|\mathscr{R}\sigma_\nu(\boldsymbol{v})| \quad \forall \boldsymbol{\Theta} \in \boldsymbol{V}\,,\ \forall \boldsymbol{v} \in \boldsymbol{K}(\boldsymbol{\Theta})\,.
\end{aligned}$$

Therefore, the problem $(\mathbf{Q})$ can be written under the form (4.107) (p. 68) and the problem $(\tilde{\mathbf{Q}})^{\mathbf{i}}_{\mathbf{n}}$ can be written under the form (4.103) (p. 68), i.e.

$$\begin{cases}
\boldsymbol{u}^{i+1} \in \boldsymbol{K}(\boldsymbol{F}^{i+1})\\
a(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i+1}) + j(\boldsymbol{F}^{i+1}, \boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i}) - j(\boldsymbol{F}^{i+1}, \boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^{i})\\
\ge b(\boldsymbol{F}^{i+1}, \boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i+1}) \quad \forall\, \boldsymbol{w} \in \boldsymbol{V},\\
b(\boldsymbol{F}^{i+1}, \boldsymbol{u}^{i+1}, \boldsymbol{z} - \boldsymbol{u}^{i+1}) \ge 0 \quad \forall\, \boldsymbol{z} \in \boldsymbol{K}.
\end{cases} \tag{9.56}$$

The hypothesis (4.105) is satisfied due to Theorem 9.2. The other hypotheses (4.83)–(4.90), (4.96)–(4.98), (4.100), and (4.101) of Theorem 4.19 are easy to prove, and so, the assertion follows. □

9.2 Discrete Approximation

This section deals with the discretization of the problem (**Q**) written under the form

$$\begin{cases} \boldsymbol{u}(0)=\boldsymbol{u}_0\,,\ \boldsymbol{u}(t)\in \boldsymbol{K}(\boldsymbol{F}(t)) \quad \forall\, t\in[0,T], \\ a(\boldsymbol{u}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t))+j(\boldsymbol{F}(t),\boldsymbol{u}(t),\boldsymbol{v})-j(\boldsymbol{F}(t),\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) \\ \geq b(f(t),u(t),v-\dot{u}(t)) \quad \forall\,\boldsymbol{v}\in\boldsymbol{V} \text{ a.e. in } (0,T), \\ b(\boldsymbol{F}(t),\boldsymbol{u}(t),\boldsymbol{z}-\boldsymbol{u}(t))\geq 0 \quad \forall \boldsymbol{z}\in\boldsymbol{K},\ \forall\, t\in[0,T]\,, \end{cases} \tag{9.57}$$

with j and b defined by (9.53), respectively, (9.55).

We shall prove a convergence result for a method based on an internal approximation in space and a backward difference scheme in time.

Let $\mathscr{T}_h=(T_j)_{j\in\mathscr{J}_h}$ be a family of regular triangulations of Ω such that

$$\overline{\Omega}=\bigcup_{j\in\mathscr{J}_h}\overline{T}_j\,,$$
$$T_i\cap T_j=\emptyset \quad \forall i,j\in\mathscr{J}_h\,,\ i\neq j\,.$$

We define the following sets

$$\boldsymbol{V}_h=\{\boldsymbol{v}_h\in(C^0(\overline{\Omega}))^d\,;\ \boldsymbol{v}_h/T_j\in(P_1(T_j))^d\,,\ \forall j\in\mathscr{J}_h\,,\ \boldsymbol{v}_h=\boldsymbol{0}\text{ on }\Gamma_0\}\,,$$
$$\boldsymbol{K}_h=\{\boldsymbol{v}_h\in\boldsymbol{V}_h\,;\ v_{h\nu}\leq 0\text{ on }\Gamma_2\}$$
$$S_h=\{\tau_h\in L^2(\Gamma_2)\,;\ \tau_h/\Gamma_{2,j}\in P_0(\Gamma_{2,j}) \quad \forall j\in\mathscr{J}_h\text{ such that }\Gamma_{2,j}\neq\emptyset\}$$

where $P_k(\omega)$ denotes the space of polynomials of degree lower or equal to k on ω and $\Gamma_{2,j}=\Gamma_2\cap\overline{T}_j$.

As in Sect. 7.3, p. 128, we consider the following semi-discrete problem.

Problem ($\mathbf{Q_h}$): Find $\boldsymbol{u}_h\in W^{1,2}(0,T;\boldsymbol{V}_h)$ such that

$$\begin{cases} \boldsymbol{u}_h(0)=\boldsymbol{u}_{0h},\ \boldsymbol{u}_h(t)\in\boldsymbol{K}_h(\boldsymbol{F}(t)) \quad \forall\, t\in[0,T], \\ a(\boldsymbol{u}_h(t),\boldsymbol{v}_h-\dot{\boldsymbol{u}}_h(t))+j(\boldsymbol{F}(t),\boldsymbol{u}_h(t),\boldsymbol{v}_h)-j(\boldsymbol{F}(t),\boldsymbol{u}_h(t),\dot{\boldsymbol{u}}_h(t)) \\ \geq b(\boldsymbol{F}(t),\boldsymbol{u}_h(t),\boldsymbol{v}_h-\dot{\boldsymbol{u}}_h(t)) \quad \forall\,\boldsymbol{v}_h\in\boldsymbol{V}_h\,,\ \text{a.e. } t\in(0,T)\,, \\ b(\boldsymbol{F}(t),\boldsymbol{u}_h(t),\boldsymbol{z}_h-\boldsymbol{u}_h(t))\geq 0 \quad \forall\,\boldsymbol{z}_h\in\boldsymbol{K}_h, \quad \forall\, t\in[0,T]\,. \end{cases} \tag{9.58}$$

and, for $i\in\{0,1,\cdots,n-1\}$, the following full discretization of Problem (**Q**).

Problem $(\mathbf{R_h})^{\mathbf{i}}_{\mathbf{n}}$: Find $\boldsymbol{u}_h^{i+1}\in\boldsymbol{K}_h^{i+1}$ such that

$$\begin{cases} a(\boldsymbol{u}_h^{i+1},\boldsymbol{w}_h-\boldsymbol{u}_h^{i+1})+j(\boldsymbol{F}^{i+1},\boldsymbol{u}_h^{i+1},\boldsymbol{w}_h-\boldsymbol{u}_h^{i}) \\ -j(\boldsymbol{F}^{i+1},\boldsymbol{u}_h^{i+1},\boldsymbol{u}_h^{i+1}-\boldsymbol{u}_h^{i})\geq 0 \quad \forall\,\boldsymbol{w}_h\in\boldsymbol{K}_h. \end{cases} \tag{9.59}$$

We also suppose that $\boldsymbol{u}_h^0=\boldsymbol{u}_{0h}$ satisfies the compatibility condition

$$\begin{cases} \boldsymbol{u}_{0h}\in\boldsymbol{K}_h(\boldsymbol{F}(0))\,, \\ a(\boldsymbol{u}_{0h},\boldsymbol{v})+j(\boldsymbol{F}(0),\boldsymbol{u}_{0h},\boldsymbol{v})\geq 0 \quad \forall\boldsymbol{v}\in\boldsymbol{K}_h\,. \end{cases}$$

Theorems 7.5 (p. 128) and 7.6 (p. 131) give convergence and existence results for these problems.

In order to solve the problem $(\mathbf{R_h})^{\mathbf{i}}_{\mathbf{n}}$, we suppose that μ is constant and we choose as a regularization mapping $\mathscr{R}$, the projection on the finite dimensional space S_{h_0} for a given h_0 (see [10]). Thus within finite element approximation, the regularization can be considered as a natural consequence of the discretization.

In the sequel, for simplicity, we shall omit the index h. We shall denote the solution $\boldsymbol{u}^{i+1}$ of $(\mathbf{R_h})^{\mathbf{i}}_{\mathbf{n}}$ by $\boldsymbol{u}_n^{i+1}$, for $i \in \{0, 1, \cdot, n-1\}$. We also remark that, from the definition of the set $\boldsymbol{K}^{i+1}$ and Remark 9.1, it follows that for the solution $\boldsymbol{u}^{i+1}$ we have

$$j(\boldsymbol{F}^{i+1}, \boldsymbol{u}_n^{i+1}, \boldsymbol{v}) = -\int_{\Gamma_2} \mu \mathscr{R} \sigma_\nu(\boldsymbol{u}_n^{i+1})\,|\boldsymbol{v}_t|\,\mathrm{d}s \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,.$$

Let us denote

$$j(\boldsymbol{u}_n^{i+1}, \boldsymbol{v}) = -\int_{\Gamma_2} \mu \mathscr{R} \sigma_\nu(\boldsymbol{u}_n^{i+1})\,|\boldsymbol{v}_t|\,\mathrm{d}s \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,.$$

Therefore, the problem to solve can be written as

$$\begin{cases} \boldsymbol{u}_n^{i+1} \in \boldsymbol{K}^{i+1}\,, \\ a(\boldsymbol{u}_n^{i+1}, \boldsymbol{w} - \boldsymbol{u}_n^{i+1}) + j(\boldsymbol{u}_n^{i+1}, \boldsymbol{w} - \boldsymbol{u}_n^{i}) - j(\boldsymbol{u}_n^{i+1}, \boldsymbol{u}_n^{i+1} - \boldsymbol{u}_n^{i}) \\ \geq (\boldsymbol{F}^{i+1}, \boldsymbol{w} - \boldsymbol{u}_n^{i}) \ \forall\, \boldsymbol{w} \in \boldsymbol{K}\,. \end{cases} \tag{9.60}$$

It is easy to see that the solution $\boldsymbol{u}_n^{i+1} \in \boldsymbol{K}^{i+1}$ of (9.60) is the fixed point of the mapping $\mathscr{T} : S \to S$ defined by $\mathscr{T}(r) = \boldsymbol{u}_n^{i+1}(r)$, for all $r \in S$, where $\boldsymbol{u}_n^{i+1}(r)$ is the unique solution of the following variational inequality:

$$\begin{cases} \boldsymbol{u}_n^{i+1}(r) \in \boldsymbol{K}^{i+1}\,, \\ a(\boldsymbol{u}_n^{i+1}(r), \boldsymbol{w} - \boldsymbol{u}_n^{i+1}(r)) + \varphi(r, \boldsymbol{w} - \boldsymbol{u}_n^{i}(r)) - \varphi(r, \boldsymbol{u}_n^{i+1}(r) - \boldsymbol{u}_n^{i}) \\ \geq (\boldsymbol{F}^{i+1}, \boldsymbol{w} - \boldsymbol{u}_n^{i+1}(r)) \ \forall\, \boldsymbol{w} \in \boldsymbol{K}\,. \end{cases} \tag{9.61}$$

where

$$\varphi(r, \boldsymbol{w}) = -\int_{\Gamma_2} \mu \mathscr{R} \sigma_\nu(r) |\boldsymbol{w}_\tau|\,\mathrm{d}s \quad \forall \boldsymbol{w} \in \boldsymbol{V}.$$

This problem is equivalent, for $r \in S$ given, to the following minimization problem under constraints:

$$\mathscr{F}(\boldsymbol{u}_n^{i+1}(r)) = \min_{\boldsymbol{v} \in \boldsymbol{K}} \mathscr{F}(\boldsymbol{v})$$

where

$$\mathscr{F}(\boldsymbol{v}) = \frac{1}{2}a(\boldsymbol{v}, \boldsymbol{v}) + \varphi(r, \boldsymbol{v} - \boldsymbol{u}_n^i(r)) - (\boldsymbol{F}^{i+1}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V} .$$

This problem is very similar to a static problem except from the fact that the known solution $\boldsymbol{u}_n^i$ of the previous step appears in the friction term. The influence of the loading history, due to the velocity formulation of the friction, is characterized by this extra term. The convex K remains unchanged from one step to the next. This minimization problem can be solved by a Gauss–Seidel method with projection. This method is robust and very easy to implement on this kind of problem when dealing with the non-differentiable part relating to the friction term. Details on the convergence of the algorithm by using an Aitken acceleration procedure can be found in [5] or [13].

9.3 Optimal Control of a Frictional Bilateral Contact Problem

We consider a mathematical model describing the quasistatic process of bilateral contact with friction between an elastic body and a rigid foundation. Our goal is to study a related optimal control problem which allows us to obtain a given profile of displacements on the contact boundary, by acting with a control on another part of the boundary of the body. Using penalization and regularization techniques, we derive the necessary conditions of optimality.

As far as we know, there are few results concerning the optimal control of quasistatic frictional contact problems. We mention here the work of Amassad et al. [2] which treats a quasistatic bilateral contact problem with given friction, and so, an optimal control problem governed by a variational inequality which has, in addition, a unique solution.

9.3.1 Setting of the Problem

Let us consider a linearly elastic body occupying a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with a Lipschitz boundary $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, where $\Gamma_0,\ \Gamma_1,\ \Gamma_2$ are open and disjoint parts of Γ, with meas $(\Gamma_0) > 0$.

The body is subjected to the action of volume forces of density $\boldsymbol{f}$ given in $\Omega \times (0, T)$ and surface tractions of density $\boldsymbol{g}$ applied on $\Gamma_1 \times (0, T)$, where $(0, T)$ is the time interval of interest. The body is clamped on $\Gamma_0 \times (0, T)$ and, so, the displacement vector $\boldsymbol{u}$ vanishes here. On $\Gamma_2 \times (0, T)$, the body is in bilateral contact with a rigid foundation, i.e. there is no loss of contact between the body and the foundation. We suppose that the contact on Γ_2 is with friction modeled by a nonlocal variant of Coulomb's law. We suppose that $\boldsymbol{f}$ and $\boldsymbol{g}$ are acting slow enough to allow us to neglect the inertial terms.

The classical formulation of this mechanical problem, with the notation of Sect. 8.1, is:
Problem ($\mathscr{S}$): Find a displacement vector $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t) : \Omega \times [0, T] \to \mathbb{R}^d$ such that

$$\begin{cases} -\operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in } \Omega \times (0, T)\,, \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u}) = \mathscr{A}\boldsymbol{\epsilon}\,, \\ \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_0 \times (0, T)\,, \\ \boldsymbol{\sigma} \cdot \boldsymbol{\nu} = \boldsymbol{g} \quad \text{on } \Gamma_1 \times (0, T)\,, \\ \begin{cases} u_\nu = 0\,, \\ |\boldsymbol{\sigma}_\tau| \le \mu |\mathscr{R}\sigma_\nu| \\ |\boldsymbol{\sigma}_\tau| < \mu |\mathscr{R}\sigma_\nu| \;\Rightarrow\; \dot{\boldsymbol{u}}_\tau = 0 \\ |\boldsymbol{\sigma}_\tau| = \mu |\mathscr{R}\sigma_\nu| \;\Rightarrow\; \exists \lambda \ge 0\,,\; \dot{\boldsymbol{u}}_\tau = -\lambda \boldsymbol{\sigma}_\tau \end{cases} \quad \text{on } \Gamma_2 \times (0, T)\,, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega\,. \end{cases} \tag{9.62}$$

with $\mathscr{A} = (a_{ijkh})$ satisfying the conditions (9.8).

