Chapter 2
Spaces of Real-Valued Functions

This chapter is a brief background on spaces of continuous functions and some Sobolev spaces including basic properties, embedding theorems and trace theorems. Hence, we recall some classical definitions and theorems of functional analysis which will be used throughout this book. These results are standard and so they are stated without proofs; for more details and proofs, we refer the readers to the monographs [1, 3–7, 10, 11, 14].

In this book we only deal with real-valued functions. We assume that the reader is familiar with the basic concepts of general topology and functional analysis.

For a point $\boldsymbol{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we denote by D_i the differential operator $\dfrac{\partial}{\partial x_i}$ $(1 \leq i \leq d)$.

If $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d)$ is a multi-index, then $D^{\boldsymbol{\alpha}}$ denotes the differential operator of order $\boldsymbol{\alpha}$, with $|\boldsymbol{\alpha}| = \displaystyle\sum_{i=1}^{d} \alpha_i$, defined by

$$D^{\boldsymbol{\alpha}} = D_1^{\alpha_1} \cdots D_d^{\alpha_d} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

Obviously, D_i^0 denotes the identity operator.

If $A \subset \mathbb{R}^d$, we denote by $C(A)$ the space of real continuous functions on A.

Let Ω be an open set in $\mathbb{R}^d$ with its boundary Γ. We denote by $\overline{\Omega} = \Omega \cup \Gamma$ the closure of Ω.

For any nonnegative integer m, let $C^m(\Omega)$, respectively $C^m(\overline{\Omega})$, be the space of real functions which, together with all their partial derivatives of orders $\boldsymbol{\alpha}$, with $|\boldsymbol{\alpha}| \leq m$, are continuous on Ω, respectively, on the closure $\overline{\Omega}$ of Ω in $\mathbb{R}^d$, i.e.

$$C^m(\Omega) = \{v \in C(\Omega);\ D^{\boldsymbol{\alpha}} v \in C(\Omega) \text{ for } |\boldsymbol{\alpha}| \leq m\}. \tag{2.1}$$

A. Capatina, *Variational Inequalities and Frictional Contact Problems*,
Advances in Mechanics and Mathematics 31, DOI 10.1007/978-3-319-10163-7_2

When $m = 0$, we abbreviate $C(\Omega) \equiv C^0(\Omega)$ and $C(\overline{\Omega}) \equiv C^0(\overline{\Omega})$. Any function in $C(\overline{\Omega})$ is bounded and uniformly continuous on Ω, thus it possesses a unique, bounded, and continuous extension to $\overline{\Omega}$.

Let

$$C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega)$$

be the space of infinitely differentiable functions on Ω.

If K is a subset of Ω, we shall write $K \subset\subset \Omega$ if $\overline{K} \subset \Omega$ and $\overline{K}$ is a compact (i.e., bounded and closed) subset of $\mathbb{R}^d$.

The support of a function $v : \Omega \to \mathbb{R}$ is defined as the closed subset

$$\text{supp}\, v = \overline{\{\boldsymbol{x} \in \Omega\, ;\; v(\boldsymbol{x}) \neq 0\}}\,. \tag{2.2}$$

We shall say that a function v has compact support in Ω if there exists a compact subset K of Ω such that $v(\boldsymbol{x}) = 0\ \forall \boldsymbol{x} \in \Omega \backslash K$ or, equivalently, $\text{supp}\, v \subset\subset \Omega$.

We shall denote by $C_0^m(\Omega)$ the subspace of $C^m(\Omega)$ consisting of all those functions which have compact support in Ω.

If $m < +\infty$ and Ω is bounded, then $C^m(\overline{\Omega})$ is a Banach space with the norm given by

$$\|v\|_{C^m(\overline{\Omega})} = \sum_{|\alpha| \le m} \max_{x \in \overline{\Omega}} |D^{\alpha} v(\boldsymbol{x})|\,. \tag{2.3}$$

In the sequel, for $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ two normed spaces with $X \subset Y$, we shall write $X \hookrightarrow Y$ to designate the continuously embedding of X in Y provided the identity operator $I : X \to Y$ is continuous. This is equivalent, since I is linear, to the existence of a constant C such that

$$\|u\|_Y \le C\,\|u\|_X \qquad \forall u \in X\,.$$

We also say that the normed space X is compactly embedded in the normed space Y and write $X \hookrightarrow_c Y$ if the identity operator I is compact, i.e. every bounded sequence in X has a subsequence converging in Y, or, equivalently, if $\{u_k\}_k$ is a sequence which converges weakly to u in X, and we write $u_k \rightharpoonup u$, then $\{u_k\}_k$ converges strongly to u in Y, and we write $u_k \to u$.

