Completions and Simple Homotopy*

Gilles Bertrand

Université Paris-Est, Laboratoire d'Informatique Gaspard-Monge,
Equipe A3SI, ESIEE Paris

Abstract. We propose an extension of simple homotopy by considering *homotopic pairs*. Intuitively, a homotopic pair is a couple of objects (X, Y) such that X is included in Y and (X, Y) may be transformed to a trivial couple by simple homotopic deformations that keep X inside Y. Thus, these objects are linked by a "relative homotopy relation".

We formalize these notions by means of completions, which are inductive properties expressed in a declarative way. In a previous work, through the notion of a *dyad*, we showed that completions were able to handle couples of objects that are linked by a certain "relative homology relation".

The main result of the paper is a theorem that makes clear the link between homotopic pairs and dyads. Thus, we prove that, in the unified framework of completions, it is possible to handle notions relative to both homotopy and homology.

Keywords: Simple homotopy, combinatorial topology, simplicial complexes, completions.

1 Introduction

Simple homotopy, introduced by J. H. C. Whitehead in the early 1930's, may be seen as a refinement of the concept of homotopy [1]. Two complexes are simple homotopy equivalent if one of them may be obtained from the other by a sequence of elementary collapses and anti-collapses.

Simple homotopy plays a fundamental role in combinatorial topology [1–7]. Also, many notions relative to homotopy in the context of computer imagery rely on the collapse operation. In particular, this is the case for the notion of a simple point, which is crucial for all image transformations that preserve the topology of the objects [8–10], see also [11–13].

In this paper, we propose an extension of simple homotopy by considering *homotopic pairs*. Intuitively, a homotopic pair is a couple of objects (X, Y) such that X is included in Y and (X, Y) may be transformed to a trivial couple by simple homotopic deformations that keep X inside Y. Thus, these objects are linked by a "relative homotopy relation".

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We formalize these notions by means of completions, which are inductive properties expressed in a declarative way [14]. In a previous work, we introduced the notions of a *dendrite* and a *dyad* [15], which were also formalized by means of completions. A dendrite is an acyclic object, a theorem asserts that an object is a dendrite if and only if it is acyclic in the sense of homology. Intuitively, a dyad is a couple of objects (X, Y), with $X \subseteq Y$, such that the cycles of X are "at the right place with respect to the ones of Y". A theorem provides a relation between dendrites and dyads. Thus, these results show that completions are able to handle couples of objects that are linked by a certain "relative homology relation".

The main result of the paper is a theorem that makes clear the link between homotopic pairs and dyads. In particular, this theorem indicates that a subset of the completions that describe dyads allows for a complete characterization of homotopic pairs. Thus, we prove that, in the unified framework of completions, it is possible to handle notions relative to both homotopy and homology.

The paper is organized as follows. First, we give some basic definitions for simplicial complexes (Sec. 2). Then, we recall some basic facts relative to the notion of a completion (Sec. 3). We recall the definitions of the completions that describe dendrites and dyads, and we introduce our notion of homotopic pairs (Sec. 4). In the following section, we introduce some tools that are necessary to prove our results (Sec. 5). We establish the theorem that provides a relation between homotopic pairs and dyads in Sec. 6. In Sec. 7, we give a result linking homotopic pairs and the more classical notion of simple homotopy.

The paper is self contained. In particular, almost all proofs are included.

2 Basic Definitions for Simplicial Complexes

Let X be a finite family composed of finite sets. The *simplicial closure of* X is the complex $X^- = \{y \subseteq x \mid x \in X\}$. The family X is a *(finite simplicial) complex* if $X = X^-$. We write $\mathbb{S}$ for the collection of all finite simplicial complexes. Note that $\emptyset \in \mathbb{S}$ and $\{\emptyset\} \in \mathbb{S}$, $\emptyset$ is the *void complex*, and $\{\emptyset\}$ is the *empty complex*.

Let $X \in \mathbb{S}$. An element of X is *a simplex of* X or *a face of* X. A *facet of* X is a simplex of X that is maximal for inclusion.

A *simplicial subcomplex* of $X \in \mathbb{S}$ is any subset Y of X that is a simplicial complex. If Y is a subcomplex of X, we write $Y \preceq X$.

Let $X \in \mathbb{S}$. The *dimension* of $x \in X$, written $dim(x)$, is the number of its elements minus one. The *dimension of* X, written $dim(X)$, is the largest dimension of its simplices, the *dimension of* $\emptyset$ being defined to be -1.

A complex $A \in \mathbb{S}$ is *a cell* if $A = \emptyset$ or if A has precisely one non-empty facet x. We write $\mathbb{C}$ for the collection of all cells. A cell $\alpha \in \mathbb{C}$ is *a vertex* if $dim(\alpha) = 0$.