In order to write a variational formulation for the problem ($\mathscr{S}$), we define the following Hilbert spaces:

$$\begin{aligned} \boldsymbol{V} &= \{\boldsymbol{v} \in [H^1(\Omega)]^d\,;\ \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_0\,;\ v_\nu = 0 \text{ a.e. on } \Gamma_2\}\,, \\ \boldsymbol{W} &= \{\boldsymbol{v} \in \boldsymbol{V}\,;\ \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{v}) \in (L^2(\Omega))^d\}\,, \end{aligned}$$

endowed with the inner products

$$\begin{aligned} (\boldsymbol{u}, \boldsymbol{v})_V &= \int_\Omega \epsilon_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v})\ \mathrm{d}x \quad \forall \boldsymbol{u},\, \boldsymbol{v} \in \boldsymbol{V}, \\ (\boldsymbol{u}, \boldsymbol{v})_W &= (\boldsymbol{u}, \boldsymbol{v})_V + (\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}), \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{v}))_{(L^2(\Omega))^d} \quad \forall \boldsymbol{u},\, \boldsymbol{v} \in \boldsymbol{W}\,. \end{aligned}$$

We make the following regularity assumptions on the data

$$\begin{cases} \boldsymbol{f} \in W^{1,2}(0, T; (L^2(\Omega))^d)\,, \\ \boldsymbol{g} \in W^{1,2}(0, T; (L^2(\Gamma_1))^d)\,, \\ a_{ijkl} \in L^\infty(\Omega),\ i, j, k, l = 1, \ldots, d\,, \\ \mu \in L^\infty(\Gamma_2),\ \mu \ge 0 \text{ a.e. on } \Gamma_2\,, \\ \mathscr{R} : H^{-1/2}(\Gamma_2) \to L^2(\Gamma_2) \text{ is a linear compact operator}\,, \\ \boldsymbol{u}_0 \in \boldsymbol{V}, \end{cases} \tag{9.63}$$

where $H^{-1/2}(\Gamma_2)$ is the dual space of $H^{1/2}(\Gamma_2) = \{v \in H^{1/2}(\Gamma)\,;\ v = 0 \text{ a.e. on } \Gamma \backslash \Gamma_2\}$.

Let $\boldsymbol{F} \in W^{1,2}(0, T; \boldsymbol{V})$, where, for all $t \in [0, T]$, $\boldsymbol{F}(t)$ is the element of $\boldsymbol{V}$ defined by (9.11) and let the symmetric, V-elliptic, continuous bilinear form $a : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ defined by $(9.10)_3$. We also denote by $j : \boldsymbol{W} \times \boldsymbol{V} \to \mathbb{R}$ the functional defined by

$$j(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_2} \mu |\mathscr{R}\sigma_\nu(\boldsymbol{u})|\, |\boldsymbol{v}_\tau|\, \mathrm{d}s \quad \forall \boldsymbol{u} \in \boldsymbol{W} \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{9.64}$$

The weak formulation of problem ($\mathscr{S}$), in terms of displacements, is the following quasi-variational inequality.

Problem (S): Find $\boldsymbol{u} \in W^{1,2}(0,T;\boldsymbol{V})$ such that

$$\begin{cases} a(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + j(\boldsymbol{u}(t), \boldsymbol{v}) - j(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{F}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_{\boldsymbol{V}} \\ \qquad\qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}\,, \text{ a.e. } t \in (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0. \end{cases}$$

We suppose that the initial displacement $\boldsymbol{u}_0 \in \boldsymbol{V}$ satisfies the following compatibility condition

$$a(\boldsymbol{u}_0, \boldsymbol{v}) + j(\boldsymbol{u}_0, \boldsymbol{v}) \geq (\boldsymbol{F}(0), \boldsymbol{v})_{\boldsymbol{V}} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}\,. \tag{9.65}$$

We have the following existence result.

Theorem 9.4. *There exists $\mu_1 > 0$ such that for all $\mu \in L^\infty(\Gamma_2)$ with $\mu \geq 0$ a.e. on Γ_2 and $\|\mu\|_{L^\infty(\Gamma_2)} \leq \mu_1$, the problem* (**S**) *has at least one solution $\boldsymbol{u} \in W^{1,2}(0,T;\boldsymbol{V})$.*

Proof. In order to apply Theorem 4.19, we put

$$\begin{aligned} &K = K(\boldsymbol{\Theta}) = \boldsymbol{W} \quad \forall \boldsymbol{\Theta} \in \boldsymbol{V}\,, \\ &D_K = \boldsymbol{W} \times \boldsymbol{V}\,, \\ &H = L^2(\Gamma_2)\,, \quad \beta(\boldsymbol{\Theta}, \boldsymbol{v}) = \mu|\mathscr{R}\sigma_\nu(\boldsymbol{v})| \quad \forall \boldsymbol{\Theta} \in \boldsymbol{V}\,, \forall \boldsymbol{v} \in \boldsymbol{W}\,, \\ &j(\boldsymbol{\Theta}, \boldsymbol{v}, \boldsymbol{w}) = j(\boldsymbol{v}, \boldsymbol{w}) - (\boldsymbol{\Theta}, \boldsymbol{w})_{\boldsymbol{V}} \quad \forall \boldsymbol{\Theta},\, \boldsymbol{w} \in \boldsymbol{V}\,, \forall \boldsymbol{v} \in \boldsymbol{W}\,, \\ &b(\boldsymbol{\Theta}, \boldsymbol{v}, \boldsymbol{w}) = 0 \quad \forall \boldsymbol{\Theta},\, \boldsymbol{w} \in \boldsymbol{V}\,, \forall \boldsymbol{v} \in \boldsymbol{W}\,. \end{aligned}$$

It is easy to verify that the hypotheses (4.83)–(4.90), (4.96)–(4.98), and (4.100) are satisfied. In addition, both the problems ($\tilde{\mathbf{Q}}^a$) and ($\tilde{\mathbf{R}}^a$), p. 68, become the following problem

$$\begin{cases} \boldsymbol{u} \in \boldsymbol{W} \\ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(\boldsymbol{u}, \boldsymbol{v} - \mathbf{d}) - j(\boldsymbol{u}, \boldsymbol{u} - \mathbf{d}) \geq (\boldsymbol{F}, \boldsymbol{v} - \boldsymbol{u}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,, \end{cases}$$

and so, the hypothesis (4.105) is satisfied. As for μ_1 sufficiently small the hypothesis (4.101) is verified, the existence of a solution of the problem (**S**) follows from Theorem 4.19. □

In the sequel we shall suppose that $\|\mu\|_{L^\infty(\Gamma_2)} \leq \mu_1$ with $\mu_1 > 0$ sufficiently small such that the problem (**S**) has at least one solution.

The following results will be frequently used.

Lemma 9.4. *The functional j, defined by (9.64), has the properties:*

$$j(\boldsymbol{w},\boldsymbol{v}) \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{W},\ \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{9.66}$$

$$j(\boldsymbol{w},\boldsymbol{v}_1) - j(\boldsymbol{w},\boldsymbol{v}_2) \leq j(\boldsymbol{w},\boldsymbol{v}_1 - \boldsymbol{v}_2) \quad \forall \boldsymbol{w} \in \boldsymbol{W},\ \forall \boldsymbol{v}_1,\boldsymbol{v}_2 \in \boldsymbol{V} \tag{9.67}$$

$$j(\boldsymbol{w},\boldsymbol{0}) = 0 \quad \forall \boldsymbol{w} \in \boldsymbol{W}. \tag{9.68}$$

Moreover, for all $s \in [0,T]$, we have

$$\begin{cases} \liminf\limits_{n\to\infty} \displaystyle\int_0^s j(\boldsymbol{w}_n(t),\boldsymbol{v}_n(t))\,\mathrm{d}t \geq \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v}(t))\,\mathrm{d}t\,, \\ \forall \boldsymbol{w}_n \rightharpoonup \boldsymbol{w} \text{ weakly in } L^2(0,T;\boldsymbol{W}),\ \forall \boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \text{ weakly in } L^2(0,T;\boldsymbol{V}), \end{cases} \tag{9.69}$$

and

$$\begin{cases} \lim\limits_{n\to\infty} \displaystyle\int_0^s j(\boldsymbol{w}_n(t),\boldsymbol{v}_n)\,\mathrm{d}t = \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v})\,\mathrm{d}t\,, \\ \forall \boldsymbol{w}_n \rightharpoonup \boldsymbol{w} \text{ weakly in } L^2(0,T;\boldsymbol{W}),\ \forall \boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \text{ weakly in } \boldsymbol{V}. \end{cases} \tag{9.70}$$

Proof. The properties (9.66), (9.67), and (9.68) are obvious.

In order to prove (9.69), we write

$$\begin{aligned} &\left| \int_0^s \left(j(\boldsymbol{w}_n(t),\boldsymbol{v}_n(t)) - j(\boldsymbol{w}(t),\boldsymbol{v}_n(t)) \right) \mathrm{d}t \right| \\ &= \left| \int_0^s \int_{\Gamma_2} \mu(|\mathscr{R}\sigma_\nu(\boldsymbol{w}_n(t))| - |\mathscr{R}\sigma_\nu(\boldsymbol{w}(t))|)|(\boldsymbol{v}_n)_\tau(t)|\,\mathrm{d}s\,\mathrm{d}t \right| \\ &\leq \int_0^s \int_{\Gamma_2} \mu\,|\mathscr{R}\sigma_\nu(\boldsymbol{w}_n(t) - \boldsymbol{w}(t))|\ |(\boldsymbol{v}_n)_\tau(t)|\,\mathrm{d}s\,\mathrm{d}t \\ &\leq \int_0^s \|\mu\|_{L^\infty(\Gamma_2)} \|\mathscr{R}\sigma_\nu(\boldsymbol{w}_n(t) - \boldsymbol{w}(t))\|_{L^2(\Gamma_2)}\ \|(\boldsymbol{v}_n)_\tau(t)\|_{(L^2(\Gamma_2))^d}\,\mathrm{d}t \\ &\leq C\mu_1 \|\mathscr{R}\sigma_\nu(\boldsymbol{w}_n - \boldsymbol{w})\|_{L^2(0,T;L^2(\Gamma_2))}\ \|\boldsymbol{v}_n\|_{L^2(0,T,V)} \leq C_1 \|\mathscr{R}\sigma_\nu(\boldsymbol{w}_n - \boldsymbol{w})\|_{L^2(0,T;L^2(\Gamma_2))}\,, \end{aligned}$$

and hence, as the operator $\mathscr{R}$ is compact, it follows that

$$\lim_{n\to\infty} \int_0^s \left(j(\boldsymbol{w}_n(t),\boldsymbol{v}_n(t)) - j(\boldsymbol{w}(t),\boldsymbol{v}_n(t)) \right) \mathrm{d}t = 0 \quad \forall s \in [0.T]\,. \tag{9.71}$$

On the other hand, for any $\boldsymbol{w} \in L^2(0,T;\boldsymbol{W})$, the mapping $\boldsymbol{v} \longmapsto \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v}(t))\,\mathrm{d}t$ is convex l.s.c. on $L^2(0,T;\boldsymbol{V})$, thus

$$\liminf_{n\to\infty} \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v}_n(t))\,\mathrm{d}t \geq \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v}(t))\,\mathrm{d}t\,. \tag{9.72}$$

By combining the relations (9.71) and (9.72), we get:

$$\liminf_{n\to\infty} \int_0^s j(\boldsymbol{w}_n(t),\boldsymbol{v}_n(t))\,\mathrm{d}t \geq \lim_{n\to\infty} \int_0^s \left(j(\boldsymbol{w}_n(t),\boldsymbol{v}_n(t)) - j(\boldsymbol{w}(t),\boldsymbol{v}_n(t))\right)\,\mathrm{d}t$$
$$+ \liminf_{n\to\infty} \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v}_n(t))\,\mathrm{d}t \geq \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v}(t))\,\mathrm{d}t\,.$$

Next, we have

$$\left|\int_0^s j(\boldsymbol{w}_n(t),\boldsymbol{v}_n)\,\mathrm{d}t - \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v})\,\mathrm{d}t\right|$$
$$\leq \left|\int_0^s j(\boldsymbol{w}_n(t),\boldsymbol{v}_n)\,\mathrm{d}t - \int_0^s j(\boldsymbol{w}_n(t),\boldsymbol{v})\,\mathrm{d}t\right| + \left|\int_0^s j(\boldsymbol{w}_n(t),\boldsymbol{v})\,\mathrm{d}t - \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v})\,\mathrm{d}t\right|$$
$$\leq C_1\|\boldsymbol{v}_n - \boldsymbol{v}\|_{(L^2(\Gamma_2))^d} + C_2\|\mathscr{R}\sigma\nu(\boldsymbol{w}_\nu - \boldsymbol{w})\|_{L^2(0,T;L^2(\Gamma_2))}$$

and hence, from the compactness of the trace map from $\boldsymbol{V}$ into $(L^2(\Gamma_2))^d$, the proof is completed. □

Now, we are interested in finding the surface tractions $\boldsymbol{g}$ acting on Γ_1 so that the resulting displacement on the contact boundary Γ_2 is as close as possible to a given profile $\boldsymbol{u}_d$, while the norm of these surface forces remains small enough. The mathematical formulation of this problem is a state-control boundary optimal control problem where the state is solution of the implicit evolutionary quasi-variational inequality (**S**).

We introduce the following control and, respectively, observation spaces:

$$\begin{aligned} \mathbf{H_g} &= W^{1,2}(0,T;(L^2(\Gamma_1))^d)\,,\\ \mathbf{H_u} &= L^2(0,T;(L^2(\Gamma_2))^d) \end{aligned} \tag{9.73}$$

and we define, for $\beta > 0$ and $\boldsymbol{u}_d \in \mathbf{H_u}$ given, the cost functional $J : \mathbf{H_g} \times W^{1,2}(0,T;\mathbf{V}) \to \mathbb{R}_+$ by:

$$J(\boldsymbol{g},\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{u} - \boldsymbol{u}_d\|^2_{\mathbf{H_u}} + \frac{\beta}{2}\|\boldsymbol{g}\|^2_{\mathbf{H_g}}\,. \tag{9.74}$$

Due to the lack of uniqueness of solution for the quasi-variational inequality (**S**), the cost functional J, instead of depending, as usual, only on the "real" control $\boldsymbol{g}$, depends also on the state $\boldsymbol{u}$. For this reason, it is convenient to rewrite the variational problem (**S**), for $\boldsymbol{g} \in \mathbf{H_g}$, in the following form.

