Chapter 10
Higher Spin Black Holes

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Abstract We review some relevant results in the context of higher spin black holes in three-dimensional spacetimes, focusing on their asymptotic behaviour and thermodynamic properties. For simplicity, we mainly discuss the case of gravity nonminimally coupled to spin-three fields, being nonperturbatively described by a Chern–Simons theory of two independent $sl(3,\mathbb{R})$ gauge fields. Since the analysis is particularly transparent in the Hamiltonian formalism, we provide a concise discussion of their basic aspects in this context; and as a warming up exercise, we briefly analyze the asymptotic behaviour of pure gravity, as well as the BTZ black hole and its thermodynamics, exclusively in terms of gauge fields. The discussion is then extended to the case of black holes endowed with higher spin fields, briefly signaling the agreements and discrepancies found through different approaches. We conclude explaining how the puzzles become resolved once the fall off of the fields is precisely specified and extended to include chemical potentials, in a way that it is compatible with the asymptotic symmetries. Hence, the global charges become completely identified in an unambiguous way, so that different sets of asymptotic conditions turn out to contain inequivalent classes of black hole solutions being characterized by a different set of global charges.

10.1 Introduction

Fundamental particles of spin greater than two are hitherto unknown, which from a purely theoretical point of view, appears to agree with the widespread belief that massless fields of spin $s > 2$ are doomed to suffer from inconsistencies. Indeed, the lore is reflected through a well-known claim in the context of supergravity (see e.g.,

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[1]), which asserts that the maximum number of local supersymmetries is bounded by eight; otherwise, since the supersymmetry generators act as raising or lowering operators for spin, a supermultiplet would contain fields of spin greater than two. In turn, through the Kaluza–Klein mechanism, this also sets an upper bound on the spacetime dimension to be at most eleven. The supposed inconsistency of higher spin fields relies on solid no-go theorems (see [2] for a good review about this subject). In particular, it is worth mentioning the result of Aragone and Deser [3], which states that the higher spin gauge symmetries of the free theory around flat spacetime, cannot be preserved once the field is minimally coupled to gravity.

A consistent way to circumvent the incompatibility of higher spin gauge symmetries with interactions was pioneered by Vasiliev [4, 5], who was able to formulate the field equations for a whole tower of nonminimally coupled fields of spin $s = 0, 1, 2, \ldots, \infty$, in presence of a cosmological constant (For recent reviews see e.g., [6, 7]). It is worth pointing out that, since the hypotheses of the Coleman–Mandula theorem are not fulfilled by Vasiliev theory, spacetime and gauge symmetries become inherently mixed in an unaccustomed form [8]. It then goes without saying that the very existence of Vasiliev theory, naturally suggests a possible reformulation of supergravity theories from scratch, which would may in turn elucidate new alternative approaches to strings and M-theory. Indeed, in eleven dimensions and in presence of a negative cosmological constant, a supergravity theory that shares some of these features, as the mixing of spacetime and gauge symmetries, is known to exist [9].

In order to gain some insights about this counterintuitive subject, one may instead follow the less ambitious approach of finding a simpler set up that still captures some of the relevant features that characterize the dynamics of higher spin fields. In this sense, the three-dimensional case turns out to be particularly appealing, since the dynamics is described through a standard field theory with a Chern–Simons action [10–12]. The generic theory can be further simplified, since it admits a consistent truncation to the case of a finite number of nonpropagating fields with spin $s = 2, 3, \ldots, N$. Hence the simplest case with the desired properties corresponds to $N = 3$, so that the theory describes gravity with negative cosmological constant, nonminimally coupled to an interacting spin-three field. The remarkable simplification of the theory then allows the possibility of finding different classes of exact black hole solutions endowed with a nontrivial spin-three field, as the ones in [13, 14], and [15], respectively. However, despite the simplicity of these solutions, the subject has not been free of controversy, mainly due to the puzzling discrepancies that have been found in the characterization of their global charges and their entropy.

The purpose of this brief review, is overviewing some of the relevant results about this ongoing subject, as well as explaining how the apparent tension between different approaches is fully resolved once the chemical potentials are suitably identified along the lines of [15, 16], so that the asymptotic symmetries, and hence the global charges, are completely characterized in an unambiguous way.

It is worth highlighting that the action principle in terms of the metric and the spin-three field is currently known as a weak field expansion of the spin-three field up to quadratic order [17]. Thus, in order to deal with the full nonperturbative treatment of the higher spin black hole solutions, it turns out to be useful to describe them only in terms of gauge fields and the topology of the manifold, without making any reference neither to the metric nor to the spin-three field.

Since the analysis becomes particularly transparent in the Hamiltonian formalism, in the next section we concisely discuss some of their basic aspects in the context of Chern–Simons theories in three dimensions. As a useful warming up exercise, in Sect. 10.3, the asymptotic behaviour of pure gravity with negative cosmological constant [18], as well as the BTZ black hole [19, 20] and its thermodynamics, are briefly analyzed exclusively in terms of gauge fields. Section 10.4 is devoted to the case of gravity coupled to spin-three fields, including the asymptotic behaviour described in [21, 22], the higher spin black hole solution of [13, 23], and its thermodynamics [24, 25], briefly signaling the agreements and discrepancies found through different approaches. We conclude with Sect. 10.5, where it is explained how these puzzling differences become fully resolved once the fall off of the fields is precisely specified, so that different sets of asymptotic conditions turn out to contain inequivalent classes of black hole solutions [15, 16] being characterized by a different set of global charges.

10.2 Basic Aspects and Hamiltonian Formulation of Chern–Simons Theories in Three Dimensions

In three-dimensional spacetimes, gauge theories described by a Chern–Simons action are much simpler than their corresponding Yang–Mills analogues, in the sense that less structure is required in order to formulate them. Indeed, the manifold M, locally described by a set of coordinates x^μ, is only endowed with a gauge field $A = A^I_\mu T_I dx^\mu$, where T_I stand for the generators of a Lie algebra $\mathfrak{g}$, which is assumed to admit an invariant nondegenerate bilinear form $g_{IJ} = \langle T_I, T_J \rangle$. These ingredients are enough to construct the action, given by

$$I_{CS}[A] = \frac{k}{4\pi} \int_M \left\langle AdA + \frac{2}{3}A^3 \right\rangle , \tag{10.1}$$

where k is a constant, and wedge product between forms has been assumed. Consequently, the action does not require the existence of a spacetime metric, but it is sensitive to the topology of M. The field equations imply the vanishing of curvature, i.e., $F = dA + A^2 = 0$, so that the connection becomes locally flat on shell, and then the theory is devoid of local propagating degrees of freedom. Note that the action (10.1) is already in Hamiltonian form. Indeed, if the topology of M is of the form $M = \Sigma \times \mathbb{R}$, where Σ stands for the spacelike section, the connection splits as $A = A_i dx^i + A_t dt$, and hence the action (10.1) reduces to

$$I_H = -\frac{k}{4\pi}\int_M dtd^2x\varepsilon^{ij}\left\langle A_i\dot{A}_j - A_tF_{ij}\right\rangle \ , \tag{10.2}$$