The *ground set* of $X \in \mathbb{S}$ is the set $\underline{X} = \cup\{x \in X \mid dim(x) = 0\}$. We say that $X \in \mathbb{S}$ and $Y \in \mathbb{S}$ are *disjoint*, or that X is *disjoint from* Y, if $\underline{X} \cap \underline{Y} = \emptyset$. Thus, X and Y are disjoint if and only if $X \cap Y = \emptyset$ or $X \cap Y = \{\emptyset\}$.

If $X \in \mathbb{S}$ and $Y \in \mathbb{S}$ are disjoint, the *join of* X *and* Y is the simplicial complex XY such that $XY = \{x \cup y \mid x \in X, y \in Y\}$. Thus, $XY = \emptyset$ if $Y = \emptyset$ and $XY = X$ if $Y = \{\emptyset\}$. The join αX of a vertex α and a complex $X \in \mathbb{S}$ is a *cone.*

Important convention. In this paper, if $X, Y \in \mathbb{S}$, we implicitly assume that X and Y have disjoint ground sets whenever we write XY.

We recall now some basic definitions related to the collapse operator introduced by J.H.C. Whitehead ([1], see also [16]).

Let $X \in \mathbb{S}$ and let x, y be two distinct faces of X. The couple (x, y) is a *free pair for* X if y is the only face of X that contains x. If (x, y) is a free pair for X, $Y = X \setminus \{x, y\}$ is *an elementary collapse of* X and X is *an elementary expansion of* Y. We say that X *collapses onto* Y, or that Y *expands onto* X, if there exists a sequence $\langle X_0, ..., X_k \rangle$ such that $X_0 = X$, $X_k = Y$, and X_i is an elementary collapse of X_{i-1}, $i \in [1, k]$. The complex X is *collapsible* if X collapses onto $\emptyset$. We say that X is *(simply) homotopic to* Y, or that X and Y are *(simply) homotopic*, if there exists a sequence $\langle X_0, ..., X_k \rangle$ such that $X_0 = X$, $X_k = Y$, and X_i is an elementary collapse or an elementary expansion of X_{i-1}, $i \in [1, k]$. The complex X is *(simply) contractible* if X is simply homotopic to $\emptyset$.

Let $X, Y \in \mathbb{S}$ and let $x, y \in Y \setminus X$. The pair (x, y) is free for $X \cup Y$ if and only if (x, y) is free for Y. Thus, by induction, we have the following proposition.

Proposition 1. *Let* $X, Y \in \mathbb{S}$*. The complex* Y *collapses onto* $X \cap Y$ *if and only if* $X \cup Y$ *collapses onto* X*.*

3 Completions

We give some basic definitions for completions, a completion may be seen as a rewriting rule that permits to derive collections of sets. See [14] for more details.

Let $\mathbf{S}$ be a given collection and let $\mathcal{K}$ be an arbitrary subcollection of $\mathbf{S}$. Thus, we have $\mathcal{K} \subseteq \mathbf{S}$. In the sequel of the paper, the symbol $\mathcal{K}$, with possible superscripts, will be a dedicated symbol (a kind of variable).

Let $\textbf{\textsc{k}}$ be a binary relation on $2^{\mathbf{S}}$, thus $\textbf{\textsc{k}} \subseteq 2^{\mathbf{S}} \times 2^{\mathbf{S}}$. We say that $\textbf{\textsc{k}}$ is *finitary,* if $\mathbf{F}$ is finite whenever $(\mathbf{F}, \mathbf{G}) \in \textbf{\textsc{k}}$.

Let $\langle \mathrm{K} \rangle$ be a property that depends on $\mathcal{K}$. We say that $\langle \mathrm{K} \rangle$ is a *completion (on* $\mathbf{S}$*)* if $\langle \mathrm{K} \rangle$ may be expressed as the following property:
–> If $\mathbf{F} \subseteq \mathcal{K}$, then $\mathbf{G} \subseteq \mathcal{K}$ whenever $(\mathbf{F}, \mathbf{G}) \in \textbf{\textsc{k}}$. $\langle \textbf{\textsc{k}} \rangle$
where $\textbf{\textsc{k}}$ is an arbitrary finitary binary relation on $2^{\mathbf{S}}$.

If $\langle \mathrm{K} \rangle$ is a property that depends on $\mathcal{K}$, we say that a given collection $\mathbf{X} \subseteq \mathbf{S}$ satisfies $\langle \mathrm{K} \rangle$ if the property $\langle \mathrm{K} \rangle$ is true for $\mathcal{K} = \mathbf{X}$.