We denote by $L^p(\Omega)$, for $1 \le p < +\infty$, the space of (equivalence classes of) real functions v defined on Ω with the p-power absolutely integrable, i.e.

$$\int_{\Omega} |v(\boldsymbol{x})|^p \,\mathrm{d}x < \infty\,,$$

where $\mathrm{d}x = \mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_d$ is the Lebesque measure. The elements of $L^p(\Omega)$, being equivalence classes of measurable functions, are identical if they are equal almost everywhere (a.e.) on Ω. Thus, we write $v = 0$ in $L^p(\Omega)$ if $v(x) = 0$ a.e. $x \in \Omega$.

We also denote by $L^\infty(\Omega)$ the space consisting of all (equivalence classes of) measurable real functions v that are essentially bounded on Ω, i.e. there exists a constant C such that $|v(\boldsymbol{x})| \leq C$ a.e. on Ω.

The space $L^p(\Omega)$ endowed with the norm

$$\|v\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |v(\boldsymbol{x})|^p \,\mathrm{d}x\right)^{1/p} & \text{if } 1 \leq p < +\infty \\ \operatorname{ess\,sup}_{\boldsymbol{x}\in\Omega} |v(\boldsymbol{x})| = \inf\{C;\ |v(\boldsymbol{x})| \leq C \text{ a.e. } \boldsymbol{x} \in \Omega\} & \text{if } p = +\infty \end{cases} \tag{2.4}$$

is a Banach space. In addition, the space $L^p(\Omega)$ is separable if $1 \leq p < +\infty$ and reflexive if $1 < p < +\infty$.

If $p \in [1, \infty]$, then the exponent conjugate to p is the number denoted by p' defined by the relation

$$\frac{1}{p} + \frac{1}{p'} = 1$$

where we used the convention

$$p' = \begin{cases} \infty & \text{if } p = 1\,, \\ 1 & \text{if } p = \infty\,. \end{cases}$$

From Riesz representation Theorem 4.1 for Hilbert spaces it follows that, for $p \in [1, +\infty)$, the dual space of $L^p(\Omega)$ is the space $(L^p(\Omega))' = L^{p'}(\Omega)$ where p' is the exponent conjugate to p. The dual space of $L^\infty(\Omega)$ is a space larger than $L^1(\Omega)$ (for more details, see [14, p. 118]).

In the case $p = 2$, the space $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(\boldsymbol{x})\, v(\boldsymbol{x})\,\mathrm{d}x\,. \tag{2.5}$$

Definition 2.1. We say that a measurable function v defined a.e. on Ω is locally p-integrable on Ω if $v \in L^p(A)$ for every measurable set $A \subset\subset \Omega$.

We shall denote by $L^p_{\mathrm{loc}}(\Omega)$ the space of all locally p-integrable functions on Ω.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set. The following assertions hold.*

1) Let $1 < p, q < \infty$.

If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^{\frac{pq}{p+q}}(\Omega)$.

If $u_n \to u$ in $L^p(\Omega)$ and $v_n \to v$ in $L^q(\Omega)$, then $u_n v_n \to uv$ in $L^{\frac{pq}{p+q}}(\Omega)$.

If $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ where p' is the exponent conjugate to p, then $uv \in L^1(\Omega)$ and the Hölder's inequality holds:

$$\int_\Omega u(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}x \le \|u\|_{L^p(\Omega)}\|v\|_{L^{p'}(\Omega)}\,. \tag{2.6}$$

When $p = p' = 2$, we get the Cauchy–Schwartz inequality.

