Chapter 6
Dual Formulations of Quasi-Variational Inequalities

The aim of this chapter is to derive dual formulations for quasi-variational inequalities. First, we present a brief background on convex analysis and, then, we recall the main ideas of the Mosco, Capuzzo-Dolcetta, and Matzeu (M–CD–M) duality theory [3] in its form adapted by Telega [14] for implicit variational inequalities.

As we saw in Lemma 4.2, for A symmetric (i.e., $\langle Au, v\rangle = \langle u, Av\rangle,\ \forall u, v \in V$), a variational inequality of the form (4.22) is equivalent to the minimization of the functional J defined by

$$J(v) = \frac{1}{2}\langle Av, v\rangle + j(v) - \langle f, v\rangle.$$

Generally speaking, the duality theory allows to associate with a minimization problem

$$\inf_{v\in K} J(v), \tag{6.1}$$

called primal problem, a maximization one, called dual problem, and to study the relationships between the two problems.

A large number of duality theories have been developed. The main idea in any duality theory is that a proper convex l.s.c. function is the upper envelope of its affine minorants, and so, we can write

$$J(v) = \sup_{\lambda\in\Lambda} \mathscr{L}(v, \lambda).$$

for various choices of $\mathscr{L}$, called the Lagrangian function, and of the set Λ of Lagrange multipliers λ. Hence, the primal problem (6.1) can be written as

$$\inf_{v\in K} \sup_{\lambda\in\Lambda} \mathscr{L}(v, \lambda). \tag{6.2}$$

A. Capatina, *Variational Inequalities and Frictional Contact Problems*,
Advances in Mechanics and Mathematics 31, DOI 10.1007/978-3-319-10163-7_6

The dual problem is defined by

$$\sup_{\lambda \in \Lambda} \inf_{v \in K} \mathscr{L}(v, \lambda). \tag{6.3}$$

The oldest of the theories of duality is that based on the classical theorems of minimax of Fan [7] and Sion [13]. They studied the existence of saddle points for the Lagrangian function $\mathscr{L}$ (a saddle point for $\mathscr{L}$ is an element $(v^*, \lambda^*) \in K \times \Lambda$ such that $\mathscr{L}(v^*, \lambda) \leq \mathscr{L}(v^*, \lambda^*) \leq \mathscr{L}(v, \lambda^*)$, $\forall v \in K$, $\forall \lambda \in \Lambda$) and they give criteria (see also [5]) which ensure that $\sup_{\lambda \in \Lambda} \inf_{v \in K} \mathscr{L}(v, \lambda) = \inf_{v \in K} \sup_{\lambda \in \Lambda} \mathscr{L}(v, \lambda)$.

Another theory has been developed by Fenchel [6] and Rockafellar [11]. In their theory, the minimization problem is approached by a family of perturbed problems and the dual problem is defined by means of the conjugate functions. More details can be found in Rockafellar [12], Céa [4], Ekeland and Temam [5].

The duality theory has many applications in mechanics, numerical analysis, control theory, game theory, or economics. In addition, the so-called primal–dual algorithms are often used in solving the primal problem. Nevertheless, classical duality approaches do not apply to quasi-variational inequalities since they cannot be formulated as extremum problems. For this reason, within this chapter we do not want to develop classical duality methods, our intention is only to recall some results of the M–CD–M [3] duality theory for the so-called implicit variational problems. In Sect. 8.5, we will use this theory to derive the so-called condensed dual formulation for a frictional contact problem.

6.1 Convex Analysis Background

We recall some definitions and standard results which will be useful in the subsequent paragraph. Let V be a reflexive Banach space with its dual V^* (we note that almost all the results remain valid if V and V^* are two topological vector spaces which are in duality; see, for instance, [1, 5, 8, 10]). We denote by $\langle \cdot, \cdot \rangle_{V^* \times V}$ the duality pairing between V^* and V.

Let $f : V \longrightarrow \overline{\mathbb{R}}$ be a function.

Let us recall that the effective domain of f, the epigraph of f and, for any $a \in \overline{\mathbb{R}}$, the level sets are defined by

$$\begin{aligned} \text{dom } f &= \{v \in V \ : \ f(v) < \infty\}, \\ \text{epi } f &= \{(v, a) \in V \times \overline{\mathbb{R}} \ : \ f(v) \leq a\}, \end{aligned}$$

and, respectively,

$$E_a(f) = \{v \in V \ : \ f(v) \leq a\}.$$

The function f is said to be proper if dom $f \neq \emptyset$ and $f(v) > -\infty, \forall v \in V$.

The convexity and the lower semicontinuity of functions can be characterized in the following way.

Proposition 6.1. *Let $f : V \longrightarrow \overline{\mathbb{R}}$ be a function. Then the following statements are equivalent:*

(i) the function f is convex and l.s.c. on V;
(ii) the set epi f is a convex and closed subset of $V \times \overline{\mathbb{R}}$.

Proof. For convexity, we only use its definition for functions and sets.