Theorem 1. [14] *Let* $\langle \mathrm{K} \rangle$ *be a completion on* $\mathbf{S}$ *and let* $\mathbf{X} \subseteq \mathbf{S}$*. There exists, under the subset ordering, a unique minimal collection that contains* $\mathbf{X}$ *and that satisfies* $\langle \mathrm{K} \rangle$*.*

If $\langle \mathrm{K} \rangle$ is a completion on $\mathbf{S}$ and if $\mathbf{X} \subseteq \mathbf{S}$, we write $\langle \mathbf{X}; \mathrm{K} \rangle$ for the unique minimal collection that contains $\mathbf{X}$ and that satisfies $\langle \mathrm{K} \rangle$.

Let $\langle \mathrm{K} \rangle$ be a completion expressed as the above property $\langle \kappa \rangle$. By a fixed point property, the collection $\langle \mathbf{X}; \mathrm{K} \rangle$ may be obtained by starting from $\mathcal{K} = \mathbf{X}$, and by iteratively adding to $\mathcal{K}$ all the sets $\mathbf{G}$ such that $(\mathbf{F}, \mathbf{G}) \in \kappa$ and $\mathbf{F} \subseteq \mathcal{K}$ (see [14]). Thus, if $\mathbf{C} = \langle \mathbf{X}; \mathrm{K} \rangle$, then $\langle \mathbf{X}; \mathrm{K} \rangle$ may be seen as a dynamic structure that describes $\mathbf{C}$, the completion $\langle \mathrm{K} \rangle$ acts as a generator, which, from $\mathbf{X}$, makes it possible to enumerate all elements in $\mathbf{C}$. We will see now that $\langle \mathrm{K} \rangle$ may in fact be composed of several completions.

Let $\langle \mathrm{K}_1 \rangle, \langle \mathrm{K}_2 \rangle, ..., \langle \mathrm{K}_k \rangle$ be completions on $\mathbf{S}$. We write $\wedge$ for the logical "and". It may be seen that $\langle \mathrm{K} \rangle = \langle \mathrm{K}_1 \rangle \wedge \langle \mathrm{K}_2 \rangle ... \wedge \langle \mathrm{K}_k \rangle$ is a completion. In the sequel, we write $\langle \mathrm{K}_1, \mathrm{K}_2, ..., \mathrm{K}_k \rangle$ for $\langle \mathrm{K} \rangle$. Thus, if $\mathbf{X} \subseteq \mathbf{S}$, the notation $\langle \mathbf{X}; \mathrm{K}_1, \mathrm{K}_2, ..., \mathrm{K}_k \rangle$ stands for the smallest collection that contains $\mathbf{X}$ and that satisfies each of the properties $\langle \mathrm{K}_1 \rangle, \langle \mathrm{K}_2 \rangle, ..., \langle \mathrm{K}_k \rangle$.

Remark 1. If $\langle \mathrm{K} \rangle$ and $\langle \mathrm{Q} \rangle$ are two completions on $\mathbf{S}$, then we have $\langle \mathbf{X}; \mathrm{K} \rangle \subseteq \langle \mathbf{X}; \mathrm{K}, \mathrm{Q} \rangle$ whenever $\mathbf{X} \subseteq \mathbf{S}$. Furthermore, we have $\langle \mathbf{X}; \mathrm{K} \rangle = \langle \mathbf{X}; \mathrm{K}, \mathrm{Q} \rangle$ if and only if the collection $\langle \mathbf{X}; \mathrm{K} \rangle$ satisfies the property $\langle \mathrm{Q} \rangle$.

4 Completions on Simplicial Complexes

The notion of a dendrite was introduced in [14] as a way for defining a collection made of acyclic complexes. Let us consider the collection $\mathbf{S} = \mathbb{S}$, and let $\mathcal{K}$ denotes an arbitrary collection of simplicial complexes.

We define the two completions $\langle \mathrm{D1} \rangle$ and $\langle \mathrm{D2} \rangle$ on $\mathbb{S}$: For any $S, T \in \mathbb{S}$,

-> If S, $T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \in \mathcal{K}$. $\langle \mathrm{D1} \rangle$

-> If S, $T \in \mathcal{K}$, then $S \cap T \in \mathcal{K}$ whenever $S \cup T \in \mathcal{K}$. $\langle \mathrm{D2} \rangle$

Let $\mathbb{D} = \langle \mathbb{C}; \mathrm{D1}, \mathrm{D2} \rangle$. Each element of $\mathbb{D}$ is a *dendrite.*

Remark 2. Let κ be the binary relation on $2^{\mathbf{S}}$ such that $(\mathbf{F}, \mathbf{G}) \in \kappa$ iff there exist $S, T \in \mathbb{S}$, with $\mathbf{F} = \{S, T, S \cap T\}$ and $\mathbf{G} = \{S \cup T\}$. We see that κ is finitary and that $\langle \mathrm{D1} \rangle$ may be expressed as the property $\langle \kappa \rangle$ given in the preceding section. Thus $\langle \mathrm{D1} \rangle$ is indeed a completion, and so is $\langle \mathrm{D2} \rangle$.