2) *For $1 \le p \le \infty$, every Cauchy sequence in $L^p(\Omega)$ has a subsequence converging pointwise a.e. on Ω.*
3) $L^p(\Omega) \subset L^1_{\mathrm{loc}}(\Omega) \quad \forall\, p$ *with* $1 \le p \le \infty$.
4) *Let $v \in L^1_{\mathrm{loc}}(\Omega)$ be such that $\int_\Omega v(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}x = 0 \quad \forall\,\varphi \in \mathscr{D}(\Omega)$. Then $v(\boldsymbol{x}) = 0$ a.e. on Ω.*
5) *$C_0^\infty(\Omega)$ is dense in $L^p(\Omega) \quad \forall\, p$ with $1 \le p < \infty$.*

The following theorem gives an embedding result for the spaces $L^p(\Omega)$ and some of its consequences.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^d$ be an open set with vol $(\Omega) = \int_\Omega \mathrm{d}x < \infty$. Then the following statements are valid.*

1) *For all p, q such that $1 \le p \le q \le \infty$, we have $L^q(\Omega) \hookrightarrow L^p(\Omega)$ and*

$$\|v\|_{L^p(\Omega)} \le (vol\ \Omega)^{\frac{1}{p}-\frac{1}{q}}\|v\|_{L^q(\Omega)} \quad \forall\, v \in L^q(\Omega)\,.$$

2) $\lim_{p\to\infty} \|v\|_{L^p(\Omega)} = \|v\|_{L^\infty(\Omega)} \quad \forall v \in L^\infty(\Omega)$.
3) *Suppose that $v \in L^p(\Omega)$ for any $1 \le p < \infty$ and that there exists a constant C such that $\|v\|_{L^p(\Omega)} \le C$. Then $v \in L^\infty(\Omega)$.*

To better understand what is the meaning of the differential operator $D^{\boldsymbol{\alpha}}v$ for functions v whose derivatives do not exist in the classical sense, we briefly remind the definition of distributions on Ω.

We denote by $\mathscr{D}(\Omega)$, called the space of test functions, the space $C_0^\infty(\Omega)$ equipped with the inductive limit topology as in the Schwartz theory of distributions [11].

Definition 2.2. A sequence $\{\varphi_k\}_k \subset C_0^\infty(\Omega)$ is said to converge to a function $\varphi \in C_0^\infty(\Omega)$ in (the sense of the space) $\mathscr{D}(\Omega)$, provided the following conditions are satisfied:

i) There exists a compact subset K of Ω such that supp $(\varphi_k - \varphi) \subset K\,,\ \forall k$
ii) $D^{\boldsymbol{\alpha}}\varphi_k \to D^{\boldsymbol{\alpha}}\varphi$ uniformly on $K\,,\ \forall \boldsymbol{\alpha}$ multi-index .

The dual space $\mathscr{D}'(\Omega)$ of $\mathscr{D}(\Omega)$ is called the space of (Schwartz) distributions (or, generalized functions). Hence, any distribution T is a linear and continuous functional on $\mathscr{D}(\Omega)$, i.e. $T(\varphi_k) \to T(\varphi)$ in $\mathbb{R}$ whenever $\varphi_k \to \varphi$ in $\mathscr{D}(\Omega)$. As dual of $\mathscr{D}(\Omega)$, the space $\mathscr{D}'(\Omega)$ is equipped with the weak-star topology: $T_k \to T$ in $\mathscr{D}'(\Omega)$ if and only if $T_k(\varphi) \to T(\varphi)$ in $\mathbb{R}$, for every $\varphi \in \mathscr{D}(\Omega)$.

Every distribution is infinitely differentiable in the following sense: if $T \in \mathscr{D}'(\Omega)$ then, for all multi-index $\boldsymbol{\alpha}$, the function $D^{\boldsymbol{\alpha}}T$ defined on $\mathscr{D}(\Omega)$ by

$$D^{\boldsymbol{\alpha}}T(\varphi) = (-1)^{|\boldsymbol{\alpha}|}T(D^{\boldsymbol{\alpha}}\varphi) \quad \forall\varphi \in \mathscr{D}(\Omega)\,. \tag{2.7}$$

is a distribution. In addition, the operator $D^{\boldsymbol{\alpha}}$ from $\mathscr{D}'(\Omega)$ into $\mathscr{D}'(\Omega)$ is continuous.

