Well-Ordering Principles and Bar Induction

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Abstract In this paper we show that the existence of ω-models of bar induction is equivalent to the principle saying that applying the Howard–Bachmann operation to any well-ordering yields again a well-ordering.

1 Introduction

This paper will be concerned with a particular Π^1_2 statement of the form

$$\mathbf{WOP}(f): \qquad \forall X\, [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(f(\mathfrak{X}))] \tag{1}$$

where f is a standard proof-theoretic function from ordinals to ordinals and $\mathrm{WO}(\mathfrak{X})$ stands for '$\mathfrak{X}$ is a well-ordering'. There are by now several examples of functions f familiar from proof theory where the statement $\mathbf{WOP}(f)$ has turned out to be equivalent to one of the theories of reverse mathematics over a weak base theory (usually $\mathbf{RCA}_0$). The first explicit example appears to be due to Girard [7, Theorem 5.4.1] (see also [8]). However, it is also implicit in Schütte's proof of cut elimination for ω-logic [15] and ultimately has its roots in Gentzen's work, namely in his first unpublished consistency proof,[1] where he introduced the notion of a "Reduziervorschrift" [6, p. 102] for a sequent. The latter is a well-founded tree built bottom-up via "Reduktionsschritte", starting with the given sequent and passing up from conclusions to premises until an axiom is reached.

Theorem 1.1 *Over* $\mathbf{RCA}_0$ *the following are equivalent:*

(i) *Arithmetical comprehension.*
(ii) $\forall \mathfrak{X}\, [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(2^{\mathfrak{X}})]$.

[1] The original German version was finally published in 1974 [6]. An earlier English translation appeared in 1969 [5].

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Another characterization from [7, Theorem 6.4.1], shows that arithmetical comprehension is equivalent to Gentzen's Hauptsatz (cut elimination) for ω-logic. Connecting statements of form (1) to cut elimination theorems for infinitary logics will also be a major tool in this paper.

There are several more recent examples of such equivalences that have been proved by recursion-theoretic as well as proof-theoretic methods. These results give characterizations of the form (1) for the theories $\mathbf{ACA}_0^+$ and $\mathbf{ATR}_0$, respectively, in terms of familiar proof-theoretic functions. $\mathbf{ACA}_0^+$ denotes the theory $\mathbf{ACA}_0$ augmented by an axiom asserting that for any set X the ω-th jump in X exists while $\mathbf{ATR}_0$ asserts the existence of sets constructed by transfinite iterations of arithmetical comprehension. $\alpha \mapsto \varepsilon_\alpha$ denotes the usual ε function while φ stands for the two-place Veblen function familiar from predicative proof theory (cf. [16]). Definitions of the familiar subsystems of reverse mathematics can be found in [17].

Theorem 1.2 (Afshari and Rathjen [1]; Marcone and Montalbán [9]) *Over* $\mathbf{RCA}_0$ *the following are equivalent:*

(i) $\mathbf{ACA}_0^+$.
(ii) $\forall \mathfrak{X}\,[\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\varepsilon_{\mathfrak{X}})]$.

Theorem 1.3 (Friedman [4]; Rathjen and Weiermann [13]; Marcone and Montalbán [9]) *Over* $\mathbf{RCA}_0$ *the following are equivalent:*

(i) $\mathbf{ATR}_0$.
(ii) $\forall \mathfrak{X}\,[\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\varphi\mathfrak{X}0)]$.

There is often another way of characterizing statements of the form (1) by means of the notion of countable coded ω-model.

Definition 1.4 Let T be a theory in the language of second order arithmetic, $\mathcal{L}_2$. A *countable coded ω-model of T* is a set $W \subseteq \mathbb{N}$, viewed as encoding the $\mathcal{L}_2$-model

$$\mathbb{M} = (\mathbb{N}, \mathcal{S}, \in, +, \cdot, 0, 1, <)$$

with $\mathcal{S} = \{(W)_n \mid n \in \mathbb{N}\}$ such that $\mathbb{M} \models T$ when the second order quantifiers are interpreted as ranging over $\mathcal{S}$ and the first order part is interpreted in the standard way (where $(W)_n = \{m \mid \langle n, m\rangle \in W\}$ with $\langle\,,\rangle$ being some primitive recursive coding function).

If T has only finitely many axioms, it is obvious how to express $\mathbb{M} \models T$ by just translating the second order quantifiers $QX \ldots X \ldots$ in the axioms by $Qx \ldots (W)_x \ldots$. If T has infinitely many axioms, one needs to formalize Tarski's truth definition for $\mathbb{M}$. This definition can be made in $\mathbf{RCA}_0$ as is shown in [17, Definitions II.8.3 and VII.2]. Some more details will be provided in Remark 1.9.

We write $X \in W$ if $\exists n\ X = (W)_n$.

The alternative characterizations alluded to above are as follows:

Theorem 1.5 *Over* $\mathbf{RCA}_0$ *the following are equivalent:*

(i) $\forall \mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\varepsilon_{\mathfrak{X}})]$ *is equivalent to the statement that every set is contained in a countable coded* ω*-model of* **ACA**.

(ii) $\forall \mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\varphi\mathfrak{X}0)]$ *is equivalent to the statement that every set is contained in a countable coded* ω*-model of* Δ^1_1-**CA** *(or* Σ^1_1-**DC***).*

Proof See [12, Corollary 1.8]. □

Whereas Theorem 1.5 has been established independently by recursion-theoretic and proof-theoretic methods, there is also a result that has a very involved proof and so far has only been shown by proof theory. It connects the well-known Γ-function (cf. [16]) with the existence of countable coded ω-models of $\mathbf{ATR}_0$.

Theorem 1.6 (Rathjen [12, Theorem 1.4]) *Over* $\mathbf{RCA}_0$ *the following are equivalent:*

(i) $\forall \mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\Gamma_{\mathfrak{X}})]$.

(ii) Every set is contained in a countable coded ω*-model of* $\mathbf{ATR}_0$.

The tools from proof theory employed in the above theorems involve search trees and Gentzen's cut elimination technique for infinitary logic with ordinal bounds. One could perhaps generalize and say that every cut elimination theorem in ordinal-theoretic proof theory encapsulates a theorem of this type.

The proof-theoretic ordinal functions that figure in the foregoing theorems are all familiar from so-called predicative or meta-predicative proof theory. Thus far a function from genuinely impredicative proof theory is missing. The first such function that comes to mind is of the Bachmann–Howard type. It was conjectured in [14] (Conjecture 7.2) that the pertaining principle (1) would be equivalent to the existence of countable coded ω-models of bar induction, **BI**. The conjecture is by and large true as will be shown in this paper, however, the relativization of the Bachmann–Howard construction allows for two different approaches, yielding principles of different strength. As it turned out, only the strongest one is equivalent to the existence of ω-models of **BI**. We now proceed to state the main result of this paper. Unexplained notions will be defined shortly.

Theorem 1.7 *Over* $\mathbf{RCA}_0$ *the following are equivalent:*

(i) $\mathbf{RCA}_0 +$ *Every set* X *is contained in a countable coded* ω*-model of* **BI**.

(ii) $\forall \mathfrak{X}\ [\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\vartheta_{\mathfrak{X}})]$.

Below we shall refer to Theorem 1.7 as the **Main Theorem**.

1.1 A Brief Outline of the Paper

Section 1.2 contains a detailed definition of the theory **BI**. Section 2 introduces a relativized version of the Howard–Bachmann ordinal representation system, i.e. given a well-ordering $\mathfrak{X}$, one defines a new well-ordering $\vartheta_{\mathfrak{X}}$ of Howard–Bachmann type which incorporates $\mathfrak{X}$. Section 3 proofs the direction $(i) \Rightarrow (ii)$ of Theorem 1.7. With Sect. 4 the proof of Theorem 1.7 $(ii) \Rightarrow (ii)$ commences. It introduces the crucial notion of a deduction chain for a given set $Q \subseteq \mathbb{N}$. The set of deduction chains forms a tree $\mathcal{D}_Q$. It is shown that from an infinite branch of this tree one can construct a countable coded ω-model of **BI** which contains Q. As a consequence, it remains to consider the case when $\mathcal{D}_Q$ does not contain an infinite branch, i.e. when $\mathcal{D}_Q$ is a well-founded tree. Then the Kleene–Brouwer ordering of $\mathcal{D}_Q$, $\mathfrak{X}$, is a well-ordering and, by the well-ordering principle (ii), $\vartheta_{\mathfrak{X}}$ is a well-ordering, too. It will then be revealed that $\mathcal{D}_Q$ can be viewed as a skeleton of a proof $\mathcal{D}^*$ of the empty sequent in an infinitary proof system T_Q^* with Buchholz' Ω-rule. However, with the help of transfinite induction over $\vartheta_{\mathfrak{X}}$ it can be shown that all cuts in $\mathcal{D}^*$ can be removed, yielding a cut-free derivation of the empty sequent. As this cannot be, the final conclusion reached is that $\mathcal{D}_Q$ must contain an infinite branch, whence there is a countable coded ω-model of **BI** containing Q, thereby completing the proof of Theorem 1.7 $(ii) \Rightarrow (i)$.

1.2 The Theory **BI**

In this subsection we introduce the theory **BI**. To set the context, we fix some notations. The language of second order arithmetic, $\mathcal{L}_2$, consists of free numerical variables $a, b, c, d, \ldots$, bound numerical variables $x, y, z, \ldots$, free set variables $U, V, W, \ldots$, bound set variables $X, Y, Z, \ldots$, the constant 0, a symbol for each primitive recursive function, and the symbols $=$ and $\in$ for equality in the first sort and the elementhood relation, respectively. The *numerical terms* of $\mathcal{L}_2$ are built up in the usual way; $r, s, t, \ldots$ are syntactic variables for them. *Formulas* are obtained from atomic formulas $s = t$, $s \in U$ and negated atomic formulas $\neg s = t, \neg s \in U$ by closing under $\wedge, \vee$ and quantification $\forall x, \exists x, \forall X, \exists X$ over both sorts; so we stipulate that formulas are in negation normal form.

The classes of Π^1_2- and Σ^1_n-*formulae* are defined as usual (with $\Pi^1_0 = \Sigma^1_0 = \cup\{\Pi^0_n : n \in \mathbb{N}\}$). $\neg A$ is defined by de Morgan's laws; $A \to B$ stands for $\neg A \vee B$. All theories in $\mathcal{L}_2$ will be assumed to contain the axioms and rules of classical two sorted predicate calculus, with equality in the first sort. In addition, it will be assumed that they comprise the system $\mathbf{ACA}_0$. $\mathbf{ACA}_0$ contains all axioms of elementary number theory, i.e. the usual axioms for 0, $'$ (successor), the defining equations for the primitive recursive functions, the *induction axiom*

$$\forall X\,[0 \in X \;\wedge\; \forall x(x \in X \to x' \in X) \to \forall x(x \in X)],$$

and all instances of *arithmetical comprehension*

$$\exists Z\,\forall x[x \in Z \leftrightarrow F(x)],$$

where $F(a)$ is an *arithmetic formula*, i.e. a formula without set quantifiers.

For a 2-place relation $\prec$ and an arbitrary formula $F(a)$ of $\mathcal{L}_2$ we define

$\mathrm{Prog}(\prec, F) := (\forall x)[\forall y(y \prec x \to F(y)) \to F(x)]$ (*progressiveness*)
$\mathbf{TI}(\prec, F) := \mathrm{Prog}(\prec, F) \to \forall x F(x)$ (*transfinite induction*)
$\mathrm{WF}(\prec) := \forall X\,\mathbf{TI}(\prec, X) :=$
$\forall X(\forall x[\forall y(y \prec x \to y \in X)) \to x \in X] \to \forall x[x \in X])$ (*well-foundedness*).

Let $\mathcal{F}$ be any collection of formulae of $\mathcal{L}_2$. For a 2-place relation $\prec$ we will write $\prec \in \mathcal{F}$, if $\prec$ is defined by a formula $Q(x, y)$ of $\mathcal{F}$ via $x \prec y := Q(x, y)$.

Definition 1.8 BI denotes the bar induction scheme, i.e. all formulae of the form

$$\mathrm{WF}(\prec) \to \mathbf{TI}(\prec, F),$$

where $\prec$ is an arithmetical relation (set parameters allowed) and F is an arbitrary formula of $\mathcal{L}_2$.

By **BI** we shall refer to the theory $\mathbf{ACA}_0 + \mathrm{BI}$.

