Chapter 7
Approximations of Variational Inequalities

This chapter is devoted to the discrete approximation of abstract elliptic and implicit evolutionary quasi-variational inequalities. We restrict ourselves to present convergence results for internal approximations in space of elliptic quasi-variational inequalities together with a backward difference scheme in time of implicit evolutionary quasi-variational inequalities. For more details we refer the reader to Glowinski, Lions and Trémolières [6], Glowinski [5], and the bibliography of these works. Here, following the works of Capatina and Cocu [7] and Capatina, Cocou and Raous [1], numerical analysis is carried out on general problems. Also, a general error estimate is derived. The results obtained in this chapter, representing generalizations of the approximations of variational inequalities of the first and second kinds, can be applied to a large variety of static and quasistatic contact problems, including unilateral and bilateral contact or normal compliance conditions with friction. In particular, static and quasistatic unilateral contact problems with nonlocal Coulomb friction in linear elasticity will be considered in Chaps. 8 and 9.

7.1 Internal Approximation of Elliptic Variational Inequalities

In this section one considers the internal approximation of the following abstract quasi-variational inequality.

Problem (P^a): Find $u \in K$ such that

$$\langle Au, v-u\rangle + j(u,v) - j(u,u) \geq \langle f, v-u\rangle \quad \forall v \in K, \tag{7.1}$$

where $(V, \|\cdot\|)$ is a real reflexive Banach space with $(V^*, \|\cdot\|_*)$ its dual and $\langle\cdot,\cdot\rangle$ the duality product between V^* and V. We denote by K a nonempty closed convex

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subset of V and let $f \in V^*$ be given. One supposes that the operator $A : V \longrightarrow V^*$ is Lipschitz continuous and strongly monotone, i.e.

$$\exists M > 0 \text{ such that } \|Au - Av\|_* \leq M\|u - v\| \quad \forall u, v \in V\,, \tag{7.2}$$

$$\exists\, \alpha > 0 \text{ such that } \langle Au - Av, u - v\rangle \geq \alpha\|u - v\|^2 \quad \forall u, v \in V\,. \tag{7.3}$$

In addition, we assume that the function $j(\cdot,\cdot) : V \times V \longrightarrow (-\infty, +\infty]$ satisfies the conditions of Theorem 4.16, so

$$\forall u \in V,\ j(u,\cdot) : V \to (-\infty,\ +\infty] \text{ is a proper convex l.s.c. function,} \tag{7.4}$$

$$\begin{cases} \exists k < \alpha \text{ such that } |j(u_1, v_1) + j(u_2, v_2) - j(u_1, v_2) - j(u_2, v_1)| \\ \leq k\|u_1 - u_2\|\,\|v_1 - v_2\| \quad \forall u_1, u_2, v_1, v_2 \in K\,. \end{cases} \tag{7.5}$$

From the existence and uniqueness proof of Theorem 4.16, the following algorithm of Bensoussan–Lions type for the numerical approximation of Problem $(\mathbf{P}^a)$ follows: let $u^0 \in K$ be arbitrary and

$$u^n = Su^{n-1}\,, \quad n \geq 1 \tag{7.6}$$

where $S : K \to K$ is the mapping which associates with every $w \in K$ the unique solution $Sw \in K$ of the following variational inequality of the second kind:

$$\langle A(Sw), v - (Sw)\rangle + j(w, v) - j(w, (Sw)) \geq \langle f, v - (Sw)\rangle \quad \forall v \in K\,.$$

The hypothesis $k < \alpha$ implies (see p. 50) that the quasi-variational inequality (7.1) has a unique solution $u = Su$ and

$$u^n \to u \qquad \text{strongly in } V \text{ as } n \to \infty\,. \tag{7.7}$$

We shall consider an internal approximation of Problem $(\mathbf{P}^a)$.

Let h be a parameter which converges to zero. Let us consider a family $\{V_h\}_h$ of closed subspaces of V (in applications, we often take V_h to be finite dimensional), and a family $\{K_h\}_h$ of nonempty convex closed subsets of V_h which approximates K in the following sense (see, e.g., [6]):

$$\begin{cases} (i) \quad \forall v \in K, \exists r_h v \in K_h \text{ such that } r_h v \to v \text{ strongly in } V\,, \\ (ii) \quad \forall v_h \in K_h \text{ with } v_h \rightharpoonup v \text{ weakly in } V, \text{ then } v \in K\,. \end{cases} \tag{7.8}$$

Often one uses approximations A_h, f_h, and j_h for A, f and j, usually obtained by a process of numerical integration. Nevertheless, since the use of approximations A_h and f_h does not bring any major change comparatively with the use of A and f, here we only consider an approximate of the function $j(\cdot,\cdot)$ by a family $\{j_h\}_h$ of functions which, for every $u \in V$, satisfies the following conditions (see also [5]):

$$\forall h,\ j_h(u,\cdot) : V_h \to (-\infty,\ +\infty] \text{ is a convex l.s.c. function,} \tag{7.9}$$

$$\begin{cases} \text{the family } \{j_h(u,\cdot)\}_h \text{ is uniformly proper, i.e.} \\ \exists\lambda = \lambda(u) \in V^*,\ \exists\mu = \mu(u) \in \mathbb{R} \text{ such that} \\ j_h(u, v_h) \geq \langle \lambda, v_h\rangle + \mu \quad \forall v_h \in V_h,\ \forall h, \end{cases} \tag{7.10}$$

$$\liminf_{h\to 0} j_h(u, v_h) \geq j(u, v) \quad \forall v_h \in V_h \text{ such that } v_h \rightharpoonup v \text{ weakly in } V\,, \tag{7.11}$$

$$\lim_{h\to 0} j_h(u, r_h v) = j(u, v) \quad \forall v \in K\,. \tag{7.12}$$

In addition, we suppose that, for every h, j_h satisfies

$$\begin{aligned} &|j_h(u_h^1, v_h^2) + j_h(u_h^2, v_h^1) - j_h(u_h^1, v_h^1) - j_h(u_h^2, v_h^2)| \\ &\leq k\|u_h^1 - u_h^2\|\,\|v_h^1 - v_h^2\| \quad \forall u_h^1, u_h^2, v_h^1, v_h^2 \in K_h\,. \end{aligned} \tag{7.13}$$

Under the previous assumptions, one formulates the following discrete problem.