If f is l.s.c., then it is easy to show that epi f is closed in $V \times \overline{\mathbb{R}}$. Conversely, if the set epi f is closed in $V \times \overline{\mathbb{R}}$, then, for any $a \in \overline{\mathbb{R}}$, the level sets $E_a(f)$ are closed in V and so, the sets $\{v \in V : f(v) > a\}$ are open, i.e. the function f is l.s.c. on V. □

Definition 6.1. The function $f^* : V^* \longrightarrow \overline{\mathbb{R}}$ defined by

$$f^*(v^*) = \sup_{v \in V}\{\langle v^*, v\rangle_{V^* \times V} - f(v)\},$$

is called the Fenchel conjugate (sometimes also called convex conjugate, conjugate function, or polar function) to f.

In the particular case $V = \mathbb{R}$, f^* is the Young conjugate function to f.

An elementary property is the following Young inequality

$$f(v) + f^*(v^*) \geq \langle v^*, v\rangle_{V^* \times V} \quad \forall v \in V,\ \forall v^* \in V^*. \tag{6.4}$$

Remark 6.1. Let $C \subset V$ be a set such that $0 \in C$. Then

$$I_C^*(v^*) = \sup_{v \in C}\{\langle v^*, v\rangle_{V^* \times V}\} = I_{C^*}(v^*)$$

where $C^* = \{v^* \in V^* : \langle v^*, v\rangle_{V^* \times V} \leq 0,\ \forall v \in C\}$ is the polar cone of C and I_A is the indicator function of the set A.

We give below a separation theorem (see, e.g., [8]) which will be frequently used in the sequel.

Theorem 6.1. *Let M be a convex closed subset of V and let be $v_0 \in V$ such that $v_0 \notin M$. Then there exists $v^* \in V^*$, $v^* \neq 0$, strictly separating M and v_0, i.e. there exists $c \in \mathbb{R}$ such that*

$$\langle v^*, v_0\rangle_{V^* \times V} > c \geq \langle v^*, v\rangle_{V^* \times V} \quad \forall v \in M.$$

Proposition 6.2. *Let $f : V \longrightarrow \overline{\mathbb{R}}$ be a function. Then*

1) The conjugate function f^* is convex l.s.c. on V^*.
2) If f is proper convex l.s.c. on V, then f^* is proper.

Proof. 1) Let $u^*, v^* \in V$ and $t \in [0, 1]$. We have

$$f^*((1-t)u^*+tv^*) = \sup_{v\in V}\{(1-t)(\langle u^*, v\rangle_{V^*\times V}-f(v))+t(\langle v^*, v\rangle_{V^*\times V}-f(v))\}$$
$$\leq (1-t)f^*(u^*) + tf^*(v^*),$$

i.e. f^* is convex.

In order to prove that f^* is l.s.c., let be the sequence $\{u_n^*\}_n \subset V^*$ and let be $u^* \in V^*$ such that $u_n^* \to u^*$ strongly in V^*. Applying Young's inequality (6.4), we get

$$f^*(u_n^*) \geq \langle u_n^*, v\rangle_{V^*\times V} - f(v) \quad \forall v \in V,$$

and hence

$$\liminf_{n\to\infty} f^*(u_n^*) \geq \langle u^*, v\rangle_{V^*\times V} - f(v) \quad \forall v \in V.$$

This yields

$$\liminf_{n\to\infty} f^*(u_n^*) \geq f^*(u^*).$$

2) As f is proper, there exists $v_0 \in V$ such that $f(v_0) < \infty$. Hence, Young's inequality (6.4) yields

$$f^*(v^*) \geq \langle v^*, v_0\rangle_{V^*\times V} - f(v_0) > -\infty \quad \forall v^* \in V^*.$$

Let $d > 0$. Since $(v_0, f(v_0)-d) \notin \text{epi } f$ and $\text{epi } f$ is convex closed in $V\times\mathbb{R}$, by the Separation Theorem 6.1, it follows that there exist $v_0^* \in V^*$, $v_0^* \neq 0$, and $\alpha \in \mathbb{R}$ such that

$$\langle v_0^*, v_0\rangle_{V^*\times V} + \alpha(f(v_0) - d) > \langle v_0^*, v\rangle_{V^*\times V} + \alpha a \quad \forall (v, a) \in \text{epi } f. \tag{6.5}$$

It is easy to prove that $\alpha < 0$. Indeed, if we suppose that $\alpha > 0$, then, for any $(v, a) \in \text{epi } f$, we can take $(v, a+n) \in \text{epi } f$ in (6.5), for any $n > 0$. Thus the right-hand side of (6.5) tends to $+\infty$ which is in contradiction with the relation (6.5). If $\alpha = 0$, then we obtain $\langle v_0^*, v_0\rangle_{V^*\times V} > \langle v_0^*, v\rangle_{V^*\times V}$, $\forall v \in V$ which contradicts $v_0 \in V$.