Remark 1.9 The statement of the main Theorem 1.7 uses the notion of a countable coded ω-model of **BI**. As the stated equivalence is claimed to be provable in $\mathbf{RCA}_0$, a few comments on how this is formalized in this weak base theory are in order. The notion of a countable coded ω-model can be formalized in $\mathbf{RCA}_0$ according to [17, Definition VII.2.1]. Let $\mathbb{M}$ be a countable coded ω-model. Since **BI** is not finitely axiomatizable we have to quantify over all axioms of **BI** to express that $\mathbb{M} \models \mathbf{BI}$. The axioms of **BI** (or rather their Gödel numbers) clearly form a primitive recursive set, $Ax(\mathbf{BI})$. To express $\mathbb{M} \models \phi$ for $\phi \in Ax(\mathbf{BI})$ we use the notion of a *valuation for* ϕ from [17, Definition VII.2.1]. A valuation f for ϕ is a function from the set of subformulae of ϕ into the set $\{0, 1\}$ obeying the usual Tarski truth conditions. Thus we write $\mathbb{M} \models \phi$, if there exists a valuation f for ϕ such that $f(\phi) = 1$. Whence $\mathbb{M} \models \mathbf{BI}$ is defined by $\forall \phi \in Ax(\mathbf{BI})\ \mathbb{M} \models \phi$.

2 Relativizing the Howard–Bachmann Ordinal

In this section we show how to relativize the construction that leads to the Howard–Bachmann ordinal to an arbitrary countable well-ordering. To begin with, mainly to foster intuitions, we provide a set-theoretic definition working in **ZFC**. This will then be followed by a purely formal definition that can be made in $\mathbf{RCA}_0$.

Throughout this section, we fix a countable well-ordering $\mathfrak{X} = (X, <_X)$ without a maximum element, i.e., an ordered pair $\mathfrak{X} = (X, <_X)$, where X is a set of natural numbers, $<_X$ is a well-ordering relation on X, and $\forall v \in X\, \exists u \in X\; v <_X u$. We write $|\mathfrak{X}|$ for X.

Firstly, we need some ordinal-theoretic background. Let ON be the class of ordinals. Let AP $:= \{\xi \in \mathrm{ON} : \exists \eta \in \mathrm{ON}[\xi = \omega^\eta]\}$ be the class of additive principal numbers and let E $:= \{\xi \in \mathrm{ON} \colon \xi = \omega^\xi\}$ be the class of ε-numbers which is enumerated by the function $\lambda\xi.\varepsilon_\xi$.

We write $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\alpha > \alpha_1 \geq \cdots \alpha_n$. Note that by Cantor's normal form theorem, for every $\alpha \notin \mathrm{E} \cup \{0\}$, there are uniquely determined ordinals $\alpha_1, \ldots, \alpha_n$ such that $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$.

Let $\Omega := \aleph_1$. For $u \in |\mathfrak{X}|$, let $\mathfrak{E}_u$ be the u^{th} ε-number $> \Omega$. Thus, if u_0 is the smallest element of $|\mathfrak{X}|$, then $\mathfrak{E}_{u_0}$ is the least ε-number $> \Omega$, and in general, for $u \in |\mathfrak{X}|$ with $u_0 <_X u$, $\mathfrak{E}_u$ is the least ε-number ρ such that $\forall v <_X u\; \mathfrak{E}_v < \rho$.

In what follows we shall only be interested in ordinals below $\sup_{u \in X} \mathfrak{E}_u$. Henceforth, unless indicated otherwise, any ordinal will be assumed to be smaller than that ordinal.

For any such α we define the set $E_\Omega(\alpha)$ which consists of the ε-numbers below Ω which are needed for the unique representation of α in Cantor normal form recursively as follows:

1. $E_\Omega(0) := E_\Omega(\Omega) := \emptyset$ and $E_\Omega(\mathfrak{E}_u) := \emptyset$ for $u \in |\mathfrak{X}|$.
2. $E_\Omega(\alpha) := \{\alpha\}$, if $\alpha \in E \cap \Omega$.
3. $E_\Omega(\alpha) := E_\Omega(\alpha_1) \cup \cdots \cup E_\Omega(\alpha_n)$ if $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$.

Let $\alpha^* := \max(E_\Omega(\alpha) \cup \{0\})$.

We define sets of ordinals $C_{\mathfrak{X}}(\alpha, \beta), C^n_{\mathfrak{X}}(\alpha, \beta)$, and ordinals $\vartheta\alpha$ by main recursion on $\alpha < \sup_{u \in X} \mathfrak{E}_u$ and subsidiary recursion on $n < \omega$ (for $\beta < \Omega$) as follows.

(C0) $\mathfrak{E}_u \in C^n_{\mathfrak{X}}(\alpha, \beta)$ for all $u \in |\mathfrak{X}|$.
(C1) $\{0, \Omega\} \cup \beta \subseteq C^n_{\mathfrak{X}}(\alpha, \beta)$.
(C2) $\gamma_1, \ldots, \gamma_n \in C^n_{\mathfrak{X}}(\alpha, \beta) \wedge \xi =_{NF} \omega^{\gamma_1} + \cdots + \omega^{\gamma_n} \implies \xi \in C^{n+1}_{\mathfrak{X}}(\alpha, \beta)$.
(C3) $\delta \in C^n_{\mathfrak{X}}(\alpha, \beta) \cap \alpha \implies \vartheta\delta \in C^{n+1}_{\mathfrak{X}}(\alpha, \beta)$.
(C4) $C_{\mathfrak{X}}(\alpha, \beta) := \bigcup\{C^n_{\mathfrak{X}}(\alpha, \beta) \colon n < \omega\}$.
(C5) $\vartheta\alpha := \min\{\xi < \Omega \colon C_{\mathfrak{X}}(\alpha, \xi) \cap \Omega \subseteq \xi \wedge \alpha \in C_{\mathfrak{X}}(\alpha, \xi)\}$ if there exists an ordinal $\xi < \Omega$ such that $C_{\mathfrak{X}}(\alpha, \xi) \cap \Omega \subseteq \xi$ and $\alpha \in C_{\mathfrak{X}}(\alpha, \xi)$. Otherwise $\vartheta\alpha$ will be undefined.
We will shortly see that $\vartheta\alpha$ is always defined (Lemma 2.2).

Remark 2.1 The definition of ϑ originated in [10]. An ordinal representation system based on ϑ was used in [11] to determine the proof-theoretic strength of fragments of Kripke–Platek set theory and in [13] it was used to characterize the strength of Kruskal's theorem.

Lemma 2.2 *$\vartheta\alpha$ is defined for every $\alpha < \sup_{u \in X} \mathfrak{E}_u$.*

Proof Let $\beta_0 := \alpha^* + 1$. Then $\alpha \in C_{\mathfrak{X}}(\alpha, \beta_0)$ via (C1) and (C2). Since the cardinality of $C_{\mathfrak{X}}(\alpha, \beta)$ is less than Ω there exists a $\beta_1 < \Omega$ such that $C_{\mathfrak{X}}(\alpha, \beta_0) \cap$

$\Omega \subseteq \beta_1$. Similarly there exists for each $\beta_n < \Omega$ (which is constructed recursively) a $\beta_{n+1} < \Omega$ such that $C_{\mathfrak{x}}(\alpha, \beta_n) \cap \Omega \subseteq \beta_{n+1}$. Let $\beta := \sup\{\beta_n : n < \omega\}$. Then $\alpha \in C_{\mathfrak{x}}(\alpha, \beta)$ and $C_{\mathfrak{x}}(\alpha, \beta) \cap \Omega \subseteq \beta < \Omega$. Therefore $\vartheta\alpha \leq \beta < \Omega$. □

Lemma 2.3

1. $\vartheta\alpha \in E$,
2. $\alpha \in C_{\mathfrak{x}}(\alpha, \vartheta\alpha)$,
3. $\vartheta\alpha = C_{\mathfrak{x}}(\alpha, \vartheta\alpha) \cap \Omega$, *and* $\vartheta\alpha \notin C_{\mathfrak{x}}(\alpha, \vartheta\alpha)$,
4. $\gamma \in C_{\mathfrak{x}}(\alpha, \beta) \iff \gamma^* \in C_{\mathfrak{x}}(\alpha, \beta)$,
5. $\alpha^* < \vartheta\alpha$,
6. $\vartheta\alpha = \vartheta\beta \implies \alpha = \beta$,
7. $$\begin{aligned} \vartheta\alpha < \vartheta\beta &\iff (\alpha < \beta \ \wedge\ \alpha^* < \vartheta\beta) \ \vee\ (\beta < \alpha \ \wedge\ \vartheta\alpha \leq \beta^*) \\ &\iff (\alpha < \beta \ \wedge\ \alpha^* < \vartheta\beta) \ \vee\ \vartheta\alpha \leq \beta^*, \end{aligned}$$
8. $\beta < \vartheta\alpha \iff \omega^\beta < \vartheta\alpha$.

Proof (1) and (8) basically follow from closure of $\vartheta\alpha$ under (C2).

(2) follows from the definition of $\vartheta\alpha$ taking Lemma 2.2 into account.

For (3), notice that $\vartheta\alpha \subset C_{\mathfrak{x}}(\alpha, \vartheta\alpha)$ is a consequence of clause (C1). Since $C_{\mathfrak{x}}(\alpha, \vartheta\alpha) \cap \Omega \subseteq \vartheta\alpha$ follows from the definition of $\vartheta\alpha$ and Lemma 2.2, we arrive at (3).

(4): If $\gamma^* \in C_{\mathfrak{x}}(\alpha, \beta)$, then $\gamma \in C_{\mathfrak{x}}(\alpha, \beta)$ by (C2). On the other hand, $\gamma \in C^n_{\mathfrak{x}}(\alpha, \beta) \implies \gamma^* \in C^n_{\mathfrak{x}}(\alpha, \beta)$ is easily seen by induction on n.

(5): $\alpha^* \in C_{\mathfrak{x}}(\alpha, \vartheta\alpha)$ holds by (4). As $\alpha^* < \Omega$, this implies $\alpha^* < \vartheta\alpha$ by (3).

(6): Suppose, aiming at a contradiction, that $\vartheta\alpha = \vartheta\beta$ and $\alpha < \beta$. Then $C_{\mathfrak{x}}(\alpha, \vartheta\alpha) \subseteq C_{\mathfrak{x}}(\beta, \vartheta\beta)$; hence $\alpha \in C_{\mathfrak{x}}(\beta, \vartheta\beta) \cap \beta$ by (2); thence $\vartheta\alpha = \vartheta\beta \in C_{\mathfrak{x}}(\beta, \vartheta\beta)$, contradicting (3).

(7): Suppose $\alpha < \beta$. Then $\vartheta\alpha < \vartheta\beta$ implies $\alpha^* < \vartheta\beta$ by (5). If $\alpha^* < \vartheta\beta$, then $\alpha \in C_{\mathfrak{x}}(\beta, \vartheta\beta)$; hence $\vartheta\alpha \in C_{\mathfrak{x}}(\beta, \vartheta\beta)$; thus, $\vartheta\alpha < \vartheta\beta$. This shows

$$(a) \quad \alpha < \beta \implies (\vartheta\alpha < \vartheta\beta \iff \alpha^* < \vartheta\beta).$$

By interchanging the roles of α and β, and employing (6) (to exclude $\vartheta\alpha = \vartheta\beta$), one obtains

$$(b) \quad \beta < \alpha \implies (\vartheta\alpha < \vartheta\beta \iff \vartheta\alpha \leq \beta^*).$$

(a) and (b) yield the first equivalence of (7) and thus the direction "⇒" of the second equivalence. Since $\vartheta\alpha \leq \beta^*$ implies $\vartheta\alpha < \vartheta\beta$ by (5), one also obtains the direction "⇐" of the second equivalence. □

Definition 2.4 Inductive definition of a set $\mathrm{OT}_{\mathfrak{x}}(\vartheta)$ of ordinals and a natural number $G_\vartheta\alpha$ for $\alpha \in \mathrm{OT}_{\mathfrak{x}}(\vartheta)$.

1. $0, \Omega \in \mathrm{OT}_{\mathfrak{x}}(\vartheta)$, $G_\vartheta 0 := G_\vartheta\Omega := 0$, $\mathfrak{E}_u \in \mathrm{OT}_{\mathfrak{x}}(\vartheta)$ and $G_\vartheta\mathfrak{E}_u = 0$ for all $u \in |\mathfrak{X}|$.