Problem $(\mathbf{P}^a)_\mathbf{h}$: Find $u_h \in K_h$ such that

$$\langle Au_h, v_h - u_h\rangle + j_h(u_h, v_h) - j_h(u_h, u_h) \geq \langle f, v_h - u_h\rangle \quad \forall v_h \in K_h\,. \tag{7.14}$$

Arguing as in the proof of Theorem 4.16, it follows that the mapping $S_h : K_h \to K_h$ defined, for every $w_h \in K_h$, as the unique element $S_h w_h \in K_h$ which verifies

$$\langle A(S_h w_h), v_h - S_h w_h\rangle + j_h(w_h, v_h) - j_h(w_h, S_h w_h) \geq \langle f, v_h - S_h w_h\rangle \quad \forall v_h \in K_h\,,$$

is a contraction:

$$\|S_h w_1 - S_h w_2\| \leq \frac{k}{\alpha}\|w_1 - w_2\| \quad \forall w_1, w_2 \in K_h\,. \tag{7.15}$$

Hence, the following existence and uniqueness result holds.

Proposition 7.1. *The discrete quasi-variational inequality (7.14) has a unique solution $u_h = S_h u_h \in K_h$.*

As in the continuous case, we approximate the discrete solution u_h by the sequence $\{u_h^n\}_{n\geq 1}$ defined by

$$u_h^n = S_h u_h^{n-1}\,, \quad n \geq 1$$

where $u_h^0 \in K_h$ is given such that the sequence $\{u_h^0\}_h$ is bounded. Obviously, we have

$$\|u_h^n - u_h\| = \|S_h u_h^{n-1} - S_h u_h\| \leq \left(\frac{k}{\alpha}\right)^n \|u_h^0 - u_h\|\,. \tag{7.16}$$

Thus, in order to prove that the sequence $\{u_h^n\}_n$ is uniformly bounded in h, it is enough to prove the following result.

Lemma 7.1. *The sequence $\{u_h\}_h$ of the solutions of the quasi-variational inequality (7.14) is bounded.*

Proof. Let $v \in K$ and $r_h v \in K_h$ such that $r_h v \to v$ strongly in V as $h \to 0$. Taking $v_h = r_h v$ in (7.14), we obtain

$$\begin{aligned}&\alpha\|u_h - r_h v\|^2 \le \langle Au_h - A(r_h v), u_h - r_h v\rangle \le \langle A(r_h v), r_h v - u_h\rangle \\ &+(j_h(u_h, r_h v) - j_h(u_h, u_h) + j_h(u, u_h) - j_h(u, r_h v)) \\ &-j_h(u, u_h) + j_h(u, r_h v) - \langle f, r_h v - u_h\rangle \,.\end{aligned} \tag{7.17}$$

From (7.12) we have

$$|j_h(u, r_h v)| \le C_1,$$

and, since the sequence $\{r_h v\}_h$ is bounded, from (7.2), we get

$$\|A(r_h v)\|_* \le C_2$$

with C_1 and C_2 positive constants independent of h. Therefore, from (7.17), (7.10) and (7.13), we obtain

$$\begin{aligned}&\alpha\|u_h - r_h v\|^2 - \|\lambda\|_*\|u_h\| - |\mu| \le \langle Au_h - A(r_h v), u_h - r_h v\rangle + j_h(u, u_h) \\ &\le C_2\|r_h v - u_h\| + k\|u_h - u\|\,\|r_h v - u_h\| + C_1 + \|f\|_*\|r_h v - u_h\|,\end{aligned} \tag{7.18}$$

hence

$$\begin{aligned}&\left(\alpha - k - \frac{k\epsilon_1 + \epsilon_2 + \epsilon_3}{2}\right)\|u_h - r_h v\|^2 \le \frac{\|\lambda\|_*}{2\epsilon_3} + \|r_h v\|\,\|\lambda\|_* \\ &+\frac{k}{2\epsilon_1}\|r_h v - u\|^2 + \frac{(C_2 + \|f\|_*)^2}{2\epsilon_2} + C_1 + |\mu| \le C\end{aligned} \tag{7.19}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ are chosen such that $\alpha - k - \dfrac{k\epsilon_1 + \epsilon_2 + \epsilon_3}{2} > 0$ (for instance, $\epsilon_1 = \dfrac{\alpha - k}{3k}$, $\epsilon_2 = \dfrac{5}{6}(\alpha - k)$, $\epsilon_3 = \dfrac{5}{12}(\alpha - k)$) and C is a positive constant independent of h. Therefore, according to the choice of $\{r_h v\}_h$, we conclude that the sequence $\{u_h - r_h v\}_h$ is bounded, and so, the sequence $\{u_h\}_h$ is. □

Now, from (7.16), the above lemma and the boundedness of $\{u_h^0\}_h$, it follows that

$$\|u_h^n - u_h\| \le Cq^n \tag{7.20}$$

with $q = \dfrac{k}{\alpha} < 1$ and C a positive constant independent of n and h, i.e. $\{u_h^n\}_n$ is uniformly bounded in h. Hence, for all $\epsilon > 0$, there exists $N = N_\epsilon$ such that

$$\|u_h^n - u_h\| \le \epsilon \quad \forall n \ge N_\epsilon\,,\ \forall h > 0\,. \tag{7.21}$$

We recall that, for any $n \ge 1$, u^n, respectively u_h^n, are defined as the unique solutions of the following problems:

Problem $(\mathbf{P}^a)_\mathbf{n}$: Find $u^n \in K$ such that

$$\langle Au^n, v - u^n\rangle + j(u^{n-1}, v) - j(u^{n-1}, u^n) \geq \langle f, v - u^n\rangle \quad \forall v \in K\,, \tag{7.22}$$

respectively,

Problem $(\mathbf{P}^a)_{\mathbf{h},\mathbf{n}}$: Find $u_h^n \in K_h$ such that

$$\langle Au_h^n, v_h - u_h^n\rangle + j_h(u_h^{n-1}, v_h) - j_h(u_h^{n-1}, u_h^n) \geq \langle f, v_h - u_h^n\rangle \quad \forall v_h \in K_h\,. \tag{7.23}$$

This means that Problem $(\mathbf{P}^a)_\mathbf{n}$ is an iterative approximation of Problem $(\mathbf{P}^a)$, while Problem $(\mathbf{P}^a)_{\mathbf{h},\mathbf{n}}$ is an iterative approximation of Problem $(\mathbf{P}^a)_\mathbf{h}$.

In order to obtain the convergence of the sequence $\{u_h\}_h$ to u, as $h \to 0$, we introduce an auxiliary sequence of problems. So, for $w_h^0 \in K_h$ given such that the sequence $\{w_h^0\}_h$ is bounded, we denote by $w_h^n \in K_h$ the solution, that there exists and is unique, of the following problem.