The collection $\mathbb{T}$ of all trees (*i.e.*, all connected acyclic graphs) provides an example of a collection of dendrites. It may be checked that $\mathbb{T}$ satisfies both $\langle \mathrm{D1} \rangle$ and $\langle \mathrm{D2} \rangle$, and that we have $\mathbb{T} \subseteq \mathbb{D}$. In fact, we have the general result [14]:

A complex is a dendrite if and only if it is acyclic in the sense of homology.

As a consequence, any contractible complex is a dendrite but there exist some dendrites that are not contractible. The punctured Poincaré homology sphere provides an example of this last fact. Note also that each complex in $\langle \mathbb{C}; \mathrm{D1} \rangle$ is contractible [6], but there exist some contractible complexes that are not in $\langle \mathbb{C}; \mathrm{D1} \rangle$. The dunce hat [18] provides an example of this last fact. It follows that it is not possible, using only the two completions $\langle \mathrm{D1} \rangle$ and $\langle \mathrm{D2} \rangle$, to characterize precisely the collection composed of all contractible complexes.

The aim of this paper is to make clear the link between (simple) homotopy and completions. By the previous remarks, we will have to consider other completions

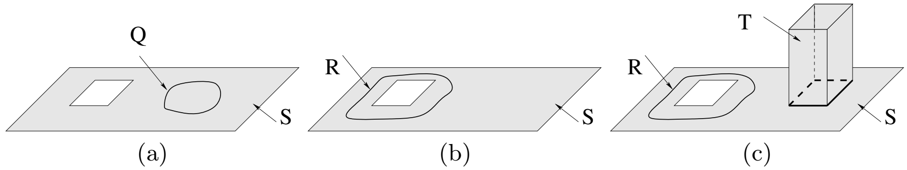

Fig. 1. (a) (Q, S) is not a dyad. (b) (R, S) is a dyad. (c) Suppose (R, S) and $(S \cap T, T)$ are dyads. Then, by $\langle \ddot{\mathrm{Y}}1 \rangle$, $(S, S \cup T)$ must be a dyad. Furthermore, by $\langle \ddot{\mathrm{T}} \rangle$, $(R, S \cup T)$ must also be a dyad.

than the two above ones. To achieve our goal, we will proceed by using some completions that describe *dyads* [15].

Intuitively, a dyad is a couple of complexes (X, Y), with $X \preceq Y$, such that the cycles of X are "at the right place with respect to the ones of Y". For example, the couple (Q, S) of Fig. 1 (a) is not a dyad, while the couple (R, S) of Fig. 1 (b) is a dyad.

We set $\ddot{\mathbb{S}} = \{(X, Y) \mid X, Y \in \mathbb{S}, X \preceq Y\}$ and $\ddot{\mathbb{C}} = \{(A, B) \in \ddot{\mathbb{S}} \mid A, B \in \mathbb{C}\}$.
In the sequel of the paper, $\ddot{\mathcal{K}}$ will denote an arbitrary subcollection of $\ddot{\mathbb{S}}$.

We define five completions on $\ddot{\mathbb{S}}$ (the symbols $\ddot{\mathrm{T}}$, $\ddot{\mathrm{U}}$, $\ddot{\mathrm{L}}$ stand respectively for "transitivity", "upper confluence", and "lower confluence"):
For any $S, T \in \mathbb{S}$,
-> If $(S \cap T, T) \in \ddot{\mathcal{K}}$, then $(S, S \cup T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{Y}}1 \rangle$
-> If $(S, S \cup T) \in \ddot{\mathcal{K}}$, then $(S \cap T, T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{Y}}2 \rangle$
For any $(R, S), (S, T), (R, T) \in \ddot{\mathbb{S}}$,
-> If $(R, S) \in \ddot{\mathcal{K}}$ and $(S, T) \in \ddot{\mathcal{K}}$, then $(R, T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{T}} \rangle$
-> If $(R, S) \in \ddot{\mathcal{K}}$ and $(R, T) \in \ddot{\mathcal{K}}$, then $(S, T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{U}} \rangle$
-> If $(R, T) \in \ddot{\mathcal{K}}$ and $(S, T) \in \ddot{\mathcal{K}}$, then $(R, S) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{L}} \rangle$
We set $\ddot{\mathbb{X}} = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{Y}}2, \ddot{\mathrm{T}}, \ddot{\mathrm{U}}, \ddot{\mathrm{L}} \rangle$. Each element of $\ddot{\mathbb{X}}$ is a *dyad* [1].

See Fig. 1 (c) for an illustration of the completions $\langle \ddot{\mathrm{Y}}1 \rangle$ and $\langle \ddot{\mathrm{T}} \rangle$. In [15], the following relation between dyads and dendrites was given.