2. If $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\alpha_1, \ldots, \alpha_n \in \mathrm{OT}_{\mathfrak{X}}(\vartheta)$ then $\alpha \in \mathrm{OT}_{\mathfrak{X}}(\vartheta)$ and $G_\vartheta\alpha := \max\{G_\vartheta\alpha_1, \ldots, G_\vartheta\alpha_n\} + 1$.
3. If $\alpha = \vartheta\alpha_1$ and $\alpha_1 \in \mathrm{OT}_{\mathfrak{X}}(\vartheta)$ then $\alpha \in \mathrm{OT}_{\mathfrak{X}}(\vartheta)$ and $G_\vartheta\alpha := G_\vartheta\alpha_1 + 1$.

Observe that according to Lemma 2.3 (1) and (6) the function G_ϑ is well-defined. Each ordinal $\alpha \in \mathrm{OT}_{\mathfrak{X}}(\vartheta)$ has a unique normal form using the symbols $0, \Omega, +, \omega, \vartheta$.

Lemma 2.5 $\mathrm{OT}_{\mathfrak{X}}(\vartheta) = \bigcup\{C_{\mathfrak{X}}(\alpha, 0)\colon \alpha < \sup_{u\in X} \mathfrak{E}_u\} = C_{\mathfrak{X}}(\sup_{u\in X} \mathfrak{E}_u, 0)$.

Proof Obviously $\beta < \sup_{u\in X} \mathfrak{E}_u$ holds for all $\beta \in \mathrm{OT}_{\mathfrak{X}}(\vartheta)$.

$$\beta \in \mathrm{OT}_{\mathfrak{X}}(\vartheta) \Rightarrow \beta \in C_{\mathfrak{X}}(\sup_{u\in X} \mathfrak{E}_u, 0)$$

is then shown by induction on $G_\vartheta\beta$.

The inclusion $C_{\mathfrak{X}}(\sup_{u\in X} \mathfrak{E}_u, 0) \subseteq \mathrm{OT}_{\mathfrak{X}}(\vartheta)$ follows from the fact that $\mathrm{OT}_{\mathfrak{X}}(\vartheta)$ is closed under the clauses (Ci) for $i = 0, 1, 2, 3$. Since $\mathfrak{X}$ is an ordering without a maximal element it is also clear that $\bigcup\{C_{\mathfrak{X}}(\alpha, 0) : \alpha < \sup_{u\in X} \mathfrak{E}_u\} = C_{\mathfrak{X}}(\sup_{u\in X} \mathfrak{E}_u, 0)$. □

If for $\alpha, \beta \in \mathrm{OT}_{\mathfrak{X}}(\vartheta)$ represented in their normal form, we wanted to determine whether $\alpha < \beta$, we could do this by deciding $\alpha_0 < \beta_0$ for ordinals α_0 and β_0 that appear in these representations and, in addition, satisfy $G_\vartheta\alpha_0 + G_\vartheta\beta_0 < G_\vartheta\alpha + G_\vartheta\beta$. This follows from Lemma 2.3 (7) and the recursive procedure for comparing ordinals in Cantor normal form. So we come to see that after a straightforward coding in the natural numbers, we may represent $\langle \mathrm{OT}_{\mathfrak{X}}(\vartheta), <\restriction \mathrm{OT}_{\mathfrak{X}}(\vartheta)\rangle$ via a primitive recursive ordinal notation system. How this ordinal representation system can be directly defined in $\mathbf{RCA}_0$ is spelled out in the next subsection.

2.1 *Defining* $\mathbf{OT}_{\mathfrak{X}}(\vartheta)$ *in* $\mathbf{RCA_0}$

We shall provide an explicit primitive recursive definition of $\mathrm{OT}_{\mathfrak{X}}(\vartheta)$ as a term structure in $\mathbf{RCA}_0$. Of course formally, terms or strings of symbols have to be treated as coded by natural numbers since $\mathbf{RCA}_0$ only talks about numbers and sets of numbers. Though, as it is well-known how to do this, we can't be bothered with these niceties.

Definition 2.6 Given a well-ordering $\mathfrak{X} = (X, <_X)$, i.e., an ordered pair $\mathfrak{X}$ in which X is a set of natural numbers and $<_X$ is a well-ordering relation on X, we define, by recursion, a binary relational structure $\vartheta_{\mathfrak{X}} = (|\vartheta_{\mathfrak{X}}|, <)$, and a function $^* : |\vartheta_{\mathfrak{X}}| \to |\vartheta_{\mathfrak{X}}|$, in the following way:

1. $0, \Omega \in |\vartheta_{\mathfrak{X}}|$, and $0^* := 0 =: \Omega^*$.
2. If $\alpha \in |\vartheta_{\mathfrak{X}}|$ and $0 \neq \alpha$ then $0 < \alpha$.

3. For every $u \in X$ there is an element $\mathfrak{E}_u \in |\vartheta_{\mathfrak{X}}|$. Moreover, $(\mathfrak{E}_u)^* := 0$, and $\Omega < \mathfrak{E}_u$. If $u, v \in X$ and $u <_X v$, then $\mathfrak{E}_u < \mathfrak{E}_v$.
4. For every $\alpha \in |\vartheta_{\mathfrak{X}}|$ there is an element $\vartheta\alpha \in |\vartheta_{\mathfrak{X}}|$; and we have $\vartheta\alpha < \Omega$, $\vartheta\alpha < \mathfrak{E}_u$ for every $u \in X$, and $(\vartheta\alpha)^* := \vartheta\alpha$.
5. If $\alpha \in |\vartheta_{\mathfrak{X}}|$ and α is not of the form Ω, $\mathfrak{E}_u$, or $\vartheta\beta$, then $\omega^\alpha \in \vartheta_{\mathfrak{X}}$ and $(\omega^\alpha)^* := \alpha^*$.
6. If $\alpha_1, \ldots, \alpha_n \in |\vartheta_{\mathfrak{X}}|$ and $\alpha_1 \geq \cdots \geq \alpha_n$ with $n \geq 2$, then $\omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_n} \in |\vartheta_{\mathfrak{X}}|$ and $(\omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_n})^* := \max\{\alpha_i^* : 1 \leq i \leq n\}$.
7. Let $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \in |\vartheta_{\mathfrak{X}}|$ and $\beta \in |\vartheta_{\mathfrak{X}}|$, where β is of one of the forms $\vartheta\gamma$, Ω, or $\mathfrak{E}_u$.

   (i) If $\alpha_1 < \beta$, then $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} < \beta$.
   (ii) If $\beta \leq \alpha_1$, then $\beta < \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$.

8. If $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n},\ \omega^{\beta_1} + \cdots + \omega^{\beta_m} \in |\vartheta_{\mathfrak{X}}|$ then
   $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} < \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ iff
   $n < m \wedge \forall i \leq n\ \alpha_i = \beta_i$ or
   $\exists i \leq \min(n, m)\ [(\forall j < i\ \alpha_j = \beta_j) \wedge (\alpha_i < \beta_i)]$.
9. If $\alpha < \beta$ and $\alpha^* < \vartheta\beta$ then $\vartheta\alpha < \vartheta\beta$.
10. If $\vartheta\beta \leq \alpha^*$ then $\vartheta\beta < \vartheta\alpha$.

Lemma 2.7

(i) The set $|\vartheta_{\mathfrak{X}}|$, the relation $<$, and the function * are primitive recursive in $\mathfrak{X} = (X, <_X)$.
(ii) $<$ is a total and linear ordering on $|\vartheta_{\mathfrak{X}}|$.

Proof Straightforward but tedious. □

Of course, $\mathbf{RCA}_0$ does not prove that $<$ is a well-ordering on $|\vartheta_{\mathfrak{X}}|$.

3 A Well-Ordering Proof

In this section we work in the background theory

$$\mathbf{RCA}_0 + \forall X \exists Y\ (X \in Y\ \wedge\ Y \text{ is an } \omega\text{-model of } \mathbf{BI})$$

and shall prove the following statement

$$\forall \mathfrak{X}\, (\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\vartheta_{\mathfrak{X}}))\,,$$

that is, the part (i) ⇒ (*ii*) of the main Theorem 1.7. Some of the proofs are similar to ones in [13, Section 10]. Note that in this theory we can deduce arithmetical comprehension and even arithmetical transfinite recursion owing to [7] and [12], respectively.

Let us fix a well-ordering $\mathfrak{X} = (X, <_X)$, an arbitrary set Y and a countable coded ω-model $\mathfrak{A}$ of **BI** which contains both $\mathfrak{X}$ and Y as elements. In the sequel $\alpha, \beta, \gamma, \delta, \ldots$ are supposed to range over $\vartheta_{\mathfrak{X}}$. $<$ will be used to denote the ordering on $\vartheta_{\mathfrak{X}}$. We are going to work informally in our background theory. A set $U \subseteq \mathbb{N}$ is said to be definable in $\mathfrak{A}$ if $U = \{n \in \mathbb{N} \mid \mathfrak{A} \models A(n)\}$ for some formula $A(x)$ of second order arithmetic which may contain parameters from $\mathfrak{A}$.

Definition 3.1

1. $\mathrm{Acc} := \{\alpha < \Omega \mid \mathfrak{A} \models \mathrm{WO}(< \restriction \alpha)\}$,
2. $\mathrm{M} := \{\alpha\colon E_\Omega(\alpha) \subseteq \mathrm{Acc}\}$,
3. $\alpha <_\Omega \beta :\iff \alpha, \beta \in \mathrm{M} \wedge \alpha < \beta$.

Lemma 3.2 $\alpha, \beta \in \mathrm{Acc} \implies \alpha + \omega^\beta \in \mathrm{Acc}$.

Proof Familiar from Gentzen's proof in Peano arithmetic. The proof just requires $\mathbf{ACA}_0$. (cf. [16, VIII.§ 21 Lemma 1]). □

Lemma 3.3 $\mathrm{Acc} = \mathrm{M} \cap \Omega$ $(:= \{\alpha \in \mathrm{M} \mid \alpha < \Omega\})$.

Proof If $\alpha \in \mathrm{Acc}$, then $E_\Omega(\alpha) \subseteq \mathrm{Acc}$ as well; hence, $\alpha \in \mathrm{M} \cap \Omega$. If $\alpha \in \mathrm{M} \cap \Omega$, then $E_\Omega(\alpha) \subseteq \mathrm{M} \cap \Omega$, so $\alpha \in Acc$ follows from Lemma 3.2. □

Lemma 3.4 *Let U be $\mathfrak{A}$ definable. Then*

$$\forall \alpha < \Omega \cap \mathrm{M}\, [\forall \beta < \alpha\ \beta \in U \to \alpha \in U] \to \mathrm{Acc} \subseteq U\,.$$

Proof This follows readily from the assumption that $\mathfrak{A}$ is a model of **BI**. □

Definition 3.5 Let $\mathrm{Prog}_\Omega(X)$ stand for

$$(\forall \alpha \in \mathrm{M})[(\forall \beta <_\Omega \alpha)(\beta \in X) \longrightarrow \alpha \in X].$$

Let $\mathrm{Acc}_\Omega := \{\alpha \in \mathrm{M}\colon \vartheta\alpha \in \mathrm{Acc}\}$.

Lemma 3.6 *If U is $\mathfrak{A}$ definable, then*

$$\mathrm{Prog}_\Omega(U) \to \Omega, \Omega + 1 \in U\,.$$

Proof This follows from Lemmas 3.3 and 3.4. □

Lemma 3.7 $\mathrm{Prog}_\Omega(\mathrm{Acc}_\Omega)$.

Proof Assume $\alpha \in \mathrm{M}$ and $(\forall \beta <_\Omega \alpha)(\beta \in \mathrm{Acc}_\Omega)$. We have to show that $\vartheta\alpha \in \mathrm{Acc}$. It suffices to show

$$\beta < \vartheta\alpha \Longrightarrow \beta \in \mathrm{Acc}. \tag{2}$$

We shall employ induction on $G_\vartheta(\beta)$, i.e., the length of (the term that represents) β. If $\beta \notin E$, then (2) follows easily by the inductive assumption and Lemma 3.2.