Problem $(\mathbf{P}^a)_{\mathbf{n},\mathbf{h}}$: Find $w_h^n \in K_h$ such that

$$\langle Aw_h^n, v_h - w_h^n\rangle + j_h(u^{n-1}, v_h) - j_h(u^{n-1}, w_h^n) \geq \langle f, v_h - w_h^n\rangle \quad \forall v_h \in K_h\,, \tag{7.24}$$

where the sequence $\{u^n\}_n \subset K$ is defined by (7.6). We note that Problem $(\mathbf{P}^a)_{\mathbf{n},\mathbf{h}}$ is an internal approximation of Problem $(\mathbf{P}^a)_\mathbf{n}$.

We have the following convergence result.

Proposition 7.2. *The sequence $\{w_h^n\}_h$, defined by (7.24), approximates the solution u^n of (7.22) in the sense*

$$w_h^n \to u^n \text{ strongly in } V \text{ as } h \to 0.$$

Moreover, we have

$$\lim_{h\to 0} j_h(u^{n-1}, w_h^n) = j(u^{n-1}, u^n).$$

Proof. Let $v \in K$ be arbitrarily chosen. Taking $v_h = r_h v$ in (7.24), it results

$$\langle Aw_h^n, w_h^n\rangle + j_h(u^{n-1}, w_h^n) \leq \langle Aw_h^n, r_h v\rangle + j_h(u^{n-1}, r_h v) - \langle f, r_h v - w_h^n\rangle\,. \tag{7.25}$$

By using the hypotheses (7.3), (7.2), (7.10), and (7.12), one gets

$$\alpha\|w_h^n\|^2 \leq \|\lambda\|_*\|w_h^n\| + |\mu| + M\|w_h^n\|\|r_h v\| + C + \|f\|_*(\|r_h v\| + \|w_h^n\|) \leq C_1\|w_h^n\| + C_2$$

with C, C_1, and C_2 positive constants independent of h. Hence, the sequence $\{w_h^n\}_h$ is bounded and we can extract a subsequence $\{w_{h_p}^n\}_p$ such that $w_{h_p}^n \rightharpoonup w_n$ weakly in V, with $w^n \in K$ (from $(7.8)_2$). Now, from (7.25), by using (7.3), (7.11), and (7.12), we obtain

$$\begin{aligned}\langle Aw^n, w^n\rangle + j(u^{n-1}, w^n) &\le \liminf_{h_p\to 0}(\langle Aw^n_{h_p}, w^n_{h_p}\rangle + j_h(u^{n-1}, w^n_{h_p}))\\ &\le \langle Aw^n, v\rangle + j(u^{n-1}, v) - \langle f, v - w^n\rangle \quad \forall v \in K\,.\end{aligned}$$

This implies $w^n = u^n$, where u^n is the unique solution of the variational inequality (7.22). Therefore, $w^n_h \rightharpoonup u^n$ weakly in V as $h \to 0$.

Finally, from (7.25) and using the hypotheses (7.11) and (7.12), we have

$$\begin{aligned}&j(u^{n-1}, u^n) \le \liminf_{h\to 0} j_h(u^{n-1}, w^n_h) \le \liminf_{h\to 0}(\alpha\|w^n_h - u^n\|^2 + j_h(u^{n-1}, w^n_h))\\ &\quad\le \limsup_{h\to 0}(\alpha\|w^n_h - u^n\|^2 + j_h(u^{n-1}, w^n_h))\\ &\quad\le \lim_{h\to 0}(\langle Aw^n_h, r_h v\rangle + j_h(u^{n-1}, r_h v) - \langle f, r_h v - w^n_h\rangle - \langle Aw^n_h, u^n\rangle - \langle Au^n, w^n_h\rangle\\ &\quad+\langle Au^n, u^n\rangle) = \langle Au^n, v - u^n\rangle + j(u^{n-1}, v) - \langle f, v - u^n\rangle \quad \forall v \in K\,.\end{aligned}$$

The proof is completed by taking $v = u^n$. □

We are now prepared to prove the main result of this section.

Theorem 7.1. *We suppose that (7.2)–(7.13) hold. Let u and u_h be the unique solutions of (7.1) and, respectively, (7.14). Then, we have*

$$u_h \to u \text{ strongly in } V \text{ as } h \to 0\,. \tag{7.26}$$

Proof. We observe that we have

$$\|u_h - u\| \le \|u_h - u^n_h\| + \|u^n_h - u^n\| + \|u^n - u\| \quad \forall n \ge 0\,. \tag{7.27}$$

First, from (7.7) and (7.21), it results that, for $\epsilon > 0$ given, there exists $N_\epsilon > 0$ such that

$$\|u^n_h - u_h\| + \|u^n - u\| \le \frac{\epsilon}{2} \quad \forall n \ge N_\epsilon\,. \tag{7.28}$$

In order to estimate the second term in the right-hand side of (7.27), we deduce, form the definitions of u^n_h and w^n_h, that

$$\begin{aligned}\alpha\|u^n_h - w^n_h\|^2 &\le \langle Aw^n_h - Au^n_h, w^n_h - u^n_h\rangle\\ &\le j_h(u^{n-1}, u^n_h) + j_h(u^{n-1}_h, w^n_h) - j_h(u^{n-1}, w^n_h) - j_h(u^{n-1}_h, u^n_h)\,,\end{aligned}$$

from which, using (7.13), we deduce

$$\|u^n_h - w^n_h\| < \|u^{n-1}_h - u^{n-1}\|\,. \tag{7.29}$$

Now, by choosing $w^0_h = u^0_h$, we shall prove by recurrence, that

$$\|u^n_h - u^n\| \le \sum_{i=0}^{n} \|w^i_h - u^i\| \quad \forall n \ge 0\,. \tag{7.30}$$

Indeed, for $n = 0$ the result is obvious. If we suppose that (7.30) holds for $n-1$, then, from (7.29), we get

$$\|u_h^n - u^n\| \le \|u_h^n - w_h^n\| + \|w_h^n - u^n\| \le \|u_h^{n-1} - u^{n-1}\| + \|w_h^n - u^n\| \le \sum_{i=0}^{n} \|w_h^i - u^i\| .$$

It follows that the relation (7.30) holds for every $n \ge 0$.