up to a boundary term. It is then apparent that A_i correspond to the dynamical fields, whose Poisson brackets are given by $\left\{A_i^I\left(x\right), A_j^J\left(x'\right)\right\} = \frac{2\pi}{k}g^{IJ}\varepsilon_{ij}\delta\left(x - x'\right)$, while A_t become Lagrange multipliers associated to the constraints $G = \frac{k}{4\pi}\varepsilon^{ij}F_{ij}$. Then, the smeared generator of the gauge transformations reads

$$G\left(\Lambda\right) = \int_\Sigma d^2x\left\langle\Lambda G\right\rangle \ ,$$

so that $\delta A_i = \left\{A_i, G\left(\Lambda\right)\right\} = \partial_i\Lambda + \left[A_i, \Lambda\right]$ (see, e.g., [26–28]). However, when Σ has a boundary, according to the Regge–Teitelboim approach [29], the generator of the gauge transformations has to be improved by a boundary term $Q\left(\Lambda\right)$, i.e.,

$$\tilde{G}\left(\Lambda\right) = G\left(\Lambda\right) + Q\left(\Lambda\right) \ , \tag{10.3}$$

being such that its functional variation is well-defined everywhere. This implies that the variation of the conserved charge associated to an asymptotic gauge symmetry, generated by a Lie algebra valued parameter Λ, is determined by the dynamical fields at a fixed time slice at the boundary, which reads

$$\delta Q\left(\Lambda\right) = -\frac{k}{2\pi}\int_{\partial\Sigma}\left\langle\Lambda\delta A_\theta\right\rangle d\theta \ , \tag{10.4}$$

where $\partial\Sigma$ stands for the boundary of the spacelike section Σ.

The transformation law of the Lagrange multipliers, $\delta A_t = \partial_t\Lambda + \left[A_t, \Lambda\right]$, is then recovered requiring the improved action to be invariant under gauge transformations. Note that on-shell, by virtue of the identity $\mathscr{L}_\xi A_\mu = \nabla_\mu\left(\xi^\nu A_\nu\right) + \xi^\nu F_{\nu\mu}$, diffeomorphisms $\delta_\xi A_\mu = -\mathscr{L}_\xi A_\mu$ are equivalent to gauge transformations with parameter $\Lambda = -\xi^\mu A_\mu$, and hence, the variation of the generator of an asymptotic symmetry spanned by an asymptotic killing vector ξ^μ, reads

$$\delta Q\left(\xi\right) = \frac{k}{2\pi}\int_{\partial\Sigma}\xi^\mu\left\langle A_\mu\delta A_\theta\right\rangle d\theta \ . \tag{10.5}$$

This means that the variation of the total energy of the system, which takes into account the contribution of all the constraints, is given by

$$\delta E = \delta Q\left(\partial_t\right) = \frac{k}{2\pi}\int_{\partial\Sigma}\left\langle A_t\delta A_\theta\right\rangle d\theta \ . \tag{10.6}$$

It should be stressed that the whole canonical structure only makes sense provided the variation of the canonical generators can be integrated. This can be generically done once a precise set of asymptotic conditions is specified, which in turn determines the asymptotic symmetries. This will be explicitly discussed in the next section for the case of pure gravity with negative cosmological constant, as well as in Sect. 10.4, and further elaborated in Sect. 10.5 in the case of gravity coupled to a spin-three field.

10.3 General Relativity with Negative Cosmological Constant in Three Dimensions

As it was shown in [30, 31] General Relativity in vacuum can be described in terms of a Chern–Simons action. In the case of negative cosmological constant the corresponding Lie algebra is of the form $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$, where $\mathfrak{g}_\pm$ stand for two independent copies of $sl(2,\mathbb{R})$, which will be assumed to be described by the same set of matrices L_i, with $i = -1, 0, 1$, given by

$$L_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad ; \quad L_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad ; \quad L_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} , \tag{10.7}$$

so that the $sl(2,\mathbb{R})$ algebra reads

$$\left[L_i, L_j\right] = (i-j)\, L_{i+j} \ . \tag{10.8}$$

The connection then splits in two independent $sl(2,\mathbb{R})$-valued gauge fields, according to $A = A^+ + A^-$, while the invariant nondegenerate bilinear form is chosen such that the action (10.1) reduces to

$$I = I_{CS}\left[A^+\right] - I_{CS}\left[A^-\right] , \tag{10.9}$$

so that the bracket now corresponds to just the trace, i.e., in the representation of (10.7), $\langle \cdots \rangle = \mathrm{tr}(\cdots)$, and the level is fixed by the AdS radius and the Newton constant as $k = \frac{l}{4G}$. The link between the gauge fields and spacetime geometry is made through

$$A^\pm = \omega \pm \frac{e}{l} , \tag{10.10}$$

where ω and e correspond to the spin connection and the dreibein, respectively. The field equations, $F^\pm = 0$, then imply that the spacetime curvature is constant and the torsion vanishes, while the metric is recovered from

$$g_{\mu\nu} = 2\mathrm{tr}\left(e_\mu e_\nu\right) , \tag{10.11}$$

which is manifestly invariant under the diagonal subgroup of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, that corresponds to the local Lorentz transformations. Note that diffeomorphisms can always be expressed in terms of the remaining gauge symmetries.

10.3.1 Brown–Henneaux Boundary Conditions

As explained in [32], the asymptotic behaviour of gravity with negative cosmological constant, as originally described by Brown and Henneaux [18], can be readily formulated in terms of the gauge fields $A^{\pm}$. The gauge can be chosen such that the radial dependence is entirely captured by the group elements

$$g_{\pm} = e^{\pm \rho L_0} \,, \tag{10.12}$$

so that the asymptotic form of the connections is given by

$$A^{\pm} = g_{\pm}^{-1} a^{\pm} g_{\pm} + g_{\pm}^{-1} d g_{\pm} \,, \tag{10.13}$$

where $a^{\pm} = a_{\theta}^{\pm} d\theta + a_t^{\pm} dt$, read

$$a^{\pm} = \pm \left(L_{\pm 1} - \frac{2\pi}{k} \mathscr{L}_{\pm} L_{\mp 1} \right) dx^{\pm} \,, \tag{10.14}$$

with $x^{\pm} = \frac{t}{l} \pm \theta$, and the functions $\mathscr{L}_{\pm}$ depend only on time and the angular coordinate.