Now suppose $\beta = \vartheta\beta_0$. According to Lemma 2.3 it suffices to consider the following two cases:

Case 1: $\beta \leq \alpha^*$. Since $\alpha \in \mathrm{M}$, we have $\alpha^* \in E_\Omega(\alpha) \subseteq \mathrm{Acc}$; therefore, $\beta \in \mathrm{Acc}$.
Case 2: $\beta_0 < \alpha$ and $\beta_0^* < \vartheta\alpha$. As the length of β_0^* is less than the length of β, we get $\beta_0^* \in \mathrm{Acc}$; thus, $E_\Omega(\beta_0) \subseteq \mathrm{Acc}$, therefore $\beta_0 \in \mathrm{M}$. By the assumption at the beginning of the proof, we then get $\beta_0 \in \mathrm{Acc}_\Omega$; hence, $\beta = \vartheta\beta_0 \in \mathrm{Acc}$. □

Definition 3.8 For every $\mathfrak{A}$ definable set U we define the "Gentzen jump"

$$U^j := \{\gamma \mid \forall\delta\,[\mathrm{M} \cap \delta \subseteq U \rightarrow \mathrm{M} \cap (\delta + \omega^\gamma) \subseteq U]\}.$$

Lemma 3.9 *Let U be $\mathfrak{A}$ definable.*

(i) $\gamma \in U^j \Rightarrow \mathrm{M} \cap \omega^\gamma \subseteq U$.
(ii) $\mathrm{Prog}_\Omega(U) \Rightarrow \mathrm{Prog}_\Omega(U^j)$.

Proof (i) is obvious. (ii) $\mathrm{M} \cap (\delta + \omega^\gamma) \subseteq U$ is to be proved under the assumptions (a) $\mathrm{Prog}_\Omega(U)$, (b) $\gamma \in \mathrm{M} \wedge \mathrm{M} \cap \gamma \subseteq U^j$ and (c) $\mathrm{M} \cap \delta \subseteq U$. So let $\eta \in \mathrm{M} \cap (\delta + \omega^\gamma)$.

1. $\eta < \delta$: Then $\eta \in U$ is a consequence of (c).
2. $\eta = \delta$: Then $\eta \in U$ follows from (c) and (a).
3. $\delta < \eta < \delta + \omega^\gamma$: Then there exist $\gamma_1, \ldots, \gamma_k < \gamma$ such that $\eta = \delta + \omega^{\gamma_1} + \cdots + \omega^{\gamma_k}$ and $\gamma_1 \geq \cdots \geq \gamma_k$. $\eta \in \mathrm{M}$ implies $\gamma_1, \ldots, \gamma_k \in \mathrm{M} \cap \gamma$. Through applying (b) and (c) we obtain $\mathrm{M} \cap (\delta + \omega^{\gamma_1}) \subseteq U$. By iterating this procedure we eventually arrive at $\delta + \omega^{\gamma_1} + \cdots + \omega^{\gamma_k} \in U$, so $\eta \in U$ holds.

□

Corollary 3.10 *Let $\mathcal{I}(\delta)$ be the statement that* $\mathrm{Prog}_\Omega(V) \rightarrow \delta \in \mathrm{M} \wedge \delta \cap \mathrm{M} \subseteq V$ *holds for all $\mathfrak{A}$ definable sets V. Assume $\mathcal{I}(\delta)$. Let $\delta_0 := \delta$ and $\delta_{n+1} := \omega^{\delta_n}$. Then*

$$\mathcal{I}(\delta_n)$$

holds for all n.

Proof We use induction on n. For $n = 0$ this is the assumption. Now suppose $\mathcal{I}(\delta_n)$ holds. Assume $\mathrm{Prog}_\Omega(U)$ for an $\mathfrak{A}$ definable U. By Lemma 3.9 we conclude $\mathrm{Prog}_\Omega(U^j)$ and hence $\delta_n \in U^j$ and $\delta_n \cap \mathrm{M} \subseteq U^j$. As clearly $\mathrm{M} \cap 0 \subseteq U$ we get $\omega^{\delta_n} \cap \mathrm{M} \subseteq U$. Since $\mathrm{Prog}_\Omega(U)$ entails $\delta \in \mathrm{M}$ we also have $\delta_{n+1} \in \mathrm{M}$. Thus $\delta_{n+1} \in \mathrm{M} \wedge \delta_{n+1} \cap \mathrm{M} \subseteq U$, showing $\mathcal{I}(\delta_{n+1})$. □

Let $\omega_0(\alpha) := \alpha$ and $\omega_{n+1}(\alpha) := \omega^{\omega_n(\alpha)}$.

Proposition 3.11 *$\mathcal{I}(\mathfrak{E}_u)$ holds for all $u \in |\mathfrak{X}|$.*

Proof Noting that in our background theory $\mathfrak{X}$ is a well-ordering, we can use induction on $\mathfrak{X}$. Note also that $\mathcal{I}(\mathfrak{E}_u)$ is a statement about all definable sets in $\mathfrak{A}$ which is not formalizable in $\mathfrak{A}$ itself. However, in our background theory quantification over all these sets is first order expressible and therefore transfinite induction along $<_X$ is available.

First observe that we have $\mathcal{I}(\Omega + 1)$ by Lemma 3.6. Let u_0 be the $<_X$-least element of $|\mathfrak{X}|$. We have $\mathfrak{E}_{u_0} \in \mathrm{M}$ and for every $\eta < \mathfrak{E}_{u_0}$ there exists n such that $\eta < \omega_n(\Omega + 1)$. As a result, using Corollary 3.10, we have

$$\mathrm{Prog}_{\Omega}(U) \to \mathfrak{E}_{u_0} \cap \mathrm{M} \subseteq U$$

for every $\mathfrak{A}$ definable set U.

Now suppose that $u \in |\mathfrak{X}|$ is not the $<_X$-least element and for all $v <_X u$ we have $\mathcal{I}(\mathfrak{E}_v)$. As for every $\delta < \mathfrak{E}_u$ there exists $v <_X u$ and n such that $\delta < \omega_n(\mathfrak{E}_v)$, the inductive assumption together with Corollary 3.10 yields

$$\mathrm{Prog}_{\Omega}(U) \to \mathfrak{E}_u \cap \mathrm{M} \subseteq U\ .$$

$\mathfrak{E}_u \in \mathrm{M}$ is obvious. □

Proposition 3.12 *For all α, $\mathcal{I}(\alpha)$.*

Proof We proceed by the induction on the term complexity of α. Clearly, $\mathcal{I}(0)$. By Lemma 3.6 we conclude that $\mathcal{I}(\Omega)$. Proposition 3.11 entails that $\mathcal{I}(\mathfrak{E}_u)$ for all $u \in |\mathfrak{X}|$.

Now let $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ be in Cantor normal form. Inductively we have $\mathcal{I}(\alpha_1), \ldots, \mathcal{I}(\alpha_n)$. Assume $\mathrm{Prog}_{\Omega}(U)$. Then $\mathrm{Prog}_{\Omega}(U^j)$ by Lemma 3.9(ii), and hence $\alpha_1 \cap \mathrm{M} \subseteq U^j, \ldots, \alpha_n \cap \mathrm{M} \subseteq U^j$ and $\alpha_1, \ldots, \alpha_n \in \mathrm{M}$. The latter implies $\alpha_1 \in U^j, \ldots, \alpha_n \in U^j$. Using the definition of U^j repeatedly we conclude $\alpha \cap \mathrm{M} \subseteq U$. Moreover, $\alpha \in \mathrm{M}$ since $\alpha_1, \ldots, \alpha_n \in \mathrm{M}$.

Now suppose that $\alpha = \vartheta\beta$. Inductively we have $\mathcal{I}(\beta)$. By Lemma 3.7 we conclude that $\beta \in \mathrm{Acc}_{\Omega}$, and hence $\alpha \in \mathrm{Acc}$. From $\mathrm{Prog}_{\Omega}(U)$ we obtain by Lemma 3.4 that $\xi \in U$ for all $\xi \leq \alpha$. As a result, $\mathcal{I}(\alpha)$. □

Corollary 3.13 *$\vartheta_{\mathfrak{X}}$ is a well-ordering.*

With the previous Corollary, the proof of Theorem 1.7 (i)⇒(ii) is finally accomplished.

4 Deduction Chains

From now on we will be concerned with the part (ii) ⇒ (i) of the main Theorem 1.7. An important tool will be the method of deduction chains. Given a sequent Γ and a set $Q \subseteq \mathbb{N}$, deduction chains starting at Γ are built by systematically decomposing Γ into its subformulas, and adding additionally at the nth step the formulas $\neg A_n$ and $\neg\bar{Q}(\bar{n})$, where $(A_n \mid n \in \mathbb{N})$ is an enumeration of the axioms of the theory **BI**, and $\bar{Q}(\bar{n})$ is the atom $\bar{n} \in U_0$ if $n \in Q$ and $\bar{n} \notin U_0$ otherwise. The set of all deduction chains that can be built from the empty sequent with respect to a given set Q forms the tree $\mathcal{D}_Q$. There are two scenarios to be considered.

(i) If there is an infinite deduction chain, i.e. $\mathcal{D}_Q$ is ill-founded, then this readily yields a model of **BI** that contains Q.
(ii) If each deduction chain is finite, then this yields a derivation of the empty sequent, $\perp$, in a corresponding infinitary system with an ω-rule. The depth of this derivation is bounded by the order-type α of the Kleene–Brouwer ordering of $\mathcal{D}_Q$. By the well-ordering principle, transfinite induction up to $\mathfrak{E}_{\alpha+1}$ is available, which allows to transform this proof into a cut-free proof of $\perp$ whose depth is less than $\vartheta\mathfrak{E}_{\alpha+1}$.

As the second alternative is impossible, the first yields the desired model.

Definition 4.1

1. We let $U_0, U_1, \ldots, U_m, \ldots$ be an enumeration of the free set variables of $\mathcal{L}_2$ and, given a closed term t, we write $t^{\mathbb{N}}$ for its numerical value.
2. Henceforth a **sequent** will be a finite list of $\mathcal{L}_2$-formulae *without* free number variables.
3. A sequent Γ is **axiomatic** if it satisfies at least one of the following conditions:
   (a) Γ contains a true **literal**, i.e., a true formula of either of the forms $R(t_1, \ldots, t_n)$ or $\neg R(t_1, \ldots, t_n)$, where R is a predicate symbol in $\mathcal{L}_2$ for a primitive recursive relation and $t_1, \ldots, t_n$ are closed terms.
   (b) Γ contains formulae $s \in U$ and $t \notin U$ for some set variable U and terms s, t with $s^{\mathbb{N}} = t^{\mathbb{N}}$.
4. A sequent is **reducible** if it is not axiomatic and contains a formula which is not a literal.

Definition 4.2 For $Q \subseteq \mathbb{N}$ we define

$$\bar{Q}(n) \Leftrightarrow \begin{cases} \bar{n} \in U_0 & \text{if } n \in Q, \\ \bar{n} \notin U_0 & \text{otherwise.} \end{cases}$$

For some of the following theorems it is convenient to have a finite axiomatization of arithmetical comprehension.

Lemma 4.3 **ACA**$_0$ *can be axiomatized via a single* Π^1_2 *sentence* $\forall X C(X)$.

Proof [17, Lemma VIII.1.5]. □

Definition 4.4 In what follows, we fix an enumeration of $A_1, A_2, A_3, \ldots$ of all the universal closures of instances of ***(BI)***. We also put $A_0 := \forall X\, C(X)$, where the latter is the sentence that axiomatizes arithmetical comprehension.

Definition 4.5 Let $Q \subseteq \mathbb{N}$. A Q**-deduction chain** is a finite string

$$\Gamma_0, \Gamma_1, \ldots, \Gamma_k$$

of sequents Γ_i constructed according to the following rules:

1. $\Gamma_0 = \neg\bar{Q}(0), \neg A_0$.
2. Γ_i is not axiomatic for $i < k$.
3. If $i < k$ and Γ_i is not reducible, then

$$\Gamma_{i+1} = \Gamma_i, \neg\bar{Q}(i+1), \neg A_{i+1}.$$

4. Every reducible Γ_i with $i < k$ is of the form

$$\Gamma_i', E, \Gamma_i''$$

where E is not a literal and Γ_i' contains only literals. E is said to be the **redex** of Γ_i.
Let $i < k$ and Γ_i be reducible. Γ_{i+1} is obtained from $\Gamma_i = \Gamma_i', E, \Gamma_i''$ as follows:

(a) If $E \equiv E_0 \vee E_1$, then

$$\Gamma_{i+1} = \Gamma_i', E_0, E_1, \Gamma_i'', \neg\bar{Q}(i+1), \neg A_{i+1}.$$

(b) If $E \equiv E_0 \wedge E_1$, then

$$\Gamma_{i+1} = \Gamma_i', E_j, \Gamma_i'', \neg\bar{Q}(i+1), \neg A_{i+1}$$

where $j = 0$ or $j = 1$.
(c) If $E \equiv \exists x F(x)$, then

$$\Gamma_{i+1} = \Gamma_i', F(\bar{m}), \Gamma_i'', \neg\bar{Q}(i+1), \neg A_{i+1}, E$$

where m is the first number such that $F(\bar{m})$ does not occur in $\Gamma_0, \ldots, \Gamma_i$.
(d) If $E \equiv \forall x F(x)$, then

$$\Gamma_{i+1} = \Gamma_i', F(\bar{m}), \Gamma_i'', \neg\bar{Q}(i+1), \neg A_{i+1}$$

for some m.
(e) If $E \equiv \exists X F(X)$, then

$$\Gamma_{i+1} = \Gamma_i', F(U_m), \Gamma_i'', \neg\bar{Q}(i+1), \neg A_{i+1}, E$$

where m is the first number such that $F(U_m)$ does not occur in $\Gamma_0, \ldots, \Gamma_i$.
(f) If $E \equiv \forall X F(X)$, then

$$\Gamma_{i+1} = \Gamma_i', F(U_m), \Gamma_i'', \neg\bar{Q}(i+1), \neg A_{i+1}$$

where m is the first number such that U_m does not occur in Γ_i.