Problem $(\mathbf{S})^{\boldsymbol{g}}$: Find $\boldsymbol{u} \in W^{1,2}(0,T;\mathbf{V})$ such that

$$\begin{cases} a(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + j(\boldsymbol{u}(t), \boldsymbol{v}) - j(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{F}^g(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \\ \qquad\qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}\,,\ \text{a.e. } t \in (0,T) \\ \boldsymbol{u}(0) = \boldsymbol{u}_0\,, \end{cases}$$

where

$$(\boldsymbol{F}^g(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{v}\ \mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}(t) \cdot \boldsymbol{v}\ \mathrm{d}s \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,.$$

We formulate now the control problem as follows:

Problem (**CS**): Find $(\boldsymbol{g}^*, \boldsymbol{u}^*) \in \mathscr{V}_{ad}$ such that

$$J(\boldsymbol{g}^*, \boldsymbol{u}^*) = \min_{(\boldsymbol{g},\boldsymbol{u}) \in \mathscr{V}_{ad}} J(\boldsymbol{g}, \boldsymbol{u})\,,$$

where

$$\mathscr{V}_{ad} = \{(\boldsymbol{g}, \boldsymbol{u}) \in \mathbf{H_g} \times W^{1,2}(0,T;\mathbf{V})\,;\boldsymbol{u} \text{ is a solution of } (\mathbf{S})^{\boldsymbol{g}}\ \}.$$

Remark 9.2. Let us assume that there exist $(\boldsymbol{g}^*, \boldsymbol{u}^*) \in \mathscr{V}_{ad}$ such that $J(\boldsymbol{g}^*, \boldsymbol{u}^*) = \min_{(\boldsymbol{g},\boldsymbol{u}) \in \mathscr{V}_{ad}} J(\boldsymbol{g}, \boldsymbol{u})$ and a function $\boldsymbol{g}_d \in \mathbf{H_g}$ such that $(\boldsymbol{g}_d, \boldsymbol{u}_d) \in \mathscr{V}_{ad}$. Then,

$$J(\boldsymbol{g}^*, \boldsymbol{u}^*) = \frac{1}{2}\|\boldsymbol{u}^* - \boldsymbol{u}_d\|^2_{\mathbf{H_u}} + \frac{\beta}{2}\|\boldsymbol{g}^*\|^2_{\mathbf{H_g}} \leq J(\boldsymbol{g}_d, \boldsymbol{u}_d) = \frac{\beta}{2}\|\boldsymbol{g}_d\|^2_{\mathbf{H_g}}$$

and, hence,

$$\|\boldsymbol{u}^* - \boldsymbol{u}_d\|^2_{\mathbf{H_u}} \leq \beta(\|\boldsymbol{g}_d\|^2_{\mathbf{H_g}} - \|\boldsymbol{g}^*\|^2_{\mathbf{H_g}})\,.$$

Therefore, for β arbitrarily small, we may hope to obtain, on the contact boundary, a displacement field $\boldsymbol{u}$ as closed as we want to the desired value $\boldsymbol{u}_d$.

As one can see, although the functional J has good properties on $\mathbf{H_g} \times W^{1,2}(0,T;\mathbf{V})$, the existence of a solution of the control problem (**CS**) cannot be obtained directly, since the correspondence control $\mapsto$ state is a multivalued mapping. In order to overcome this difficulty, we approximate the optimal control problem (**CS**) by a family of penalized optimal control problems, governed by a variational inequality.

We start by introducing a new control space:

$$\mathbf{H_w} = L^2(0, T; \boldsymbol{W}) .$$

Now, for $(\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}$, we consider the variational inequality which models the problem $(\mathscr{S})$ in the case of Tresca friction.

Problem $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}$: Find $\boldsymbol{u} \in W^{1,2}(0, T; \mathbf{V})$ such that

$$\begin{cases} a(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + j(\boldsymbol{w}(t), \boldsymbol{v}) - j(\boldsymbol{w}(t), \dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{F}^{\boldsymbol{g}}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \\ \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V} \,, \text{ a.e. } t \in (0, T) \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \,. \end{cases}$$

Using the same techniques as in [7] or Sect. 4.3 and taking into account the positivity of j, one can prove the following existence result.

Proposition 9.3. *For* $(\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}$ *given, there exists a unique solution* $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}$ *of Problem* $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}$. *Moreover, we have*

$$\|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}\|_{L^2(0,T;V)} \leq C(\|\dot{\boldsymbol{F}}^{\boldsymbol{g}}\|_{L^2(0,T;V)} + \|\boldsymbol{w}\|_{L^2(0,T;V)}) ,$$

with C *a positive constant.*

In the sequel, for $(\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}$ given, we will denote by $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}$ the unique solution of Problem $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}$.

Let us fix $\epsilon > 0$. We introduce the penalized functional $J_\epsilon : \mathbf{H_g} \times \mathbf{H_w} \to \mathbb{R}_+$ by

$$J_\epsilon(\boldsymbol{g}, \boldsymbol{w}) = J(\boldsymbol{g}, \boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}) + \frac{1}{2\epsilon}\|\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}} - \boldsymbol{w}\|^2_{\mathbf{H_w}} \tag{9.75}$$

and we consider the control problem

Problem $(\mathbf{CS})_\epsilon$: Find $(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon) \in \mathbf{H_g} \times \mathbf{H_w}$ such that

$$J_\epsilon(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon) = \min\{J_\epsilon(\boldsymbol{g}, \boldsymbol{w})\,;\ (\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}\} .$$

The following result establishes the existence of an optimal solution for this penalized control problem.

Proposition 9.4. *Let (9.63) and (9.65) hold. Then, for all* $\epsilon > 0$, *there exists a solution* $(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon)$ *of problem* $(\mathbf{CS})_\epsilon$.

Proof. Let $\{(\boldsymbol{g}^n_\epsilon, \boldsymbol{w}^n_\epsilon)\}_n \subset \mathbf{H_g} \times \mathbf{H_w}$ be a minimizing sequence for the functional J_ϵ. Then, from the definition (9.75) of J_ϵ, we deduce

$$\lim_{n\to\infty} J_\epsilon(\boldsymbol{g}^n_\epsilon, \boldsymbol{w}^n_\epsilon) = \inf\{J_\epsilon(\boldsymbol{g}, \boldsymbol{w}),\ (\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}\} \in [0, +\infty) , \tag{9.76}$$

which implies that the sequence $\{\boldsymbol{g}_\epsilon^n\}_n$ is bounded in $\mathbf{H_g}$. Obviously, the sequence $\{\boldsymbol{F}_\epsilon^n\}_n$ defined by

$$(\boldsymbol{F}_\epsilon^n(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}_\epsilon^n(t) \cdot \boldsymbol{v} \, \mathrm{d}s \tag{9.77}$$

is also bounded in $W^{1,2}(0, T; \mathbf{V})$.

Thus, there exists $(\boldsymbol{g}_\epsilon^*, \boldsymbol{F}_\epsilon^*) \in \mathbf{H_g} \times W^{1,2}(0, T; \mathbf{V})$ such that, passing to a subsequence still denoted in the same way, we have

$$\boldsymbol{g}_\epsilon^n \rightharpoonup \boldsymbol{g}_\epsilon^* \text{ weakly in } \mathbf{H_g}\,, \tag{9.78}$$

$$\boldsymbol{F}_\epsilon^n \rightharpoonup \boldsymbol{F}_\epsilon^* \text{ weakly in } W^{1,2}(0, T; \mathbf{V})\,, \tag{9.79}$$

where

$$(\boldsymbol{F}_\epsilon^*(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}_\epsilon^*(t) \cdot \boldsymbol{v} \, \mathrm{d}s\,.$$

Let $\boldsymbol{u}_\epsilon^n = \boldsymbol{u}^{\boldsymbol{g}_\epsilon^n, \boldsymbol{w}_\epsilon^n}$. Taking $\boldsymbol{v} = \boldsymbol{0}$ in $(\mathbf{S})^{\boldsymbol{g}_\epsilon^n, \boldsymbol{w}_\epsilon^n}$, integrating by parts on $[0, s]$ with $s \in [0, T]$ and taking into account the properties (9.66), (9.68) of the functional j, we have

$$\int_0^s a(\boldsymbol{u}_\epsilon^n(t), \dot{\boldsymbol{u}}_\epsilon^n(t)) \, \mathrm{d}t \le \int_0^s (\boldsymbol{F}_\epsilon^n(t), \dot{\boldsymbol{u}}_\epsilon^n(t))_V \, \mathrm{d}t\,. \tag{9.80}$$

By using the V-ellipticity of $a(\cdot, \cdot)$, we obviously obtain

$$\begin{aligned} \int_0^s a(\boldsymbol{u}_\epsilon^n(t), \dot{\boldsymbol{u}}_\epsilon^n(t)) \, \mathrm{d}t &= \frac{1}{2} \int_0^s \frac{\mathrm{d}}{\mathrm{d}t} a(\boldsymbol{u}_\epsilon^n(t), \boldsymbol{u}_\epsilon^n(t)) \, \mathrm{d}t \\ &= \frac{a(\boldsymbol{u}_\epsilon^n(s), \boldsymbol{u}_\epsilon^n(s)) - a(\boldsymbol{u}_0, \boldsymbol{u}_0)}{2} \ge \frac{\alpha \|\boldsymbol{u}_\epsilon^n(s)\|_V^2 - a(\boldsymbol{u}_0, \boldsymbol{u}_0)}{2}. \end{aligned} \tag{9.81}$$

On the other hand, we have

$$\begin{aligned} &\left| \int_0^s (\boldsymbol{F}_\epsilon^n(t), \dot{\boldsymbol{u}}_\epsilon^n(t))_V \, \mathrm{d}t \right| = \left| \int_0^s \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{F}_\epsilon^n(t), \boldsymbol{u}_\epsilon^n(t))_V \, \mathrm{d}t - \int_0^s (\dot{\boldsymbol{F}}_\epsilon^n(t), \boldsymbol{u}_\epsilon^n(t))_V \, \mathrm{d}t \right| \\ &\le C \left(|(\boldsymbol{F}_\epsilon^n(s), \boldsymbol{u}_\epsilon^n(s))_V - (\boldsymbol{F}_\epsilon^n(0), \boldsymbol{u}_\epsilon^n(0))_V| + \int_0^s \|\dot{\boldsymbol{F}}_\epsilon^n(t)\|_V^2 \, \mathrm{d}t + \int_0^s \|\boldsymbol{u}_\epsilon^n(t)\|_V^2 \, \mathrm{d}t \right) \\ &\le C \left(\frac{\|\boldsymbol{F}_\epsilon^n(s)\|_V^2}{2\delta} + \frac{\delta \|\boldsymbol{u}_\epsilon^n(s)\|_V^2}{2} + \frac{\|\boldsymbol{F}_\epsilon^n(0)\|_V^2}{2} + \frac{\|\boldsymbol{u}_\epsilon^n(0)\|_V^2}{2} \right. \\ &\quad \left. + \int_0^s \|\dot{\boldsymbol{F}}_\epsilon^n(t)\|_V^2 \, \mathrm{d}t + \int_0^s \|\boldsymbol{u}_\epsilon^n(t)\|_V^2 \, \mathrm{d}t \right). \end{aligned}$$

By choosing $0 < \delta < \dfrac{\alpha}{C}$ in the last relation, from (9.80), (9.81) and Young's inequality, we get

$$\|\boldsymbol{u}_\epsilon^n(s)\|_V^2 \le C\left(\|\boldsymbol{u}_0\|_V^2 + \|\boldsymbol{F}_\epsilon^n(s)\|_V^2 + \|\boldsymbol{F}_\epsilon^n(0)\|_V^2 + \int_0^s \|\dot{\boldsymbol{F}}_\epsilon^n(t)\|_V^2\, \mathrm{d}t \right.$$
$$\left. + \int_0^s \|\boldsymbol{u}_\epsilon^n(t)\|_V^2\, \mathrm{d}t\right),$$

and hence, by using Gronwall's inequality and the boundedness of $\{\boldsymbol{F}_\epsilon^n\}_n$, it follows that

$$\|\boldsymbol{u}_\epsilon^n(s)\|_V^2 \le C\left(1 + \|\boldsymbol{F}_\epsilon^n(0)\|_V^2 + \|\dot{\boldsymbol{F}}_\epsilon^n\|_{L^2(0,T;V)}\right) \le C \quad \forall s \in [0,T]. \tag{9.82}$$

Therefore, the sequence $\{\boldsymbol{u}_\epsilon^n\}_n$ is bounded in $L^\infty(0,T;\boldsymbol{V})$. In addition, from $(\mathbf{S})^{\boldsymbol{g}_\epsilon^n, \boldsymbol{w}_\epsilon^n}$, we have

$$\|\boldsymbol{u}_\epsilon^n\|_{\mathbf{H_w}}^2 = \|\boldsymbol{u}_\epsilon^n\|_{L^2(0,T;V)}^2 + \|\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{u}_\epsilon^n)\|_{L^2(0,T;(L^2(\Omega))^d)}^2$$
$$= \|\boldsymbol{u}_\epsilon^n\|_{L^2(0,T;V)}^2 + \|\boldsymbol{f}\|_{L^2(0,T;(L^2(\Omega))^d)}^2 \le C,$$

which, from the definition of J_ϵ and the boundedness (9.76) of J_ϵ, implies that the sequence $\{\boldsymbol{w}_\epsilon^n\}_n$ is bounded in $\mathbf{H_w}$.

Now, from Proposition 9.3, we obtain

$$\|\dot{\boldsymbol{u}}_\epsilon^n\|_{L^2(0,T;V)} \le C. \tag{9.83}$$

Thus, we deduce that there exist the elements $\boldsymbol{u}_\epsilon^* \in W^{1,2}(0,T;\boldsymbol{V})$ and $\boldsymbol{w}_\epsilon^* \in \mathbf{H_w}$ and the subsequences, still denoted by $\{\boldsymbol{u}_\epsilon^n\}_n$ and $\{\boldsymbol{w}_\epsilon^n\}_n$, such that

$$\boldsymbol{w}_\epsilon^n \rightharpoonup \boldsymbol{w}_\epsilon^* \text{ weakly in } \mathbf{H_w}, \tag{9.84}$$

$$\begin{cases} \boldsymbol{u}_\epsilon^n \rightharpoonup \boldsymbol{u}_\epsilon^* \text{ weakly * in } L^\infty(0,T;\boldsymbol{V}), \\ \dot{\boldsymbol{u}}_\epsilon^n \rightharpoonup \dot{\boldsymbol{u}}_\epsilon^* \text{ weakly in } L^2(0,T;\boldsymbol{V}). \end{cases} \tag{9.85}$$

Using the embedding $W^{1,2}(0,T;\boldsymbol{V}) \hookrightarrow C([0,T];\boldsymbol{V})$, we also have

$$\boldsymbol{u}_\epsilon^n(t) \rightharpoonup \boldsymbol{u}_\epsilon^*(t) \text{ weakly in } \boldsymbol{V} \quad \forall t \in [0,T]. \tag{9.86}$$

Now, we shall prove the strong convergence of $\boldsymbol{u}_\epsilon^n$ to $\boldsymbol{u}$ in $L^2(0,T;\boldsymbol{V})$. Putting $\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v} = 2\dot{\boldsymbol{u}}_\epsilon^n(t)$ in $(\mathbf{S})^{\boldsymbol{g}_\epsilon^n, \boldsymbol{w}_\epsilon^n}$, one obtains:

$$a(\boldsymbol{u}_\epsilon^n(t), \boldsymbol{v}) + j(\boldsymbol{w}_\epsilon^n(t), \boldsymbol{v}) \ge (\boldsymbol{F}_\epsilon^n(t), \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \text{ a.e. } t \in (0,T),$$