Therefore, if we put $v_1^* = -\dfrac{1}{\alpha}v_0^*$ in (6.5), in particular we deduce that

$$\langle v_1^*, v_0\rangle_{V^*\times V} - f(v_0) + d > \langle v_1^*, v\rangle_{V^*\times V} - f(v) \quad \forall v \in V,$$

and so, as $v_0 \in \text{dom}\,(f)$, we get

$$+\infty > \langle v_1^*, v_0\rangle_{V^*\times V} - f(v_0) + d > \sup_{v\in V}\{\langle v_1^*, v\rangle_{V^*\times V} - f(v)\} = f^*(v_1^*),$$

and hence, f^* is proper. □

Definition 6.2. Let $f^* : V^* \to \overline{\mathbb{R}}$ be the conjugate function to f. Then the function $f^{**} : V \to \overline{\mathbb{R}}$ defined by

$$f^{**}(v) = \sup_{v^*\in V^*}\{\langle v^*, v\rangle_{V^*\times V} - f^*(v^*)\},$$

is called the biconjugate function to f.

From Young's inequality (6.4), we always have $f^{**}(v) \le \sup_{v^*\in V^*}\{\langle v^*, v\rangle_{V^*\times V} - \langle v^*, v\rangle_{V^*\times V} + f(v)\} = f(v)$, i.e.

$$f^{**}(v) \le f(v) \quad \forall v \in V. \tag{6.6}$$

The following statement gives conditions which ensure the equality between a function and its biconjugate.

Theorem 6.2 (Fenchel–Moreau Duality Theorem). *Let $f : V \to \overline{\mathbb{R}}$ be a proper function. Then, f is l.s.c. and convex if and only if $f^{**} = f$.*

Proof. Suppose that f is l.s.c. and convex. By Proposition 6.2, it follows that f^* is a proper l.s.c. convex function, and so, f^{**} is a proper l.s.c. convex function.

As we always have $f^{**}(v) \le f(v)$, suppose that there exists $v_0 \in V$ such that $f^{**}(v_0) < f(v_0)$. Thus $(v_0, f^{**}(v_0)) \notin \text{epi}\,(f)$. Applying the Separation Theorem 6.1, it follows that there exist $v_0^* \in V^*$, $v_0^* \ne 0$, and $\alpha \in \mathbb{R}$ such that

$$\langle v_0^*, v_0\rangle_{V^*\times V} + \alpha f^{**}(v_0) > \langle v_0^*), v\rangle_{V^*\times V} + \alpha a \quad \forall (v, a) \in \text{epi}\ f.$$

Proceeding as in the proof of Proposition 6.2 we conclude that $\alpha < 0$. If we put $v_1^* = -\frac{1}{\alpha} v_0^*$, then we deduce

$$\begin{aligned}\langle v_1^*, v_0\rangle_{V^*\times V} - f^{**}(v_0) &> \sup_{(v,a)\in\text{epi}\ f}\{\langle v_1^*), v\rangle_{V^*\times V} - a\}\\ &\ge \sup_{v\in V}\{\langle v_1^*), v\rangle_{V^*\times V} - f(v)\} = f^*(v_1^*)\end{aligned}$$

which contradicts the definition of $f^{**}(v_0)$.

Conversely, if $f = f^{**}$ then, by Proposition 6.2, it follows that f, as the conjugate to f^*, is l.s.c. convex on V. □

Remark 6.2. If $f : V \to \overline{\mathbb{R}}$ is a convex l.s.c. function which takes the value $-\infty$, then f is identically equal to $-\infty$. Therefore it is natural to consider convex l.s.c. functions $f : V \to (-\infty, +\infty]$.

Definition 6.3. Let $f : V \to (-\infty, +\infty]$ be a proper function and $u \in$dom (f). An element $u^* \in V^*$ is said to be subgradient of f at u (according to e.g., [9]) if

$$f(v) - f(u) \geq \langle u^*, v - u \rangle_{V^* \times V}, \quad \forall v \in V.$$

The set of all subgradients of f at u is called the subdifferential of f at u and is denoted by $\partial f(u)$,

$$\partial f(u) = \{u^* \in V^*;\ f(v) - f(u) \geq \langle u^*, v - u \rangle_{V^* \times V}, \quad \forall v \in V\}.$$

So, the subdifferential of f is the multivalued mapping $\partial f : V \to 2^{V^*}$ which associates with every $u \in V$ the subset $\partial f(u)$ of V^*.

The function f is said to be subdifferentiable at u, respectively, on V, if $\partial f(u) \neq \emptyset$, respectively, $\partial f(u) \neq \emptyset$, $\forall u \in V$.

The next result follows immediately from the definitions.

Theorem 6.3. *Let $f : V \to (-\infty, +\infty]$ be a proper function. Then, the following two conditions are equivalent:*

(1) $f(u) = \min_{v \in V} f(v)$,
(2) $0 \in \partial f(u)$

Theorem 6.4. *Let $f : V \to (-\infty, +\infty]$ be a function. Then the following two conditions are equivalent:*

(1) $f(u) + f^*(u^*) = \langle u^*, u \rangle$,
(2) $u^* \in \partial f(u)$.
Moreover, any of the above conditions implies
(3) $u \in \partial f^*(u^*)$.
In addition, if f is proper l.s.c. and convex, then the three above conditions are equivalent.