Choosing $n = N_\epsilon$ in (7.27) and taking into account (7.30) and (7.28), we obtain

$$\|u_h - u\| \le \frac{\epsilon}{2} + \sum_{i=0}^{N_\epsilon} \|w_h^i - u^i\| . \tag{7.31}$$

But, from Proposition 7.2, it follows that, for every i, there exists $H_\epsilon^i > 0$ such that

$$\|w_h^i - u^i\| \le \frac{\epsilon}{2(N_\epsilon + 1)} \quad \forall h \le H_\epsilon^i . \tag{7.32}$$

Concluding, from (7.31) and (7.32), for $\epsilon > 0$ given, there exists $H_\epsilon = \min_{i=0}^{N_\epsilon} H_\epsilon^i$ such that

$$\|u_h - u\| \le \epsilon \quad \forall h \le H_\epsilon ,$$

hence $u_h \to u$ strongly in V as $h \to 0$. □

7.2 Abstract Error Estimate

The purpose of this section is to obtain a priori error estimate for the approximation (7.14) of the quasi-variational inequality (7.1). This estimate generalizes the estimates obtained by Cea [2, 3] and Falk [4] for the approximation of variational equations and, respectively, variational inequalities of the first kind.

Theorem 7.2. *Let u and u_h be the unique solutions of the quasi-variational inequality (7.1) and, respectively, (7.14).*

We suppose that (7.2)–(7.13) hold. Moreover, we assume that there exists a Hilbert space $(H, \|\cdot\|_H)$ and a Banach space $(U, \|\cdot\|_U)$ such that $V \hookrightarrow H$ dense, $V \subset U$ and

$$Au - f \in H , \tag{7.33}$$

$$|j_h(u, v_h) - j(u, v)| \le C_1 \|v_h - v\|_U \quad \forall v_h \in K_h , \ \forall v \in K , \tag{7.34}$$

where C_1 is a positive constant independent of h. Then, there exists a positive constant C, independent of h, such that the estimate

$$\|u_h - u\| \le C \left\{ \inf_{v_h \in K_h} \left(\|u - v_h\|^2 + \|Au - f\|_H \|u - v_h\|_H + C_1 \|u - v_h\|_U\right) + \inf_{v \in K} (\|Au - f\|_H \|u_h - v\|_H + C_1 \|u_h - v\|_U) \right\}^{1/2} \tag{7.35}$$

holds.

Proof. From (7.1) and (7.14), we get

$$\langle Au_h - Au, u_h - u\rangle \le \langle Au - f, v - u_h + v_h - u\rangle + \langle Au_h - Au, v_h - u\rangle + j_h(u_h, v_h) - j_h(u_h, u_h) + j(u, v) - j(u, u) \quad \forall v \in K \quad \forall v_h \in K_h \, . \tag{7.36}$$

Evaluating each term in the right-hand side, we have

$$\langle Au - f, v - u_h + v_h - u\rangle \le \|Au - f\|_H (\|v - u_h\|_H + \|v_h - u\|_H)\, , \tag{7.37}$$

$$\langle Au_h - Au, v_h - u\rangle \le M \|u_h - u\| \, \|v_h - u\| \tag{7.38}$$

and

$$\begin{aligned} &j_h(u_h, v_h) - j_h(u_h, u_h) + j(u, v) - j(u, u) \\ &\le |j_h(u_h, v_h) - j_h(u_h, u_h) + j_h(u, u_h) - j_h(u, v_h)| + |j_h(u, v_h) - j(u, u)| \\ &+ |j(u, v) - j_h(u, u_h)| \le k \|u_h - u\| \, \|v_h - u_h\| + C_1 (\|v_h - u\|_U + \|v - u_h\|_U) \\ &\le k \|u_h - u\|^2 + k \|u_h - u\| \, \|v_h - u\| + C_1 (\|v_h - u\|_U + \|v - u_h\|_U)\, . \end{aligned} \tag{7.39}$$

By using (7.37)–(7.39) in (7.36), with (7.3), it follows

$$(\alpha - k) \|u_h - u\|^2 \le (M + k) \|u_h - u\| \, \|v_h - u\| + \|Au - f\|_H (\|v - u_h\|_H + \|v_h - u\|_H) + C_1 (\|v_h - u\|_U + \|v - u_h\|_U) \quad \forall v \in K\, , \ \forall v_h \in K_h\, , \tag{7.40}$$

which, by Young'inequality : $ab \le \dfrac{\epsilon a^2}{2} + \dfrac{b^2}{2\epsilon}$ for $\epsilon = \dfrac{\alpha - k}{M + k}$, $a = \|u_h - u\|$ and $b = \|v_h - u\|$, implies

$$\frac{\alpha - k}{2} \|u_h - u\|^2 \le \frac{M + k}{2(\alpha - k)} \|v_h - u\|^2 + \|Au - f\|_H (\|v - u_h\|_H + \|v_h - u\|_H) + C_1 (\|v_h - u\|_U + \|v - u_h\|_U) \quad \forall v \in K\, , \ \forall v_h \in K_h\, , \tag{7.41}$$

i.e. (7.35). □

Remark 7.1. If $K_h \subset K$, then the term

$$\inf_{v \in K} \left(\|Au - f\|_H \|u_h - v\|_H + C_1 \|u_h - v\|_U \right) ,$$

which is expected to have the highest weight in (7.35), vanishes, thus one obtains

$$\|u_h - u\| \leq C \left\{ \inf_{v_h \in K_h} \left(\|u - v_h\|^2 + \|Au - f\|_H \|u - v_h\|_H + C_1 \|u - v_h\|_U \right) \right\}^{1/2} .$$

This means that an optimal error estimate $\|u_h - u\|$ depends of the distance between the exact solution u and the finite dimensional subspace V_h of V. Hence, the more suitable construction of the space V_h is, the better order of the error estimate will be. As we shall see on concrete examples in Sect. 8.6, the order of approximation essentially depends on the chosen type of finite element approximation for the space V.

Remark 7.2. If $j(\cdot, \cdot) \equiv 0$, therefore, by taking $C_1 = 0$, we deduce

$$\|u_h - u\| \leq C \left\{ \inf_{v_h \in K_h} \left(\|u - v_h\|^2 + \|Au - f\|_H \|u - v_h\|_H \right) \right. \\ \left. + \|Au - f\|_H \inf_{v \in K} \|u_h - v\|_H \right\}^{1/2} ,$$

so, the estimate obtained by Falk [4] for the internal approximation of variational inequalities of first kind with A a linear and continuous operator.