Taking $\boldsymbol{v} = -\boldsymbol{v}$, it follows that

$$a(\boldsymbol{u}_\epsilon^n(t), \boldsymbol{v}) - j(\boldsymbol{w}_\epsilon^n(t), \boldsymbol{v}) \le (\boldsymbol{F}_\epsilon^n(t), \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \text{ a.e. } t \in (0,T). \tag{9.87}$$

Passing to the limit with $n \to \infty$ in this inequality and taking into account the convergences (9.85), (9.84), and (9.79), we obtain

$$a(\boldsymbol{u}_\epsilon^*(t), \boldsymbol{v}) - j(\boldsymbol{w}_\epsilon^*(t), \boldsymbol{v}) \leq (\boldsymbol{F}_\epsilon^*(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}\,, \text{ a.e. } t \in (0, T)\,. \tag{9.88}$$

Setting $\boldsymbol{v} = \boldsymbol{u}_\epsilon^n(t) - \boldsymbol{u}_\epsilon^*(t)$ in (9.87) and $\boldsymbol{v} = \boldsymbol{u}_\epsilon^*(t) - \boldsymbol{u}_\epsilon^n(t)$ in (9.88), we get

$$\begin{aligned}
&\alpha \|\boldsymbol{u}_\epsilon^n(t) - \boldsymbol{u}_\epsilon^*(t)\|_V^2 \leq a(\boldsymbol{u}_\epsilon^n(t) - \boldsymbol{u}_\epsilon^*(t), \boldsymbol{u}_\epsilon^n(t) - \boldsymbol{u}_\epsilon^*(t)) \\
&\leq C\|\mu\|_{L^\infty(\Gamma_2)} \big(\|\mathscr{R}\sigma_\nu(\boldsymbol{w}_\epsilon^n(t))\|_{L^2(\Gamma_2)} + \|\mathscr{R}\sigma_\nu(\boldsymbol{w}_\epsilon^*(t))\|_{L^2(\Gamma_2)}\big) \|\boldsymbol{u}_\epsilon^n(t) - \boldsymbol{u}_\epsilon^*(t)\|_{(L^2(\Gamma_2))^d} \\
&+\|\boldsymbol{g}_\epsilon^n(t) - \boldsymbol{g}_\epsilon^*(t)\|_{(L^2(\Gamma_1))^d} \|\boldsymbol{u}_\epsilon^n(t) - \boldsymbol{u}_\epsilon^*(t)\|_{(L^2(\Gamma_1))^d} \leq C\|\boldsymbol{u}_\epsilon^n(t) - \boldsymbol{u}_\epsilon^*(t)\|_{(L^2(\Gamma))^d}\,,
\end{aligned}$$

which, with (9.86) and the compactness of the trace map from $\boldsymbol{V}$ to $(L^2(\Gamma))^d$, implies

$$\boldsymbol{u}_\epsilon^n(t) \to \boldsymbol{u}_\epsilon^*(t) \text{ strongly in } \boldsymbol{V} \quad \forall t \in [0, T]\,. \tag{9.89}$$

Hence, by Lebesgue's Theorem 3.4, we obtain the strong convergence:

$$\boldsymbol{u}_\epsilon^n \to \boldsymbol{u}_\epsilon^* \text{ strongly in } L^2(0, T; \boldsymbol{V})\,. \tag{9.90}$$

We shall prove that $\boldsymbol{u}_\epsilon^* = \boldsymbol{u}^{\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*}$ and, from the uniqueness of the solution, we shall conclude that the convergences (9.78), (9.84), (9.85), and (9.89) hold true for the whole sequences.

For $s \in [0, T]$, from the convergences (9.85), (9.90), (9.84), (9.79) and the properties (9.69), (9.70), we have

$$\lim_{n\to\infty} \int_0^s a(\boldsymbol{u}_\epsilon^n(t), \dot{\boldsymbol{u}}_\epsilon^n(t))\,\mathrm{d}t = \int_0^s a(\boldsymbol{u}_\epsilon^*(t), \dot{\boldsymbol{u}}_\epsilon^*(t))\,\mathrm{d}t\,, \tag{9.91}$$

$$\lim_{n\to\infty} \int_0^s a(\boldsymbol{u}_\epsilon^n(t), \boldsymbol{v}(t))\,\mathrm{d}t = \int_0^s a(\boldsymbol{u}_\epsilon^*(t), \boldsymbol{v}(t))\,\mathrm{d}t \quad \forall \boldsymbol{v} \in L^2(0, T; \boldsymbol{V})\,, \tag{9.92}$$

$$\lim_{n\to\infty} \int_0^s (\boldsymbol{F}_\epsilon^n(t), \boldsymbol{v}(t))_V\,\mathrm{d}t = \int_0^s (\boldsymbol{F}_\epsilon^*(t), \boldsymbol{v}(t))_V\,\mathrm{d}t \quad \forall \boldsymbol{v} \in L^2(0, T; \boldsymbol{V})\,, \tag{9.93}$$

$$\lim_{n\to\infty} \int_0^s j(\boldsymbol{w}_\epsilon^n(t), \boldsymbol{v}(t))\,\mathrm{d}t = \int_0^s j(\boldsymbol{w}_\epsilon^*(t), \boldsymbol{v}(t))\,\mathrm{d}t \quad \forall \boldsymbol{v} \in L^2(0, T; \boldsymbol{V})\,, \tag{9.94}$$

$$\liminf_{n\to\infty} \int_0^s j(\boldsymbol{w}_\epsilon^n(t), \dot{\boldsymbol{u}}_\epsilon^n(t))\,\mathrm{d}t \geq \int_0^s j(\boldsymbol{w}_\epsilon^*(t), \dot{\boldsymbol{u}}_\epsilon^*(t))\,\mathrm{d}t\,. \tag{9.95}$$

Next, since we can write

$$\int_0^s (\boldsymbol{F}_\epsilon^n(t), \dot{\boldsymbol{u}}_\epsilon^n(t))_V \, \mathrm{d}t = (\boldsymbol{F}_\epsilon^n(s), \boldsymbol{u}_\epsilon^n(s))_V - (\boldsymbol{F}_\epsilon^n(0), \boldsymbol{u}_0)_V \, \mathrm{d}t - \int_0^s (\dot{\boldsymbol{F}}_\epsilon^n(t), \boldsymbol{u}_\epsilon^n(t))_V \, \mathrm{d}t ,$$

it follows that

$$\lim_{n\to\infty} \int_0^s (\boldsymbol{F}_\epsilon^n(t), \dot{\boldsymbol{u}}_\epsilon^n(t))_V \, \mathrm{d}t = \int_0^s (\boldsymbol{F}_\epsilon^*(t), \dot{\boldsymbol{u}}_\epsilon^*(t))_V \, \mathrm{d}t . \tag{9.96}$$

Now, by passing to the limit in $(\mathbf{S})^{g_\epsilon^n, w_\epsilon^n}$ with $n \to \infty$, one obtains

$$\begin{aligned} &\int_0^s a(\boldsymbol{u}_\epsilon^*(t), \boldsymbol{v}(t) - \dot{\boldsymbol{u}}_\epsilon^*(t)) \, \mathrm{d}t + \int_0^s j(\boldsymbol{w}_\epsilon^*(t), \boldsymbol{v}(t)) \, \mathrm{d}t - \int_0^s j(\boldsymbol{w}_\epsilon^*(t), \dot{\boldsymbol{u}}_\epsilon^*(t)) \, \mathrm{d}t \\ &\geq \int_0^s (\boldsymbol{F}_\epsilon^*(t), \boldsymbol{v}(t) - \dot{\boldsymbol{u}}_\epsilon^*(t))_V \, \mathrm{d}t \quad \forall\, \boldsymbol{v} \in L^2(0, T; \boldsymbol{V}) , \ \forall s \in [0, T] . \end{aligned} \tag{9.97}$$

Then, as usually, taking $\boldsymbol{v} \in L^2(0, T; \boldsymbol{V})$ defined by

$$\boldsymbol{v}(t) = \begin{cases} \boldsymbol{z} & \text{for } t \in [s, s+h] , \\ \dot{\boldsymbol{u}}_\epsilon^*(t) & \text{otherwise} , \end{cases}$$

with an arbitrary $\boldsymbol{z} \in \boldsymbol{V}$ and $h > 0$ such that $s + h \leq T$, one obtains

$$\begin{aligned} &\int_s^{s+h} a(\boldsymbol{u}_\epsilon^*(t), \boldsymbol{z} - \dot{\boldsymbol{u}}_\epsilon^*(t)) \, \mathrm{d}t + \int_s^{s+h} j(\boldsymbol{w}_\epsilon^*(t), \boldsymbol{z}) \, \mathrm{d}t - \int_s^{s+h} j(\boldsymbol{w}_\epsilon^*(t), \dot{\boldsymbol{u}}_\epsilon^*(t)) \, \mathrm{d}t \\ &\geq \int_s^{s+h} (\boldsymbol{F}_\epsilon^*(t), \boldsymbol{z} - \dot{\boldsymbol{u}}_\epsilon^*(t))_V \, \mathrm{d}t \quad \forall \boldsymbol{z} \in \boldsymbol{V} , \ \forall s \in [0, T) , \end{aligned} \tag{9.98}$$

which leads us, by passing to the limit with $h \to 0$, to the following inequality

$$\begin{aligned} &a(\boldsymbol{u}_\epsilon^*(t), \boldsymbol{z} - \dot{\boldsymbol{u}}_\epsilon^*(t)) + j(\boldsymbol{w}_\epsilon^*(t), \boldsymbol{z}) - j(\boldsymbol{w}_\epsilon^*(t), \dot{\boldsymbol{u}}_\epsilon^*(t)) \\ &\geq (\boldsymbol{F}_\epsilon^*(t), \boldsymbol{z} - \dot{\boldsymbol{u}}_\epsilon^*(t))_V \quad \forall \boldsymbol{z} \in \boldsymbol{V} \ \text{ a.e. } t \in (0, T) . \end{aligned} \tag{9.99}$$

Moreover, the pointwise convergence (9.89) and the initial condition $\boldsymbol{u}_\epsilon^n(0) = \boldsymbol{u}_0$ give us $\boldsymbol{u}_\epsilon^*(0) = \boldsymbol{u}_0$ and, so, $\boldsymbol{u}_\epsilon^* = \boldsymbol{u}^{g_\epsilon^*, w_\epsilon^*}$, i.e. $\boldsymbol{u}_\epsilon^*$ is the unique solution of problem $(\mathbf{S})^{g_\epsilon^*, w_\epsilon^n}$.

In order to end the proof of our existence result, let us notice that, from $(\mathbf{S})^{g^n_\epsilon, w^n_\epsilon}$ and (9.99), it follows that

$$\|\boldsymbol{u}^n_\epsilon - \boldsymbol{u}^*_\epsilon\|_{\mathbf{H_w}} = \|\boldsymbol{u}^n_\epsilon - \boldsymbol{u}^*_\epsilon\|_{L^2(0,T;V)},$$

which obviously, from (9.90), gives

$$\boldsymbol{u}^n_\epsilon \to \boldsymbol{u}^*_\epsilon \text{ strongly in } \mathbf{H_w}\,.$$

Therefore, since the norm is weakly lower semicontinuous, from the convergence (9.84), we get

$$\liminf_{n\to\infty} \frac{1}{2\epsilon}\|\boldsymbol{w}^n_\epsilon - \boldsymbol{u}^n_\epsilon\|^2_{\mathbf{H_w}} \geq \frac{1}{2\epsilon}\|\boldsymbol{w}^*_\epsilon - \boldsymbol{u}^*_\epsilon\|^2_{\mathbf{H_w}}\,. \tag{9.100}$$

Finally, by using the convergences (9.90), (9.78) and the relation (9.100), we have

$$\begin{aligned}&\inf\{J_\epsilon(\boldsymbol{g},\boldsymbol{w})\,;\ (\boldsymbol{g},\boldsymbol{w}) \in \mathbf{H_g}\times\mathbf{H_w}\}\\ &= \lim_{n\to\infty} J_\epsilon(\boldsymbol{g}^n_\epsilon, \boldsymbol{w}^n_\epsilon) \geq \liminf_{n\to\infty} J_\epsilon(\boldsymbol{g}^n_\epsilon, \boldsymbol{w}^n_\epsilon) \geq J_\epsilon(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon)\end{aligned}$$

and hence, we conclude

$$J_\epsilon(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon) = \min\{J_\epsilon(\boldsymbol{g},\boldsymbol{w})\,;\ (\boldsymbol{g},\boldsymbol{w}) \in \mathbf{H_g}\times\mathbf{H_w}\}\,.$$

□

Lemma 9.5. *If* $(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon)$ *is an optimal control for* $(\mathbf{CS})_\epsilon$ *and* $\boldsymbol{u}^*_\epsilon = \boldsymbol{u}^{\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon}$, *then*

$$\lim_{\epsilon\to 0} \|\boldsymbol{w}^*_\epsilon - \boldsymbol{u}^*_\epsilon\|_{\mathbf{H_w}} = 0. \tag{9.101}$$

Proof. Indeed, if $(\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{u}}) \in \mathcal{V}_{ad}$, then $\tilde{\boldsymbol{u}} \in \mathbf{H_w}$, $\tilde{\boldsymbol{u}} = \boldsymbol{u}^{\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}}$ and, hence,

$$J_\epsilon(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon) \leq J_\epsilon(\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{u}}) = J(\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{u}})\,. \tag{9.102}$$

Consequently, from the definition of J_ϵ, we get

$$\|\boldsymbol{w}^*_\epsilon - \boldsymbol{u}^*_\epsilon\|^2_{\mathbf{H_w}} \leq 2\epsilon J_\epsilon(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon) \leq 2\epsilon J(\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{u}})\,,$$

which implies (9.101). □

We are now in the position to prove the main result of this section, the existence of a solution to the optimal control problem $(\mathbf{CS})$.