Proof. "(1) $\Rightarrow$ (2)" By using the hypothesis (1) and Young's inequality (6.4), we obtain

$$\langle u^*, u \rangle_{V^* \times V} - f(u) = f^*(u^*) \geq \langle u^*, v \rangle_{V^* \times V} - f(v) \quad \forall v \in V,$$

i.e. the condition (2).

"(2) $\Rightarrow$ (1)" If $u^* \in \partial f(u)$, then

$$\langle u^*, u \rangle_{V^* \times V} - f(u) \geq \langle u^*, v \rangle_{V^* \times V} - f(v) \quad \forall v \in V$$

and so,

$$\langle u^*, u\rangle_{V^*\times V} - f(u) \geq \sup_{\forall v\in V}\{\langle u^*, v\rangle_{V^*\times V} - f(v)\} = f^*(u^*).$$

Therefore, by Young's inequality (6.4), the assertion follows.

"(1) $\Rightarrow$ (3)" By the definition of f^* and the hypothesis (1), we have

$$\begin{aligned} f^*(v^*) - f^*(u^*) &= \sup_{\forall v\in V}\{\langle v^*, v\rangle_{V^*\times V} - f(v)\} + f(u) - \langle u^*, u\rangle_{V^*\times V} \\ &\geq \langle v^*, u\rangle_{V^*\times V} - f(u) + f(u) - \langle u^*, u\rangle_{V^*\times V} \\ &= \langle v^* - u^*, u\rangle_{V^*\times V} \quad \forall v^* \in V^*, \end{aligned}$$

i.e. $u \in \partial f^*(u^*)$.

Suppose now that $f : V \to (-\infty, +\infty]$ is a proper l.s.c. convex function.

"(3) $\Rightarrow$ (1)" If $u \in \partial f^*(u^*)$, then we have

$$\langle u^*, u\rangle_{V^*\times V} - f^*(u^*) \geq \langle v^*, u\rangle_{V^*\times V} - f^*(v^*) \quad \forall v^* \in V^*,$$

which implies

$$\langle u^*, u\rangle_{V^*\times V} - f^*(u^*) \geq \sup_{\forall v^*\in V^*}\{\langle v^*, u\rangle_{V^*\times V} - f^*(v^*)\} = f^{**}(u),$$

As Theorem 6.2 provides $f^{**}(u) = f(u)$, by Young's inequality (6.4), we conclude that $f^*(u^*) + f(u) = \langle u^*, u\rangle_{V^*\times V}$. □

Let $f_1, f_2 : V \to (-\infty, +\infty]$ be two proper functions.

Definition 6.4. The infimal convolution of functions f_1 and f_2, denoted by $f_1 \nabla f_2$, is the function defined by

$$(f_1\nabla f_2)(u) = \inf_{v\in V}\{f_1(v) + f_2(u-v)\} = \inf_{\substack{v_1+v_2=u \\ v_1, v_2\in V}}\{f_1(v_1) + f_2(v_2)\} \quad \forall u \in V.$$

Definition 6.5. We say that the infimal convolution $f_1\nabla f_2$ is exact at u if there exists $v \in V$ such that $(f_1\nabla f_2)(u) = f_1(v) + f_2(u-v)$ or, equivalent, if there exist $v_1, v_2 \in V$ such that $v_1 + v_2 = u$ and $(f_1\nabla f_2)(u) = f_1(v_1) + f_2(v_2)$.

Proposition 6.3. *Let $f_1, f_2 : V \to (-\infty, +\infty]$ be proper functions. Then*

(1) $(f_1\nabla f_2)^* = f_1^* + f_2^*$,
(2) If $f_1\nabla f_2$ is exact at u, i.e. there exists $u_1, u_2 \in V$ such that $u_1 + u_2 = u$ and $(f_1\nabla f_2)(u) = f_1(u_1) + f_2(u_2)$, then $\partial(f_1\nabla f_2)(u) = \partial f(u_1) \cap \partial f_2(u_2)$.
(3) If f_1, f_2 are convex, then $f_1\nabla f_2$ is convex.

Proof. (1) By definitions, we have

$$\begin{aligned}(f_1 \nabla f_2)^*(u^*) &= \sup_{v\in V}\{\langle u^*, v\rangle_{V^*\times V} - \inf_{u\in V}\{f_1(u) + f_2(v-u)\}\}\\ &= \sup_{v\in V}\{\langle u^*, v\rangle_{V^*\times V} + \sup_{u\in V}\{-f_1(u) - f_2(v-u)\}\}\\ &= \sup_{u,v\in V}\{\langle u^*, v\rangle_{V^*\times V} - f_1(u) - f_2(v-u)\}\\ &= \sup_{u\in V}\{\langle u^*, u\rangle_{V^*\times V} - f_1(u) + \sup_{v\in V}\{\langle u^*, v-u\rangle_{V^*\times V} - f_2(v-u)\}\}\\ &= \sup_{u\in V}\{\langle u^*, u\rangle_{V^*\times V} - f_1(u)\} + f_2^*(u^*) = f_1^*(u^*) + f_2^*(u^*).\end{aligned}$$