The set of Q-deduction chains forms a tree $\mathcal{D}_Q$ labeled with strings of sequents.

We will now consider two cases.

Case I: $\mathcal{D}_Q$ is not well-founded. Then $\mathcal{D}_Q$ contains an infinite path $\mathbb{P}$. Now define a set M via

$$(M)_i = \{k \mid \bar{k} \notin U_i \text{ occurs in } \mathbb{P}\}.$$

Set $\mathbb{M} = (\mathbb{N}; \{(M)_i \mid i \in \mathbb{N}\}, \in, +, \cdot, 0, 1, <)$.

For a formula F, let $F \in \mathbb{P}$ mean that F occurs in $\mathbb{P}$, i.e. $F \in \Gamma$ for some $\Gamma \in \mathbb{P}$.
Claim: Under the assignment $U_i \mapsto (M)_i$ we have

$$F \in \mathbb{P} \quad \Rightarrow \quad \mathbb{M} \models \neg F. \tag{3}$$

The Claim will imply that $\mathbb{M}$ is an ω-model of **BI**. Also note that $(M)_0 = Q$, thus Q is in $\mathbb{M}$. The proof of (3) follows by induction on F using Lemma 4.6 below. The upshot of the foregoing is that we can prove Theorem 1.7 under the assumption that $\mathcal{D}_Q$ is ill-founded for all sets $Q \subseteq \mathbb{N}$.

Lemma 4.6 *Let Q be an arbitrary subset of $\mathbb{N}$ and $\mathcal{D}_Q$ be the corresponding deduction tree. Moreover, suppose $\mathcal{D}_Q$ is not well-founded. Then $\mathcal{D}_Q$ has an infinite path $\mathbb{P}$. $\mathbb{P}$ has the following properties:*

1. *$\mathbb{P}$ does not contain literals which are true in $\mathbb{N}$.*
2. *$\mathbb{P}$ does not contain formulas $s \in U_i$ and $t \notin U_i$ for constant terms s and t such that $s^{\mathbb{N}} = t^{\mathbb{N}}$.*
3. *If $\mathbb{P}$ contains $E_0 \vee E_1$, then $\mathbb{P}$ contains E_0 and E_1.*
4. *If $\mathbb{P}$ contains $E_0 \wedge E_1$, then $\mathbb{P}$ contains E_0 or E_1.*
5. *If $\mathbb{P}$ contains $\exists x F(x)$, then $\mathbb{P}$ contains $F(\bar{n})$ for all n.*
6. *If $\mathbb{P}$ contains $\forall x F(x)$, then $\mathbb{P}$ contains $F(\bar{n})$ for some n.*
7. *If $\mathbb{P}$ contains $\exists X F(X)$, then $\mathbb{P}$ contains $F(U_m)$ for all m.*
8. *If $\mathbb{P}$ contains $\forall X F(X)$, then $\mathbb{P}$ contains $F(U_m)$ for some m.*
9. *$\mathbb{P}$ contains $\neg C(U_m)$ for all m.*
10. *$\mathbb{P}$ contains $\neg \bar{Q}(m)$ for all m.*

Proof Standard. □

Corollary 4.7 *If $\mathcal{D}_Q$ is ill-founded, then there exists a countable coded ω-model of* **BI** *which contains Q.*

For our purposes it is important that Corollary 4.7 can be proved in $T_0 := \mathbf{RCA}_0 + \forall \mathfrak{X}\, (\mathrm{WO}(\mathfrak{X}) \to \mathrm{WO}(\vartheta_{\mathfrak{X}}))$. To this end we need to show that the semantics of ω-models can be handled in the latter theory, i.e. for every formula F of $\mathcal{L}_2$ there exists a valuation for F in the sense of [17, VII.2.1]. It is easily seen that the principle $\forall \mathfrak{X}\, (\mathrm{WO}(\mathfrak{X}) \to \mathrm{WO}(\vartheta_{\mathfrak{X}}))$ implies

$$\forall \mathfrak{X}\, (\mathrm{WO}(\mathfrak{X}) \to \mathrm{WO}(\varepsilon_{\mathfrak{X}}))$$

(see [1, Definition 2.1]) and thus, by [1, Theorem 4.1], T_0 proves that every set is contained in an ω-model of **ACA**. Now take an ω-model containing $\mathcal{D}_Q$ and an infinite branch of $\mathcal{D}_Q$. In this ω-model we find a valuation for every formula by [17, VII.2.2]. And hence Corollary 4.7 holds in the model, but then it also holds in the world at large by absoluteness.

5 Proof of the Main Theorem: The Hard Direction Part 2

The remainder of the paper will be devoted to ruling out the possibility that for some Q, $\mathcal{D}_Q$ could be a well-founded tree. This is the place where the principle $\forall \mathfrak{X}\,(\mathrm{WO}(\mathfrak{X}) \rightarrow \mathrm{WO}(\vartheta_{\mathfrak{X}}))$ in the guise of cut elimination for an infinitary proof system enters the stage. Aiming at a contradiction, suppose that $\mathcal{D}_Q$ is a well-founded tree. Let $\mathfrak{X}$ be the Kleene–Brouwer ordering on $\mathcal{D}_Q$ (see [17, Definition V.1.2]). Then $\mathfrak{X}$ is a well-ordering. In a nutshell, the idea is that a well-founded $\mathcal{D}_Q$ gives rise to a derivation of the empty sequent (contradiction) in an infinitary proof system.

5.1 Majorization and Fundamental Functions

In this section we introduce the concepts of majorization and fundamental function. They are needed for carrying through the ordinal analysis of bar induction. More details can be found in [13, Section 4] and [3, I.4] to which we refer for proofs. The missing proofs are actually straightforward consequences of Definition 2.6.

Definition 5.1 1. $\alpha \lhd \beta$ means $\alpha < \beta$ and $\vartheta\alpha < \vartheta\beta$.
2. $\alpha \unlhd \beta :\iff (\alpha \lhd \beta \vee \alpha = \beta)$.

Lemma 5.2 *1.* $\alpha \lhd \beta \;\wedge\; \beta \lhd \gamma \implies \alpha \lhd \gamma$.
2. $0 < \beta < \varepsilon_0 \implies \alpha \lhd \alpha + \beta$.
3. $\alpha < \beta < \Omega \implies \alpha \lhd \beta$.
4. $\alpha \lhd \beta \implies \alpha + 1 \unlhd \beta$.
5. $\alpha \lhd \beta \implies \vartheta\alpha \lhd \vartheta\beta$.
6. $\alpha = \alpha_0 + 1 \implies \vartheta\alpha_0 \lhd \vartheta\alpha$.

Lemma 5.3 $\alpha \lhd \beta,\ \beta < \omega^{\gamma+1} \implies \omega^{\gamma} + \alpha \lhd \omega^{\gamma} + \beta$.

Corollary 5.4 $\omega^{\alpha} \cdot n \lhd \omega^{\alpha} \cdot (n+1)$.

Lemma 5.5 $\alpha \lhd \beta \implies \omega^{\alpha} \cdot n \lhd \omega^{\beta}$.

Definition 5.6 Let $D_\Omega := (\mathrm{OT}_{\mathfrak{X}}(\vartheta) \cap \Omega) \cup \{\Omega\}$. A function $f : D_\Omega \rightarrow \mathrm{OT}_{\mathfrak{X}}(\vartheta)$ will be called a *fundamental function* if it is generated by the following clauses:

F1. $Id : D_\Omega \rightarrow D_\Omega$ with $Id(\alpha) = \alpha$ is a fundamental function.

F2. If f is a fundamental function, $\gamma \in \mathrm{OT}_{\mathfrak{X}}(\vartheta)$ and $f(\Omega) < \omega^{\gamma+1}$, then $\omega^{\gamma} + f$ is a fundamental function, where $(\omega^{\gamma} + f)(\alpha) := \omega^{\gamma} + f(\alpha)$ for all $\alpha \in D_{\Omega}$.
F3. If f is a fundamental function, then so is ω^{f} with $(\omega^{f})(\alpha) := \omega^{f(\alpha)}$ for all $\alpha \in D_{\Omega}$.

Lemma 5.7 *Let f be a fundamental function and $\beta \leq \Omega$.*

(i) If $\alpha < \beta$, then $f(\alpha) < f(\beta)$.
(ii) If $\alpha \lhd \beta$, then $f(\alpha) \lhd f(\beta)$.
(iii) $(f(\beta))^* \leq \max((f(0))^*, \beta^*)$.

Proof (i) is obvious by induction on the generation of fundamental functions.
(ii) also follows by induction on the generation of fundamental functions, using Lemmas 5.3 and 5.5.
(iii) as well follows by induction on the generation of fundamental functions.

□

Lemma 5.8 *For every fundamental function f we have $f(\vartheta(f(0))) \lhd f(\Omega)$.*

Proof Since $\vartheta(f(0)) < \Omega$, we clearly have $f(\vartheta(f(0))) < f(\Omega)$. Since $0 \lhd \Omega$ and f is a fundamental function, we have $\vartheta(f(0)) < \vartheta(f(\Omega))$ by Lemma 5.7 (ii). Invoking Lemma 5.7 (iii), the latter entails that $(f(\vartheta(f(0))))^* < \vartheta(f(\Omega))$, so that in conjunction with $f(\vartheta(f(0))) < f(\Omega)$ it follows that $\vartheta(f(\vartheta(f(0)))) \lhd \vartheta(f(\Omega))$. As a result, $f(\vartheta(f(0))) \lhd f(\Omega)$. □

5.2 *The Infinitary Calculus T_Q^**

The calculus T_Q^* to be introduced stems from [13, Section 6]. We fix a set $Q \subseteq \mathbb{N}$. Let $\mathcal{L}_2^Q$ be the language of second order arithmetic augmented by a unary predicate $\bar{Q}$. The *formulas* of T_Q^* arise from $\mathcal{L}_2^Q$-formulas by replacing free numerical variables by numerals, i.e. terms of the form $0, 0', 0'', \ldots$ Especially, every formula A of T_Q^* is an $\mathcal{L}_2^Q$-formula. We are going to measure the length of derivations by ordinals. We are going to use the set of ordinals $\mathrm{OT}_{\mathfrak{X}}(\vartheta)$ of Sect. 3.

Definition 5.9

1. A formula B is said to be *weak* if it belongs to $\Pi^1_0 \cup \Pi^1_1$.
2. Two closed terms s and t are said to be equivalent if they yield the same value when computed.
3. A formula is called constant if it contains no set variables. The truth or falsity of such a formula is understood with respect to the standard structure of the integers.
4. $\overline{0} := 0$, $\overline{m+1} := \overline{m}'$.

In the sequent calculus T_Q^* below we shall use the following rules of inference:

$$(\wedge) \quad \vdash \Gamma, A \text{ and } \vdash \Gamma, B \implies \vdash \Gamma, A \wedge B,$$

$$(\vee) \quad \vdash \Gamma, A_i \implies \vdash \Gamma, A_0 \vee A_1 \quad \text{if } i \in \{0, 1\},$$

$$(\forall_2) \quad \vdash \Gamma, F(U) \implies \vdash \Gamma, \forall X F(X),$$

$$(\exists_1) \quad \vdash \Gamma, F(t) \implies \vdash \Gamma, \exists x F(x),$$

$$(Cut) \vdash \Gamma, A \text{ and } \vdash \Gamma, \neg A \implies \vdash \Gamma,$$

where in $(\forall_2)$ the free variable U is not to occur in the conclusion.

The most important feature of sequent calculi is cut-elimination. To state this fact concisely, let us introduce a measure of complexity, $gr(A)$, the *grade of a formula* A, for $\mathcal{L}_2^Q$-formulae.