Theorem 9.5. *For $\epsilon > 0$, let $(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*) \in \mathbf{H_g} \times \mathbf{H_w}$ be an optimal control of* $(\mathbf{CS})_\epsilon$ *and $\boldsymbol{u}_\epsilon^* = \boldsymbol{u}^{\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*}$. Then, there exist the elements $\boldsymbol{u}^* \in W^{1,2}(0,T;\mathbf{V})$ and $\boldsymbol{g}^* \in \mathbf{H_g}$ such that*

$$\begin{aligned} &\boldsymbol{g}_\epsilon^* \rightarrow \boldsymbol{g}^* \ \textit{weakly in } \mathbf{H_g}\,, \\ &\boldsymbol{w}_\epsilon^* \rightharpoonup \boldsymbol{u}^* \ \textit{strongly in } \mathbf{H_w}\,, \\ &\boldsymbol{u}_\epsilon^* \rightharpoonup \boldsymbol{u}^* \ \textit{weakly in } W^{1,2}(0,T;\mathbf{V})\,, \\ &\boldsymbol{u}_\epsilon^* \rightarrow \boldsymbol{u}^* \ \textit{strongly in } L^2(0,T;V)\,. \end{aligned} \tag{9.103}$$

Moreover, $(\boldsymbol{g}^*, \boldsymbol{u}^*) \in \mathscr{V}_{ad}$ and

$$\lim_{\epsilon\to 0} J_\epsilon(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*) = J(\boldsymbol{g}^*, \boldsymbol{u}^*) = \min_{(\boldsymbol{g},\boldsymbol{u})\in\mathscr{V}_{ad}} J(\boldsymbol{g}, \boldsymbol{u})\,. \tag{9.104}$$

Proof. From the definition and the boundedness (9.102) of $J_\epsilon(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*)$, it follows that the sequence $\{\boldsymbol{g}_\epsilon^*\}_\epsilon$ is bounded in $\mathbf{H_g}$. Therefore, there exists $\boldsymbol{g}^* \in \mathbf{H_g}$ such that, up to a subsequence, we have

$$\boldsymbol{g}_\epsilon^* \rightharpoonup \boldsymbol{g}^* \text{ weakly in } \mathbf{H_g}\,. \tag{9.105}$$

So,

$$\boldsymbol{F}_\epsilon^* \rightharpoonup \boldsymbol{F}^* \text{ weakly in } W^{1,2}(0,T,\boldsymbol{V})\,, \tag{9.106}$$

where

$$(\boldsymbol{F}_\epsilon^*(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\ \mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}_\epsilon^*(t)\cdot\boldsymbol{v}\ \mathrm{d}s \tag{9.107}$$

and

$$(\boldsymbol{F}^*(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\ \mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}^*(t)\cdot\boldsymbol{v}\ \mathrm{d}s\,.$$

Using the same arguments as in the proof of Proposition 9.4, we deduce

$$\begin{cases} \boldsymbol{u}_\epsilon^* \rightharpoonup \boldsymbol{u}^* \text{ weakly } * \text{ in } L^\infty(0,T;V)\,, \\ \dot{\boldsymbol{u}}_\epsilon^* \rightharpoonup \dot{\boldsymbol{u}}^* \text{ weakly in } L^2(0,T;V)\,, \\ \boldsymbol{u}_\epsilon^* \rightarrow \boldsymbol{u}^* \text{ strongly in } L^2(0,T;V)\,, \\ \boldsymbol{w}_\epsilon^* \rightharpoonup \boldsymbol{w}^* \text{ weakly in } \mathbf{H_w}\,, \end{cases} \tag{9.108}$$

with $\boldsymbol{u}^* \in W^{1,2}(0,T;\mathbf{V})$ and $\boldsymbol{w}^* \in \mathbf{H_w}$.

Passing to the limit with $\epsilon \to 0$ in the integral form of $(\mathbf{S})^{g_\epsilon^*, w_\epsilon^n}$, we deduce that $\boldsymbol{u}^* = \boldsymbol{u}^{\boldsymbol{g}^*, \boldsymbol{w}^*}$. As

$$\|\boldsymbol{u}_\epsilon^* - \boldsymbol{u}^*\|_{\mathbf{H_w}} = \|\boldsymbol{u}_\epsilon^* - \boldsymbol{u}^*\|_{L^2(0,T;V)}\,,$$

we have

$$\boldsymbol{u}_\epsilon^* \to \boldsymbol{u}^* \text{ strongly in } \mathbf{H_w}\,,$$

and thus, from (9.101), we get $(9.103)_2$, $\boldsymbol{w}^* = \boldsymbol{u}^*$ and $(\boldsymbol{g}^*, \boldsymbol{u}^*) \in \mathscr{V}_{ad}$.

Next, from the definition of J_ϵ, we have

$$\begin{aligned}&\frac{1}{2\epsilon}\|\boldsymbol{w}_\epsilon^* - \boldsymbol{u}_\epsilon^*\|^2_{\mathbf{H_w}} = J_\epsilon(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*) - J(\boldsymbol{g}_\epsilon^*, \boldsymbol{u}_\epsilon^*)\\ &\le J_\epsilon(\boldsymbol{g}^*, \boldsymbol{u}^*) - J(\boldsymbol{g}_\epsilon^*, \boldsymbol{u}_\epsilon^*) = J(\boldsymbol{g}^*, \boldsymbol{u}^*) - J(\boldsymbol{g}_\epsilon^*, \boldsymbol{u}_\epsilon^*)\,,\end{aligned}$$

so,

$$0 \le \limsup_{\epsilon\to 0} \frac{1}{2\epsilon}\|\boldsymbol{w}_\epsilon^* - \boldsymbol{u}_\epsilon^*\|^2_{\mathbf{H_w}} \le J(\boldsymbol{g}^*, \boldsymbol{u}^*) - \liminf_{\epsilon\to 0} J(\boldsymbol{g}_\epsilon^*, \boldsymbol{u}_\epsilon^*) \le 0\,,$$

i.e.

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\|\boldsymbol{w}_\epsilon^* - \boldsymbol{u}_\epsilon^*\|^2_{\mathbf{H_w}} = 0\,. \tag{9.109}$$

Finally, it is easy to see that

$$\begin{aligned}J(\boldsymbol{g}^*, \boldsymbol{u}^*) &\le \liminf_{\epsilon\to 0} J_\epsilon(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*) \le \limsup_{\epsilon\to 0} J_\epsilon(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*) \le \limsup_{\epsilon\to 0} J_\epsilon(\boldsymbol{g}^*, \boldsymbol{u}^*)\\ &= J(\boldsymbol{g}^*, \boldsymbol{u}^*)\end{aligned}$$

and

$$J_\epsilon(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*) \le J_\epsilon(\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{u}}) = J(\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{u}}) \quad \forall (\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{u}}) \in \mathscr{V}_{ad}\,,$$

which give us

$$J(\boldsymbol{g}^*, \boldsymbol{u}^*) = \lim_{\epsilon\to 0} J_\epsilon(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*) \le J(\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{u}}) \quad \forall (\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{u}}) \in \mathscr{V}_{ad}\,.$$

So, $(\boldsymbol{g}^*, \boldsymbol{u}^*)$ is an optimal control for the cost functional J and the minimal value of J_ϵ converges to the minimal value of J. □

9.3.2 Regularized Problems and Optimality Conditions

Until now, we have reduced our optimal control problem to one governed by a variational inequality of the second kind. Unfortunately, the problem $(\mathbf{CS})_\epsilon$, despite the fact that it is simpler than the initial one, still involves a non-differentiable functional J_ϵ. Therefore, to attain our main goal, the obtaining of the optimality conditions, we shall consider a family of regularized problems associated with $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}$, defined, for $\rho > 0$, by

Problem $(\mathbf{S})_\rho^{g,w}$: Find $\boldsymbol{u} \in W^{1,2}(0,T;\mathbf{V})$ such that

$$\begin{cases} \rho(\dot{\boldsymbol{u}}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V + a(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + j^\rho(\boldsymbol{w}(t), \boldsymbol{v}) - j^\rho(\boldsymbol{w}(t), \dot{\boldsymbol{u}}(t)) \\ \qquad \geq (\boldsymbol{F}^g(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \text{ a.e. } t \in (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \end{cases}$$

where, for $\boldsymbol{w} \in \boldsymbol{W}$, $\{j^\rho(\boldsymbol{w},\cdot)\}_\rho$ is a family of convex functionals $j^\rho(\boldsymbol{w},\cdot) : \boldsymbol{V} \to \mathbb{R}_+$, of class C^2, i.e. the gradients with respect to the second variable, $\nabla_2 j^\rho(\boldsymbol{w},\cdot) : \boldsymbol{V} \to \boldsymbol{V}^*$ and $\nabla_2^2 j^\rho(\boldsymbol{w},\cdot) : \boldsymbol{V} \to \mathscr{L}(\boldsymbol{V}, \boldsymbol{V}^*)$, are continuous. In addition, we suppose that the following conditions hold true:

$$j^\rho(\boldsymbol{w}, \boldsymbol{0}) = 0 \quad \forall \boldsymbol{w} \in \boldsymbol{W}, \tag{9.110}$$

$$|j^\rho(\boldsymbol{w}, \boldsymbol{v}) - j(\boldsymbol{w}, \boldsymbol{v})| \leq C\rho\|\boldsymbol{w}\|_V \quad \forall \boldsymbol{w} \in \boldsymbol{W},\ \forall \boldsymbol{v} \in \boldsymbol{V} \tag{9.111}$$

$$\begin{cases} \lim_{n\to\infty} \int_0^T \langle \nabla_2 j^\rho(\boldsymbol{w}_n(t), \boldsymbol{u}_n(t)), \boldsymbol{v}\rangle \,\mathrm{d}t = \int_0^T \langle \nabla_2 j^\rho(\boldsymbol{w}(t), \boldsymbol{u}(t)), \boldsymbol{v}\rangle \,\mathrm{d}t \\ \forall (\boldsymbol{w}_n, \boldsymbol{u}_n) \rightharpoonup (\boldsymbol{w}, \boldsymbol{u}) \text{ weakly in } \mathbf{H_w} \times L^2(0,T;\boldsymbol{V}),\ \forall \boldsymbol{v} \in \boldsymbol{V}, \end{cases} \tag{9.112}$$

where C is a constant independent of $\boldsymbol{v}$ and $\langle\cdot,\cdot\rangle$ denotes the duality pair between $\boldsymbol{V}^*$ and $\boldsymbol{V}$.

Remark 9.3. We can choose

$$j^\rho(\boldsymbol{w}, \boldsymbol{v}) = \int_{\Gamma_2} \mu|\mathscr{R}\sigma_\nu(\boldsymbol{w})|\,\theta_\rho(\boldsymbol{v}_\tau)\,\mathrm{d}s \quad \forall (\boldsymbol{w}, \boldsymbol{v}) \in \boldsymbol{W} \times \boldsymbol{V}, \tag{9.113}$$

where the function $\theta_\rho : \mathbb{R}^p \to \mathbb{R}$ is an approximation (see [12] or [1]) of the function $|\cdot| : \mathbb{R}^p \to \mathbb{R}$, satisfying the following properties:

$$\begin{cases} \theta_\rho \text{ is a convex, nonnegative function of class } C^2, \\ \theta_\rho(\boldsymbol{0}) = 0, \\ \big|\theta_\rho(\boldsymbol{u}) - |\boldsymbol{u}|\big| \leq C_0\rho, \\ \big|\theta'_\rho(\boldsymbol{u}) \cdot \boldsymbol{v}\big| \leq C_1|\boldsymbol{v}|, \\ \big|\theta''_\rho(\boldsymbol{u})(\boldsymbol{v}\cdot\boldsymbol{q})\big| \leq C_2(\rho)|\boldsymbol{v}|\,|\boldsymbol{q}|, \end{cases} \tag{9.114}$$

with C_0, C_1, and $C_2(\rho)$ positive constants.

Then, after some computations, it follows that

$$\langle \nabla_2 j^\rho(\boldsymbol{w}, \boldsymbol{u}), \boldsymbol{v}\rangle = \int_{\Gamma_2} \mu|\mathscr{R}(\sigma_\nu(\boldsymbol{w}))|\theta'_\rho(\boldsymbol{u}_\tau) \cdot \boldsymbol{v}_\tau \,\mathrm{d}s,$$

$$\langle \nabla_2^2 j^\rho(\boldsymbol{w}, \boldsymbol{u})\boldsymbol{v}, \boldsymbol{q}\rangle = \int_{\Gamma_2} \mu|\mathscr{R}(\sigma_\nu(\boldsymbol{w}))|\theta''_\rho(\boldsymbol{u}_\tau)(\boldsymbol{v}_\tau \cdot \boldsymbol{q}_\tau)\,\mathrm{d}s.$$

For instance, if we take

$$\theta_\rho(\boldsymbol{v}) = \begin{cases} \dfrac{|\boldsymbol{v}|^2}{\rho}\left(1 - \dfrac{|\boldsymbol{v}|}{3\rho}\right) & \text{if } |\boldsymbol{v}| \le \rho\,, \\ \rho\left(\dfrac{|\boldsymbol{v}|}{\rho} - \dfrac{1}{3}\right) & \text{if } |\boldsymbol{v}| \ge \rho\,, \end{cases} \tag{9.115}$$

then θ'_ρ and θ''_ρ are defined by (8.156) and (8.157) (p. 182), and if we choose

$$\theta_\rho(\boldsymbol{v}) = \sqrt{\rho^2 + |\boldsymbol{v}|^2} - \rho\,, \tag{9.116}$$

then one has:

$$\theta'_\rho(\boldsymbol{u}) = \frac{\boldsymbol{u}}{\sqrt{\rho^2 + |\boldsymbol{u}(x)|^2}},$$

and

$$\theta''_\rho(\boldsymbol{u})(\boldsymbol{v}) = \frac{1}{\sqrt{\rho^2 + |\boldsymbol{u}(x)|^2}}\left(\boldsymbol{v} - \frac{(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{u}}{\rho^2 + |\boldsymbol{u}(x)|^2}\right).$$

It is easy to see that, in both cases, the functional j_ρ, defined by (9.113), satisfies the properties (9.110)–(9.112) and, in addition, we have

$$\begin{cases} |\nabla_2 j^\rho(\boldsymbol{w},\boldsymbol{u})\cdot\boldsymbol{v}| \le C_1\|\boldsymbol{v}\| & \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}\,, \\ |\langle\nabla_2^2 j^\rho(\boldsymbol{w},\boldsymbol{u})\cdot\boldsymbol{v},\boldsymbol{q}\rangle| \le C_2\|\boldsymbol{v}\|\,\|\boldsymbol{q}\| & \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{q} \in \boldsymbol{V}\,, \end{cases}$$

with $C_1 = C_1(\boldsymbol{w}) > 0$ and $C_2 = C_2(\boldsymbol{w},\rho) > 0$.

Obviously, the regularized problem $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$ can be equivalently written as the following variational equality.

Problem $(\mathscr{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$: Find $\boldsymbol{u} \in W^{1,2}(0,T;\boldsymbol{V})$ such that

$$\begin{cases} \rho(\dot{\boldsymbol{u}}(t),\boldsymbol{v})_V + a(\boldsymbol{u}(t),\boldsymbol{v}) + \langle\nabla_2 j^\rho(\boldsymbol{w}(t),\dot{\boldsymbol{u}}(t)),\boldsymbol{v}\rangle \\ = (\boldsymbol{F}^g(t),\boldsymbol{v})_V\,, \forall\, \boldsymbol{v} \in \boldsymbol{V}\,, \text{ a.e. } t \in (0,T)\,, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0\,. \end{cases}$$

We have the following existence and uniqueness result.

Proposition 9.5. *Let* $(\boldsymbol{g},\boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}$ *and* $\rho > 0$. *Then, there exists a unique solution* $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho \in W^{1,2}(0,T;\mathbf{V})$ *of Problem* $(\mathscr{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$.

Proof. Arguing as in [2], one can prove the following main steps of the proof.

(1) For any $\boldsymbol{\alpha} \in W^{1,2}(0,T;\boldsymbol{V})$, the problem

$$\begin{cases} \boldsymbol{v}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}} \in W^{1,2}(0,T;\boldsymbol{V}) \\ \rho(\boldsymbol{v}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}}(t),\boldsymbol{v})_V + \langle\nabla_2 j^\rho(\boldsymbol{w}(t),\boldsymbol{v}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}}(t)),\boldsymbol{v}\rangle = (\boldsymbol{F}^{g(t)},\boldsymbol{v}) \\ -a(\boldsymbol{\alpha}(t),\boldsymbol{v})\ \forall\, \boldsymbol{v} \in \boldsymbol{V}\,,\ \forall t \in (0,T)\,, \end{cases} \tag{9.117}$$

has a unique solution $\boldsymbol{v}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}} \in W^{1,2}(0,T;\boldsymbol{V})$.