The asymptotic form of the dynamical fields $a_{\theta}^{\pm}$ is preserved under gauge transformations, $\delta a_{\theta}^{\pm} = \partial_{\theta} \Lambda^{\pm} + \left[a_{\theta}^{\pm}, \Lambda^{\pm} \right]$, generated by

$$\Lambda^{\pm} \left(\varepsilon_{\pm} \right) = \varepsilon_{\pm} L_{\pm 1} \mp \varepsilon_{\pm}' L_0 + \frac{1}{2} \left(\varepsilon_{\pm}'' - \frac{4\pi}{k} \varepsilon_{\pm} \mathscr{L}_{\pm} \right) L_{\mp 1} \,, \tag{10.15}$$

where $\varepsilon_{\pm}$ are arbitrary functions of t, θ, provided the functions $\mathscr{L}_{\pm}$ transform as

$$\delta \mathscr{L}_{\pm} = \varepsilon_{\pm} \mathscr{L}_{\pm}' + 2 \mathscr{L}_{\pm} \varepsilon_{\pm}' - \frac{k}{4\pi} \varepsilon_{\pm}''' \,. \tag{10.16}$$

Hereafter, prime denotes the derivative with respect to θ. Furthermore, requiring the components of the gauge fields along time, $a_t^{\pm}$, to be mapped into themselves under the same gauge transformations, together with the transformation laws in (10.16), implies that the functions $\mathscr{L}_{\pm}$ and the parameters $\varepsilon_{\pm}$ are chiral, i.e.,

$$\partial_{\mp} \mathscr{L}_{\pm} = 0 \;\; , \;\; \partial_{\mp} \varepsilon_{\pm} = 0 \,. \tag{10.17}$$

Note that the first condition in (10.17) means that the field equations have to be fulfilled in the asymptotic region.

Consequently, according to (10.4), the variation of the canonical generators associated to the asymptotic gauge symmetries generated by $\Lambda = \Lambda^{+} + \Lambda^{-}$, in this case reduces to

$$\delta Q\left(\Lambda\right)=\delta Q_{+}\left(\Lambda^{+}\right)-\delta Q_{-}\left(\Lambda^{-}\right)\ , \tag{10.18}$$

with

$$\delta Q_{\pm}\left(\Lambda^{\pm}\right)=-\frac{k}{2\pi}\int\left\langle \Lambda^{\pm}\delta a_{\theta}^{\pm}\right\rangle d\theta=-\int\varepsilon_{\pm}\delta\mathscr{L}_{\pm}d\theta\ , \tag{10.19}$$

which can be readily integrated as

$$Q_{\pm}\left(\Lambda^{\pm}\right)=-\int\varepsilon_{\pm}\mathscr{L}_{\pm}d\theta\ . \tag{10.20}$$

Therefore, since the canonical generators fulfill $\delta_{\Lambda_1}Q\left[\Lambda_2\right]=\left\{Q\left[\Lambda_2\right],Q\left[\Lambda_1\right]\right\}$, their algebra can be directly obtained by virtue of (10.16), which reduces to two copies of the Virasoro algebra with the same central extension $c=\frac{3l}{2G}$ [18]. Expanding in Fourier modes, according to $\mathscr{L}=\frac{1}{2\pi}\sum_m\mathscr{L}_m e^{im\theta}$, the algebra explicitly reads

$$i\left\{\mathscr{L}_m,\mathscr{L}_n\right\}=\left(m-n\right)\mathscr{L}_{m+n}+\frac{k}{2}m^3\delta_{m+n,0}\ , \tag{10.21}$$

for both copies.

10.3.2 BTZ Black Hole and Its Thermodynamics

The asymptotic conditions described above, manifestly contain the BTZ black hole solution [19, 20], being described by

$$a^{\pm}=\pm\left(L_{\pm1}-\frac{2\pi}{k}\mathscr{L}_{\pm}L_{\mp1}\right)dx^{\pm}\ , \tag{10.22}$$

when $\mathscr{L}_{\pm}$ are nonnegative constants. Indeed, by virtue of Eqs. (10.10) and (10.11), the spacetime metric is recovered in normal coordinates:

$$\begin{aligned}ds^2=l^2\Bigg[&d\rho^2+\frac{2\pi}{k}\left(\mathscr{L}_{+}\left(dx^{+}\right)^2+\mathscr{L}_{-}\left(dx^{-}\right)^2\right)\\&-\left(e^{2\rho}+\frac{4\pi^2}{k^2}\mathscr{L}_{+}\mathscr{L}_{-}e^{-2\rho}\right)dx^{+}dx^{-}\Bigg]\ .\end{aligned} \tag{10.23}$$

As shown in [33] (see also [34]), the topology of the Euclidean black hole corresponds to $\mathbb{R}^2\times S^1$, where $\mathbb{R}^2$ stands for the one of the $\rho-\tau$ plane, and $\tau=-it$ is the Euclidean time, fulfilling $0\le\tau<\beta$, where $\beta=T^{-1}$ is the inverse of the Hawking temperature. Since $\mathbb{R}^2$ can be mapped into a disk through a conformal compactification, the black hole topology is then equivalent to the one of a solid torus.

As explained in the introduction, and for later purposes, afterwards we will perform the remaining analysis exclusively in terms of the gauge fields (10.22) and the topology of the manifold, without making any reference to the spacetime metric.

The simplest gauge covariant object that is sensitive to the global properties of the manifold turns out to be the holonomy of the gauge field around a closed cycle $\mathscr{C}$, defined as

$$\mathscr{H}_{\mathscr{C}} = P \exp\left(\int_{\mathscr{C}} A_\mu dx^\mu\right) , \tag{10.24}$$

which is an element of the gauge group. Hence, since in this case the gauge group corresponds to $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, the holonomy around $\mathscr{C}$ is of the form $\mathscr{H}_{\mathscr{C}} = \mathscr{H}_{\mathscr{C}}^{+} \otimes \mathscr{H}_{\mathscr{C}}^{-}$, with

$$\mathscr{H}_{\mathscr{C}}^{\pm} = P \exp\left(\int_{\mathscr{C}} A_\mu^{\pm} dx^\mu\right) . \tag{10.25}$$

As the topology of the manifold is the one of a solid torus, there are two inequivalent classes of cycles: (I) the ones that wind around the handle, and (II) those that do not. This means that the former ones are noncontractible, while the latter can be continuously shrunk to a point. Then, the holonomies along contractible cycles are trivial, i.e.,

$$\mathscr{H}_{\mathscr{C}_{II}}^{\pm} = -1 , \tag{10.26}$$

where the negative sign is due to the fact that, according to (10.35), we are dealing with the fundamental (spinorial) representation of $SL(2,\mathbb{R})$; while the holonomies along noncontractible cycles $\mathscr{H}_{\mathscr{C}_I}^{\pm}$ are necessarily nontrivial. Indeed, it is easy to verify that this is the case for the gauge fields that describe the BTZ black hole (10.22). For simplicity, we explicitly carry out the computation in the static case, i.e., for $\mathscr{L} := \mathscr{L}_{\pm}$, since the inclusion of rotation is straightforward.

A simple noncontractible cycle in this case is parameterized by $\rho = \rho_0$, and $\tau = \tau_0$, with ρ_0, τ_0 constants, so that the corresponding holonomies around it read

$$\mathscr{H}_{\theta}^{\pm} = e^{2\pi a_\theta^{\pm}} . \tag{10.27}$$

These holonomies are then fully characterized, up to conjugacy by elements of $SL(2,\mathbb{R})$, by the eigenvalues of $2\pi a_\theta^{\pm}$, given by

$$\lambda_{\pm}^2 = 2\pi^2 \mathrm{tr}\left[\left(a_\theta^{\pm}\right)^2\right] = \frac{8\pi^3}{k}\mathscr{L} , \tag{10.28}$$

and hence, since $\mathscr{L}$ is nonnegative, they are manifestly nontrivial.