Any function $u \in L^1_{\text{loc}}(\Omega)$ generates a distribution $T_u \in \mathscr{D}'(\Omega)$ defined by

$$T_u(\varphi) = \int_\Omega u(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}x \quad \forall\varphi \in \mathscr{D}(\Omega)\,. \tag{2.8}$$

Therefore, for any multi-index $\boldsymbol{\alpha}$, there exists the $\boldsymbol{\alpha}$-th derivative of T_u, namely the distribution $D^{\alpha}T_u \in \mathscr{D}'(\Omega)$ defined by (2.7), i.e.

$$D^{\boldsymbol{\alpha}}T_u(\varphi) = (-1)^{|\boldsymbol{\alpha}|}T_u(D^{\boldsymbol{\alpha}}\varphi) \quad \forall\varphi \in \mathscr{D}(\Omega)\,.$$

But not any distribution is generated by a locally integrable function.

Definition 2.3. We shall say that the function $u \in L^1_{\text{loc}}(\Omega)$ possesses the distributional (or generalized or weak) partial derivative of order $\boldsymbol{\alpha}$ on Ω, denoted by $D^{\boldsymbol{\alpha}}u$, if there exists a function $v_\alpha \in L^1_{\text{loc}}(\Omega)$ which generates the distribution $D^{\alpha}T_u \in \mathscr{D}'(\Omega)$, i.e.

$$D^{\boldsymbol{\alpha}}T_u = T_{v_\alpha}\,.$$

Thus, from the last three relations, it follows that $D^{\boldsymbol{\alpha}}u = v_\alpha$ is the distributional partial derivative of u if $v_\alpha \in L^1_{\text{loc}}(\Omega)$ satisfies

$$\int_\Omega u(\boldsymbol{x})\, D^{\boldsymbol{\alpha}}\varphi(\boldsymbol{x})\,\mathrm{d}x = (-1)^{|\boldsymbol{\alpha}|}\int_\Omega v_\alpha\, \varphi(\boldsymbol{x})\,\mathrm{d}x \quad \forall\varphi \in \mathscr{D}(\Omega)\,. \tag{2.9}$$

Obviously, the distributional derivative is uniquely defined up to a set of measure zero.

In fact, this definition generalizes the classical partial derivative, obtained, for a function $u \in C^{|\boldsymbol{\alpha}|}(\Omega)$, by integrating by parts $|\boldsymbol{\alpha}|$ times

$$\int_\Omega D^{\boldsymbol{\alpha}}u(\boldsymbol{x})\,\varphi(\boldsymbol{x})\,\mathrm{d}x = (-1)^{|\boldsymbol{\alpha}|}\int_\Omega u(\boldsymbol{x})\, D^{\boldsymbol{\alpha}}\varphi(\boldsymbol{x})\,\mathrm{d}x \quad \forall\varphi \in \mathscr{D}(\Omega)\,. \tag{2.10}$$

Of course, in this case, $D^{\boldsymbol{\alpha}} u$ is also a distributional partial derivative of u. However, it should be noted that the derivative in the sense of distributions of a function, even sufficiently smooth, may exist, even if it does not exist in the classical sense.

In particular, the relation (2.8) brings out a linear and continuous mapping $u \mapsto T_u$ from $L^p(\Omega)$ into $\mathscr{D}'(\Omega)$ and so, we may identify the distribution T_u with the integrable function u. The same identification may be made for $\mathscr{D}(\Omega)$. Thus, we have

$$\mathscr{D}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathscr{D}'(\Omega) .$$

Using this result and the definition (2.9), Sobolev [12] expanded in a natural way the space $L^p(\Omega)$ by considering those functions which, for some nonnegative integer m, possess distributional partial derivatives of all orders $|\boldsymbol{\alpha}| \leq m$ in $L^p(\Omega)$. This is the definition of the Sobolev space

$$W^{m,p}(\Omega) = \{v\, ;\; D^{\boldsymbol{\alpha}} v \in L^p(\Omega)\, , \;\; \text{for } |\boldsymbol{\alpha}| \leq m\} .$$

The space $W^{m,p}(\Omega)$ is a Banach space with the norm

$$\|v\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\displaystyle\sum_{|\boldsymbol{\alpha}| \leq m} \|D^{\boldsymbol{\alpha}} v\|^p_{L^p(\Omega)} \right)^{1/p} & \text{if } p \in [1, \infty)\, , \\ \displaystyle\max_{|\boldsymbol{\alpha}| \leq m} \|D^{\boldsymbol{\alpha}} v\|_{L^\infty(\Omega)} & \text{if } p = \infty\, . \end{cases} \tag{2.11}$$