Theorem 2. *Let $(X, Y) \in \ddot{\mathbb{S}}$ and let α be a vertex such that $\alpha X \cap Y = X$. The couple (X, Y) is a dyad if and only if $\alpha X \cup Y$ is a dendrite. In particular, a complex $Y \in \mathbb{S}$ is a dendrite if and only if $(\emptyset, Y)$ is a dyad.*

Intuitively, this result indicates that (X, Y) is a dyad if we cancel out all cycles of Y (*i.e.*, we obtain an acyclic complex), whenever we cancel out those of X (by the way of a cone).

In Fig. 1 (b), we see that it is possible to continuously deform R onto S, this deformation keeping R inside S. It follows the idea to introduce the following notions in order to make a link between dyads and simple homotopy.

[1] In [15], a different but equivalent definition of a dyad was given. See Th. 2 of [15].

If $X, Y \in \mathbb{S}$, we write $X \overset{E}{\longmapsto} Y$, whenever Y is an elementary expansion of X. We define four completions on $\ddot{\mathbb{S}}$: For any (R,S), (R,T), (S,T) in $\ddot{\mathbb{S}}$,

-> If $(R,S) \in \ddot{\mathcal{K}}$ and $S \overset{E}{\longmapsto} T$, then $(R,T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{H}}1 \rangle$
-> If $(R,T) \in \ddot{\mathcal{K}}$ and $S \overset{E}{\longmapsto} T$, then $(R,S) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{H}}2 \rangle$
-> If $(R,T) \in \ddot{\mathcal{K}}$ and $R \overset{E}{\longmapsto} S$, then $(S,T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{H}}3 \rangle$
-> If $(S,T) \in \ddot{\mathcal{K}}$ and $R \overset{E}{\longmapsto} S$, then $(R,T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{H}}4 \rangle$

We set $\ddot{\mathbb{I}} = \{(X,X) \mid X \in \mathbb{S}\}$ and $\ddot{\mathbb{H}} = \langle \ddot{\mathbb{I}}; \ddot{\mathrm{H}}1, \ddot{\mathrm{H}}2, \ddot{\mathrm{H}}3, \ddot{\mathrm{H}}4 \rangle$. Each element of $\ddot{\mathbb{H}}$ is a *homotopic pair*.

Note that we have $\langle \ddot{\mathbb{I}}; \ddot{\mathrm{H}}1 \rangle = \langle \ddot{\mathbb{I}}; \ddot{\mathrm{H}}1, \ddot{\mathrm{H}}4 \rangle$. Furthermore, $(X,Y) \in \langle \ddot{\mathbb{I}}; \ddot{\mathrm{H}}1 \rangle$ if and only if Y collapses onto X.

If (X', Y') is obtained from (X,Y) by applying one of the completions $\langle \ddot{\mathrm{H}}1 \rangle$, $\langle \ddot{\mathrm{H}}2 \rangle$, $\langle \ddot{\mathrm{H}}3 \rangle$, $\langle \ddot{\mathrm{H}}4 \rangle$, then X' is homotopic to X, and Y' is homotopic to Y. Since, for generating the collection $\ddot{\mathbb{H}}$, we start from $\ddot{\mathbb{I}}$, we have the following.

Proposition 2. *If $(X,Y) \in \ddot{\mathbb{H}}$, then X is homotopic to Y.*

Observe that, if X is homotopic to Y and if $X \preceq Y$, then we have not necessarily $(X,Y) \in \ddot{\mathbb{H}}$. See the couple (Q,S) of Fig. 1 (a).

5 Product

In this section we give some notions that are essential for the proofs of the main results of this paper. In particular, we introduce the notion of a product of a simplicial complex by a copy of this complex. Intuitively, this product has the structure of a Cartesian product of an object by the unit interval.

Let $Z, Z' \in \mathbb{S}$. We say that Z and Z' are *isomorphic* if there exists a bijection $\lambda : Z \rightarrow Z'$ such that, for all $x, y \in Z$, we have $\lambda(x) \subseteq \lambda(y)$ if and only if $x \subseteq y$. In this case, we also say that Z' is a *copy of* Z, we write $\lambda x = \lambda(x)$, and we set $\lambda X = \{\lambda x \mid x \in X\}$ whenever $X \preceq Z$. Thus, λZ stands for Z'. If $T \preceq Z$, we say that λZ is a *copy of Z with T fixed* if $\lambda T = T$.
In the sequel, we denote by $Cop(Z)$ the collection of all copies of a complex Z, and we denote by $Cop(Z;T)$ the collection of all copies of Z with T fixed.
Let $Z \in \mathbb{S}$ and let $\lambda Z \in Cop(Z)$. If $X \preceq Z$ and $Y \preceq Z$, we note that $\lambda(X \cup Y) = \lambda X \cup \lambda Y$, $\lambda(X \cap Y) = \lambda X \cap \lambda Y$. If $XY \preceq Z$, we also have $\lambda(XY) = \lambda X \lambda Y$.