Analogously, a simple contractible cycle is parameterized by $\rho = \rho_0$, and $\theta = \theta_0$. Since the holonomies around this cycle are trivial, the conditions in (10.26) reduce to

$$\mathscr{H}_\tau^\pm = e^{\beta a_\tau^\pm} = e^{i\beta a_t^\pm} = -1 \,, \tag{10.29}$$

and since the cycle winds once, the eigenvalues of $i\beta a_t$ are given by $\pm i\pi$, which equivalently implies that

$$\beta^2 \mathrm{tr}\left[\left(a_t^\pm\right)^2\right] = 2\pi^2 \,. \tag{10.30}$$

Therefore, the triviality of the holonomies around this cycle amounts to fix the Euclidean time period as

$$\beta = l\sqrt{\frac{\pi k}{2\mathscr{L}}} \,, \tag{10.31}$$

in full agreement with the Hawking temperature.

Note that the variation of the total energy (10.6) in this case reads

$$\delta E = \frac{k}{2\pi}\int_{\partial\Sigma} \left(\langle a_t^+ \delta a_\theta^+\rangle - \langle a_t^- \delta a_\theta^-\rangle\right) d\theta = \frac{4\pi}{l}\delta\mathscr{L} \,, \tag{10.32}$$

from which, by virtue of (10.31) and the first law, implies that

$$\delta S = \beta\delta E = \delta\left(4\pi\sqrt{2\pi k\mathscr{L}}\right) \,, \tag{10.33}$$

which means that the entropy can be expressed in terms of the global charges (10.20), as

$$S = 4\pi\sqrt{2\pi k\mathscr{L}} \,. \tag{10.34}$$

The black hole entropy found in this way agrees with the standard result obtained in the metric formalism. Indeed, according to (10.23), in the static case the event horizon is located at $e^{2\rho+} = \frac{2\pi}{k}\mathscr{L}$, so that its area is given by $A = 4\pi l\sqrt{\frac{2\pi}{k}\mathscr{L}}$, and hence (10.34) is equivalent to the Bekenstein–Hawking formula $S = \frac{A}{4G}$.

10.4 Higher Spin Gravity in 3D

As explained in the introduction, gravity with negative cosmological constant, nonminimally coupled to an interacting spin-three field can be described in terms of a Chern–Simons theory [10–12]. The action is then of the form (10.1), and as in the case of pure gravity, the corresponding Lie algebra is of the form $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$, but where now $\mathfrak{g}_\pm$ are enlarged to two independent copies of $sl(3, \mathbb{R})$. Both copies of the algebra will be assumed to be spanned by the same set of matrices L_i, W_m, with $i = -1, 0, 1$, and $m = -2, -1, 0, 1, 2$, given by (see e.g., [22])

$$L_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad ; \quad L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} ,$$

$$W_{-2} = \begin{pmatrix} 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad W_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad W_0 = \frac{2}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{10.35}$$

$$W_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad ; \quad W_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} ,$$

whose commutation relations read

$$\begin{aligned} \left[L_i, L_j\right] &= (i - j)\, L_{i+j} \ , \\ \left[L_i, W_m\right] &= (2i - m)\, W_{i+m} \ , \\ \left[W_m, W_n\right] &= -\frac{1}{3}\,(m - n)\left(2m^2 + 2n^2 - mn - 8\right) L_{m+n} \ , \end{aligned} \tag{10.36}$$

so that the subset of generators L_i span the algebra $sl(2, \mathbb{R})$ in the so-called principal embedding.

The invariant nondegenerate bilinear form can also be chosen so that the action (10.1) reads

$$I = I_{CS}\left[A^+\right] - I_{CS}\left[A^-\right] \ , \tag{10.37}$$

where $A^\pm$ stand for the gauge fields that correspond to both copies of $sl(3, \mathbb{R})$, and now the bracket is given by a quarter of the trace in the representation of (10.35), i.e., $\langle \cdots \rangle = \frac{1}{4}\mathrm{tr}(\cdots)$. As in the case of pure gravity, the level is also chosen as $k = \frac{l}{4G}$.

It is useful to introduce a generalization of the dreibein and the spin connection, which relate with the gauge fields according to

$$A^\pm = \omega \pm \frac{e}{l} \ , \tag{10.38}$$

so that the spacetime metric and the spin-three field can be recovered as

$$g_{\mu\nu} = \frac{1}{2}\mathrm{tr}\left(e_\mu e_\nu\right) \; ; \; \varphi_{\mu\nu\rho} = \frac{1}{3!}\mathrm{tr}\left(e_{(\mu} e_\nu e_{\rho)}\right) \; , \tag{10.39}$$

being manifestly invariant under the diagonal subgroup of $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$, which corresponds to an extension of the local Lorentz group. The remaining gauge symmetries are then not only related to diffeomorphisms, but also with the higher spin gauge transformations. It is worth pointing out that, since the metric transforms in a nontrivial way under the action of the higher spin gauge symmetries, some standard geometric and physical notions turn out to be ambiguous, since they are no longer invariant. This last observation can be regarded as an additional motivation to explore the physical properties of the theory directly in terms of its original variables, given by the gauge fields $A^{\pm}$.

10.4.1 Asymptotic Conditions with W_3 Symmetries

A consistent set of asymptotic conditions for the theory described above was found in [21, 22]. Using the gauge choice as in [32], the radial dependence can be completely absorbed by $SL(3,\mathbb{R})$ group elements of the form (10.12), so that the asymptotic behaviour of the gauge fields can be written as in Eq. (10.13), where $a^{\pm}$ are now given by

$$a^{\pm} = \pm\left(L_{\pm1} - \frac{2\pi}{k}\mathscr{L}_{\pm}L_{\mp1} - \frac{\pi}{2k}\mathscr{W}_{\pm}W_{\mp2}\right)dx^{\pm} \; , \tag{10.40}$$

and $\mathscr{L}_{\pm}$, $\mathscr{W}_{\pm}$ stand for arbitrary functions of t, θ. The asymptotic symmetries can then be readily found following the same steps as in the case of pure gravity, previously discussed in Sect. 10.3.1.