Obviously, $W^{0,p}(\Omega) = L^p(\Omega)$ for $p \in [1, \infty)$. The seminorm over $W^{m,p}(\Omega)$ is defined by

$$|v|_{W^{m,p}(\Omega)} = \begin{cases} \left(\displaystyle\sum_{|\boldsymbol{\alpha}| = m} \|D^{\boldsymbol{\alpha}} v\|^p_{L^p(\Omega)} \right)^{1/p} & \text{if } p \in [1, \infty)\, , \\ \displaystyle\max_{|\boldsymbol{\alpha}| = m} \|D^{\boldsymbol{\alpha}} v\|_{L^\infty(\Omega)} & \text{if } p = \infty\, . \end{cases} \tag{2.12}$$

We denote by $W_0^{m,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$ for the norm $\|\cdot\|_{W^{m,p}(\Omega)}$. For $p \in [1, \infty)$, we have the following chain of embeddings

$$W_0^{m,p}(\Omega) \hookrightarrow W^{m,p}(\Omega) \hookrightarrow L^p(\Omega)$$

and, since $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$, it is clear that $W_0^{0,p}(\Omega) = L^p(\Omega)$.

It is easy to see that, if the open set Ω is bounded, the seminorm $|\cdot|_{W^{m,p}(\Omega)}$ is a norm over $W_0^{m,p}(\Omega)$ equivalent to the norm $\|\cdot\|_{W^{m,p}(\Omega)}$.

In the case $p = 2$, we use the notation

$$H^m(\Omega) = W^{m,2}(\Omega) .$$

Endowed with the scalar product

$$(u, v)_{H^m(\Omega)} = \sum_{\alpha \le m} (D^{\alpha} u, D^{\alpha} v)_{L^2(\Omega)}\,, \tag{2.13}$$

the Sobolev space $H^m(\Omega)$ is a Hilbert space. Also we denote $H_0^m(\Omega) = W_0^{m,2}(\Omega)$.

If Ω is bounded, then, without any hypothesis on the regularity of Ω, we have

$$H_0^1(\Omega) \hookrightarrow_c L^2(\Omega)\,.$$

Many different symbols are being used to denote these norms, when no confusion may occur: $\|\cdot\|_{m,p,\Omega}$ or $\|\cdot\|_{m,p}$ instead of $\|\cdot\|_{W^{m,p}(\Omega)}$, $\|\cdot\|_{m,\Omega}$ or $\|\cdot\|_m$ instead of $\|\cdot\|_{H^m(\Omega)}$ and $\|\cdot\|_{0,\Omega}$ or $\|\cdot\|_0$ instead of $\|\cdot\|_{L^2(\Omega)}$.

If $m \ge 1$ and $1 \le p < \infty$, we denote by $W^{-m,p'}(\Omega)$ the dual space of $W_0^{m,p}(\Omega)$, p' being the exponent conjugate to p (in fact, $W^{-m,p'}(\Omega)$ is the notation for a space of some distributions on Ω which is isometrically isomorphic to the dual space $(W_0^{m,p}(\Omega))'$; for details, see [1]). Endowed with the norm

$$\|f\|_{W^{-m,p'}(\Omega)} = \sup_{\substack{u \in W_0^{m,p}(\Omega) \\ u \neq 0}} \frac{\langle f, u\rangle}{\|u\|_{W^{m,p}(\Omega)}}\,,$$

the space $W^{-m,p'}(\Omega)$ is a Banach space which is separable and reflexive if $1 < p < \infty$. Here $\langle\cdot,\cdot\rangle$ is the duality pairing between $W^{-m,p'}(\Omega)$ and $W_0^{m,p}(\Omega)$.

We note that if X, Y are two Hilbert spaces such that $X \hookrightarrow Y$ dense, then (see, for instance, [2, p. 51]) $Y^* \hookrightarrow X^*$ dense, where Y^* and X^* denote their dual spaces.