Definition 5.10

1. $gr(A) = 0$ if A is a prime formula or negated prime formula.
2. $gr(\forall X F(X)) = gr(\exists X F(X)) = \omega$ if $F(U)$ is arithmetic.
3. $gr(A \wedge B) = gr(A \vee B) = max\{gr(A), gr(B)\} + 1$.
4. $gr(\forall x H(x)) = gr(\exists x H(x)) = gr(H(0)) + 1$.
5. $gr(\forall X G(X)) = gr(\exists X G(X)) = gr(G(U)) + 1$, if G is not arithmetic.

Definition 5.11 Inductive definition of $T_Q^* \mathrel{|}\!\frac{\alpha}{\varrho}\, \Gamma$ for $\alpha \in \mathrm{OT}_{\mathfrak{X}}(\vartheta)$ and $\varrho{<}\omega{+}\omega$.

1. If A is a true constant prime formula or negated prime formula and $A \in \Gamma$, then $T_Q^* \mathrel{|}\!\frac{\alpha}{\varrho}\, \Gamma$.
2. If $n \in Q$ and t is a closed term with value n and $\bar{Q}(t)$ is in Γ, then $T_Q^* \mathrel{|}\!\frac{\alpha}{\varrho}\, \Gamma$.
3. If $n \notin Q$ and t is a closed term with value n and $\neg\bar{Q}(t)$ is in Γ, then $T_Q^* \mathrel{|}\!\frac{\alpha}{\varrho}\, \Gamma$.
4. If Γ contains formulas $A(s_1, \ldots, s_n)$ and $\neg A(t_1, \ldots, t_n)$ of grade 0 or ω, where s_i and t_i $(1 \le i \le n)$ are equivalent terms, then $T_Q^* \mathrel{|}\!\frac{\alpha}{\varrho}\, \Gamma$.
5. If $T_Q^* \mathrel{|}\!\frac{\beta}{\varrho}\, \Gamma_i$ and $\beta \lhd \alpha$ hold for every premiss Γ_i of an inference $(\wedge), (\vee), (\exists_1), (\forall_2)$ or (Cut) with a cut formula having grade $< \varrho$, and conclusion Γ, then $T_Q^* \mathrel{|}\!\frac{\alpha}{\varrho}\, \Gamma$.
6. If $T_Q^* \mathrel{|}\!\frac{\alpha_0}{\varrho}\, \Gamma, F(U)$ holds for some $\alpha_0 \lhd \alpha$ and a non-arithmetic formula $F(U)$ (i.e., $gr(F(U)) \ge \omega$), then $T_Q^* \mathrel{|}\!\frac{\alpha}{\varrho}\, \Gamma, \exists X F(X)$.
7. (ω-rule). If $T_Q^* \mathrel{|}\!\frac{\beta}{\varrho}\, \Gamma, A(\overline{m})$ is true for every $m < \omega$, $\forall x A(x) \in \Gamma$, and $\beta \lhd \alpha$, then $T_Q^* \mathrel{|}\!\frac{\alpha}{\varrho}\, \Gamma$.
8. (Ω-rule). Let f be a fundamental function satisfying
   (a) $f(\Omega) \trianglelefteq \alpha$,
   (b) $T_Q^* \mathrel{|}\!\frac{f(0)}{\varrho}\, \Gamma, \forall X F(X)$, where $\forall X F(X) \in \Pi_1^1$, and
   (c) $T_Q^* \mathrel{|}\!\frac{\beta}{0}\, \Xi, \forall X F(X)$ implies $T_Q^* \mathrel{|}\!\frac{f(\beta)}{\varrho}\, \Xi, \Gamma$ for every set of weak formulas Ξ and $\beta < \Omega$.

Then $T^*_Q \mid\frac{\alpha}{\varrho} \Gamma$ holds.

Remark 5.12 The derivability relation $T^*_Q \mid\frac{\alpha}{\varrho} \Gamma$ is from [13] and is modelled upon the relation $PB^* \mid\frac{\alpha}{n} F$ of [3], the main difference being the sequent calculus setting instead of P- and N-forms and a different assignment of cut-degrees. The allowance for transfinite cut-degrees will enable us to deal with arithmetical comprehension.

Remark 5.13 If one ruminates on the definition of the derivability predicate $T^*_Q \mid\frac{\alpha}{\varrho} \Xi$ the question arises whether it is actually a proper inductive definition. The critical point is obviously the condition (c) of the Ω-rule. Note that $T^*_Q \mid\frac{\beta}{0} \Xi, \forall X F(X)$ occurs negatively in clause (c). However, since $\beta < \Omega$, the pertaining derivation does not contain any applications of the Ω-rule. Thus the definition of $T^*_Q \mid\frac{\alpha}{\varrho} \Xi$ proceeds via an iterated inductive definition. First one defines a derivability predicate without involvement of the Ω-rule via an ordinary inductive definition, and in a second step defines $T^*_Q \mid\frac{\alpha}{\varrho} \Gamma$ inductively referring to the first derivability predicate in the Ω-rule.

It will actually be a non-trivial issue how to handle such inductive definitions in a weak background theory.

Lemma 5.14

1. $T^*_Q \mid\frac{\alpha}{\delta} \Gamma \ \& \ \Gamma \subseteq \Delta \ \& \ \alpha \trianglelefteq \beta \ \& \ \delta \leq \varrho \implies T^*_Q \mid\frac{\beta}{\varrho} \Delta$,
2. $T^*_Q \mid\frac{\alpha}{\varrho} \Gamma, A \wedge B \implies T^*_Q \mid\frac{\alpha}{\varrho} \Gamma, A \ \& \ T^*_Q \mid\frac{\alpha}{\varrho} \Gamma, B$,
3. $T^*_Q \mid\frac{\alpha}{\varrho} \Gamma, A \vee B \implies T^*_Q \mid\frac{\alpha}{\varrho} \Gamma, A, B$
4. $T^*_Q \mid\frac{\alpha}{\varrho} \Gamma, F(t) \implies T^*_Q \mid\frac{\alpha}{\varrho} \Gamma, F(s)$ *if t and s are equivalent,*
5. $T^*_Q \mid\frac{\alpha}{\varrho} \Gamma, \forall x F(x) \implies T^*_Q \mid\frac{\alpha}{\varrho} \Gamma, F(s)$ *for every term s.*
6. *If* $T^*_Q \mid\frac{\alpha}{\varrho} \Gamma, \forall X G(X)$ *and* $gr(G(U)) \geq \omega$, *then* $T^*_Q \mid\frac{\alpha}{\varrho} \Gamma, G(U)$.

Proof Proceed by induction on α. These can be carried out straightforwardly. (5) requires (4). As to (6), observe that $\forall X G(X)$ cannot be the main formula of an axiom. □

Lemma 5.15 $T^*_Q \mid\frac{2\cdot\alpha}{0} \Gamma, A(s_1, \ldots, s_k), \neg A(t_1, \ldots, t_k)$ *if* $\alpha \geq gr(A(s_1, \ldots, s_k))$ *and s_i and t_i are equivalent terms.*

Proof Proceed by induction on $gr(A(s_1, \ldots, s_k))$. Crucially note that if $gr(A(s_1, \ldots, s_k)) = \omega$ then $\Gamma, A(s_1, \ldots, s_k), \neg A(t_1, \ldots, t_k)$ is an axiom according to Definition 5.11 clause (4). □

Lemma 5.16

1. $T^*_Q \mid\frac{2m}{0} \neg(0 \in U), (\exists x)[x \in U \wedge \neg(x' \in U)], \overline{m} \in U$,
2. $T^*_Q \mid\frac{\omega+5}{0} \forall X[0 \in X \wedge \forall x(x \in X \to x' \in X) \to \forall x(x \in X)]$.

Proof For (1) use induction on m. (2) is an immediate consequence of (1) using Lemma 5.14 (1), the ω-rule, $(\vee)$, and $(\forall_2)$. □

Definition 5.17 For formulas $F(U)$ and $A(a)$, $F(A)$ denotes the result of replacing each occurrence of the form $e \in U$ in $F(U)$ by $A(e)$. The expression $F(A)$ is a formula if the bound variables in $A(a)$ are chosen in an appropriate way, in particular, if $F(U)$ and $A(a)$ have no bound variables in common.

Lemma 5.18 *Suppose* $\alpha < \Omega$ *and let* $\Delta(U) = \{F_1(U), \ldots, F_k(U)\}$ *be a set of weak formulas such that* U *doesn't occur in* $\forall X F_i(X)$ $(1 \leq i \leq k)$*. For an arbitrary formula* $A(a)$ *we then have:*

$$T_Q^* \mid\frac{\alpha}{0} \Delta(U) \implies T_Q^* \mid\frac{\Omega+\alpha}{0} \Delta(A)\,.$$

Proof Proceed by induction on α. Suppose $\Delta(U)$ is an axiom. Then either $\Delta(A)$ is an axiom too, or $T_Q^* \mid\frac{\omega+\omega}{0} \Delta(A)$ can be obtained through use of Lemma 5.15. Therefore $T_Q^* \mid\frac{\Omega+\alpha}{0} \Delta(A)$ by Lemma 5.14 (1). If $T_Q^* \mid\frac{\alpha}{0} \Delta(U)$ is the result of an inference, then this inference must be different from $(\exists_2)$, (Cut), and the $(\Omega - rule)$ since $\Delta(U)$ consists of weak formulas, the derivation is cut-free and $\alpha < \Omega$. For the remaining possible inference rules the assertion follows easily from the induction hypothesis. □

Lemma 5.19 *Let* $\Gamma, \forall X F(X)$ *be a set of weak formulas. If* $T_Q^* \mid\frac{\alpha}{0} \Gamma, \forall X F(X)$ *and* $\alpha < \Omega$*, then* $T_Q^* \mid\frac{\alpha}{0} \Gamma, F(U)$*.*

Proof Use induction on α. Note that $\forall X F(X)$ cannot be a principal formula of an axiom, since $\exists X \neg F(X)$ does not surface in such a derivation. Also, due to $\alpha < \Omega$, the derivation doesn't involve instances of the Ω-rule. Therefore the proof is straightforward. □

The role of the Ω-rule in our calculus T_Q^* is enshrined in the next lemma.

Lemma 5.20 $T_Q^* \mid\frac{\Omega\cdot 2}{0} \exists X F(X), \neg F(A)$ *for every arithmetic formula* $F(U)$ *and arbitrary formula* $A(a)$*.*

Proof Let $f(\alpha) := \Omega + \alpha$ with $dom(f) := \{\alpha \in OT(\psi) \colon \alpha \leq \Omega\}$. Then

$$T_Q^* \mid\frac{f(0)}{0} \forall X \neg F(X), \exists X F(X), \neg F(A) \tag{4}$$

according to Lemma 5.15. For $\alpha < \Omega$ and every set of weak formulas Θ, we have by Lemmas 5.18 and 5.19,

$$T_Q^* \mid\frac{\alpha}{0} \Theta, \forall X \neg F(X) \implies T_Q^* \mid\frac{f(\alpha)}{0} \Theta, \neg F(A).$$

Therefore, by Lemma 5.14 (1),

$$T_Q^* \mid\frac{\alpha}{0} \Theta, \forall X \neg F(X) \implies T_Q^* \mid\frac{f(\alpha)}{0} \Theta, \exists X F(X), \neg F(A). \tag{5}$$

The assertion now follows from (4) and (5) by the Ω-rule. □

Corollary 5.21 $T^*_{\varrho} \mid\!\frac{\Omega\cdot 2+1}{\omega}\, \exists X\, \forall y\, (y \in X \leftrightarrow B(y))$ *for every arithmetic formula* $B(a)$.

Proof Owing to Lemma 5.20 we have

$$T^*_{\varrho} \mid\!\frac{\Omega\cdot 2}{0}\, \exists X\, \forall y\, (y \in X \leftrightarrow B(y)),\ \neg\forall y\, (B(y) \leftrightarrow B(y))\,. \tag{6}$$

As Lemma 5.15 yields $T^*_{\varrho} \mid\!\frac{k}{0}\, \forall y\, (B(y) \leftrightarrow B(y))$ for some $k < \omega$, cutting with (6) yields $T^*_{\varrho} \mid\!\frac{\Omega\cdot 2+1}{\omega}\, \exists X\, \forall y\, (y \in X \leftrightarrow B(x))\,.$ □

Corollary 5.22 *For every arithmetic relation* $\prec$ *(parameters allowed) and arbitrary formula* $A(a)$ *we have* $T^*_{\varrho} \mid\!\frac{\Omega\cdot 2+\omega}{0}\, \forall\vec{X}\, \forall\vec{x}(\mathrm{WF}(\prec) \to \mathrm{TI}(\prec, A))$ *where the quantifiers* $\forall\vec{X}\, \forall\vec{x}$ *bind all free variables in* $\mathrm{WF}(\prec) \to \mathrm{TI}(\prec, A)$.