The asymptotic form of the fields $a_\theta^{\pm}$ is maintained under gauge transformations generated by

$$\begin{aligned}
\Lambda^{\pm}(\varepsilon_{\pm},\chi_{\pm}) &= \varepsilon_{\pm}L_{\pm1} + \chi_{\pm}W_{\pm2} \mp \varepsilon'_{\pm}L_0 \mp \chi'_{\pm}W_{\pm1} + \frac{1}{2}\left(\chi''_{\pm} - \frac{8\pi}{k}\mathscr{L}_{\pm}\chi_{\pm}\right)W_0 \\
&\quad + \frac{1}{2}\left(\varepsilon''_{\pm} - \frac{4\pi}{k}\varepsilon_{\pm}\mathscr{L}_{\pm} + \frac{8\pi}{k}\mathscr{W}_{\pm}\chi_{\pm}\right)L_{\mp1} - \left(\frac{\pi}{2k}\mathscr{W}_{\pm}\varepsilon_{\pm} + \frac{7\pi}{6k}\mathscr{L}'_{\pm}\chi'_{\pm}\right. \\
&\quad \left. + \frac{\pi}{3k}\chi_{\pm}\mathscr{L}''_{\pm} + \frac{4\pi}{3k}\mathscr{L}_{\pm}\chi''_{\pm} - \frac{4\pi^2}{k^2}\mathscr{L}^2_{\pm}\chi_{\pm} - \frac{1}{24}\chi''''_{\pm}\right)W_{\mp2} \\
&\quad \mp \frac{1}{6}\left(\chi'''_{\pm} - \frac{8\pi}{k}\chi_{\pm}\mathscr{L}'_{\pm} - \frac{20\pi}{k}\mathscr{L}_{\pm}\chi'_{\pm}\right)W_{\mp1} \; ,
\end{aligned} \tag{10.41}$$

which depend on two arbitrary parameters per copy, $\varepsilon_{\pm}$, $\chi_{\pm}$, being functions of t and θ, provided the transformation law of the fields $\mathscr{L}_{\pm}$, $\mathscr{W}_{\pm}$ reads

$$\delta\mathscr{L}_{\pm}=\varepsilon_{\pm}\mathscr{L}_{\pm}'+2\mathscr{L}_{\pm}\varepsilon_{\pm}'-\frac{k}{4\pi}\varepsilon_{\pm}'''-2\chi_{\pm}\mathscr{W}_{\pm}'-3\mathscr{W}_{\pm}\chi_{\pm}' \ , \tag{10.42}$$

$$\delta\mathscr{W}_{\pm}=\varepsilon_{\pm}\mathscr{W}_{\pm}'+3\mathscr{W}_{\pm}\varepsilon_{\pm}'-\frac{64\pi}{3k}\mathscr{L}_{\pm}^{2}\chi_{\pm}'+3\chi_{\pm}'\mathscr{L}_{\pm}''+5\mathscr{L}_{\pm}'\chi_{\pm}''+\frac{2}{3}\chi_{\pm}\mathscr{L}_{\pm}'''$$
$$-\frac{k}{12\pi}\chi_{\pm}'''''-\frac{64\pi}{3k}\left(\chi_{\pm}\mathscr{L}_{\pm}'-\frac{5k}{32\pi}\chi_{\pm}'''\right)\mathscr{L}_{\pm} \ . \tag{10.43}$$

Then, the time component of the gauge fields $a_t^{\pm}$, is preserved under the gauge transformations generated by (10.41), with the transformation rules in (10.42), (10.43), provided the fields and the parameters are chiral:

$$\partial_{\mp}\mathscr{L}_{\pm}=\partial_{\mp}\mathscr{W}_{\pm}=0 \ , \tag{10.44}$$

$$\partial_{\mp}\varepsilon_{\pm}=\partial_{\mp}\chi_{\pm}=0 \ . \tag{10.45}$$

As in the case of pure gravity, the chirality of the fields in Eq. (10.44) reflects the fact that the field equations in the asymptotic region are satisfied.

The variation of the canonical generators that correspond to the asymptotic symmetries spanned by (10.41) now reads

$$\delta Q_{\pm}\left(\Lambda^{\pm}\right)=-\frac{k}{2\pi}\int\left\langle\Lambda^{\pm}\delta a_{\theta}^{\pm}\right\rangle d\theta=-\int\left(\varepsilon_{\pm}\delta\mathscr{L}_{\pm}-\chi_{\pm}\delta\mathscr{W}_{\pm}\right)d\theta \ , \tag{10.46}$$

and then integrates as

$$Q_{\pm}\left(\Lambda^{\pm}\right)=-\int\left(\varepsilon_{\pm}\mathscr{L}_{\pm}-\chi_{\pm}\mathscr{W}_{\pm}\right)d\theta \ . \tag{10.47}$$

This means that generic gauge fields that fulfill the asymptotic conditions described here, do not only carry spin-two charges associated to $\mathscr{L}_{\pm}$, whose zero modes are related to the energy and the angular momentum, but they also possess spin-three charges corresponding to $\mathscr{W}_{\pm}$.

The algebra of the canonical generators can be straightforwardly recovered from the transformation law of the fields in (10.42), (10.43) and it is found to be given by two copies of the W_3 algebra with the same central extension as in pure gravity, i.e., $c=\frac{3l}{2G}$. Once the fields are expanded in modes, the Poisson bracket algebra is such that both copies fulfill

$$\begin{aligned}
i\left\{\mathscr{L}_m,\mathscr{L}_n\right\}&=(m-n)\,\mathscr{L}_{m+n}+\frac{k}{2}m^3\delta_{m+n,0}\ ,\\
i\left\{\mathscr{L}_m,\mathscr{W}_n\right\}&=(2m-n)\,\mathscr{W}_{m+n}\ ,\\
i\left\{\mathscr{W}_m,\mathscr{W}_n\right\}&=\frac{1}{3}(m-n)\left(2m^2-mn+2n^2\right)\mathscr{L}_{m+n}+\frac{16}{3k}(m-n)\,\Lambda_{m+n}\\
&\quad+\frac{k}{6}m^5\delta_{m+n,0}\ ,
\end{aligned} \tag{10.48}$$

where

$$\Lambda_n = \sum_m \mathscr{L}_{n-m}\mathscr{L}_m \ , \tag{10.49}$$

so that the algebra is manifestly nonlinear.

It has also been shown that once the asymptotic conditions (10.40) are expressed in a suitable "decoupling" gauge choice, they admit a consistent vanishing cosmological constant limit, so that the asymptotic symmetries are spanned by a higher spin extension of the BMS_3 algebra with an appropriate central extension [35] (see also [36]). Related results along these lines, including Hamiltonian reduction [37], unitarity [38], and the analysis of cosmologies endowed with higher spin fields have been discussed in [39–42].