Remark 7.3. If $j(\cdot, , \cdot) \equiv 0$ and $K = V$, then, by taking $K_h = V_h$, from (7.35), we get

$$\|u_h - u\| \leq C \inf_{v_h \in V_h} \|u - v_h\|$$

so, the result given by Céa [3] for the operator equation $Au = f$ with A a linear and continuous operator.

Finally, the following form of the error estimate is obvious.

Theorem 7.3. *We suppose that the hypotheses of Theorem 7.2 are satisfied but with the condition (7.33) replaced by*

$$\langle Au - f, v \rangle \leq C_2 \|v\|_U \quad \forall v \in V . \tag{7.42}$$

Therefore, we have the estimate

$$\|u_h - u\| \leq C \left\{ \inf_{v_h \in K_h} \left(\|u - v_h\|^2 + (C_1 + C_2) \|u - v_h\|_U \right) \right. \\ \left. + (C_1 + C_2) \inf_{v \in K} \|u_h - v\|_U \right\}^{1/2} \tag{7.43}$$

with C a positive constant independent of h.

7.3 Discrete Approximation of Implicit Evolutionary Inequalities

This section is concerned with the numerical analysis of a class of abstract implicit evolutionary variational inequalities. Convergence results are proved using a method based on a semi-discrete internal approximation and an implicit time discretization scheme.

More precisely, for $f \in W^{1,2}(0,T;V)$ given, one considers the problem (4.107) (p. 68), i.e.
Problem ($\mathbf{Q}^a$): Find $u \in W^{1,2}(0,T;V)$ such that

$$\begin{cases} u(0) = u_0\,,\ u(t) \in K(f(t)) \quad \forall\, t \in [0,T\,], \\ a(u(t), v - \dot{u}(t)) + j(f(t), u(t), v) - j(f(t), u(t), \dot{u}(t)) \\ \geq b(f(t), u(t), v - \dot{u}(t)) \quad \forall\, v \in V \text{ a.e. in }]0,T[, \\ b(f(t), u(t), z - u(t)) \geq 0 \quad \forall\, z \in K,\ \forall\, t \in [0,T\,], \end{cases} \tag{7.44}$$

where $(V, (\cdot,\cdot))$ is a real Hilbert space with the associated norm $\|\cdot\|$ and $K \subset V$ is a closed convex cone with its vertex at 0.

We suppose that $a(\cdot,\cdot)$, $j(\cdot,\cdot,\cdot)$, $b(\cdot,\cdot,\cdot)$ and $K(g)$ satisfy the hypotheses (4.83)–(4.90), (4.96)–(4.98), (4.100), (4.101), and (4.105). We recall that $u_0 \in K(f(0))$ is the unique solution of the following elliptic variational inequality

$$a(u_0, w - u_0) + j(f(0), u_0, w) - j(f(0), u_0, u_0) \geq 0 \quad \forall\, w \in K. \tag{7.45}$$

In order to obtain the discretization of Problem ($\mathbf{Q}^a$), we first consider a semi-discrete approximation of it. For a positive parameter h converging to 0, let $\{V_h\}_h$ be a family of finite dimensional subspaces of V and let $\{K_h\}_h$ be a family of closed convex cones with their vertices at 0 such that $K_h \subset V_h$ and $(K_h)_h$ is an internal approximation of K in the sense specified in Sect. 7.1, i.e.

$$\begin{cases} (i) \quad \forall v \in K,\ \exists r_h v \in K_h \text{ such that } r_h v \to v \text{ strongly in } V\,, \\ (ii) \quad \forall v_h \in K_h \text{ avec } v_h \rightharpoonup v \text{ weakly in } V, \text{ then } v \in K\,. \end{cases} \tag{7.46}$$

For any $h > 0$, let $\{K_h(g)\}_{g \in V}$ be a family of nonempty convex subsets of K_h such that $0 \in K_h(0)$. We put $D_{K_h} = \{(g, v_h) \in V \times K_h\,;\ v_h \in K_h(g)\}$ and we assume the following conditions hold:

$$\left.\begin{array}{l} \forall (g_n, v_{hn}) \in D_{K_h} \text{ such that} \\ g_n \to g \text{ strongly in } V\,,\ v_{hn} \rightharpoonup v_h \text{ weakly in } V \end{array}\right\} \Longrightarrow (g, v_h) \in D_{K_h} \tag{7.47}$$

$$\forall (g, v_h) \in D_{K_h} \text{ such that } v_h \rightharpoonup v \text{ weakly in } V \Longrightarrow (g, v) \in D_K \tag{7.48}$$

We assume that the functional $j : D_K \times V \to \mathbb{R}$ is approximated by a family $\{j_h\}_h$ of functionals $j_h : D_{K_h} \times V_h \to \mathbb{R}$ satisfying

$$\left.\begin{array}{l}\forall g \in C([0,T];V)\,,\ \forall v_h \rightharpoonup v \text{ weakly in } W^{1,2}(0,T,V) \text{ such that}\\ (g(t), v_h(t)) \in D_{K_h}\,,\ \forall t \in [0,T]\end{array}\right\}$$
$$\Rightarrow \liminf_{h\to 0} \int_0^s j_h(g(t), v_h(t), \dot{v}_h(t))\,\mathrm{d}t \geq \int_0^s j(g(t), v(t), \dot{v}(t))\,\mathrm{d}t\,,\ \forall s \in [0,T]\,, \tag{7.49}$$

and

$$\left.\begin{array}{l}\forall (g, v_h) \in D_{K_h},\ \forall w_h \in V_h \text{ such that}\\ v_h \rightharpoonup v \text{ weakly in } V\,,\\ w_h \to w \text{ strongly in } V\end{array}\right\} \Rightarrow \lim_{h\to 0} j_h(g, v_h, w_h) = j(g, v, w). \tag{7.50}$$

Furthermore, we suppose that, for all h, the following conditions are fulfilled:

$$\begin{array}{l}\forall (g, v_h) \in D_{K_h},\ j_h(g, v_h, \cdot) : V_h \to \mathbb{R} \text{ is a sub-additive and}\\ \text{positively homogeneous functional,}\end{array} \tag{7.51}$$

$$j_h(0,0,w_h) = 0 \quad \forall w_h \in V_h \tag{7.52}$$

$$\begin{array}{l}|j_h(g_1, v_{1h}, w_{1h}) + j_h(g_2, v_{2h}, w_{2h}) - j_h(g_1, v_{1h}, w_{2h}) - j_h(g_2, v_{2h}, w_{1h})|\\ \leq k_2(\|g_1 - g_2\| + \|\beta_h(g_1, v_{1h}) - \beta_h(g_2, v_{2h})\|_H)\|w_{1h} - w_{2h}\|\\ \forall (g_i, v_{ih}) \in D_{K_h}, \forall w_{ih} \in V_h,\ i = 1,2\end{array} \tag{7.53}$$

where the operator $\beta_h : D_{K_h} \to H$ is such that

$$\begin{array}{l}\|\beta_h(g_1, v_{1h}) - \beta_h(g_2, v_{2h})\|_H \leq k_1(\|g_1 - g_2\| + \|v_{1h} - v_{2h}\|)\\ \qquad\qquad \forall (g_1, v_{1h}),\ (g_2, v_{2h}) \in D_{K_h},\end{array} \tag{7.54}$$

with k_1, k_2 the positive constants from (4.86), (4.90) such that $k_1 k_2 < \alpha$ (i.e., condition (4.101) from p. 65).