(2) Let $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}} : [0,T] \to \boldsymbol{V}$ be the function defined by

$$\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}}(t) = \int_0^t \boldsymbol{v}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}}(s)\,\mathrm{d}s + \boldsymbol{u}_0\,. \tag{9.118}$$

Then $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}} \in W^{2,2}(0,T;\boldsymbol{V})$ and $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}}(0) = \boldsymbol{u}_0$.

(3) We denote by $\Lambda_\rho : W^{1,2}(0,T;\boldsymbol{V}) \to W^{1,2}(0,T;\boldsymbol{V})$ the mapping defined by

$$\Lambda_\rho(\boldsymbol{\alpha})(t) = \boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}}(t) \qquad \forall \boldsymbol{\alpha} \in W^{1,2}(0,T;\boldsymbol{V}),\ \forall t \in [0,T]\,. \tag{9.119}$$

One can prove that the map Λ_ρ has a unique fixed point $\boldsymbol{\alpha}^*$. Therefore, the function $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}^*}$ defined by (9.118), is a solution of Problem $(\mathscr{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$. Finally, by using Gronwall' inequality and the properties (9.110)–(9.112) of the function j_ρ, from the formulation $(\mathscr{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$, the uniqueness follows. □

The regularized problem $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$ approximates the penalized problem $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}$ in the following sense.

Proposition 9.6. *Let $(\boldsymbol{g},\boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}$. For $\rho > 0$, let $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho$ be the unique solution of problem $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$. Then*

$$\begin{aligned} \boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho &\to \boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}} \quad \textit{strongly in } L^\infty(0,T;\boldsymbol{V})\,,\\ \dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho &\rightharpoonup \dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}} \quad \textit{weakly in } L^2(0,T;\boldsymbol{V})\,, \end{aligned} \tag{9.120}$$

$\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}$ being the unique solution of $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}$. Moreover, there exists a constant $C > 0$, independent of ρ, such that

$$\|\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho - \boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}\|_{L^\infty(0,T;\boldsymbol{V})} \le C\sqrt{\rho}\left(1 + \|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}\|^2_{L^2(0,T;\boldsymbol{V})}\right)\,. \tag{9.121}$$

Proof. Using the property (9.111) of j^ρ and taking $\boldsymbol{v} = \dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho$ in $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}$ and $\boldsymbol{v} = \dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}$ in $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$, we get

$$\begin{aligned}
&\rho \int_0^s \|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho(t)\|^2_V\,\mathrm{d}t + \frac{\alpha}{2}\|\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}(s) - \boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho(s)\|^2_V\\
&\le \int_0^s |j^\rho(\boldsymbol{w}(t),\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}(t)) - j(\boldsymbol{w}(t),\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}(t))|\,\mathrm{d}t\\
&\quad + \int_0^s |j(\boldsymbol{w}(t),\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho(t)) - j^\rho(\boldsymbol{w}(t),\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho(t))|\,\mathrm{d}t + \rho\int_0^s (\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho(t),\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}(t))_V\,\mathrm{d}t\\
&\le C\rho\int_0^s \|\boldsymbol{w}(t)\|_V\,\mathrm{d}t + \rho\int_0^s \|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho(t)\|_V\,\|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}(t)\|_V\,\mathrm{d}t\\
&\le \rho\left(C_0 + \frac{\nu}{2}\int_0^s \|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho(t)\|^2_V\,\mathrm{d}t + \frac{1}{2\nu}\int_0^s \|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}(t)\|^2_V\,\mathrm{d}t\right)\,,\ \forall s \in [0,T]\,,
\end{aligned}$$

which implies, for $\nu > 0$ conveniently chosen, that

$$\|\dot{\boldsymbol{u}}_\rho^{\boldsymbol{g},\boldsymbol{w}}\|^2_{L^2(0,T;V)} \le C(1 + \|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}\|^2_{L^2(0,T;V)}) \tag{9.122}$$

and

$$\|\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}(s) - \boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}}(s)\|^2_V \le C\rho(1 + \|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}\|^2_{L^2(0,T;V)}) \quad \forall s \in [0,T]\,.$$

□

Now, we formulate an optimal control problem, governed by the regularized problem $(\mathbf{S})_\rho^{\boldsymbol{g},\boldsymbol{w}}$, in which the cost functional is defined similarly to J_ϵ, the only difference being that the state is, in this case, the solution of an equation. More precisely, we introduce the regularized functional:

$$\begin{aligned} J_{\epsilon\rho}(\boldsymbol{g},\boldsymbol{w}) =& J(\boldsymbol{g},\boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}}) + \frac{1}{2\epsilon}\|\boldsymbol{w} - \boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}}\|^2_{\mathbf{H_w}} \\ =& \frac{1}{2}\|\boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}} - \boldsymbol{u}_d\|^2_{\mathbf{H_u}} + \frac{\beta}{2}\|\boldsymbol{g}\|^2_{\mathbf{H_g}} + \frac{1}{2\epsilon}\|\boldsymbol{w} - \boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}}\|^2_{\mathbf{H_w}}\,, \end{aligned} \tag{9.123}$$

$\boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}}$ being the unique solution of the regularized problem $(\mathbf{S})_\rho^{\boldsymbol{g},\boldsymbol{w}}$ or, equivalently, of the variational equation $(\mathscr{S})_\rho^{\boldsymbol{g},\boldsymbol{w}}$.

For any $\rho > 0$, we consider the corresponding regularized optimal control problem.

Problem $(\mathbf{CS})_{\epsilon\rho}$: Find $(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) \in \mathbf{H}_{\boldsymbol{g}} \times \mathbf{H}_{\boldsymbol{w}}$ such that

$$J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) = \min\{J_{\epsilon\rho}(\boldsymbol{g},\boldsymbol{w})\,;\,(\boldsymbol{g},\boldsymbol{w}) \in \mathbf{H}_{\boldsymbol{g}} \times \mathbf{H}_{\boldsymbol{w}}\}.$$

Theorem 9.6. *For $\rho > 0$, there exists a solution $(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho})$ of Problem $(\mathbf{CS})_{\epsilon\rho}$.*

Proof. Let $\{(\boldsymbol{g}^n_{\epsilon\rho}, \boldsymbol{w}^n_{\epsilon\rho})\}_n$ be a minimizing sequence for the functional $J_{\epsilon\rho}$. From the definition of $J_{\epsilon\rho}$, it follows that there exists $\boldsymbol{g}^*_{\epsilon\rho} \in \mathbf{H_g}$ such that, up to a subsequence, we have

$$\boldsymbol{g}^n_{\epsilon\rho} \rightharpoonup \boldsymbol{g}^*_{\epsilon\rho} \text{ weakly in } \mathbf{H_g}\,. \tag{9.124}$$

Let $\boldsymbol{u}^n_{\epsilon\rho} = u^{\boldsymbol{g}^n_{\epsilon\rho},\boldsymbol{w}^n_{\epsilon\rho}}$. Putting $\boldsymbol{v} = \dot{\boldsymbol{u}}^n_{\epsilon\rho}(t)$ in $(\mathscr{S})_\rho^{\boldsymbol{g}^n_{\epsilon\rho},\boldsymbol{w}^n_{\epsilon\rho}}$ and taking into account that (9.110) implies

$$\langle \nabla_2 j^\rho(\boldsymbol{w},\boldsymbol{u}),\boldsymbol{u}\rangle \ge 0\,,\ \forall(\boldsymbol{w},\boldsymbol{u}) \in \boldsymbol{W} \times \boldsymbol{V}\,, \tag{9.125}$$

we get

$$\begin{aligned} &\rho\int_0^s \|\dot{\boldsymbol{u}}^n_{\epsilon\rho}(t)\|^2_V\,\mathrm{d}t + \frac{\alpha}{2}\|\boldsymbol{u}^n_{\epsilon\rho}(s)\|^2_V \le \frac{1}{2}a(\boldsymbol{u}_0,\boldsymbol{u}_0) + \int_0^s (\boldsymbol{F}^n_{\epsilon\rho}(t), \dot{\boldsymbol{u}}^n_{\epsilon\rho}(t))_V\,\mathrm{d}t \\ &\le C + \frac{\nu}{2}\int_0^s \|\dot{\boldsymbol{u}}^n_{\epsilon\rho}(t)\|^2_V\,\mathrm{d}t + \frac{1}{2\nu}\int_0^s \|\boldsymbol{F}^n_{\epsilon\rho}(t)\|^2_V\,\mathrm{d}t\,, \end{aligned}$$

where

$$(\boldsymbol{F}^n_{\epsilon\rho}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}^n_{\epsilon\rho}(t)\cdot\boldsymbol{v}\,\mathrm{d}s\,. \tag{9.126}$$

Thus, with (9.124), it follows that

$$\begin{aligned}
&\|\boldsymbol{u}^n_{\epsilon\rho}(s)\|^2_V \le C \quad \forall s\in[0,T]\,,\\
&\|\boldsymbol{u}^n_{\epsilon\rho}\|^2_{L^\infty(0,T;V)} \le C\,,\\
&\|\dot{\boldsymbol{u}}^n_{\epsilon\rho}\|^2_{L^2(0,T;V)} \le C_\rho\,,
\end{aligned} \tag{9.127}$$

with C and C_ρ positive constants. So, up to a subsequence, we have

$$\begin{cases}
\boldsymbol{u}^n_{\epsilon\rho} \rightharpoonup \boldsymbol{u}^*_{\epsilon\rho} \text{ weakly * in } L^\infty(0,T;V)\,,\\
\boldsymbol{u}^n_{\epsilon\rho}(t) \rightharpoonup \boldsymbol{u}^*_{\epsilon\rho}(t) \text{ weakly in } V \quad \forall t\in[0,T]\,,\\
\dot{\boldsymbol{u}}^n_{\epsilon\rho} \rightharpoonup \dot{\boldsymbol{u}}^*_{\epsilon\rho} \text{ weakly in } L^2(0,T;V)\,.
\end{cases} \tag{9.128}$$

Therefore, since

$$\|\boldsymbol{u}^n_{\epsilon\rho}\|^2_{\mathbf{H_w}} = \|\boldsymbol{u}^n_{\epsilon\rho}\|^2_{L^2(0,T;V)} + \|\rho\dot{\boldsymbol{u}}^n_{\epsilon\rho} - \boldsymbol{f}\|^2_{L^2(0,T;(L^2(\Omega))^d)}\,,$$

we conclude that the sequence $\{\boldsymbol{u}^n_{\epsilon\rho}\}_n$ is also bounded in $\mathbf{H_w}$ and, from the definition and the boundedness of $\{J_{\epsilon\rho}(\boldsymbol{g}^n_{\epsilon\rho}, \boldsymbol{w}^n_{\epsilon\rho})\}_n$, it follows that the sequence $\{\boldsymbol{w}^n_{\epsilon\rho}\}_n$ is bounded in $\mathbf{H_w}$. So, up to a subsequence, we have

$$\boldsymbol{w}^n_{\epsilon\rho} \rightharpoonup \boldsymbol{w}^*_{\epsilon\rho} \text{ weakly in } \mathbf{H_w}\,, \tag{9.129}$$

with $\boldsymbol{w}^*_{\epsilon\rho}\in\mathbf{H_w}$.

Now, passing to the limit with $n\to\infty$ in $(\mathscr{S})_\rho^{\boldsymbol{g}^n_{\epsilon\rho},\boldsymbol{w}^n_{\epsilon\rho}}$ and using the convergences (9.124), (9.129), (9.128), and (9.112), we obtain that $\boldsymbol{u}^*_{\epsilon\rho} = \boldsymbol{u}^{\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}$. From the uniqueness of the solution, we deduce that all the above convergences hold on the whole sequences.

Next, from $(\mathbf{S})^{\boldsymbol{g}^n_{\epsilon\rho},\boldsymbol{w}^n_{\epsilon\rho}}$ and $(\mathbf{S})^{\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}$, we obtain

$$(\boldsymbol{u}^n_{\epsilon\rho}-\boldsymbol{u}^*_{\epsilon\rho},\varphi)_{\mathbf{H_w}} = (\boldsymbol{u}^n_{\epsilon\rho}-\boldsymbol{u}^*_{\epsilon\rho},\varphi)_{L^2(0,T;V)} + \rho(\dot{\boldsymbol{u}}^n_{\epsilon\rho}-\dot{\boldsymbol{u}}^*_{\epsilon\rho},\varphi)_{L^2(0,T;(L^2(\Omega))^d)}\ \forall\varphi\in\mathbf{H_w}\,,$$

which, together with $(9.128)_{1,3}$, implies

$$\boldsymbol{u}^n_{\epsilon\rho} \rightharpoonup \boldsymbol{u}^*_{\epsilon\rho} \text{ weakly in } \mathbf{H_w}\,.$$

Therefore, by using the convergence (9.129), one gets

$$\liminf_{n\to\infty} \frac{1}{2\epsilon}\|\boldsymbol{w}^n_{\epsilon\rho}-\boldsymbol{u}^n_{\epsilon\rho}\|^2_{\mathbf{H_w}} \geq \frac{1}{2\epsilon}\|\boldsymbol{w}^*_{\epsilon\rho}-\boldsymbol{u}^*_{\epsilon\rho}\|^2_{\mathbf{H_w}}\,. \tag{9.130}$$

Finally, using the weakly lower semi-continuity of $J_{\epsilon\rho}$ and (9.130), one deduces

$$\begin{aligned}&\inf\{J_{\epsilon\rho}(\boldsymbol{g},\boldsymbol{w})\,;\ (\boldsymbol{g},\boldsymbol{w})\in \mathbf{H_g}\times\mathbf{H_w}\}\\ &=\lim_{n\to\infty} J_{\epsilon\rho}(\boldsymbol{g}^n_{\epsilon\rho},\boldsymbol{w}^n_{\epsilon\rho}) \geq \liminf_{n\to\infty} J_{\epsilon\rho}(\boldsymbol{g}^n_{\epsilon\rho},\boldsymbol{w}^n_{\epsilon\rho}) \geq J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho})\end{aligned}$$

and so, we conclude

$$J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}) = \min\{J_{\epsilon\rho}(\boldsymbol{g},\boldsymbol{w})\,;\ (\boldsymbol{g},\boldsymbol{w})\in \boldsymbol{H}_g\times\boldsymbol{H}_w\}\,.$$

□

The following property of the solution of the regularized problem $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$ will allow us to prove an important result of this section, stated in Theorem 9.7, which gives the asymptotic behavior of the regularized optimal controls of problem $(\mathbf{CS})_{\epsilon\rho}$.