10.4.2 Higher Spin Black Hole Proposal and Its Thermodynamics

It is simple to verify that, for the case of constant functions $\mathscr{L}_\pm$ and $\mathscr{W}_\pm$, the asymptotic conditions described in the previous subsection do not accommodate black holes carrying nontrivial spin-three charges. This is because once the holonomies along a thermal cycle are required to be trivial, the spin-three charges $\mathscr{W}_\pm$ are forced to vanish. Thus, with the aim of finding black holes solutions which could in principle be endowed with spin-three charges, a different set of asymptotic conditions was proposed in [13] (see Sect. 10.5) and further analyzed in [43, 44]. Indeed, this set includes interesting new black holes solutions, which in the static case are described by three constants, and the gauge fields are of the form (10.13), with

$$\begin{aligned} a^{\pm} &= \pm\left(L_{\pm1} - \frac{2\pi}{k}\tilde{\mathscr{L}} L_{\mp1} \mp \frac{\pi}{2k}\tilde{\mathscr{W}} W_{\mp2}\right) dx^{\pm} \\ &\quad + \tilde{\mu}\left(W_{\pm2} - \frac{4\pi}{k}\tilde{\mathscr{L}} W_0 + \frac{4\pi^2}{k^2}\tilde{\mathscr{L}}^2 W_{\mp2} \pm \frac{4\pi}{k}\tilde{\mathscr{W}} L_{\mp1}\right) dx^{\mp} \ . \end{aligned} \tag{10.50}$$

The precise form of the $SL(3,\mathbb{R})$ group elements $g_\pm = g_\pm(\rho)$, which was further specified in [23], would be needed in order to reconstruct the metric and the spin-three field according to Eq. (10.39). In the case of $sl(3,\mathbb{R})$ gauge fields, the conditions that guarantee the triviality of their holonomies around the thermal circle, since the representation in (10.35) is vectorial, now read

$$\mathscr{H}_\tau^{\pm} = e^{i\beta a_\tau^{\pm}} = 1 \ , \tag{10.51}$$

which turn out to be equivalent to

$$\mathrm{tr}\left[\left(a_t^{\pm}\right)^3\right]=0\ ;\ \beta^2\mathrm{tr}\left[\left(a_t^{\pm}\right)^2\right]=8\pi^2\,. \tag{10.52}$$

For the gauge fields (10.50), conditions (10.52) reduce to

$$64\pi\tilde{\mathscr{L}}^2\tilde{\mu}\left(32\pi\tilde{\mathscr{L}}\tilde{\mu}^2-9k\right)+27k\tilde{\mathscr{W}}\left(32\pi\tilde{\mathscr{L}}\tilde{\mu}^2+k\right)-864\pi k\tilde{\mathscr{W}}^2\tilde{\mu}^3=0\,, \tag{10.53}$$

$$\frac{l^2\pi k}{2}\left(\tilde{\mathscr{L}}-3\tilde{\mu}\tilde{\mathscr{W}}+\frac{32\pi}{3k}\tilde{\mu}^2\tilde{\mathscr{L}}^2\right)^{-1}=\beta^2\,, \tag{10.54}$$

respectively, which for the branch that is connected to the BTZ black hole, being such that $\tilde{\mu}\to 0$ when $\tilde{\mathscr{W}}\to 0$, can be solved for β and $\tilde{\mu}$ in terms of $\tilde{\mathscr{L}}$ and $\tilde{\mathscr{W}}$, according to

$$\beta=\frac{l}{2}\sqrt{\frac{\pi k}{2\tilde{\mathscr{L}}}}\,\frac{2C-3}{C-3}\left(1-\frac{3}{4C}\right)^{-1/2}\,, \tag{10.55}$$

$$\tilde{\mu}=\frac{3}{4}\sqrt{\frac{kC}{2\pi\tilde{\mathscr{L}}}}\,\frac{1}{2C-3}\,, \tag{10.56}$$

where the constant C is defined through

$$\frac{C-1}{C^{3/2}}=\sqrt{\frac{k}{32\pi\tilde{\mathscr{L}}^3}}\tilde{\mathscr{W}}\,. \tag{10.57}$$

A proposal to deal with the global charges and the thermodynamics of this black hole solution, being based on a holographic approach, was put forward in [13, 23]. The bulk field equations were identified with the Ward identities for the stress tensor and the spin-three current of an underlying dual CFT in two dimensions, so that the integration constant $\tilde{\mathscr{L}}$ was interpreted as the stress tensor, while $\tilde{\mathscr{W}}$ and $\tilde{\mu}$ were associated to the spin-three current and its source, respectively. According to this prescription, the first law of thermodynamics implies that the variation of the entropy should be given by

$$\delta\tilde{S}=\frac{4\pi}{l}\beta\left(\delta\tilde{\mathscr{L}}-\tilde{\mu}\delta\tilde{\mathscr{W}}\right)\,, \tag{10.58}$$

which by virtue of (10.55), (10.56) integrates as

$$\tilde{S}=4\pi\sqrt{2\pi k\tilde{\mathscr{L}}}\sqrt{1-\frac{3}{4C}}\,, \tag{10.59}$$

so that the trivial holonomy conditions around the thermal circle agree with the integrability conditions of thermodynamics.

It is worth mentioning that the black hole entropy formula (10.59) remarkably agrees with the result found in [45], which was obtained from a completely different approach. Indeed, the computation of the free energy was carried out directly in the dual CFT with extended conformal symmetry in two dimensions, exploiting the properties of the partition function under the S-modular transformation, making then no reference to the holonomies in the bulk.

These approaches have been reviewed in [46–48], and further results about black hole thermodynamics along these lines have been found in [49–58].

However, it should be stressed that identifying the integration constants $\tilde{\mathscr{L}}$ and $\tilde{\mathscr{W}}$ with global charges, appears to be very counterintuitive from the point of view of the canonical formalism. This is because, in spite of the fact that the components of the gauge fields along $dx^{\pm}$ for the black hole solution (10.50) agree with the ones of the asymptotic fall-off in (10.40), once a nonvanishing constant $\tilde{\mu}$ is included, the additional terms along $dx^{\mp}$ amount to a severe modification of the asymptotic form of the dynamical fields $a_\theta^{\pm}$, so that the expression for the global charges in Eq. (10.47) no longer applies for this class of black hole solutions. Hence, as shown in [24], in full analogy with what occurs in the case of three-dimensional General Relativity coupled to scalar fields with slow fall-off at infinity [59, 60], the effect of modifying the asymptotic behaviour is such that the total energy acquires additional nonlinear contributions in the deviation of the fields with respect to the reference background. Indeed, the variation of the total energy can be obtained directly from (10.6), which for the case of the black hole solution (10.50), reads

$$\begin{aligned}\delta E &= \frac{k}{2\pi}\int \left(\langle a_t^+ \delta a_\theta^+\rangle - \langle a_t^- \delta a_\theta^-\rangle\right) d\theta \ , \\ &= \frac{4\pi}{l}\left[\delta\tilde{\mathscr{L}} - \frac{32\pi}{3k}\delta(\tilde{\mathscr{L}}^2\mu^2) + \tilde{\mu}\delta\tilde{\mathscr{W}} + 3\tilde{\mathscr{W}}\delta\tilde{\mu}\right] . \end{aligned} \tag{10.60}$$