Remark 3. Let $T, Z \in \mathbb{S}$, with $T \preceq Z$, and let $\lambda Z \in Cop(Z)$. Suppose that Z collapses onto T. Then, the complex λZ collapses onto λT. Nevertheless, if $T \preceq \lambda Z$, then λZ does not necessarily collapses onto T (T may be not "at the right place" w.r.t λZ). Of course if $\lambda Z \in Cop(Z;T)$, then λZ collapses onto T.

Let $X \in \mathbb{S}$ and let $\lambda X \in Cop(X)$ disjoint from X. The *product of X by λX* is the simplicial complex $X \otimes \lambda X$ such that $X \otimes \lambda X = \{x \cup \lambda y \mid x \cup y \in X\}$. Observe that z is a facet of $X \otimes \lambda X$ if and only if there exists a facet x of X such that $z = x \cup \lambda x$. Note also that we have $dim(X \otimes \lambda X) = 2dim(X) + 1$.

Let $Z \in \mathbb{S}$ and $\lambda Z \in Cop(Z)$ disjoint from Z. If $X \preceq Z$, $Y \preceq Z$, then:
- $(X \cup Y) \otimes \lambda(X \cup Y) = (X \otimes \lambda X) \cup (Y \otimes \lambda Y)$; and

- $(X \cap Y) \otimes \lambda(X \cap Y) = (X \otimes \lambda X) \cap (Y \otimes \lambda Y)$.
If $XY \preceq Z$, we have $(XY) \otimes \lambda(XY) = (X \otimes \lambda X)(Y \otimes \lambda Y)$.
If $A \in \mathbb{C}$ and if $A \preceq Z$, then $A \otimes \lambda A = A\lambda A$.

The proofs of the two following propositions will be given in an extended version of this paper.

Proposition 3. *Let* $(X, Y) \in \ddot{\mathbb{S}}$*, let* $\lambda X \in Cop(X)$ *disjoint from* Y*, and let* $Z \preceq X$*. The complex* $(X \otimes \lambda X) \cup Y$ *collapses onto* $(Z \otimes \lambda Z) \cup Y$*. In particular,* $(X \otimes \lambda X) \cup Y$ *collapses onto* Y *and* $(X \otimes \lambda X)$ *collapses onto* X*.*

Proposition 4. *Let* $(X, Y) \in \ddot{\mathbb{S}}$ *and let* $\lambda X \in Cop(X)$ *disjoint from* Y*.*
Let $(\lambda X, Z) \in \ddot{\mathbb{S}}$ *such that* Z *is disjoint from* Y*. If* X *collapses onto* X'*, then* $(X \otimes \lambda X) \cup Y \cup Z$ *collapses onto* $(X' \otimes \lambda X') \cup Y \cup Z$*.*

Let $X, Y \in \mathbb{S}$. We say that Y is *independent from* X if a simplex $x \in Y$ is necessarily in X whenever $x \subseteq \underline{X}$. In other words, Y is independent from X if any cell that is included in Y but not in X, contains a vertex that is included in Y but not in X.

Observe that Y is independent from X if and only if $X \cup Y$ is independent from X. Also, a product such that $X \otimes \lambda X$ is independent from X and from λX.

The proof of the following proposition is easy.

Proposition 5. *Let* $X, Y \in \mathbb{S}$ *and* $(X, Z) \in \ddot{\mathbb{S}}$*. If* Z *is independent from* X*, then there exists* $\lambda Z \in Cop(Z; X)$ *such that* $\lambda Z \cap Y = X \cap Y$*.*

Remark 4. Let $X, Y \in \mathbb{S}$ such that X and Y are disjoint, and let $(X, Z) \in \ddot{\mathbb{S}}$. Then, there exists $\lambda Z \in Cop(Z; X)$ such that λZ and Y are disjoint. In other words, in this particular case, Prop. 5 is satisfied even if the complex Z is not independent from X.

6 Completions and Homotopic Pairs

In this section, we establish a link between dyads and homotopic pairs (Th. 3). For that purpose, we give first the following characterization of $\ddot{\mathbb{H}}$.

If $X, Y \in \mathbb{S}$, we write $X \overset{*E}{\longmapsto} Y$, whenever X expands onto Y.