If Ω is bounded, then $\mathscr{D}(\Omega)$ is dense in $H_0^m(\Omega)$, and so, we can identify the dual space $H^{-m}(\Omega)$ of $H_0^m(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$:

$$\mathscr{D}(\Omega) \subset H_0^m(\Omega) \subset L^2(\Omega) \subset H^{-m}(\Omega) \subset \mathscr{D}'(\Omega)\,.$$

Now, we notice that most of the important results involving Sobolev spaces are first obtained for regular functions and then extended to Sobolev spaces. The density theorems and the embedding theorems show how and whether an element of a Sobolev space can be approximated by smooth functions. Since these theorems require additional regularity properties for the open set Ω, we recall some definitions of them. Later, in Chaps. 5 and 8, we will use some of these assumptions on Ω for getting regularity properties of the solutions of some concrete variational inequalities.

Definition 2.4. We say that the open subset Ω of $\mathbb{R}^d$ has the cone property if there exists a finite open bounded cover $\{O_j\}_{j\in J}$ of the boundary Γ of Ω and, for any j, there exists a cone C_j with the vertex at 0, such that, for all $\boldsymbol{x} \in O_j \cap \Omega$, $\boldsymbol{x} + C_j$ do not intersect $O_j \cap \Gamma$.

Definition 2.5. We say that the open set $\Omega \subset \mathbb{R}^d$ has the segment property if there exists a locally finite open cover $\{U_j\}_j$ of the boundary Γ of Ω and a corresponding sequence $\{\boldsymbol{y}_j\}_j$ of nonzero vectors such that if $\boldsymbol{x} \in \overline{\Omega} \cap U_j$ for some j, then $\boldsymbol{x} + t\boldsymbol{y}_j \in \Omega$ for $0 < t < 1$. In this case, Ω must have $(d-1)$-dimensional boundary and cannot simlutaneously lie on both sides of its boundary.

Definition 2.6. Let $r \geq 1$ an integer. An open bounded set $\Omega \subset \mathbb{R}^d$ is said to be $\mathscr{C}^r$-smooth (or, of class $\mathscr{C}^r$) if there exists a covering of the boundary Γ of Ω by a finite number of bounded open subsets $\{U_j\}_{j\in J} \subset \mathbb{R}^d$ and, for any $j \in J$, there exists a C^r-homeomorphisms θ_j such that:

(i) $\theta_j(U_j) = S = \{\boldsymbol{y} = (\boldsymbol{y}', y_d) \in \mathbb{R}^d;\ |\boldsymbol{y}'| < 1,\ |y_d| < 1\}$,
(ii) $\theta_j(U_j \cap \Omega) = S_+ = \{\boldsymbol{y} \in S;\ y_d > 0\}$,
(iii) $\theta_j(U_j \cap \Gamma) = S_0 = \{\boldsymbol{y} \in S;\ y_d = 0\}$.

Concerning the approximation by smooth functions, we have the following results (see, for instance, [13, p. 11], [9, p. 44], or [8, p. 40]).

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set. Then, the following approximation results are true.*

1) $C_0^\infty(\Omega)$ is dense in $W_0^{m,p}(\Omega)$.
2) If Ω has the cone property, then $C^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$.
3) If Ω is $\mathscr{C}^\infty$-smooth, then $\mathscr{D}(\overline{\Omega})$ is dense in $H^m(\Omega)$.

We now recall the following Sobolev embedding theorem (see [1,9,13] for more details and proofs) which will be used frequently in this book.

Theorem 2.4 (Sobolev Embedding Theorem). *Suppose that the open bounded set Ω has the cone property and $1 \leq p < \infty$. Then, the following assertions hold.*

1) If $mp < d$, then

i) $W^{m,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ where $p^* = \dfrac{\mathrm{d}p}{\mathrm{d} - mp}$.
ii) $W^{m,p}(\Omega) \hookrightarrow_c L^q(\Omega)$ for any q with $1 \leq q < p^*$.

2) If $mp = d$, then
$W^{m,p}(\Omega) \hookrightarrow_c L^q(\Omega)$ for any $1 \leq q < \infty$.
3) If $mp > d$, then

i) $W^{m,p}(\Omega) \hookrightarrow_c L^q(\Omega)$ for any $1 \leq q < \infty$.
ii) $W^{m,p}(\Omega) \hookrightarrow C^k(\overline{\Omega})$ for any integer k with $\dfrac{mp-d}{p} - 1 \leq k < \dfrac{mp-d}{p}$.