Note that (10.60) is not an exact differential. This is natural because the variation of the total energy not only includes the variation of the mass, but also the contribution coming from all the constraints. Therefore, in order to suitably disentangle the mass (internal energy) from the work terms, one should provide a consistent set of asymptotic conditions that allows the precise identification of the global charges as well as the chemical potentials. This is discussed in Sect. 10.5. Nonetheless, the expression (10.60) provides a nice shortcut to compute the black hole entropy, circumventing the explicit computation of higher spin charges and their chemical potentials [24, 25]. This is because, by virtue of the first law, the inverse temperature β acts as an integrating factor, being such that the product $\beta\delta E$ becomes an exact differential that corresponds to the variation of the entropy, i.e.,

$$\delta S = \beta\delta E = \delta\left[4\pi\sqrt{2\pi k\tilde{\mathscr{L}}}\left(1-\frac{3}{2C}\right)^{-1}\sqrt{1-\frac{3}{4C}}\right] , \tag{10.61}$$

so that the black hole entropy is given by

$$S = 4\pi\sqrt{2\pi k\tilde{\mathscr{L}}}\left(1-\frac{3}{2C}\right)^{-1}\sqrt{1-\frac{3}{4C}}\ . \tag{10.62}$$

As explained in [25], the entropy (10.62) can be recovered from a suitable generalization of the Bekenstein–Hawking formula, given by

$$S = \frac{A}{4G}\cos\left[\frac{1}{3}\arcsin\left(3^{3/2}\frac{\varphi_+}{A^3}\right)\right]\ , \tag{10.63}$$

which depends on the reparameterization invariant integrals of the pullback of the metric and the spin-3 field at the spacelike section of the horizon, i.e., on the horizon area A and its spin-3 analogue:

$$\varphi_+^{1/3} := \int_{\partial\Sigma_+}\left(\varphi_{\mu\nu\rho}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\frac{dx^\rho}{d\sigma}\right)^{1/3} d\sigma\ . \tag{10.64}$$

It is worth highlighting that, for the static case, and in the weak spin-three field limit, our expression for the entropy (10.63) reduces to

$$S = \frac{A}{4G}\left(1-\frac{3}{2}\left(g^{\theta\theta}\right)^3\varphi_{\theta\theta\theta}^2+\mathscr{O}\left(\varphi^4\right)\right)\Big|_{\rho_+}\ , \tag{10.65}$$

in full agreement with the result found in [17], which was obtained from a completely different approach. Indeed, in [17] the action was written in terms of the metric and the perturbative expansion of the spin-three field up to quadratic order, so that the correction to the area law in (10.65) was found by means of Wald's formula [61].

Further results about black hole thermodynamics and along these lines have been found in [53, 54, 62–65], and the variation of the total energy (10.60) has also been recovered through different methods in [43, 44].

Since the entropy is expected to be an intrinsic property of the black hole, the fact that the nonperturbative expression for the entropy S in Eq. (10.62) differs from $\tilde{S}$ in (10.59) by a factor that characterizes the presence of the spin-three field, i.e., $S = \tilde{S}\left(1-\frac{3}{2C}\right)^{-1}$, is certainly disturbing. Indeed, curiously, a variety of different approaches either lead to $\tilde{S}$ or S, in [13, 45, 52, 58], and [25, 62–64], respectively, or even to both results [53, 54] for the black hole entropy.

As explained in [24,25], the discrepancy of these results stems from the mismatch in the definition of global charges aforementioned, which turns out to be inherited by the entropy once computed through the first law, even in the weak spin-three field limit.

Nonetheless, some puzzles still remain to be clarified, as it is the question about how the entropy (10.62) fulfills the first law of thermodynamics in the grand canonical ensemble, which is related to whether the black hole solution (10.50) actually carries or not a global a spin-three charge. This is discussed in the next Sect. 10.5.

10.5 Solving the Puzzles: Asymptotic Conditions Revisited and Different Classes of Black Holes

As explained in [15, 16], the puzzles mentioned above become resolved once the asymptotic conditions are extended so as to admit a generic choice of chemical potentials associated to the higher spin charges, so that the original asymptotic W_3 symmetries are manifestly preserved by construction. In this way, any possible ambiguity is removed. This can be seen as follows. At a slice of fixed time, according to (10.40), the asymptotic behaviour of the dynamical fields is of the form

$$a_\theta^\pm = \left(L_{\pm 1} - \frac{2\pi}{k} \mathscr{L}_\pm L_{\mp 1} - \frac{\pi}{2k} \mathscr{W}_\pm W_{\mp 2} \right) d\theta \ , \tag{10.66}$$

which is maintained under the gauge transformations $\Lambda^\pm$, defined through (10.41), with (10.42) and (10.43). In order to determine the asymptotic form of the gauge fields along time evolution, note that the field equations $F_{ti} = 0$ read

$$\dot{A}_i = \partial_i A_t + [A_i, A_t] \ ,$$

which implies that the time evolution of the dynamical fields corresponds to a gauge transformation parameterized by A_t. Hence, in order to preserve the asymptotic symmetries along the evolution in time, the Lagrange multipliers must be of the allowed form (10.41), i.e., $a_t^\pm = \Lambda^\pm$. Thus, following [15], the chemical potentials are included in the time component of the gauge fields only, so that the asymptotic form of the gauge fields is given by

$$a^\pm = \pm \left(L_{\pm 1} - \frac{2\pi}{k} \mathscr{L}_\pm L_{\mp 1} - \frac{\pi}{2k} \mathscr{W}_\pm W_{\mp 2} \right) dx^\pm \pm \frac{1}{l} \Lambda^\pm(\nu_\pm, \mu_\pm) dt \ , \tag{10.67}$$

where $\nu_\pm$, $\mu_\pm$ stand for arbitrary fixed functions of t, θ without variation ($\delta\nu_\pm = \delta\mu_\pm = 0$), that correspond to the chemical potentials. Note that, since the asymptotic form of the dynamical fields (10.66) is unchanged as compared with (10.40), the expression for the global charges remains the same, i.e., at a fixed t slice, the global charges are again given by (10.47), so that the asymptotic symmetries are still generated by two copies of the W_3 algebra.

Consistency then requires that the asymptotic form of $a_t^\pm$, should also be preserved under the asymptotic symmetries, which implies that the field equations have to be fulfilled in the asymptotic region, and the parameters of the asymptotic symmetries satisfy “deformed chirality conditions”, which read

$$l\dot{\mathscr{L}}_{\pm} = \pm (1+\nu_{\pm})\,\mathscr{L}'_{\pm} \mp 2\mu_{\pm}\mathscr{W}'_{\pm}\,,$$
$$l\dot{\mathscr{W}}_{\pm} = \pm (1+\nu_{\pm})\,\mathscr{W}'_{\pm} \pm \frac{2}{3}\mu_{\pm}\left(\mathscr{L}'''_{\pm} - \frac{16\pi}{k}\left(\mathscr{L}^{2}_{\pm}\right)'\right)\,, \tag{10.68}$$

and

$$l\dot{\chi}_{\pm} = \pm (1+\nu_{\pm})\,\chi'_{\pm} \pm 2\mu_{\pm}\varepsilon'_{\pm}\,,$$
$$l\dot{\varepsilon}_{\pm} = \pm (1+\nu_{\pm})\,\varepsilon'_{\pm} \mp \frac{2}{3}\mu_{\pm}\left(\chi'''_{\pm} - \frac{32\pi}{k}\chi'_{\pm}\mathscr{L}_{\pm}\right)\,, \tag{10.69}$$

respectively, where for simplicity, in Eqs. (10.68), (10.69), the chemical potentials associated to the spin-two and spin-three charges, given by $\nu_{\pm}$ and $\mu_{\pm}$, were assumed to be constants.