(2) Theorem 6.4, the relation (1) and the hypothesis yield that we have the following sequence of equivalent assertions

$$\begin{aligned}&u^* \in \partial(f_1 \nabla f_2)(u)\\ &\iff (f_1 \nabla f_2)^*(u^*) + (f_1 \nabla f_2)(u) = \langle u^*, u\rangle_{V^*\times V}\\ &\iff f_1^*(u^*) + f_2^*(u^*) + f_1(u_1) + f_2(u_2) = \langle u^*, u_1\rangle_{V^*\times V} + \langle u^*, u_2\rangle_{V^*\times V}\end{aligned}$$

As from the Young inequality (6.4) we have

$$\begin{aligned}f_1^*(u^*) + f_1(u_1) \geq \langle u^*, u_1\rangle_{V^*\times V},\\ f_2^*(u^*) + f_2(u_2) \geq \langle u^*, u_2\rangle_{V^*\times V},\end{aligned}$$

it follows that we must have $f_i^*(u^*) + f_i(u_i) = \langle u^*, u_i\rangle_{V^*\times V}$, for $i = 1, 2$. Again Theorem 6.4 provides $u^* \in \partial f_i(u_i)$, for $i = 1, 2$, i.e. $u^* \in \partial f_1(u_1) \cap \partial f_2(u_2)$.

(3) As f_1, f_2 are convex, it follows that epi f_1 and epi f_2 are convex sets in $V \times \overline{\mathbb{R}}$. We prove that

$$\text{epi } (f_1 \nabla f_2) = \text{epi } (f_1) + \text{epi } (f_2),$$

from which the assertion follows. Indeed, we have

$$\begin{aligned}&(u, a) \in \text{epi } (f_1 \nabla f_2)\\ &\iff \inf_{\substack{v_1+v_2=u\\ v_1, v_2\in V}}\{f_1(v_1) + f_2(v_2)\} \leq a\\ &\iff \exists u_1, u_2 \in V,\ u_1 + u_1 = u \text{ s.t. } f_1(u_1) + f_2(u_2) \leq a\\ &\iff f_1(u_1) \leq a_1,\ f_2(u_2) \leq a_2,\ u_1 + u_2 = u,\ a_1 + a_2 = a\\ &\iff (u_1, a_1) \in \text{epi } f_1,\ (u_2, a_2) \in \text{epi } f_2,\ u_1 + u_2 = u,\ a_1 + a_2 = a\\ &\iff (u, a) = (u_1, a_1) + (u_2, a_2) \in \text{epi } f_1 + \text{epi } f_2.\end{aligned}$$

□

We now recall the Fenchel's duality theorem. The proof is available in [9], so we omit here.

Theorem 6.5 (Fenchel's Duality Theorem). *Let $f, -g : V \to (-\infty, +\infty]$ be two proper convex l.s.c. functions. Suppose that there exists $u_0 \in dom(f) \cap dom(-g)$ such that f or g is continuous at u_0. Then*

$$\inf_{v \in V}\{f(v) - g(v)\} = \max_{v^* \in V^*}\{g_*(v^*) - f^*(v^*)\}, \tag{6.7}$$

where g_* is the concave conjugate function to g, i.e.

$$g_*(v^*) = \inf_{v \in V}\{\langle v^*, v\rangle_{V^* \times V} - g(v)\}.$$

Proposition 6.4. *Let $f_1, f_2 : V \to (-\infty, +\infty]$ be two proper convex l.s.c. functions. If there exists $u_0 \in dom(f_1) \cap dom(f_2)$ such that f_1 or f_2 is continuous at u_0, then*

$$(f_1+f_2)^*(u^*) = (f_1^* \nabla f_2^*)(u^*) = f_1^*(u_1^*) + f_2^*(u_2^*) \quad \forall u^* \in V^* \text{ with } u_1^* + u_2^* = u^*$$

i.e. $f_1^* \nabla f_2^*$ is exact on V^*.

Proof. Let $u^* \in V^*$. We apply Fenchel's Duality Theorem 6.5 for

$$f(v) = f_2(v), \; g(v) = \langle u^*, v\rangle_{V^* \times V} - f_1(v) \quad \forall v \in V.$$

It is easy to verify that

$$\inf_{v \in V}\{f(v) - g(v)\} = -\sup_{v \in V}\{\langle u^*, v\rangle_{V^* \times V} - (f_1 + f_2)(v)\} = -(f_1^* + f_2^*)(u^*)$$

and

$$\begin{aligned}\max_{v^* \in V^*}\{g_*(v^*) - f^*(v^*)\} &= -\min_{v^* \in V^*}\{f_1^*(u^* - v^*) + f_2^*(v^*)\} \\ &= -f_1^*(u_1^*) - f_2^*(u_2^*), \quad u_1^* + u_2^* = u^*.\end{aligned}$$