As a consequence of this theorem we have the following particular cases that we shall often use:

$$
\begin{aligned}
&H^1(\Omega) \hookrightarrow_c C(\overline{\Omega}) \quad \text{if } d = 1\,,\\
&H^1(\Omega) \hookrightarrow_c L^q(\Omega) \quad \text{where} \quad \begin{cases} q \in [1,\infty) & \text{if } d = 2\,,\\ q = 6 & \text{if } d = 3\,,\end{cases}\\
&H^2(\Omega) \hookrightarrow_c C(\overline{\Omega}) \quad \text{if } d \in \{1,2\}\,.
\end{aligned}
$$

We note that a function $v \in H^1(\Omega)$ is not necessary continuous on Ω, neither on $\overline{\Omega}$, and so, we may not define, in the classical sense, the values of v on the boundary Γ of Ω. The trace theorems show how one can define, in the trace sense, the restriction on the boundary Γ of a function which is not necessary continuous. Their purpose is to determine the space of functions defined on the boundary Γ of Ω containing the traces of functions in $W^{m,p}(\Omega)$.

The next theorem (see [8, p. 40] or [13, p. 9]) allows to define every function $v \in H^1(\Omega)$ almost everywhere on Γ.

Theorem 2.5 (Trace Theorem for $H^1(\Omega)$). *Let Ω be an open bounded set in $\mathbb{R}^d$ of class $\mathscr{C}^1$ with its boundary Γ. Then, one can uniquely define the trace $\gamma_0 v$ of $v \in H^1(\Omega)$ on Γ such that $\gamma_0 v$ coincides with the usual definition*

$$\gamma_0 v(x) = v(x) \quad x \in \Gamma\,, \tag{2.14}$$

if $v \in C^1(\overline{\Omega})$. Moreover, the mapping $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$ is linear continuous and the range of $\gamma_0(H^1(\Omega))$ is a space smaller than $L^2(\Gamma)$ denoted by $H^{1/2}(\Gamma)$.

Now, if $v \in C^m(\overline{\Omega})$, let $\boldsymbol{\gamma} v$ be the linear mapping defined by

$$\boldsymbol{\gamma} v = (\gamma_0 v, \gamma_1 v, \dots, \gamma_{m-1} v)$$

where $\gamma_0 v$ is "the trace of v" on Γ and $\gamma_j v,\ j = 1,\dots,m-1$ is "the trace of order j of v"' defined as the j-th order derivative in the direction of the outward unit normal $\boldsymbol{\nu}$ to Γ, i.e.

$$
\begin{aligned}
\gamma_0 v(x) &= v(x) \quad x \in \Gamma\,,\\
\gamma_j v(x) &= \frac{\partial^j v}{\partial \nu^j}(x) \quad x \in \Gamma\,.
\end{aligned} \tag{2.15}
$$

The problem of characterizing the image of the space $H^m(\Omega)$ under the trace operator involves Sobolev spaces of fractional order. These spaces can be defined in different ways but, for Ω sufficiently smooth, these definitions give the same space.

A compact definition (see [1]) of $H^s(\Omega)$, for s a real number, is the space obtained as the closure of $C^\infty(\Omega)$ in the norm

$$\|v\|^2_{H^s(\Omega)} = \|v\|^2_{H^{[s]}(\Omega)} + \sum_{|\boldsymbol{\alpha}|=[s]} \int_{\Omega\times\Omega} \frac{|D^{\boldsymbol{\alpha}} v(\boldsymbol{y}) - D^{\boldsymbol{\alpha}} v(\boldsymbol{x})|^2}{|\boldsymbol{y}-\boldsymbol{x}|^{d+2\{s\}}}\,\mathrm{d}y\,\mathrm{d}x\,,$$

where $s = [s] + \{s\}$ with $[s]$ an integer and $0 < \{s\} < 1$.

Another approach (see [2, 10]) for the definition of the space $H^s(\Omega)$ uses the Fourier transformation of a function.

Definition 2.7. The Fourier transformation of a function $v \in L^2(\mathbb{R}^d)$ is the function $\hat{v} \in L^2(\mathbb{R}^d)$ defined by

$$\hat{v}(\boldsymbol{y}) = \frac{1}{(2\pi)^{d/2}} \int\limits_{\mathbb{R}^d} v(\boldsymbol{x}) \exp(-i\, \boldsymbol{x}\, \boldsymbol{y})\, \mathrm{d}x$$

where $i = \sqrt{-1}$.