Therefore, by construction, the functions $\mathscr{L}_{\pm}$, $\mathscr{W}_{\pm}$ are really what they mean, since their Poisson brackets fulfill the W_3 algebra with the same central extension. Note that this is so regardless the choice of chemical potentials, because the canonical generators do no depend on the Lagrange multipliers.

The asymptotic conditions given by (10.67) then provide the required extension of the ones in [21, 22], since the latter are recovered when the chemical potentials are switched off, i.e., for $\nu_{\pm} = 0$, $\mu_{\pm} = 0$. In this case, Eqs. (10.68) and (10.69) reduce to (10.44) and (10.45), respectively, expressing the fact that the fields and the parameters become chiral.

From a different perspective, the case of $\nu_{\pm} = -1$, $\mu_{\pm} = 1$ has also been discussed in [66].

It is worth emphasizing that since the Lagrange multipliers appear in the improved action through the improved generators (10.3), the interpretation of $\nu_{\pm}$, $\mu_{\pm}$ as chemical potentials, is also guaranteed by construction. Note that this corresponds to the standard procedure one follows in the case of Reissner–Nordstrm black holes, where the chemical potential associated to the electric charge corresponds to the time component of the electromagnetic field, being the Lagrange multiplier of the $U(1)$ constraint.

The extended asymptotic conditions (10.67), in the case of constant functions $\mathscr{L}_{\pm}$, $\mathscr{W}_{\pm}$ and chemical potentials $\nu_{\pm}$, $\mu_{\pm}$, then accommodate a new class of black hole solutions, endowed not only with mass and angular momentum, but also with nontrivial well-defined spin-three charges [15]. Their asymptotic and thermodynamical properties are further discussed in [16], where it is explicitly shown that for this solution, there is no tension between the different approaches mentioned above.

Note that in the standard approach for black hole thermodynamics, the temperature and the chemical potential for the angular momentum do not explicitly appear in the fields. Instead, they enter through the identifications involving the Euclidean time and the angle, so that the range of the coordinates is not fixed and depends on the solution. The presence of nonvanishing chemical potentials $\nu_\pm$ associated to the spin-two charges, then allows performing the description keeping the range of the coordinates fixed once and for all, i.e., $0 \leq \theta < 2\pi$ and $0 \leq \tau < 2\pi l$, which amounts to introduce a non trivial lapse and shift in the metric formalism. Both approaches are indeed equivalent, but in the case of higher spin black holes, since the chemical potentials that correspond to the spin-three charges cannot be absorbed into the modular parameter of the torus, it becomes conceptually safer to follow the latter approach, since all the chemical potentials become introduced and treated unambiguously in the same footing.

Otherwise, for instance, if the chemical potentials were not introduced along the thermal circles, but instead along additional non-vanishing components of the gauge fields along the conjugate null directions, as in the case of [13], the asymptotic form of the gauge fields would be given by

$$a^{\pm} = \pm\left(L_{\pm 1} - \frac{2\pi}{k}\tilde{\mathscr{L}}_{\pm}L_{\mp 1} - \frac{\pi}{2k}\tilde{\mathscr{W}}_{\pm}W_{\mp 2}\right)dx^{\pm} \pm \Lambda^{\pm}\left(\tilde{\nu}_{\pm}, \tilde{\mu}_{\pm}\right)dx^{\mp}, \tag{10.70}$$

which severely modifies the components of the dynamical fields $a_\theta^\pm$, in a way that is incompatible with the asymptotic W_3 symmetry. This is because at a fixed t slice, the terms proportional to $\tilde{\mu}_\pm$ contribute to $a_\theta^\pm$ with additional terms of the form

$$\begin{aligned} a_\theta^{\pm} &= \left(L_{\pm 1} - \frac{2\pi}{k}\tilde{\mathscr{L}}_{\pm}L_{\mp 1} - \frac{\pi}{2k}\tilde{\mathscr{W}}_{\pm}W_{\mp 2}\right) + \left(\tilde{\nu}_{\pm}L_{\pm 1} + \tilde{\mu}_{\pm}W_{\pm 2}\right) \\ &\quad + \left[\frac{1}{2}\left(-\frac{4\pi}{k}\tilde{\nu}_{\pm}\tilde{\mathscr{L}}_{\pm} + \frac{8\pi}{k}\tilde{\mathscr{W}}_{\pm}\tilde{\mu}_{\pm}\right)L_{\mp 1} - \left(\frac{\pi}{2k}\tilde{\mathscr{W}}_{\pm}\tilde{\nu}_{\pm} - \frac{4\pi^2}{k^2}\tilde{\mathscr{L}}_{\pm}^2\tilde{\mu}_{\pm}\right)W_{\mp 2}\right] \\ &\quad - \frac{4\pi}{k}\tilde{\mathscr{L}}_{\pm}\tilde{\mu}_{\pm}W_0\,, \end{aligned} \tag{10.71}$$

that are not of highest (or lowest) weight, and hence incompatible with the asymptotic conditions (10.67) that implement the Hamiltonian reduction of the current algebra associated to $sl(3,\mathbb{R})$ to the W_3 algebra. Indeed, in this case, the asymptotic symmetries that preserve the asymptotic form of a_θ are shown to be spanned by two copies of the Bershardsky–Polyakov algebra W_3^2 [67,68], corresponding to the other non trivial (so-called diagonal) embedding of $sl\,(2,\mathbb{R})$ into $sl\,(3,\mathbb{R})$ [16]. Therefore, in spite of dealing with the same action, the effect of this drastic modification of the boundary conditions amounts to deal with a completely different theory, being characterized by a different field content, and hence with an inequivalent spectrum, so that their corresponding black hole solutions, as the one in (10.50), are characterized by another set of global charges of lower spin.

It is worth pointing out that our procedure to incorporate chemical potentials can be straightforwardly extended to the case of $\mathfrak{g}_\pm = sl\,(N,\mathbb{R})$, regardless the way in which $sl\,(2,\mathbb{R})$ is embedded, as well as to the case of infinite-dimensional higher spin algebras.

Some closing remarks are in order. It should be mentioned that the case of three-dimensional gravity nonminimally coupled with spin-three fields, also appears to be consistently formulated in the second-order formalism by introducing a suitable set of auxiliary fields [69]. Besides, in the case of spin-three and higher, consistent sets of asymptotic conditions have also been proposed in [21, 70, 71], while exact solutions and their properties have been explored in [72–76]. In the context of higher spin supergravity in three dimensions, the asymptotic structure was analyzed in [77], and exact solutions have also been found in [78–80]. Moreover, along the lines of holography and the corresponding dual CFT theory with extended conformal symmetry at the boundary [81–83], further interesting results can also be found in [84–91].

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