Proof By Lemma 5.20 we have $T^*_{\varrho} \mid\!\frac{\Omega\cdot 2}{0}\, \neg(\mathrm{WF}(\prec))', (\mathrm{TI}(\prec, A))'$ where $'$ denotes any assignment of free numerical variables to numerals. Hence

$$T^*_{\varrho} \mid\!\frac{\Omega\cdot 2+2}{0}\, (\mathrm{WF}(\prec) \to \mathrm{TI}(\prec, A))'$$

by two applications of $(\vee)$. Applying the ω-rule the right number of times followed by the right number of $(\forall_2)$ inferences, one arrives at the desired conclusion. □

5.3 The Reduction Procedure for T^*_{ϱ}

Below we follow [13, Section 7].

Lemma 5.23 *Let* C *be a formula of grade* ϱ. *Suppose* C *is a prime formula or of either form* $\exists X H(X)$, $\exists x G(x)$ *or* $A \vee B$. *Let* $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_k}$ *with* $\delta \le \omega^{\alpha_k} \le \cdots \le \omega^{\alpha_1}$. *Then we have* $T^*_{\varrho} \mid\!\frac{\alpha}{\varrho}\, \Delta, \neg C\ \ \&\ T^*_{\varrho} \mid\!\frac{\delta}{\varrho}\, \Gamma, C\ \ \Longrightarrow T^*_{\varrho} \mid\!\frac{\alpha+\delta}{\varrho}\, \Delta, \Gamma\,.$

Proof We proceed by induction on δ.

1. Let Γ, C be an axiom. Then there are three cases to consider.

1.1. Γ is an axiom. Then so is Δ, Γ. Hence $T^*_{\varrho} \mid\!\frac{\alpha+\delta}{\varrho}\, \Delta, \Gamma\,.$

1.2. C is a true constant prime formula or negated prime formula. A straightforward induction on α then yields $T^*_{\varrho} \mid\!\frac{\alpha}{\varrho}\, \Delta$, and thus $T^*_{\varrho} \mid\!\frac{\alpha+\delta}{\varrho}\, \Delta, \Gamma$ by Lemma 5.14 (1).

1.3. $C \equiv A(s_1, \ldots, s_n)$ and Γ contains a formula $\neg A(t_1, \ldots, t_n)$ where s_i and t_i are equivalent terms. From $T^*_{\varrho} \mid\!\frac{\alpha}{\varrho}\, \Delta, \neg A(s_1, \ldots, s_n)$ one receives $T^*_{\varrho} \mid\!\frac{\alpha}{\varrho}\, \Delta, \neg A(t_1, \ldots, t_n)$ by use of Lemma 5.14 (4). Thence $T^*_{\varrho} \mid\!\frac{\alpha+\delta}{\varrho}\, \Delta, \Gamma$ follows by use of Lemma 5.14 (1), since $\neg A(t_1, \ldots, t_n) \in \Gamma$.

2. Suppose $C \equiv A \vee B$ and $T^*_Q \overset{\delta_0}{\underset{\varrho}{\vdash}} \Gamma, C, A_0$ with $A_0 \in \{A, B\}$ and $\delta_0 \lhd \delta$. Inductively we get

$$T^*_Q \overset{\alpha+\delta_0}{\underset{\varrho}{\vdash}} \Delta, \Gamma, A_0 \,. \tag{7}$$

Next use Lemma 5.14 (2) on $T^*_Q \overset{\alpha}{\underset{\varrho}{\vdash}} \Delta, \neg A \wedge \neg B$ to obtain

$$T^*_Q \overset{\alpha+\delta_0}{\underset{\varrho}{\vdash}} \Delta, \Gamma, \neg A_0 \,. \tag{8}$$

Whence use a cut on (7) and (8) to get the assertion.

3. Suppose $C \equiv \exists x G(x)$ and $T^*_Q \overset{\delta_0}{\underset{\varrho}{\vdash}} \Gamma, C, G(t)$ with $\delta_0 \lhd \delta$. Inductively we get

$$T^*_Q \overset{\alpha+\delta_0}{\underset{\varrho}{\vdash}} \Delta, \Gamma, G(t) \,. \tag{9}$$

By Lemma 5.14 (1) and (5), we also get

$$T^*_Q \overset{\alpha+\delta_0}{\underset{\varrho}{\vdash}} \Delta, \Gamma, \neg G(t) \,; \tag{10}$$

thus (9) and (10) yield $T^*_Q \overset{\alpha+\delta}{\underset{\varrho}{\vdash}} \Delta, \Gamma$ by (*Cut*).

4. Suppose the last inference was $(\exists_2)$ with principal formula C. Then $C \equiv \exists X H(X)$ and $T^*_Q \overset{\delta_0}{\underset{\varrho}{\vdash}} \Gamma, C, H(U)$ for some $\delta_0 \lhd \delta$ and $gr(H(U)) \geq \omega$. Inductively we get

$$T^*_Q \overset{\alpha+\delta_0}{\underset{\varrho}{\vdash}} \Delta, \Gamma, H(U). \tag{11}$$

By Lemma 5.14 (1) and (6) we also get

$$T^*_Q \overset{\alpha+\delta_0}{\underset{\varrho}{\vdash}} \Delta, \Gamma, \neg H(U). \tag{12}$$

From (11) and (12) we obtain

$$T^*_Q \overset{\alpha+\delta}{\underset{\varrho}{\vdash}} \Delta, \Gamma.$$

5. Let $T^*_Q \overset{\delta}{\underset{\varrho}{\vdash}} \Gamma, C$ be derived by the Ω-rule with fundamental function f. Then the assertion follows from the I. H. by the Ω-rule using the fundamental function $\alpha + f$.
6. In the remaining cases the assertion follows from the I. H. used on the premises and by reapplying the same inference. □

Lemma 5.24 $T_Q^* \,\big|\!\frac{\alpha}{\eta+1}\, \Gamma \implies T_Q^* \,\big|\!\frac{\omega^\alpha}{\eta}\, \Gamma$.

Proof We proceed by induction on α. We only treat the crucial case when $T_Q^* \,\big|\!\frac{\alpha_0}{\eta+1}\, \Gamma, D$ and $T_Q^* \,\big|\!\frac{\alpha_0}{\eta+1}\, \Gamma, \neg D$, where $\alpha_0 \lhd \alpha$, and $gr(D) = \eta$. Inductively this becomes $T_Q^* \,\big|\!\frac{\omega^{\alpha_0}}{\eta}\, \Gamma, D$ and $T_Q^* \,\big|\!\frac{\omega^{\alpha_0}}{\eta}\, \Gamma, \neg D$. Since D or $\neg D$ must be one of the forms exhibited in Lemma 5.23, we obtain $T_Q^* \,\big|\!\frac{\omega^{\alpha_0}+\omega^{\alpha_0}}{\eta}\, \Gamma$ by Lemma 5.23. As $\omega^{\alpha_0} + \omega^{\alpha_0} \lhd \omega^\alpha$, we can use Lemma 5.14 (1) to get the assertion. □

Theorem 5.25 (Collapsing Theorem) *Let* Γ *be a set of weak formulas. We have*

$$T_Q^* \,\big|\!\frac{\alpha}{\omega}\, \Gamma \implies T_Q^* \,\big|\!\frac{\vartheta\alpha}{0}\, \Gamma.$$

Proof We proceed by induction on α. Observe that for $\beta < \delta < \Omega$, we always have $\beta \lhd \delta$.

1. If Γ is an axiom, then the assertion is trivial.
2. Let $T_Q^* \,\big|\!\frac{\alpha}{\omega}\, \Gamma$ be the result of an inference other than (*Cut*) and Ω-rule. Then we have $T_Q^* \,\big|\!\frac{\alpha_0}{\omega}\, \Gamma_i$ with $\alpha_0 \lhd \alpha$ and Γ_i being the i-th premiss of that inference. $\alpha_0 \lhd \alpha$ implies $\vartheta\alpha_0 \lhd \vartheta\alpha$. Therefore $T_Q^* \,\big|\!\frac{\vartheta\alpha_0}{0}\, \Gamma_0$ by the I. H., hence $T_Q^* \,\big|\!\frac{\vartheta\alpha}{0}\, \Gamma$ by reapplying the same inference.
3. Suppose $T_Q^* \,\big|\!\frac{\alpha}{\omega}\, \Gamma$ results by the Ω-rule with respect to a Π^1_1-formula $\forall X F(X)$ and a fundamental function f. Then $f(\Omega) \unlhd \alpha$ and

$$T_Q^* \,\big|\!\frac{f(0)}{\omega}\, \Gamma, \forall X F(X), \tag{13}$$

and, for every set of weak formulas Ξ and $\beta < \Omega$,

$$T_Q^* \,\big|\!\frac{\beta}{0}\, \Xi, \forall X F(X) \implies T_Q^* \,\big|\!\frac{f(\beta)}{\omega}\, \Xi, \Gamma. \tag{14}$$

The I. H. used on (13) supplies us with $T_Q^* \,\big|\!\frac{\vartheta(f(0))}{0}\, \Gamma, \forall X F(X)$. Hence with $\Xi = \Gamma$ we get

$$T_Q^* \,\big|\!\frac{f(\vartheta(f(0)))}{\omega}\, \Gamma \tag{15}$$

from (14). Now Lemma 5.8 ensures that $f(\beta) \lhd f(\Omega)$, where $\beta = \vartheta(f(0))$. So using the I. H. on (15), we obtain

$$T_Q^* \,\big|\!\frac{\vartheta(f(\beta))}{0}\, \Gamma, \tag{16}$$

thus $T_Q^* \,\big|\!\frac{\vartheta\alpha}{0}\, \Gamma$ as $f(\beta) \lhd \alpha$.

4. Suppose $T_Q^* \mid\!\frac{\alpha_0}{\omega}\, \Gamma, A$ and $T_Q^* \mid\!\frac{\alpha_0}{\omega}\, \Gamma, \neg A$, where $\alpha_0 \lhd \alpha$ and $gr(A) < \omega$. Inductively we then get $T_Q^* \mid\!\frac{\vartheta\alpha_0}{0}\, \Gamma, A$ and $T_Q^* \mid\!\frac{\vartheta\alpha_0}{0}\, \Gamma, \neg A$. Let $gr(A) = n - 1$. Then (Cut) yields

$$T_Q^* \mid\!\frac{\beta_1}{n}\, \Gamma \tag{17}$$

with $\beta_1 = (\vartheta\alpha_0) + 1$. Applying Lemma 5.24, we get $T_Q^* \mid\!\frac{\omega^{\beta_1}}{n-1}\, \Gamma$, and by repeating this process we arrive at

$$T_Q^* \mid\!\frac{\beta_n}{0}\, \Gamma\,,$$

where $\beta_{k+1} := \omega^{\beta_k}$ $(1 \leq k < n)$. Since $\vartheta\alpha_0 < \vartheta\alpha$, we have $\beta_n < \vartheta\alpha$; thus, $T_Q^* \mid\!\frac{\vartheta\alpha}{0}\, \Gamma$. □

5.4 Embedding $\mathcal{D}_Q$ into T_Q^*

Assuming that $\mathcal{D}_Q$ is well-founded tree, the objective of this section is to embed $\mathcal{D}_Q$ into T_Q^*, so as to obtain a contradiction. Let $\mathfrak{X}$ be the Kleene–Brouwer ordering of $\mathcal{D}_Q$. We write $\mathcal{D}_Q \mid\!\frac{\tau}{}\, \Gamma$ if Γ is the sequent attached to the node τ in $\mathcal{D}_Q$.

Theorem 5.26 $\mathcal{D}_Q \mid\!\frac{\tau}{}\, \Xi \Rightarrow \exists k < \omega\, T_Q^* \mid\!\frac{\mathfrak{E}_\tau + k}{\omega}\, \Xi$.

Proof We proceed by induction on τ, i.e., the Kleene–Brouwer ordering of $\mathcal{D}_Q$.

Suppose τ is an end-node of $\mathcal{D}_Q$. Then Ξ must be axiomatic and therefore is an axiom of T_Q^*, and hence $T_Q^* \mid\!\frac{\mathfrak{E}_\tau}{\omega}\, \Xi$.