For any real number $s \geq 0$, the space $\mathscr{S}(\mathbb{R}^d)$, of test functions of rapid decay, is defined by:

$$\mathscr{S}(\mathbb{R}^d) = \{\varphi\,;\; \boldsymbol{x}^{\boldsymbol{\alpha}} D^{\boldsymbol{\beta}} \varphi \in L^2(\mathbb{R}^d)\,,\; \forall \boldsymbol{\alpha},\, \boldsymbol{\beta} \text{ multi-indices}\}\,,$$

and $\mathscr{S}(\mathbb{R}^d)'$, called the space of temperated distributions, is the dual space of $\mathscr{S}(\mathbb{R}^d)$.

Then, the fractional order Sobolev space $H^s(\mathbb{R}^d)$ is defined by

$$H^s(\mathbb{R}^d) = \{v \in \mathscr{S}(\mathbb{R}^d)'\,;\; (1 + |\boldsymbol{y}|^2)^{s/2} \hat{v} \in L^2(\mathbb{R}^d)\}$$

with the norm

$$\|v\|_{H^s(\mathbb{R}^d)} = \|(1 + |\boldsymbol{y}|^2)^{s/2} \hat{v}\|_{L^2(\mathbb{R}^d)}\,.$$

If $s < 0$, one denotes by $H^s(\mathbb{R}^d)$ the dual space of $H^{-s}(\mathbb{R}^d)$.

If Ω is sufficiently smooth, then we define $H^s(\Omega)$ to be the space of restrictions to Ω of functions of $H^s(\mathbb{R}^d)$. The boundary Γ of Ω can be identified, by means of local coordinates, to $\mathbb{R}^{d-1}$, and we can define $H^s(\Gamma)$ to be isomorphic to the Sobolev space $H^s(\mathbb{R}^{d-1})$.

If the open bounded set Ω of $\mathbb{R}^d$ is $\mathscr{C}^\infty$-smooth, then $\mathscr{D}(\overline{\Omega})$ is dense in $H^m(\Omega)$ and so, it is possible to extend by continuity the classical definition (2.15) to a generalized one $\boldsymbol{\gamma} v$ for $v \in H^m(\Omega)$ (see, for instance, [9, p. 44], [10, p. 142]).

Theorem 2.6 (Trace Theorem for $H^m(\Omega)$). *Suppose that the open bounded set Ω is C^∞-smooth. Then, for any $m > 0$ integer, the trace operator*

$$\boldsymbol{\gamma} : \mathscr{D}(\overline{\Omega}) \to (\mathscr{D}(\Gamma))^m$$

can be extended to the continuous linear and surjective operator

$$\boldsymbol{\gamma} : H^m(\Omega) \to \prod_{j=0}^{m-1} H^{m-j-1/2}(\Gamma)\,. \tag{2.16}$$

Moreover, there exists a continuous linear inverse operator

$$\boldsymbol{\gamma}^{-1} : \prod_{j=0}^{m-1} H^{m-j-1/2}(\Gamma) \to H^m(\Omega)$$

such that

$$\boldsymbol{\gamma}_j(\boldsymbol{\gamma}^{-1}\mathbf{g}) = g_j \quad 0 \le j \le m-1\,, \quad \forall \mathbf{g} \in \prod_{j=0}^{m-1} H^{m-j-1/2}(\Gamma)\,.$$

Therefore, the space $H^{m-j-1/2}(\Gamma)$ can be seen as the space of traces of order j of $H^m(\Omega)$. In addition, the kernel of the operator $\boldsymbol{\gamma}$ is the space $H_0^m(\Omega)$, the completion of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{H^m(\Omega)}$.

Finally, we recall the following result (see [1, p. 114]).

Theorem 2.7. *Suppose that* Ω *is sufficiently smooth. Then*

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Gamma)$$

where $q = \dfrac{\mathrm{d}p - p}{\mathrm{d} - mp}$ *if* $mp < d$, *and* $1 \le q < \infty$ *if* $mp = d$.

In particular, we have the following frequently useful results.

$$H^1(\Omega) \hookrightarrow L^q(\Gamma) \quad \text{where} \quad \begin{cases} q \in [1,\infty) & \text{if } d = 2\,, \\ q = \dfrac{2(n-1)}{n-2} & \text{if } d \ge 3\,. \end{cases}$$

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