From the properties of a, j_h and K_h and proceeding as in the continuous case, it follows that, for any $g \in V$, $d_h \in K_h$, $w_h \in K_h(g)$, the elliptic variational inequality

$$\left\{\begin{array}{l}\text{Find } u_h \in K_h \text{ such that}\\ a(u_h, v_h - u_h) + j_h(g, w_h, v_h - d_h) - j_h(g, w_h, u_h - d_h) \geq 0 \quad \forall\, v_h \in K_h\end{array}\right. \tag{7.55}$$

has a unique solution $u_h = u_h(g, d_h, w_h)$. Hence, we can define the mapping

$$S^h_{g,d_h} : K_h(g) \to K_h \text{ by } S^h_{g,d_h}(w_h) = u_h \tag{7.56}$$

and, as in Remark 4.8, one obtains that it is a contraction.

We suppose that, for all $g \in V$ and $d_h \in K_h$

$$S^h_{g,d_h}(K_h(g)) \subset K_h(g). \tag{7.57}$$

Let u_{0h} be the unique fixed point of the mapping $S^h_{f(0),0}$, so

$$\begin{cases} u_{0h} \in K_h(f(0))\,, \\ a(u_{0h}, w_h - u_{0h}) + j_h(f(0), u_{0h}, w_h) - j_h(f(0), u_{0h}, u_{0h}) \geq 0 \;\forall\, w_h \in K_h. \end{cases} \tag{7.58}$$

From Theorem 7.1, it follows that

$$u_{0h} \to u_0 \text{ strongly in } V\,, \tag{7.59}$$

as $h \to 0$, u_0 being the unique solution of (7.45).

Now, for all $g \in V$ and $d_h \in K_h$, we introduce the following two auxiliary problems.

Problem ($\tilde{\mathbf{Q}}^a_{\mathbf{h}}$): Find $u_h \in K_h(g)$ such that

$$\begin{cases} a(u_h, v_h - u_h) + j_h(g, u_h, v_h - d_h) - j_h(g, u_h, u_h - d_h) \\ \qquad \geq b(g, u_h, v_h - u_h) \quad \forall\, v_h \in V_h, \\ b(g, u_h, z_h - u_h) \geq 0 \quad \forall\, z_h \in K_h, \end{cases} \tag{7.60}$$

and

Problem ($\tilde{\mathbf{R}}^a_{\mathbf{h}}$): Find $u_h \in K_h(g)$ such that

$$a(u_h, v_h - u_h) + j_h(g, u_h, v_h - d_h) - j_h(g, u_h, u_h - d_h) \geq 0 \quad \forall\, v_h \in K_h. \tag{7.61}$$

We will suppose that

$$\text{If } u_h \text{ is a solution of } (\tilde{\mathbf{R}}^a_{\mathbf{h}}), \text{ then } u_h \text{ is a solution of } (\tilde{\mathbf{Q}}^a_{\mathbf{h}}). \tag{7.62}$$

Remark 7.4. It is obvious that, if u_h satisfies ($\tilde{\mathbf{Q}}^a_{\mathbf{h}}$), then u_h satisfies also ($\tilde{\mathbf{R}}^a_{\mathbf{h}}$).

Let us consider the following semi-discrete problem.

Problem ($\mathbf{Q}^a_{\mathbf{h}}$): Find $u_h \in W^{1,2}(0,T;V_h)$ such that

$$\begin{cases} u_h(0) = u_{0h},\; u_h(t) \in K_h(f(t)) \quad \forall\, t \in [0,T], \\ a(u_h(t), v_h - \dot{u}_h(t)) + j_h(f(t), u_h(t), v_h) - j_h(f(t), u_h(t), \dot{u}_h(t)) \\ \qquad \geq b(f(t), u_h(t), v_h - \dot{u}_h(t)) \quad \forall\, v_h \in V_h \text{ a.e. in }]0,T[, \\ b(f(t), u_h(t), z_h - u_h(t)) \geq 0 \quad \forall\, z_h \in K_h \quad \forall\, t \in [0,T]\,. \end{cases} \tag{7.63}$$

The full discretization of $(\mathbf{Q_h^a})$ is obtained by using a backward difference scheme as in Sect. 4.3 for $(\mathbf{Q}^a)$: for $u_h^0 = u_{0h}$ and $i \in \{0, 1, \ldots, n-1\}$, we define u_h^{i+1} as the unique solution of the following problem.

Problem $(\mathbf{Q_h^a})_\mathbf{n}^\mathbf{i}$: Find $u_h^{i+1} \in K_h^{i+1}$ such that

$$\begin{cases} a(u_h^{i+1}, v_h - \partial u_h^i) + j_h(f^{i+1}, u_h^{i+1}, v_h) - j_h(f^{i+1}, u_h^{i+1}, \partial u_h^i) \\ \qquad\qquad \geq b(f^{i+1}, u_h^{i+1}, v_h - \partial u_h^i) \quad \forall\, v_h \in V_h\,, \\ b(f^{i+1}, u_h^{i+1}, z_h - u_h^{i+1}) \geq 0 \quad \forall\, z_h \in K_h\,, \end{cases} \tag{7.64}$$

where $K_h^{i+1} = K_h(f^{i+1})$.

By (7.62) and Remark 7.4, it is easy to see that Problem $(\mathbf{Q_h^a})_\mathbf{n}^\mathbf{i}$ is equivalent to the following quasi-variational inequality.