On the other hand, from the definition of the infinimal convolution, we have

$$\min_{v^* \in V^*}\{f_1^*(u^* - v^*) + f_2^*(v^*)\} = (f_1^* \nabla f_2^*)(u^*).$$

Therefore, by (6.7), we get $(f_1^* + f_2^*)(u^*) = (f_1^* \nabla f_2^*)(u^*) = f_1^*(u_1^*) + f_2^*(u_2^*)$, $\forall u^* \in V^*$, and $u_1^* + u_2^* = u^*$, which completes the proof. □

Theorem 6.6. *Let $f_1, f_2 : V \to (-\infty, +\infty]$ be two proper convex l.s.c. functions. Suppose that there exists $u_0 \in dom(f_1) \cap dom(f_2)$ such that f_1 is continuous at u_0. Then*

$$\partial(f_1 + f_2)(u) = \partial f_1(u) + \partial f_2(u) \quad \forall u \in V.$$

Proof. Let $u \in V$.

We first prove that $\partial(f_1 + f_2)(u) \subset \partial f_1(u) + \partial f_2(u)$. Let $u^* \in \partial(f_1 + f_2)(u)$. Applying Theorem 6.4 and Proposition 6.4, we get

$$\begin{aligned}\langle u^*, u\rangle_{V^*\times V} &= (f_1 + f_2)(u) + (f_1 + f_2)^*(u^*) \\ &= f_1(u) + f_1^*(u_1^*) + f_2(u) + f_2^*(u_2^*) \quad \text{with } u_1^* + u_2^* = u^*.\end{aligned} \tag{6.8}$$

Since from the Young inequality we have

$$\begin{aligned}f_1(u) + f_1^*(u_1^*) \geq \langle u_1^*, u\rangle_{V^*\times V},\\ f_2(u) + f_2^*(u_2^*) \geq \langle u_2^*, u\rangle_{V^*\times V},\end{aligned}$$

the relation (6.8) implies

$$\begin{aligned}f_1(u) + f_1^*(u_1^*) = \langle u_1^*, u\rangle_{V^*\times V},\\ f_2(u) + f_2^*(u_2^*) = \langle u_2^*, u\rangle_{V^*\times V},\end{aligned}$$

and so, again by Theorem 6.4, $u_1^* \in \partial f_1(u)$ and $u_2^* \in \partial f_2(u)$ with $u_1^* + u_2^* = u^*$, i.e. $u^* \in \partial f_1(u) + \partial f_2(u)$.

The reverse $\partial f_1(u) + \partial f_2(u) \subset \partial(f_1 + f_2)(u)$ holds without any hypotheses on f_1 or f_2. Indeed, if $u^* \in \partial f_1(u) + \partial f_2(u)$, then there exist $u_1^*, u_2^* \in V^*$ such that $u^* = u_1^* + u_2^*$, $u_1^* \in \partial f_1(u)$ and $u_2^* \in \partial f_2(u)$, i.e.

$$\begin{aligned}f_1(v) - f_1(u) \geq \langle u_1^*, v - u\rangle_{V^*\times V} \quad \forall v \in V,\\ f_2(v) - f_2(u) \geq \langle u_2^*, v - u\rangle_{V^*\times V} \quad \forall v \in V.\end{aligned}$$

By adding them, we have

$$(f_1 + f_2)(v) - (f_1 + f_2)(u) \geq \langle u^*, v - u\rangle_{V^*\times V} \quad \forall v \in V,$$

which means $u^* \in \partial(f_1 + f_2)(u)$. □

6.2 M–CD–M Theory of Duality

We present here the main ideas for obtaining a dual formulation in the sense of M–CD–M (see [3, 14]) of an abstract problem.

Let $(V, V^*, \langle\cdot,\cdot\rangle_{V^*\times V})$ and $(Y, Y^*, \langle\cdot,\cdot\rangle_{Y^*\times Y})$ be two reflexive Banach spaces with their duals and their duality pairings. We consider the following primal problem:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ \varphi(Lu, u) + \psi(u, u) \leq \varphi(Lu, v) + \psi(u, v) \quad \forall v \in V \end{cases} \tag{6.9}$$

where the operator $L : V \to Y$ and the functions $\varphi : Y \times V \to (-\infty, +\infty]$ and $\psi : V \times V \to \mathbb{R}$ satisfy the following hypotheses:

$$L \text{ is a linear continuous operator,} \tag{6.10}$$

$$\forall u \in V,\ \varphi(Lu, \cdot) \text{ is proper convex l.s.c.} \tag{6.11}$$

$$\forall u \in V,\ \psi(u, \cdot) \text{ is convex and } \psi(u, u) \text{ is continuous} \tag{6.12}$$

$$\begin{cases} \forall u \in V, \text{ the mapping } v \mapsto \psi(u, v) \text{ has a Gâteaux derivative } D_2\psi(u, v) \\ \text{with respect to the second variable at } v = u \text{ such that, for any} \\ v^* \in V^*, \text{ the set } \{u \in V\,;\ D_2\psi(u, u) = v^*\} \text{ contains at most one} \\ \text{element denoted by } (D_2\psi)^{-1}(v^*). \end{cases} \tag{6.13}$$