Now assume that τ is not an end-node of $\mathcal{D}_Q$. Then Ξ is not axiomatic.

If Ξ is not reducible, then there is a node τ_0 immediately above τ in $\mathcal{D}_Q$ such that $\mathcal{D}_Q \mid\!\frac{\tau_0}{}\, \Xi, \neg\bar{Q}(i), \neg A_i$ for some i. Inductively we have

$$T_Q^* \mid\!\frac{\mathfrak{E}_{\tau_0} + k_0}{\omega}\, \Xi, \neg\bar{Q}(i), \neg A_i$$

for some $k_0 < \omega$. We also have $T_Q^* \mid\!\frac{0}{0}\, \bar{Q}(i)$ and, using Corollaries 5.21 (if $i = 0$) and 5.22 (if $i > 0$), $T_Q^* \mid\!\frac{\Omega\cdot 2+\omega}{\omega}\, A_i$. Thus, noting that $\Omega \cdot 2 + \omega \lhd \mathfrak{E}_{\tau_0} + k_0$, and by employing two cuts we arrive at

$$T_Q^* \mid\!\frac{\mathfrak{E}_{\tau_0} + k_0 + 2}{\omega + n}\, \Xi$$

for some $n < \omega$. By Lemma 5.24 we get $T_Q^* \mid\frac{\omega_n(\mathfrak{E}_{\tau_0}+k_0+2)}{\omega} \Xi$, and hence $T_Q^* \mid\frac{\mathfrak{E}_\tau}{\omega} \Xi$ since $\omega_n(\mathfrak{E}_{\tau_0} + k_0 + 2) \lhd \mathfrak{E}_\tau$.

Now suppose that Ξ is reducible. Ξ will be of the form

$$\Xi',\ E,\ \Xi''$$

where E is not a literal and Ξ' contains only literals.

First assume E to be of the form $\forall x\, F(x)$. Then, for each m, there is a node τ_m immediately above τ in $\mathcal{D}_Q$ such that

$$\mathcal{D}_Q \stackrel{\tau_n}{\models} \Xi', F(\bar{m}), \Xi'', \neg\bar{Q}(i), \neg A_i$$

for some i. Inductively we have

$$T_Q^* \mid\frac{\mathfrak{E}_{\tau_m}+k_m}{\omega} \Xi', F(\bar{m}), \Xi'', \neg\bar{Q}(i), \neg A_i$$

for all m, where $k_m < \omega$. We also have $T_Q^* \mid\frac{0}{0} \bar{Q}(i)$ and, using Lemma 5.22, $T_Q^* \mid\frac{\Omega\cdot 2+\omega}{0} A_i$. Thus, noting that $\Omega \cdot 2 + \omega \lhd \mathfrak{E}_{\tau_m} + k_m$, and by employing two cuts there is an n such that

$$T_Q^* \mid\frac{\mathfrak{E}_{\tau_m}+k_m+2}{\omega+n} \Xi', F(\bar{m}), \Xi''$$

holds for all m. By Lemma 5.24 we get

$$T_Q^* \mid\frac{\omega_n(\mathfrak{E}_{\tau_m}+k_m+2)}{\omega} \Xi', F(\bar{m}), \Xi''$$

for all m. Whence

$$T_Q^* \mid\frac{\mathfrak{E}_\tau}{\omega} \Xi', F(\bar{m}), \Xi''$$

since $\omega_n(\mathfrak{E}_{\tau_m} + k_m + 2) \lhd \mathfrak{E}_\tau$. A final application of the ω-rule yields

$$T_Q^* \mid\frac{\mathfrak{E}_\tau+1}{\omega} \Xi', \forall x\, F(x), F(\bar{m}), \Xi''$$

i.e., $T_Q^* \mid\frac{\mathfrak{E}_\tau+1}{\omega} \Xi$.

If E is a redex of another type but not of the form $\exists X B(X)$ with $B(U)$ arithmetic, then one proceeds in a similar way as in the previous case.

Now assume E to be of the form $\exists X\ B(X)$ with $B(U)$ arithmetic. Then there is a node τ_0 immediately above τ in $\mathcal{D}_Q$ such that

$$\mathcal{D}_Q \stackrel{\tau_0}{\models} \Xi', B(U), \Xi'', \neg\bar{Q}(i), \neg A_i$$

for some i and set variable U. Inductively we have

$$T_Q^* \mathrel{\Big|\frac{\mathfrak{E}_{\tau_0}+k_0}{\omega}} \Xi', B(U), \Xi'', \neg\bar{Q}(i), \neg A_i$$

for some $k_0 < \omega$. We also have $T_Q^* \mathrel{\Big|\frac{0}{0}} \bar{Q}(i)$ and, using Lemma 5.22, $T_Q^* \mathrel{\Big|\frac{\Omega\cdot 2+\omega}{0}} A_i$. Thus, noting that $\Omega \cdot 2 + \omega \lhd \mathfrak{E}_{\tau_0} + k_0$, and by employing two cuts there is an n such that

$$T_Q^* \mathrel{\Big|\frac{\mathfrak{E}_{\tau_0}+k_0+2}{\omega+n}} \Xi', B(U), \Xi''.$$

By Lemma 5.24 we get

$$T_Q^* \mathrel{\Big|\frac{\omega_n(\mathfrak{E}_{\tau_0}+k_0+2)}{\omega}} \Xi', B(U), \Xi''. \tag{18}$$

Lemma 5.20 yields

$$T_Q^* \mathrel{\Big|\frac{\Omega\cdot 2}{0}} \exists X B(X), \neg B(U). \tag{19}$$

Cutting $B(U)$ and $\neg B(U)$ out of (18) and (19) we arrive at

$$T_Q^* \mathrel{\Big|\frac{\omega_n(\mathfrak{E}_{\tau_0}+k_0+2)+1}{\omega}} \Xi', \exists X B(X), \Xi''.$$

Since $\omega_n(\mathfrak{E}_{\tau_0} + k_0 + 2) + 1 \lhd \mathfrak{E}_\tau$ we get $T_Q^* \mathrel{\Big|\frac{\mathfrak{E}_\tau}{\omega}} \Xi', \exists X B(X), \Xi''$, i.e., $T_Q^* \mathrel{\Big|\frac{\mathfrak{E}_\tau}{\omega}} \Xi$. □

Below $\emptyset$ stands for the empty sequent and τ_0 denotes the bottom node of $\mathcal{D}_Q$ which is the maximum element of the pertaining Kleene–Brouwer ordering.

Corollary 5.27 *If $\mathcal{D}_Q$ is well-founded, then $T_Q^* \mathrel{\Big|\frac{\vartheta(\omega_n(\mathfrak{E}_{\tau_0}+m))}{0}} \emptyset$ for some $n, m < \omega$.*

Proof We have $\mathcal{D}_Q \mathrel{\Big|\frac{\tau_0}{}} \neg\bar{Q}(0), \neg A_0$. Thus there is a $k < \omega$ such that

$$T_Q^* \mathrel{\Big|\frac{\mathfrak{E}_{\tau_0}+k}{\omega}} \neg\bar{Q}(0), \neg A_0$$

holds by Theorem 5.26. We also have $T_Q^* \mathrel{\Big|\frac{0}{0}} \bar{Q}(0)$ and, using Corollary 5.22, $T_Q^* \mathrel{\Big|\frac{\Omega\cdot 2+\omega}{0}} A_0$. Thus, noting that $\Omega \cdot 2 + \omega \lhd \mathfrak{E}_{\tau_0} + k$, and by employing two cuts we arrive at

$$T_Q^* \mathrel{\Big|\frac{\mathfrak{E}_{\tau_0}+k+2}{\omega+n}} \emptyset$$

for some $n < \omega$. Via Lemma 5.24 we deduce $T_Q^* \mid\frac{\omega_n(\mathfrak{E}_{\tau_0}+k+2)}{\omega} \emptyset$, so that by Theorem 5.25 we conclude $T_Q^* \mid\frac{\vartheta(\omega_n(\mathfrak{E}_{\tau_0}+m))}{0} \emptyset$ with $m = k + 2$. □

Corollary 5.28 *$\mathcal{D}_Q$ is not well-founded.*

Proof If $\mathcal{D}_Q$ were well-founded, we would have

$$T_Q^* \mid\frac{\vartheta(\omega_n(\mathfrak{E}_{\tau_0}+m))}{0} \emptyset \tag{20}$$

for some $n, m < \omega$ by Corollary 5.27. But a straightforward induction on $\alpha < \Omega$ shows that

$$T_Q^* \mid\frac{\alpha}{0} \Gamma \;\Rightarrow\; \Gamma \neq \emptyset,$$

yielding that (20) is impossible. □

It remains to show that the result of Corollary 5.28 is provable in $\mathbf{ACA}_0$ from

$$\forall \mathfrak{X}\, (\mathrm{WO}(\mathfrak{X}) \to \mathrm{WO}(\vartheta_{\mathfrak{X}}))\,.$$

Let $\mathbf{S}$ be the theory $\mathbf{ACA}_0$ plus the latter axiom. The main issue is how to formalize the derivability predicate $T_Q^* \mid\frac{\alpha}{\rho} \Gamma$ in the background theory $\mathbf{S}$. We elaborated earlier in Remark 5.13 that this seems to require an iterated inductive definition, something apparently not available in $\mathbf{S}$. However, all we need is a fixed point not a proper inductive definition, i.e., to capture the notion of derivability in T_Q^* without the Ω-rule it suffices to find a predicate $\mathcal{D}$ of α, ρ, Γ such that

(∗) $\mathcal{D}(\alpha, \rho, \Gamma)$ if and only if $\alpha \in |\vartheta_{\mathfrak{X}}|$, $\rho \leq \omega + \omega$, Γ is a sequent, and either Γ contains an axiom of T_Q^* or Γ is the conclusion of an inference of T_Q^* other than (Ω) with premisses $(\Gamma_i)_{i \in I}$ such that for every $i \in I$ there exists $\beta_i \lhd \alpha$ with $\mathcal{D}(\beta_i, \rho, \Gamma_i)$, and if the inference is a cut it has rank $< \rho$.

(∗) can be viewed as a fixed-point axiom which together with transfinite induction for $\vartheta_{\mathfrak{X}}$ defines T_Q^*-derivability (without (Ω)-rule) implicitly.

How can we find a fixed point as described in (∗)? As it turns out, it follows from [12] that $\mathbf{S}$ proves that every set is contained in a countable coded ω-model of the theory $\mathbf{ATR}_0$. It is also known that $\mathbf{ATR}_0$ proves the Σ^1_1 axiom of choice, Σ^1_1-$\mathbf{AC}$ (see [17, Theorem V.8.3]). Moreover, in $\mathbf{ACA}_0 + \Sigma^1_1$-$\mathbf{AC}$ one can prove for every P-positive arithmetical formula $A(u, P)$ that there is a Σ^1_1 formula $F(u)$ such that $\forall x[F(x) \leftrightarrow A(x, F)]$, where $A(x, F)$ arises from $A(x, P)$ by replacing every occurrence of the form $P(t)$ in the first formula by $F(t)$. This is known as the Second Recursion Theorem (see [2, V.2.3]). Arguing in $\mathbf{S}$, we find a countable coded ω model $\mathfrak{B}$ with $\mathfrak{X} \in \mathfrak{B}$ such that $\mathfrak{B}$ is a model of $\mathbf{ATR}$. As a result, there is a predicate $\mathcal{D}$ definable in $\mathfrak{B}$ that satisfies (∗). As a result, $\mathcal{D}$ is a set in $\mathbf{S}$. To obtain the full derivability relation $T_Q^* \mid\frac{\alpha}{\rho} \Gamma$ we have to take the Ω-rule into account. We do this by taking a countable coded ω-model $\mathfrak{C}$ of **ATR** that contains both $\mathfrak{X}$ and $\mathcal{D}$. We then define an appropriate fixed point predicate $\mathcal{D}_\Omega$ using the clauses for defining $T^*_\varrho \mid\frac{\alpha}{\rho}\, \Gamma$ and $\mathcal{D}$ for the negative occurrences in the Ω-rule.

The upshot is that we can formalize all of this in **S**.

Remark 5.29 When giving talks about the material of this article, the first author was asked what the proof-theoretic ordinal of the theories that Theorem 1.7 is concerned with might be. He conjectures that it is the ordinal

$$\vartheta(\varphi 2(\Omega + 1))$$

(or $\psi(\varphi 2(\Omega + 1))$ in the representation system based on the ψ-function; see [13, Section 3]), i.e. the collapse of the first fixed point of the epsilon function above Ω.

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