Problem $(\mathbf{R_h^a})_\mathbf{n}^\mathbf{i}$: Find $u_h^{i+1} \in K_h^{i+1}$ such that

$$\begin{cases} a(u_h^{i+1}, w_h - u_h^{i+1}) + j_h(f^{i+1}, u_h^{i+1}, w_h - u_h^i) \\ \qquad\qquad -j_h(f^{i+1}, u_h^{i+1}, u_h^{i+1} - u_h^i) \geq 0 \quad \forall\, w_h \in K_h. \end{cases} \tag{7.65}$$

From (4.83), (4.86), (4.90), (4.101), and (7.57), it follows that the mapping

$$S^h_{f^{i+1}, u_h^i} : K_h^{i+1} \to K_h^{i+1},$$

defined by (7.56), is a contraction, so that $(\mathbf{R_h^a})_\mathbf{n}^\mathbf{i}$ has a unique solution.

We now define, as in the continuous case, the functions

$$\begin{cases} u_{hn}(0) = \hat{u}_{hn}(0) = u_{0h}\,, \\ \left.\begin{array}{l} u_{hn}(t) = u_h^{i+1} \\ \hat{u}_{hn}(t) = u_h^i + (t - t_i)\partial u_h^i \end{array}\right\} \forall\, i \in \{0, 1, \ldots, n-1\} \quad \forall\, t \in (t_i, t_{i+1}]\,. \end{cases} \tag{7.66}$$

Then, the functions $u_{hn} \in L^2(0, T; V_h)$ and $\hat{u}_{hn} \in W^{1,2}(0, T; V_h)$ satisfy the following problem.

Problem $(\mathbf{Q_h^a})_\mathbf{n}$: Find $u_{hn}(t) \in K(f_n(t))$ such that

$$\begin{cases} a\left(u_{hn}(t), v_h - \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{u}_{hn}(t)\right) + j_h(f_n(t), u_{hn}(t), v_h) \\ -j_h\left(f_n(t), u_{hn}(t), \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{u}_{hn}(t)\right) \geq b\left(f_n(t), u_{hn}(t), v_h - \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{u}_{hn}(t)\right) \\ \quad \forall\, v_h \in V_h, \\ b(f_n(t), u_{hn}(t), z_h - u_{hn}(t)) \geq 0 \quad \forall\, z_h \in K_h\,. \end{cases} \tag{7.67}$$

Moreover, we have the analogues of Lemmas 4.12 and 4.13. Hence, we conclude, as in Theorem 4.19, that the following convergence and existence result holds.

Theorem 7.4. *Assume that the hypotheses (4.83)–(4.90), (4.96)–(4.98),(4.100), (4.101), (4.105), (7.46), (7.57), and (7.62) hold. Then, the problem* $(\mathbf{Q}_\mathbf{h}^a)$ *has at least one solution. In addition, there exists a subsequence of* $\{(u_{hn}, \hat{u}_{hn})\}_{n\in N^*}$*, still denoted by* $\{(u_{hn}, \hat{u}_{hn})\}_{n\in N^*}$*, such that*

$$u_{hn}(t) \to u_h(t) \quad \text{in} \quad V \quad \forall\, t \in [0, T] \quad \text{as } n \to \infty\,, \tag{7.68}$$

$$\hat{u}_{hn} \to u_h \quad \text{in} \quad W^{1,2}(0, T; V) \quad \text{as } n \to \infty\,, \tag{7.69}$$

where $u_h \in W^{1,2}(0, T; V_h)$ *is a solution of* $(\mathbf{Q}_\mathbf{h}^a)$.

We now proceed to find a priori estimates for the solutions of u_h of $(\mathbf{Q}_\mathbf{h}^a)$ which are limits of subsequences of $\{u_{hn}\}_n$.

Lemma 7.2. *For* $h > 0$*, let* u_h *be the solution of* $(\mathbf{Q}_\mathbf{h}^a)$ *given by Lemma 7.4. Then,*

$$\|u_h(t)\| \le C_0 \|f\|_{C([0,T];V)} \quad \forall\, t \in [0, T]\,, \tag{7.70}$$

$$\|u_h(s) - u_h(t)\| \le C_0 \int_s^t \|\dot{f}(\tau)\|\, \mathrm{d}\tau \quad \forall\, s,\, t \in [0, T],\ s < t, \tag{7.71}$$

$$\|u_h\|_{W^{1,2}(0,T;V)} \le C_0 \sqrt{T \|f\|^2_{C([0,T];V)} + \|\dot{f}\|^2_{L^2(0,T;V)}}\,, \tag{7.72}$$

where C_0 *is the constant, independent of h, given by the relation (4.116).*

Proof. Using the same arguments as in the proof of Lemma 4.12, we obtain the estimates

$$\|u_{hn}(t)\| \le C_0 \|f\|_{C([0,T];V)} \quad \forall\, t \in [0, T],$$

$$\|u_{hn}(s) - u_{hn}(t)\| \le C_0 \int_s^{\min\{t+\Delta t,\, T\}} \|\dot{f}(\tau)\|\, \mathrm{d}\tau \quad \forall\, s,\, t \in [0, T],\ s < t,$$

$$\|\hat{u}_{hn}\|^2_{W^{1,2}(0,T;V)} \le C_0^2 (T \|f\|^2_{C([0,T];V)} + \|\dot{f}\|^2_{L^2(0,T;V)})\,.$$

Combining these results with (7.68), (7.69) and taking into account that the norm is weakly lower semicontinuous, the estimates (7.70)–(7.72) follow. □

Now, we have in position to prove the following convergence result.

Theorem 7.5. *Under the assumptions (4.83)–(4.90), (4.96)–(4.98), (4.100), (4.101), (4.105), (7.46), (7.57), and (7.62), there exists a subsequence of* $\{u_h\}_h$*, still denoted by* $\{u_h\}_h$*, such that*

$$u_h(t) \to u(t) \quad \textit{strongly in} \quad V \quad \forall\, t \in [\,0, T\,] \quad \textit{as } h \to 0\,, \tag{7.73}$$

$$\dot{u}_h \rightharpoonup \dot{u} \quad \textit{weakly in} \quad L^2(0, T; V) \quad \textit{as } h \to 0\,, \tag{7.74}$$

where $u \in W^{1,2}(0, T; V)$ *is a solution of* $(\mathbf{Q}^a)$.