Proposition 9.7. *Let* $\{(\boldsymbol{g}_n,\boldsymbol{w}_n)\}_n\subset \mathbf{H_g}\times\mathbf{H_w}$ *be such that*

$$(\boldsymbol{g}_n,\boldsymbol{w}_n)\rightharpoonup(\boldsymbol{g},\boldsymbol{w}) \text{ weakly in } \mathbf{H_g}\times\mathbf{H_w}\,.$$

Then,

$$\boldsymbol{u}^{\boldsymbol{g}_n,\boldsymbol{w}_n}_\rho \to \boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho \quad \text{weakly in } W^{1,2}(0,T;\boldsymbol{V})\,,$$

$\boldsymbol{u}^{\boldsymbol{g}_n,\boldsymbol{w}_n}_\rho$ *being the unique solution of* $(\mathbf{S})^{\boldsymbol{g}_n,\boldsymbol{w}_n}_\rho$ *and* $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho$ *the unique solution of* $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$.

Proof. Let $\boldsymbol{u}_n=\boldsymbol{u}^{\boldsymbol{w}_n,\boldsymbol{g}_n}_\rho$. Taking $\boldsymbol{v}=\dot{\boldsymbol{u}}_n$ in $(\mathscr{S})^{\boldsymbol{g}_n,\boldsymbol{w}_n}_\rho$ and using the positivity (9.125), we deduce, for all $s\in[0,T]$, that

$$\rho\int_0^s \|\dot{\boldsymbol{u}}_n(t)\|^2_V\,\mathrm{d}t + \frac{\alpha}{2}\|\boldsymbol{u}_n(s)\|^2_V \leq \frac{1}{2\nu}\int_0^s \|\boldsymbol{F}^{\boldsymbol{g}_n}(t)\|^2_V + \frac{\nu}{2}\int_0^s \|\,\dot{\boldsymbol{u}}_n(t)\|^2_V\,\mathrm{d}t + C\,,$$

which, for $\nu>0$ conveniently chosen, implies

$$\|\boldsymbol{u}_n(s)\|^2_V \leq C(1+\|\boldsymbol{F}^{\boldsymbol{g}_n}\|^2_{L^2(0,T;\boldsymbol{V})}) \quad \forall s\in[0,T]\,,$$

$$\|\dot{\boldsymbol{u}}_n\|^2_{L^2(0,T;\boldsymbol{V})} \leq C_\rho(1+\|\boldsymbol{F}^{\boldsymbol{g}_n}\|^2_{L^2(0,T;\boldsymbol{V})})\,.$$

Thus, there exists $\boldsymbol{u}\in W^{1,2}(0,T;\boldsymbol{V})$ such that, up to a subsequence, we have

$$\begin{aligned}&\boldsymbol{u}_n\to\boldsymbol{u} \text{ weakly * in } L^\infty(0,T;\boldsymbol{V})\,,\\ &\boldsymbol{u}_n\to\boldsymbol{u} \text{ weakly in } W^{1,2}(0,T;\boldsymbol{V})\,.\end{aligned} \tag{9.131}$$

Finally, by passing to the limit in $(\mathscr{S})_\rho^{\boldsymbol{g}_n,\boldsymbol{w}_n}$, with $n \to \infty$, and using (9.131), (9.112) and the hypotheses on $\{\boldsymbol{g}_n\}_n$ and $\{\boldsymbol{w}_n\}_n$, we get $\boldsymbol{u} = \boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}}$. □

Now, we state the following convergence result.

Theorem 9.7. *Let $(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho})$ be a solution of problem $(\mathbf{CS})_{\epsilon\rho}$ and $\boldsymbol{u}^*_{\epsilon\rho} = \boldsymbol{u}_\rho^{\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}$. Then,*

$$\begin{cases} \boldsymbol{g}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{g}^*_\epsilon \text{ weakly in } \mathbf{H}_\mathbf{g}\,, \\ \boldsymbol{w}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{w}^*_\epsilon \text{ weakly in } \mathbf{H}_\mathbf{w}\,, \\ \boldsymbol{u}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{u}^*_\epsilon \text{ weakly in } W^{1,2}(0,T;V)\,, \end{cases} \tag{9.132}$$

where $\boldsymbol{u}^*_\epsilon = \boldsymbol{u}^{\boldsymbol{g}^*_\epsilon,\boldsymbol{w}^*_\epsilon}$. Moreover, $(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon)$ is an optimal control for J_ϵ and

$$\lim_{\rho\to 0} J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) = J_\epsilon(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon) = \min_{(\boldsymbol{g},\boldsymbol{w})\in \mathbf{H}_\mathbf{g}\times\mathbf{H}_\mathbf{w}} J_\epsilon(\boldsymbol{g},\boldsymbol{w})\,.$$

Proof. Let $(\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}) \in \mathscr{V}_{ad}$. Obviously, $\tilde{\boldsymbol{u}} = \boldsymbol{u}^{\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}}$ and, from Proposition 9.6, we have

$$\begin{aligned} \boldsymbol{u}_\rho^{\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}} &\to \tilde{\boldsymbol{u}} \quad \text{strongly in } L^\infty(0,T;V)\,,\\ \dot{\boldsymbol{u}}_\rho^{\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}} &\rightharpoonup \dot{\tilde{\boldsymbol{u}}} \quad \text{weakly in } L^2(0,T;V)\,. \end{aligned}$$

Therefore, we obtain

$$\lim_{\rho\to 0} J_{\epsilon\rho}(\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}) = \lim_{\rho\to 0}\left(\frac{1}{2}\|\boldsymbol{u}_\rho^{\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}} - \boldsymbol{u}_d\|^2_{\mathbf{H_u}} + \frac{1}{2\epsilon}\|\boldsymbol{u}_\rho^{\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}} - \tilde{\boldsymbol{u}}\|^2_{\mathbf{H_w}} + \frac{\beta}{2}\|\tilde{\boldsymbol{g}}\|^2_{\mathbf{H_g}}\right) = J(\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}})\,.$$

Since

$$J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) \le J_{\epsilon\rho}(\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}})\,,$$

it follows that the sequence $\{J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho})\}_\rho$ is bounded. Hence, the sequence $\{\boldsymbol{g}^*_{\epsilon\rho}\}_\rho$ is bounded in $\boldsymbol{H}_g$.

Next, putting $\boldsymbol{v} = \mathbf{0}$ in $(\mathbf{S})^{\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}$, integrating by parts on $[0,s]$ with $s \in [0,T]$ and taking into account the positivity and the property (9.110) of j_ρ, we get

$$\rho\int_0^s \|\dot{\boldsymbol{u}}^*_{\epsilon\rho}(t)\|^2\,\mathrm{d}t + \int_0^s a(\boldsymbol{u}^*_{\epsilon\rho}(t), \dot{\boldsymbol{u}}^*_{\epsilon\rho}(t))\,\mathrm{d}t \le \int_0^s (\boldsymbol{F}^*_{\epsilon\rho}(t), \dot{\boldsymbol{u}}^*_{\epsilon\rho}(t))_V\,\mathrm{d}t\,, \tag{9.133}$$

where

$$(\boldsymbol{F}^*_{\epsilon\rho}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}^*_{\epsilon\rho}(t)\cdot\boldsymbol{v}\,\mathrm{d}s\,.$$

Proceeding like in the proof of Proposition 9.4, we deduce that the sequence $\{(\boldsymbol{u}^*_{\epsilon\rho}, \rho\dot{\boldsymbol{u}}^*_{\epsilon\rho})\}_\rho$ is bounded in $L^\infty(0,T;\boldsymbol{V}) \times L^2(0,T;\boldsymbol{V})$.

Thus, since

$$\|\boldsymbol{u}^*_{\epsilon\rho}\|^2_{\mathbf{H_w}} = \|\boldsymbol{u}^*_{\epsilon\rho}\|^2_{L^2(0,T;V)} + \|\rho\dot{\boldsymbol{u}}^*_{\epsilon\rho} - \boldsymbol{f}\|^2_{L^2(0,T;(L^2(\Omega))^d)},$$

it follows that the sequence $\{\boldsymbol{u}^*_{\epsilon\rho}\}_\rho$ is also bounded in $\mathbf{H_w}$. From the definition of $J_{\epsilon\rho}$ and the boundedness of the sequence $\{J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho})\}_\rho$, it follows that the sequence $\{\boldsymbol{w}^*_{\epsilon\rho}\}_\rho$ is bounded in $\mathbf{H_w}$. Thus, there exist the elements $\boldsymbol{g}^*_\epsilon \in \mathbf{H_g}$ and $\boldsymbol{w}^*_\epsilon \in \mathbf{H_w}$ and the subsequences, still denoted by $\{\boldsymbol{g}^*_{\epsilon\rho}\}_\rho$ and $\{\boldsymbol{w}^*_{\epsilon\rho}\}_\rho$, such that

$$\begin{aligned} \boldsymbol{g}^*_{\epsilon\rho} &\rightharpoonup \boldsymbol{g}^*_\epsilon \text{ weakly in } \mathbf{H_g}, \\ \boldsymbol{w}^*_{\epsilon\rho} &\rightharpoonup \boldsymbol{w}^*_\epsilon \text{ weakly in } \mathbf{H_w}. \end{aligned} \tag{9.134}$$

Applying Propositions 9.6 and 9.7, we deduce

$$\boldsymbol{u}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{u}^*_\epsilon \quad \text{weakly in } W^{1,2}(0,T;\mathbf{V}), \tag{9.135}$$

where $\boldsymbol{u}^*_\epsilon = \boldsymbol{u}^{\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon}$. An easy computation gives

$$\boldsymbol{u}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{u}^*_\epsilon \quad \text{weakly in } \mathbf{H_w}. \tag{9.136}$$

Let $(\bar{\boldsymbol{g}}_\epsilon, \bar{\boldsymbol{w}}_\epsilon)$ be a solution of problem $(\mathbf{CS})_\epsilon$, $\bar{\boldsymbol{u}}_\epsilon = \boldsymbol{u}^{\bar{\boldsymbol{g}}_\epsilon, \bar{\boldsymbol{w}}_\epsilon}$ and $\bar{\boldsymbol{u}}_{\epsilon\rho} = \boldsymbol{u}_\rho^{\bar{\boldsymbol{g}}_\epsilon, \bar{\boldsymbol{w}}_\epsilon}$. From Proposition 9.6, we get

$$\begin{aligned} \bar{\boldsymbol{u}}_{\epsilon\rho} &\to \bar{\boldsymbol{u}}_\epsilon \text{ strongly in } L^\infty(0,T;\boldsymbol{V}), \\ \dot{\bar{\boldsymbol{u}}}_{\epsilon\rho} &\to \dot{\bar{\boldsymbol{u}}}_\epsilon \text{ weakly in } L^2(0,T;\boldsymbol{V}), \end{aligned} \tag{9.137}$$

which, using $(\mathbf{S})_\rho^{\bar{\boldsymbol{g}}_\epsilon, \bar{\boldsymbol{w}}_\epsilon}$ and $(\mathbf{S})^{\bar{\boldsymbol{g}}_\epsilon, \bar{\boldsymbol{w}}_\epsilon}$, give

$$\bar{\boldsymbol{u}}_{\epsilon\rho} \rightharpoonup \bar{\boldsymbol{u}}_\epsilon \quad \text{strongly in } \mathbf{H_w}. \tag{9.138}$$

Therefore, the convergences (9.134)–(9.138) lead us

$$\begin{aligned} J_\epsilon(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon) &\le \liminf_{\rho\to 0} J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) \le \limsup_{\rho\to 0} J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) \\ &\le \limsup_{\rho\to 0} J_{\epsilon\rho}(\bar{\boldsymbol{g}}_\epsilon, \bar{\boldsymbol{w}}_\epsilon) = \lim_{\rho\to 0} J_{\epsilon\rho}(\bar{\boldsymbol{g}}_\epsilon, \bar{\boldsymbol{w}}_\epsilon) = J_\epsilon(\bar{\boldsymbol{g}}_\epsilon, \bar{\boldsymbol{w}}_\epsilon) \le J_\epsilon(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon), \end{aligned} \tag{9.139}$$

i.e.

$$\lim_{\rho\to 0} J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) = J_\epsilon(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon) = \min\{J_\epsilon(\boldsymbol{g}, \boldsymbol{w});\ (\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}\}.$$

□

Finally, coupling the results proven in Theorems 9.7 and 9.5, we conclude that the regularized optimal problems represent a good approximation for the initial control problem.

Corollary 9.1. *Let* ϵ , $\rho > 0$ *and* $\{\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}\}_{\epsilon\rho}$ *be the sequence of solutions for problems* $(\mathbf{CS})_{\epsilon\rho}$. *Then, there exists* $(\boldsymbol{g}^*, \boldsymbol{u}^*) \in \mathscr{V}_{ad}$*, such that, up to a subsequence, for* ϵ , $\rho \to 0$*, we have*

$$\begin{cases} \boldsymbol{g}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{g}^* \textit{ weakly in } \mathbf{H_g}\,, \\ \boldsymbol{w}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{u}^* \textit{ weakly in } \mathbf{H_w}\,, \\ \boldsymbol{u}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{u}^* \textit{ weakly in } W^{1,2}(0,T;\boldsymbol{V})\,, \end{cases} \tag{9.140}$$

where $\boldsymbol{u}^*_{\epsilon\rho} = \boldsymbol{u}^{\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}}$. *Moreover,*

$$\lim_{\epsilon,\rho\to 0} J(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) = J(\boldsymbol{g}^*, \boldsymbol{u}^*) = \min_{(\boldsymbol{g},\boldsymbol{u})\in\mathscr{V}_{ad}} J(\boldsymbol{g}, \boldsymbol{u})\,. \tag{9.141}$$

In the sequel, we are concerned with the obtaining of the optimality conditions for the problem $(\mathbf{CS})_{\epsilon\rho}$, which means to derive the equations characterizing an optimal control from the fact that the differential of $J_{\epsilon\rho}$ vanishes at an extremum. We shall use the following result due to Lions [11].

Theorem 9.8. *Let* $\mathscr{B}$ *be a Banach space and* $\mathbf{X}$, $\mathbf{Y}$ *two reflexive Banach spaces. We consider two functions of class* C^1, $\mathscr{F} : \mathscr{B} \times \mathbf{X} \longrightarrow \mathbf{Y}$, *and* $\mathscr{J} : \mathscr{B} \times \mathbf{X} \longrightarrow \mathbb{R}$.