We recall that the Gâteaux derivative with respect to the second variable of $\psi(u, \cdot)$ at v is defined by

$$\langle D_2\psi(u, v), w\rangle_{V^*\times V} = \lim_{t\to 0+} \frac{\psi(u, v + tw) - \psi(u, v)}{t}.$$

The dual problem of (6.9) is constructed by means of Fenchel conjugates of φ^* and ψ^* with respect to the second variable, defined by

$$\varphi^* : Y \times V^* \to (-\infty, +\infty], \quad \varphi^*(Lu, v^*) = \sup_{v\in V} \left(\langle v^*, v\rangle_{V^*\times V} - \varphi(Lu, v)\right),$$
$$\psi^* : V \times V^* \to (-\infty, +\infty], \quad \psi^*(u, v^*) = \sup_{v\in V} \left(\langle v^*, v\rangle_{V^*\times V} - \psi(u, v)\right).$$

We also denote, for all $u \in V$, the subdifferentials of $\psi(u, \cdot)$ and $\varphi^*(Lu, \cdot)$ with respect to the second variable by $\partial_2\psi(u, \cdot)$, and respectively, by $\partial_2\varphi^*(Lu, \cdot)$, where

$$\partial_2\psi(u, z) = \{v^* \in V\,;\ \psi(u, v) - \psi(u, z) \geq \langle v^*, v - z\rangle_{V^*\times V},\ \forall v \in V\} \quad \forall z \in V,$$
$$\partial_2\varphi^*(Lu, z^*) = \{v \in V\,;\ \varphi^*(Lu, v^*) - \varphi^*(Lu, z^*)$$
$$\geq \langle v^* - z^*, v\rangle_{V^*\times V},\ \forall v^* \in V^*\} \quad \forall z^* \in V^*.$$

With the above notation, the dual problem of (6.9) is

$$\begin{cases} \text{Find } (u,u^*) \in V \times V^* \text{ such that} \\ -u^* \in \partial_2 \psi(u,u) \\ u \in \partial_2 \varphi^*(Lu,u^*) \end{cases} \tag{6.14}$$

or, equivalently

$$\begin{cases} \text{Find } (u,u^*) \in V \times V^* \text{ such that} \\ \psi(u,v) - \psi(u,u) \geq \langle -u^*, v-u \rangle_{V^* \times V} \qquad \forall v \in V, \\ \varphi^*(Lu,v^*) - \varphi^*(Lu,u^*) \geq \langle v^* - u^*, u \rangle_{V^* \times V} \qquad \forall v^* \in V^*. \end{cases} \tag{6.15}$$

The relationship between the primal problem and the dual problem is given by the next result (see, e.g., [2, 15]).

Theorem 6.7. *Suppose the hypotheses (6.10)–(6.12) are satisfied.*

(i) If u is a solution of the primal problem (6.9), then there exists $u^* \in V^*$ such that (u,u^*) is a solution of the dual problem (6.14).
(ii) If (u,u^*) is a solution of the dual problem (6.14), then u is a solution of the primal problem (6.9).

In addition, the following extremality conditions hold:

$$\begin{cases} \varphi(Lu,u) + \varphi^*(Lu,u^*) = \langle u^*, u \rangle_{V^* \times V}, \\ \psi(u,u) + \psi^*(u,-u^*) = -\langle u^*, u \rangle_{V^* \times V}. \end{cases} \tag{6.16}$$

Proof. (i) Let u be a solution of (6.9) and $f(v) = \varphi(Lu,v) + \psi(u,v)$. It follows that

$$f(u) \leq f(v) \qquad \forall v \in V,$$

and so, by using Theorems 6.3, 6.4 and Proposition 6.4, we get

$$0 \in \partial f(u) \iff u \in \partial f^*(0) = \partial(f_1^* \nabla f_2^*)(0), \tag{6.17}$$

where $f_1(v) = \varphi(Lu,v)$ and $f_2(v) = \psi(u,v)$.

On the other hand, from Proposition 6.4, the infimal convolution $f_1^* \nabla f_2^*$ is exact at 0. Hence, by Proposition 6.3_2, we deduce that $\partial(f_1^* \nabla f_2^*)$ is exact at 0, i.e. there exists $u^* \in V^*$ such that

$$\partial(f_1^* \nabla f_2^*)(0) = \partial f_1^*(u^*) \cap \partial f_2^*(-u^*). \tag{6.18}$$

Now, the relations (6.17) and (6.18) yield that there exists $u^* \in V^*$ such that $u \in \partial_2 \varphi^*(Lu,u^*) \cap \partial_2 \psi^*(u,-u^*)$. We conclude, by Theorem 6.4, that $u \in \partial_2 \varphi^*(Lu,u^*)$ and $-u^* \in \partial_2 \psi(u,u)$, i.e. (u,u^*) is a solution of (6.14).