Proof. From Lemma 7.2, it follows that there exists a subsequence of $\{u_h\}_h$ and an element $u \in W^{1,2}(0, T; V)$ such that

$$u_h(t) \to u(t) \quad \text{strongly in} \quad V \quad \forall\, t \in [\,0, T\,]\,, \tag{7.75}$$

$$u_h \rightharpoonup u \quad \text{weakly in} \quad W^{1,2}(0, T; V)\,.. \tag{7.76}$$

Moreover, from (7.75) and (7.59), we get

$$\begin{aligned} &\liminf_{h\to 0} \int_0^s a(u_h(t), \dot{u}_h(t))\,\mathrm{d}t \ge \frac{1}{2}(\liminf_{h\to 0} a(u_h(s), u_h(s)) - \lim_{h\to 0} a(u_{0h}, u_{0h})) \\ &\ge \frac{1}{2}(a(u(s), u(s)) - a(u_0, u_0)) = \int_0^s a(u(t), \dot{u}(t))\,\mathrm{d}t \quad \forall\, s \in [\,0, T\,]. \end{aligned} \tag{7.77}$$

On the other hand, from the hypothesis (7.49), we have

$$\liminf_{h\to 0} \int_0^s j_h(f(t), u_h(t), \dot{u}_h(t))\,\mathrm{d}t \ge \int_0^s j(f(t), u(t), \dot{u}(t))\,\mathrm{d}t\,. \tag{7.78}$$

Next, we prove that u satisfies (7.44). In order to pass to the limit in $(\mathbf{Q}^a_\mathbf{h})$, we will make a convenable choice of v_h in V_h. Let $\pi_h : L^2(0, T; V) \to L^2(0, T; V_h)$ be the projection operator defined by $a(\pi_h v, w_h) = a(v, w_h)$ $\forall v \in L^2(0, T; V)\,,\ \forall w_h \in V_h$. Obviously, the operator π_h is well defined and $\pi_h v(t) \to v(t)$ in V a.e. on $[\,0, T\,]$, hence, by (7.49) and (4.97), it follows that, for all $s \in [0, T]$, we have

$$\lim_{h\to 0} \int_0^s j_h(f(t), u_h(t), \pi_h v(t))\,\mathrm{d}t = \int_0^s j(f(t), u(t), v(t))\,\mathrm{d}t \quad \forall v \in L^2(0, T; V)$$

and

$$\lim_{h\to 0} \int_0^s b(f(t), u_h(t), \pi_h v(t))\,\mathrm{d}t = \int_0^s b(f(t), u(t), v(t))\,\mathrm{d}t \quad \forall v \in L^2(0, T; V)\,.$$

Since $b(f(t), u_h(t), \dot{u}_h(t)) = 0$ a.e. on $[0, T]$, by integrating $(\mathbf{Q}_\mathbf{h}^a)$ over $[0, s]$ for $v_h = \pi_h \dot{u}$ and passing to the limit, we obtain that u satisfies the first inequality of (7.44).

Now, we prove the strong convergence (7.73). Using the same argument as in the proof of Theorem 4.19, by taking $v = 0$, $v = 2\dot{u}$ in (7.44), $v_h = 0$, $v_h = 2\dot{u}_h(t)$ in $(\mathbf{Q}_\mathbf{h}^a)$ and using (7.77), (7.78), for all $s \in [0, T]$, we have

$$\liminf_{h\to 0} \int_0^s a(u_h(t), \dot{u}_h(t))\,\mathrm{d}t = \int_0^s a(u(t), \dot{u}(t))\,\mathrm{d}t, \tag{7.79}$$

$$\liminf_{h\to 0} \int_0^s j_h(f(t), u_h(t), \dot{u}_h(t))\,\mathrm{d}t = \int_0^s j(f(t), u(t), \dot{u}(t))\,\mathrm{d}t. \tag{7.80}$$

and, by taking $v_h = \pi_h \dot{u}(t)$ in $(\mathbf{Q}_\mathbf{h}^a)$, we obtain

$$\limsup_{h\to 0} \int_0^s a(u_h(t), \dot{u}_h(t))\,\mathrm{d}t \le \int_0^s a(u(t), \dot{u}(t))\,\mathrm{d}t \quad \forall\, s \in [0, T]. \tag{7.81}$$

From (7.79) and (7.81), it follows

$$\lim_{h\to 0} \int_0^s a(u_h(t), \dot{u}_h(t))\,\mathrm{d}t = \int_0^s a(u(t), \dot{u}(t))\,\mathrm{d}t,$$

or

$$\lim_{h\to 0} (a(u_h(s), u_h(s)) - a(u_h(0), u_h(0))) = a(u(s), u(s)) - a(u_0, u_0)\,.$$

We recall that $u_h(0) = u_{0h}$ and $u_{0h} \longrightarrow u_0$ strongly in V. Hence, we conclude

$$\lim_{h\to 0} a(u_h(s), u_h(s)) = a(u(s), u(s)) \quad \forall s \in [0, T]$$

which, with the ellipticity of a, implies the strong convergence (7.73).

Finally, we prove that u satisfies the second inequality of (7.44). From $(\mathbf{Q}_\mathbf{h}^a)$, as $j_h(f(t), u_h(t), \cdot)$ is sub-additive, we deduce that, for all $t \in [0, T]$, we have

$$a(u_h(t), v_h - u_h(t)) + j_h(f(t), u_h(t), v_h - u_h(t)) \ge 0 \quad \forall\, v_h \in K_h. \tag{7.82}$$

Let $v \in K$ be arbitrarily chosen. Then, from (7.46), there exists $r_h v \in K_h$ such that $r_h v \to v$ strongly in V. By passing to the limit in (7.82) for $v_h = r_h v$ and using (7.73) and (7.50), we get that u satisfies

$$a(u(t), v - u(t)) + j(f(t), u(t), v - u(t)) \ge 0 \quad \forall v \in K$$

which, by the hypothesis (4.105), implies that u satisfies the second inequality of (7.44). From (7.73) and (7.48), it results that $u \in K(f)$ which completes the proof. □

Using Theorems 7.4 and 7.5, we conclude with the following main approximation result.

Theorem 7.6. *Under the assumptions of Theorem 7.5, the sequence* $\{u_{hn}\}_{hn}$ *of all solutions of complete discrete Problem* $(\mathbf{Q_h^a})_\mathbf{n}$ *has a subsequence, still denoted by* $\{u_{hn}\}_{hn}$*, such that*

$$u_{hn}(t) \to u(t) \quad \textit{strongly in} \quad V \quad \forall\, t \in [0, T] \quad \textit{as } h \to 0\,, n \to \infty\,, \tag{7.83}$$

$$\dot{u}_{hn} \rightharpoonup \dot{u} \quad \textit{weakly in} \quad L^2(0, T; V) \quad \textit{as } h \to 0\,, n \to \infty, \tag{7.84}$$

where $u \in W^{1,2}(0, T; V)$ *is a solution of Problem* $(\mathbf{Q}^a)$.

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