We suppose that, for all $\mathbf{h} \in \mathscr{B}$,

(i) there exists a unique solution $\boldsymbol{u}^h \in \mathbf{X}$ *of equation* $\mathscr{F}(\mathbf{h}, \boldsymbol{u}^h) = 0$;

(ii) the operator $\dfrac{\partial \mathscr{F}}{\partial \boldsymbol{u}}(\mathbf{h}, \boldsymbol{u}^h) : \mathbf{X} \longrightarrow \mathbf{Y}$ *is an isomorphism.*

Then, the function $J : \mathscr{B} \longrightarrow \mathbb{R}$*, defined by* $J(\mathbf{h}) = \mathscr{J}(\mathbf{h}, \boldsymbol{u}^h)$*, is differentiable and*

$$\frac{dJ}{d\mathbf{h}}(\mathbf{h})(\delta\mathbf{h}) = \frac{\partial \mathscr{J}}{\partial \mathbf{h}}(\mathbf{h}, \boldsymbol{u}^h)(\delta\mathbf{h}) - \left\langle \boldsymbol{q}^h, \frac{\partial \mathscr{F}}{\partial \mathbf{h}}(\mathbf{h}, \boldsymbol{u}^h)(\delta\mathbf{h}) \right\rangle_{Y^*,Y} \quad \forall\, \delta\mathbf{h} \in \mathscr{B}, \tag{9.142}$$

where the adjoint state $\boldsymbol{q}^h \in \mathbf{Y}^*$ *is the unique solution of*

$$\left\langle \left[\frac{\partial \mathscr{F}}{\partial \boldsymbol{u}}(\mathbf{h}, \boldsymbol{u}^h)\right]^* \cdot \boldsymbol{q}^h, \boldsymbol{v} \right\rangle_{X^*,X} = \frac{\partial \mathscr{J}}{\partial \boldsymbol{u}}(\mathbf{h}, \boldsymbol{u}^h)(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{X}. \tag{9.143}$$

First, let us remark that, for $(\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}$, the regularized problem $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$ has a unique solution $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho \in W^{1,2}(0,T;\boldsymbol{V})$ satisfying $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho(0) = \boldsymbol{u}_0$. Then, $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho = \boldsymbol{u}_0 + \tilde{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho$, where $\tilde{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho \in W^{1,2}(0,T;\boldsymbol{V})$ satisfies

$$\begin{cases} \rho(\dot{\tilde{\boldsymbol{u}}}_\rho^{\boldsymbol{g},\boldsymbol{w}}(t), \boldsymbol{v})_V + a(\tilde{\boldsymbol{u}}_\rho^{\boldsymbol{g},\boldsymbol{w}}(t) + \boldsymbol{u}_0, \boldsymbol{v}) + \langle \nabla_2 j^\rho(\boldsymbol{w}(t), \dot{\tilde{\boldsymbol{u}}}_\rho^{\boldsymbol{g},\boldsymbol{w}}(t)), \boldsymbol{v}\rangle \\ = (\boldsymbol{F}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}\,, \quad \text{a.e. } t \in (0,T)\,, \\ \tilde{\boldsymbol{u}}_\rho^{\boldsymbol{g},\boldsymbol{w}}(0) = \boldsymbol{0}\,. \end{cases} \tag{9.144}$$

In order to apply Theorem 9.8, we take

$$\begin{aligned}
&\mathscr{B} = \mathbf{H_g} \times \mathbf{H_w}\,,\\
&\mathbf{X} = \{\boldsymbol{v} \in W^{1,2}(0,T;\boldsymbol{V}) \cap L^2(0,T;\boldsymbol{W})\,;\ \boldsymbol{v}(0) = \boldsymbol{0}\}\,,\\
&Y = L^2(0,T;\boldsymbol{V}^*)\,,\\
&\mathscr{F} : \mathscr{B} \times X \to Y\,,\\
&\langle \mathscr{F}(\boldsymbol{g},\boldsymbol{w},\boldsymbol{u}), \boldsymbol{v}\rangle = \int_0^T \rho(\dot{\boldsymbol{u}}(t), \boldsymbol{v}(t))_V \,\mathrm{d}t + \int_0^T a(\boldsymbol{u}(t) + \boldsymbol{u}_0, \boldsymbol{v}(t))\,\mathrm{d}t\\
&+\langle \nabla_2 j^\rho(\boldsymbol{w}(t), \dot{\boldsymbol{u}}(t)), \boldsymbol{v}(t)\rangle \,\mathrm{d}t - \int_0^T (\boldsymbol{f}(t), \boldsymbol{v}(t))_{(L^2(\Omega))^d}\,\mathrm{d}t\\
&-\int_0^T (\boldsymbol{g}(t), \boldsymbol{v}(t))_{(L^2(\Gamma_1))^d}\,\mathrm{d}t \ \forall\, \boldsymbol{v} \in L^2(0,T;\boldsymbol{V})\,,\\
&\mathscr{J} : \mathscr{B} \times X \to \mathbb{R}\,,\\
&\mathscr{J}(\boldsymbol{g},\boldsymbol{w},\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{u} + \boldsymbol{u}_0 - \boldsymbol{u}_d\|^2_{\mathbf{H_u}} + \frac{\beta}{2}\|\boldsymbol{g}\|^2_{\mathbf{H_g}} + \frac{1}{2\epsilon}\|\boldsymbol{u} + \boldsymbol{u}_0 - \boldsymbol{w}\|^2_{\mathbf{H_w}}\,.
\end{aligned}$$

We remark that

$$\mathscr{J}(\boldsymbol{g},\boldsymbol{w},\tilde{\boldsymbol{u}}_\rho^{g,w}) = J_{\epsilon\rho}(\boldsymbol{g},\boldsymbol{w}) \quad \forall (\boldsymbol{g},\boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}\,.$$

In the sequel, to simplify the notation, we shall omit to write explicitly the indices ϵ, ρ, $\boldsymbol{g}$, and $\boldsymbol{w}$.

We state now the main result of this section.

Theorem 9.9. *Let $(\boldsymbol{g}^*, \boldsymbol{w}^*) \in \mathbf{H_g} \times \mathbf{H_w}$ be a solution of the optimal control problem $(\mathbf{CS})_{\epsilon\rho}$. Then, there exist the unique elements $\boldsymbol{u}^* \in \mathbf{X}$ and $\boldsymbol{q}^* \in \mathbf{Y}^*$ such that*

$$\begin{cases} \rho \displaystyle\int_0^T (\dot{\boldsymbol{u}}^*(t), \boldsymbol{v}(t))_V\,\mathrm{d}t + \int_0^T a(\boldsymbol{u}^*(t) + \boldsymbol{u}_0, \boldsymbol{v}(t))\,\mathrm{d}t \\ + \displaystyle\int_0^T \langle \nabla_2 j(\boldsymbol{w}^*(t), \dot{\boldsymbol{u}}^*(t)), \boldsymbol{v}(t)\rangle\,\mathrm{d}t = \int_0^T (\boldsymbol{f}(t), \boldsymbol{v}(t))_{(L^2(\Omega))^d}\,\mathrm{d}t \\ + \displaystyle\int_0^T (\boldsymbol{g}^*(t), \boldsymbol{v}(t))_{(L^2(\Gamma_1))^d}\,\mathrm{d}t \quad \forall\, \boldsymbol{v} \in L^2(0,T;\boldsymbol{V})\,, \end{cases} \tag{9.145}$$

$$\begin{cases} \int_0^T \rho(\dot{\boldsymbol{v}}(t), \boldsymbol{q}^*(t))_V \, \mathrm{d}t + \int_0^T a(\boldsymbol{v}(t), \boldsymbol{q}^*(t)) \, \mathrm{d}t \\ + \int_0^T \langle \nabla_2^2 j(\boldsymbol{w}^*(t), \dot{\boldsymbol{u}}^*(t)) \dot{\boldsymbol{v}}(t) - \nabla_2 j(\boldsymbol{v}(t), \dot{\boldsymbol{u}}^*(t)), \boldsymbol{q}^*(t) \rangle \, \mathrm{d}t \\ = \int_0^T (\boldsymbol{u}^*(t) + \boldsymbol{u}_0 - \boldsymbol{u}_d, \boldsymbol{v}(t))_{(L^2(\Gamma_2))^d} \, \mathrm{d}t \quad \forall \boldsymbol{v} \in \mathbf{X} \end{cases} \tag{9.146}$$

and

$$\beta(\boldsymbol{g}^*, \boldsymbol{g})_{\mathbf{H_g}} = (\boldsymbol{q}^*, \boldsymbol{g})_{L^2(0,T;(L^2(\Gamma_1))^p)} \quad \forall \boldsymbol{g} \in \mathbf{H_g}\,. \tag{9.147}$$

Proof. Let $\boldsymbol{u}^*$ be the unique solution of (9.144) corresponding to $(\boldsymbol{g}^*, \boldsymbol{w}^*)$. Some easy computations give:

$$\frac{\partial \mathscr{J}}{\partial \boldsymbol{w}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*)(\boldsymbol{w}) = \frac{1}{\epsilon}(\boldsymbol{u}^* + \boldsymbol{u}_0 - \boldsymbol{w}^*, \boldsymbol{w})_{\mathbf{H_w}} \quad \forall \boldsymbol{w} \in \mathbf{H_w}\,,$$
$$\frac{\partial \mathscr{J}}{\partial \boldsymbol{g}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*)(\boldsymbol{g}) = \beta(\boldsymbol{g}^*, \boldsymbol{g})_{\mathbf{H_g}} \ \forall \boldsymbol{g} \in \mathbf{H_g}\,,$$
$$\frac{\partial \mathscr{J}}{\partial \boldsymbol{u}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*)(\boldsymbol{u}) = (\boldsymbol{u}^* + \boldsymbol{u}_0 - \boldsymbol{u}_d, \boldsymbol{u})_{\mathbf{H_u}} + \frac{1}{\epsilon}(\boldsymbol{u}^* + \boldsymbol{u}_0 - \boldsymbol{w}^*, \boldsymbol{u})_{\mathbf{H_w}} \quad \boldsymbol{u} \in \mathbf{X}\,,$$
$$\left\langle \frac{\partial \mathscr{F}}{\partial \boldsymbol{w}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*)(\boldsymbol{w}), \boldsymbol{v} \right\rangle = \int_0^T \langle \nabla_2 j(\boldsymbol{w}(t), \dot{\boldsymbol{u}}^*(t)), \boldsymbol{v}(t) \rangle \, \mathrm{d}t \ \forall (\boldsymbol{w}, \boldsymbol{v}) \in \mathbf{H_w} \times L^2(0,T;\boldsymbol{V})\,,$$
$$\left\langle \frac{\partial \mathscr{F}}{\partial \boldsymbol{g}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*)(\boldsymbol{g}), \boldsymbol{v} \right\rangle = -\int_0^T (\boldsymbol{g}(t), \boldsymbol{v}(t))_{(L^2(\Gamma_1))^d} \, \mathrm{d}t \ \forall \boldsymbol{g} \in \mathbf{H_g} \ \forall \boldsymbol{v} \in L^2(0,T;\boldsymbol{V})\,,$$
$$\left\langle \frac{\partial \mathscr{F}}{\partial \boldsymbol{u}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*)(\boldsymbol{u}), \boldsymbol{v} \right\rangle = \rho \int_0^T (\dot{\boldsymbol{u}}(t), \boldsymbol{v}(t))_V \, \mathrm{d}t + \int_0^T a(\boldsymbol{u}(t), \boldsymbol{v}(t)) \, \mathrm{d}t$$
$$+ \int_0^T \langle \nabla_2^2 j(\boldsymbol{w}(t), \dot{\boldsymbol{u}}^*(t)) \dot{\boldsymbol{u}}(t), \boldsymbol{v}(t) \rangle \, \mathrm{d}t \quad \forall \boldsymbol{u} \in \mathbf{X},\ \forall \boldsymbol{v} \in L^2(0,T;\boldsymbol{V})\,.$$

Thus, the operator $\frac{\partial \mathscr{F}}{\partial \boldsymbol{u}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*) : \mathbf{X} \to \mathbf{Y}$ is an isomorphism.

Using Theorem 9.8, the adjoint state $\boldsymbol{q}^* \in \mathbf{Y}^*$ is defined as being the unique solution of the following equation:

$$\left\langle \left[\frac{\partial \mathscr{F}}{\partial \boldsymbol{u}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*) \right]^* \cdot \boldsymbol{q}^*, \boldsymbol{v} \right\rangle = \frac{\partial \mathscr{J}}{\partial \boldsymbol{u}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*)(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{X}\,.$$

Therefore, we have

$$\int_0^T \left[\rho(\dot{\boldsymbol{v}}(t), \boldsymbol{q}^*(t)) + a(\boldsymbol{v}(t), \boldsymbol{q}^*(t)) + \langle \nabla_2^2 j(\boldsymbol{w}^*(t), \dot{\boldsymbol{u}}^*(t))(\dot{\boldsymbol{v}}(t), \boldsymbol{q}^*(t)\rangle\right] \mathrm{d}t$$
$$= \int_0^T \left[(\boldsymbol{u}^*(t) + \boldsymbol{u}_0 - \boldsymbol{u}_d, \boldsymbol{v}(t))_{(L^2(\Gamma_2))^d} + \frac{1}{\epsilon}(\boldsymbol{u}_\rho^h + \boldsymbol{u}_0 - \boldsymbol{w}(t), \boldsymbol{v}(t))_W\right] \mathrm{d}t \;\; \forall \boldsymbol{v} \in \mathbf{X}.$$

Next, since $\mathbf{h}^* = (\boldsymbol{g}^*, \boldsymbol{w}^*)$ is a solution of the optimal control problem $(\mathbf{CS})_{\epsilon\rho}$, using Theorem 9.8, we obtain

$$\frac{dJ}{d\mathbf{h}}(\mathbf{h}^*)(\mathbf{h}) = \frac{\partial \mathscr{J}}{\partial \mathbf{h}}(\mathbf{h}^*, \boldsymbol{u}^*)(\mathbf{h}) - \left\langle \boldsymbol{q}^*, \frac{\partial \mathscr{F}}{\partial \mathbf{h}}(\mathbf{h}^*, \boldsymbol{u}^*)(\mathbf{h})\right\rangle = 0 \;\; \forall \mathbf{h} = (\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}\,,$$

which gives

$$\int_0^T \frac{1}{\epsilon}(\boldsymbol{u}^*(t) + \boldsymbol{u}_0 - \boldsymbol{w}^*(t), \boldsymbol{w}(t))_W \,\mathrm{d}t + \beta(\boldsymbol{g}^*, \boldsymbol{g})_{\mathbf{H_g}} = \int_0^T \langle \boldsymbol{q}^*(t), \nabla_2 j(\boldsymbol{w}(t), \dot{\boldsymbol{u}}^*(t))\rangle \,\mathrm{d}t$$
$$-(\boldsymbol{q}^*, \boldsymbol{g})_{L^2(0,T;(L^2(\Gamma_1))^d)} \quad \forall (\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}\,.$$

Taking $\boldsymbol{g} = \mathbf{0}$, we deduce

$$\int_0^T \frac{1}{\epsilon}(\boldsymbol{u}^*(t) + \boldsymbol{u}_0 - \boldsymbol{w}^*(t), \boldsymbol{v}(t))_W \,\mathrm{d}t = \int_0^T \langle \boldsymbol{q}^*(t), \nabla_2 j(\boldsymbol{v}(t)(\dot{\boldsymbol{u}}^*(t))\rangle \,\mathrm{d}t \;\; \forall \boldsymbol{v} \in L^2(0,T;\boldsymbol{W})$$

and, so, we obtain (9.146) and (9.147). □

The asymptotic analysis (Corollary 9.1) of smoother problems $(\mathbf{CS})_{\epsilon\rho}$ provides that the sequence of optimal regularized controls $\{\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{u}^*_{\epsilon\rho}\}_{\epsilon\rho}$ converges to an optimal control $(\boldsymbol{g}^*, \boldsymbol{u}^*)$ of the initial problem $(\mathbf{CS})$. Therefore, the system (9.145)–(9.147) can be useful in the numerical analysis of an optimal control.

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