Proposition 6. *Let* $(X, Y) \in \ddot{\mathbb{S}}$*. We have* $(X, Y) \in \ddot{\mathbb{H}}$ *if and only if there exists a complex* Z *independent from* Y *such that* $X \overset{*E}{\longmapsto} Z$ *and* $Y \overset{*E}{\longmapsto} Z$*.*

Proof. Let $(X, Y) \in \ddot{\mathbb{S}}$.
i) Suppose $X \overset{*E}{\longmapsto} Z$ and $Y \overset{*E}{\longmapsto} Z$. Then, we may derive (X, Z) from (Z, Z) by repeated applications of $\langle\ddot{\text{H}}4\rangle$ and we may derive (X, Y) from (X, Z) by repeated applications of $\langle\ddot{\text{H}}2\rangle$. Thus, $(X, Y) \in \ddot{\mathbb{H}}$.
ii) We proceed by induction on the four completions that describe $\ddot{\mathbb{H}}$. If $Y = X$, then Y is independent from X, $X \overset{*E}{\longmapsto} Y$, and $Y \overset{*E}{\longmapsto} Y$. Suppose $Y \neq X$ and suppose there exists a complex Z independent from Y such that $X \overset{*E}{\longmapsto} Z$ and

$Y \overset{*E}{\longmapsto} Z$. Thus, we have $\lambda X \overset{*E}{\longmapsto} \lambda Z$ and $\lambda Y \overset{*E}{\longmapsto} \lambda Z$, whenever $\lambda Z \in Cop(Z)$.

1) Let T such that $Y \overset{E}{\longmapsto} T$. Let $\lambda Z \in Cop(Z)$ disjoint from T. We consider the complex $Z' = T \cup (Y \otimes \lambda Y) \cup \lambda Z$, Z' is independent from T. By Prop. 1 and 3, we have:
$T \overset{*E}{\longmapsto} T \cup (Y \otimes \lambda Y) \overset{*E}{\longmapsto} T \cup (Y \otimes \lambda Y) \cup \lambda Z$, and
$X \overset{*E}{\longmapsto} (X \otimes \lambda X) \overset{*E}{\longmapsto} (X \otimes \lambda X) \cup \lambda Z \overset{*E}{\longmapsto} (Y \otimes \lambda Y) \cup \lambda Z \overset{*E}{\longmapsto} T \cup (Y \otimes \lambda Y) \cup \lambda Z$.
Thus, $T \overset{*E}{\longmapsto} Z'$ and $X \overset{*E}{\longmapsto} Z'$.

2) Let T such that $T \overset{E}{\longmapsto} Y$ and $X \preceq T$. Let $\lambda Z \in Cop(Z)$ disjoint from Y. Thus, we have $\lambda T \overset{*E}{\longmapsto} \lambda Y$. We consider the complex $Z' = (T \otimes \lambda T) \cup \lambda Z$, Z' is independent from T. By Prop. 1 and 3, we have:
$T \overset{*E}{\longmapsto} (T \otimes \lambda T) \overset{*E}{\longmapsto} (T \otimes \lambda T) \cup \lambda Y \overset{*E}{\longmapsto} (T \otimes \lambda T) \cup \lambda Z$, and
$X \overset{*E}{\longmapsto} (X \otimes \lambda X) \overset{*E}{\longmapsto} (X \otimes \lambda X) \cup \lambda Z \overset{*E}{\longmapsto} (T \otimes \lambda T) \cup \lambda Z$.
Thus, $T \overset{*E}{\longmapsto} Z'$ and $X \overset{*E}{\longmapsto} Z'$.

3) Let T such that $X \overset{E}{\longmapsto} T$ and $T \preceq Y$. Let $\lambda Z \in Cop(Z)$ disjoint from Y. We consider the complex $Z' = (Y \otimes \lambda Y) \cup \lambda Z$, Z' is independent from Y. By Prop. 1, 3, 4, we have:
$Y \overset{*E}{\longmapsto} (Y \otimes \lambda Y) \overset{*E}{\longmapsto} (Y \otimes \lambda Y) \cup \lambda Z$, and
$T \overset{*E}{\longmapsto} T \cup (X \otimes \lambda X) \overset{*E}{\longmapsto} T \cup (X \otimes \lambda X) \cup \lambda Z \overset{*E}{\longmapsto} (T \otimes \lambda T) \cup \lambda Z \overset{*E}{\longmapsto} (Y \otimes \lambda Y) \cup \lambda Z$.
Thus, $Y \overset{*E}{\longmapsto} Z'$ and $T \overset{*E}{\longmapsto} Z'$.

4) Let T such that $T \overset{E}{\longmapsto} X$. The complex Z is independent from Y, and we have $Y \overset{*E}{\longmapsto} Z$ and $T \overset{*E}{\longmapsto} X \overset{*E}{\longmapsto} Z$. □

As a direct consequence of Prop. 6, we have the following result. Observe that the expression $\ddot{\mathbb{H}} = \langle \ddot{\mathbb{I}}; \ddot{\mathrm{H}}1, \ddot{\mathrm{H}}2 \rangle$ means that a pair (X, Y) may be detected as a homotopic pair by using only transformations that keep the complex X fixed.

Proposition 7. *We have* $\ddot{\mathbb{H}} = \langle \ddot{\mathbb{I}}; \ddot{\mathrm{H}}1, \ddot{\mathrm{H}}2 \rangle = \langle \ddot{\mathbb{I}}; \ddot{\mathrm{H}}2, \ddot{\mathrm{H}}4 \rangle$.