(ii) If (u, u^*) is a solution of (6.14), then, from Theorem 6.4, we obtain

$$u \in \partial_2 \varphi^*(Lu, u^*) \cap \partial_2 \psi^*(u, -u^*)$$

and, proceeding as in the first part (i), the assertion follows.

Finally, as

$$-u^* \in \partial_2 \psi(u, u) \quad \text{and} \quad u \in \partial_2 \varphi^*(Lu, u^*),$$

Theorem 6.4 provides the extremality conditions (6.16).

□

The first variable u from the solution (u, u^*) of the dual problem (6.14) is eliminated by using the assumption (6.13).

Theorem 6.8 (M–CD–M Theorem). *Let the hypotheses (6.10)–(6.13) be satisfied. Then, u is a solution of the primal problem (6.9) if and only if $u^* = -D_2\psi(u, u)$ is a solution of the following dual problem*

$$\begin{cases} \text{Find } u^* \in V^* \text{ such that} \\ \varphi^*(L(D_2\psi)^{-1}(-u^*), v^*) - \varphi^*(L(D_2\psi)^{-1}(-u^*), u^*) \\ \qquad \geq \langle v^* - u^*, (D_2\psi)^{-1}(-u^*)\rangle_{V^* \times V} \quad \forall v^* \in V^*. \end{cases} \tag{6.19}$$

Moreover, the extremality conditions (6.16) hold.

Proof. We first remark that the hypothesis (6.13) implies

$$-u^* = D_2\psi(u, u) \iff u = (D_2\psi)^{-1}(-u^*). \tag{6.20}$$

Now, if u is a solution of the primal problem (6.9), then, by Theorem 6.7, one has $u \in \partial_2 \varphi^*(Lu, u^*)$, and hence, by the characterization (6.20), one obtains

$$(D_2\psi)^{-1}(-u^*) \in \partial_2 \varphi^*(L(D_2\psi)^{-1}(-u^*), u^*).$$

Therefore, from the definition of the subdifferential of ψ with respect to the second variable, we conclude that u^* solves (6.19).

Conversely, if $u^* = -D_2\psi(u, u)$ is a solution of the dual problem (6.19), then $(D_2\psi)^{-1}(-u^*) \in \partial_2 \varphi^*(L(D_2\psi)^{-1}(-u^*), u^*)$ which, together with (6.20), gives

$$\begin{cases} u \in \partial_2 \varphi^*(Lu, u^*), \\ -u^* = D_2\psi(u, u) = \partial_2 \psi(u, u), \end{cases}$$

that is (u, u^*) is a solution of (6.14). Finally, from Theorem 6.7, we conclude the proof. □

References

1. Barbu, V., Precupanu T.: Convexity and Optimization in Banach Spaces. D. Reidel, Dordrecht (1986)
2. Capatina, A., Lebon, F.: Remarks on the equilibrium finite element method for frictional contact problems. In: Mihailescu-Suliciu, M., (ed.) New Trends in Continuum Mechanics, The Theta Foundation, Conference proceedings, Constanta, 2003, pp. 25–33 (2005)
3. Capuzzo-Dolcetta, I., Matzeu, M.: Duality for implicit variational problems and numerical applications. Numer. Funct. Anal. Optim. **2**, 231–265 (1980)
4. Céa, J.: Optimisation, Théorie et Algorithmes. Dunod, Paris (1971)
5. Ekeland, I., Temam, R.: Analyse Convexe et Problèmes Variationnells. Dunod/Gauthier-Villars, Paris (1974)
6. Fenchel, W.: On conjugate convex functions. Canad. J. Math. **1**, 73–77 (1949)
7. Fan, K.: Minimax theorems. Proc. Natl. Acad. Sci. USA **39**, 42–47 (1953)
8. Laurent, P.J.: Approximation et Optimisation. Hermann, Paris (1972)
9. Moreau, J.J.: Proximité et dualité dans un espace hilbertien. Bull. Soc. Math. France **93**, 273–299 (1965)
10. Moreau, J.J.: Fonctionnelles convexes. Sém. sur les équations aux dérivées partielles, Collége de France (1967)
11. Rockafellar, R.T.: Convex functions and duality in optimization problems and dynamics. Lecture Notes in Operation Research and Mathematical Economics 11/12, pp. 117–141, Springer, Berlin (1969)
12. Rockafellar, R.T.: Convex Analysis. Princeton Landmarks in Mathematics and Physics series (1997)
13. Sion, M.: On general minimax theorems. Pac. J. Math. **8**, 171–176 (1958)
14. Telega, J.J.: Topics on unilateral contact problems of elasticity and inelasticity. In: Moreau, J.J., Panagiotopoulos P.D., (eds.) Nonsmooth Mechanics and Applications; CSIM Lect. Notes, vol. 302, pp. 341–462, Springer, Wien-New York (1988)
15. Telega, J.J.: Quasi-static Signorini's contact problem with friction and duality. Int. Ser. Numer. Math. **101**, 199–214 (1991)