Proof. Let $\ddot{\mathbb{H}}' = \langle \ddot{\mathbb{I}}; \ddot{\mathrm{H}}1, \ddot{\mathrm{H}}2 \rangle$ and $\ddot{\mathbb{H}}'' = \langle \ddot{\mathbb{I}}; \ddot{\mathrm{H}}2, \ddot{\mathrm{H}}4 \rangle$. We have $\ddot{\mathbb{H}}' \subseteq \ddot{\mathbb{H}}$ and $\ddot{\mathbb{H}}'' \subseteq \ddot{\mathbb{H}}$ (see Remark 1). Let $(X, Y) \in \ddot{\mathbb{H}}$. By Prop. 6, there exists Z such that $X \overset{*E}{\longmapsto} Z$ and $Y \overset{*E}{\longmapsto} Z$. We have $(X, X) \in \ddot{\mathbb{H}}'$. Thus, by $\langle \ddot{\mathrm{H}}1 \rangle$, $(X, Z) \in \ddot{\mathbb{H}}'$ and, by $\langle \ddot{\mathrm{H}}2 \rangle$, $(X, Y) \in \ddot{\mathbb{H}}'$. We also have $(Z, Z) \in \ddot{\mathbb{H}}''$. Thus, by $\langle \ddot{\mathrm{H}}4 \rangle$, $(X, Z) \in \ddot{\mathbb{H}}''$ and, by $\langle \ddot{\mathrm{H}}2 \rangle$, $(X, Y) \in \ddot{\mathbb{H}}''$. It follows that $\ddot{\mathbb{H}} \subseteq \ddot{\mathbb{H}}'$ and $\ddot{\mathbb{H}} \subseteq \ddot{\mathbb{H}}''$. □

Lemma 1. *If* $X \in \mathbb{S}$ *and* α *is a vertex, then* $(\emptyset, \alpha X) \in \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{T}}, \ddot{\mathrm{U}} \rangle$.

Proof. Let $\ddot{\mathbb{H}}' = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{T}}, \ddot{\mathrm{U}} \rangle$. If $Card(X) \leq 2$, then αX is a cell. In this case the property is true since $(\emptyset, \alpha X) \in \ddot{\mathbb{C}}$. Let $k \geq 3$. Suppose the property is true whenever $Card(X) < k$ and let X such that $Card(X) = k$. If X has a single facet then, again, αX is a cell and $(\emptyset, \alpha X) \in \ddot{\mathbb{C}}$. If X has more than one facet, then there exists X' and X'' such that $\alpha X = \alpha X' \cup \alpha X''$ with $Card(X') < k$, $Card(X'') < k$, and $Card(X' \cap X'') < k$. By the induction hypothesis, we have $(\emptyset, \alpha X') \in \ddot{\mathbb{H}}'$, $(\emptyset, \alpha X'') \in \ddot{\mathbb{H}}'$, and $(\emptyset, \alpha(X' \cap X'')) \in \ddot{\mathbb{H}}'$. By $\langle \ddot{\mathrm{U}} \rangle$, we have

$(\alpha(X' \cap X''), \alpha X'') = (\alpha X' \cap \alpha X'', \alpha X'') \in \ddot{\mathbb{H}}'$. By $\langle \ddot{\mathrm{Y}}1 \rangle$, we obtain $(\alpha X', \alpha X) \in \ddot{\mathbb{H}}'$. Now, by $\langle \ddot{\mathrm{T}} \rangle$, we conclude that $(\emptyset, \alpha X) \in \ddot{\mathbb{H}}'$. □

Lemma 2. *Let $X, Y \in \mathbb{S}$. If $X \overset{E}{\longmapsto} Y$, then $(X, Y) \in \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{T}}, \ddot{\mathrm{U}} \rangle$.*

Proof. If A is a cell, with $A \neq \emptyset$, we set $\partial A = A \setminus \{x\}$, where x is the unique facet of A. Let $\ddot{\mathbb{H}}' = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{T}}, \ddot{\mathrm{U}} \rangle$. Suppose $X \overset{E}{\longmapsto} Y$. If $X = \emptyset$, then Y is a vertex, and $(X, Y) \in \ddot{\mathbb{C}}$. Otherwise, there exists a vertex α and a cell $A \in \mathbb{C}$, with $A \neq \emptyset$, such that $Y = X \cup \alpha A$ and $X \cap \alpha A = \alpha \partial A$ (the free pair is $(\underline{A}, \underline{\alpha A})$). By Lemma 1, we have $(\emptyset, \alpha \partial A) \in \ddot{\mathbb{H}}'$ and $(\emptyset, \alpha A) \in \ddot{\mathbb{H}}'$. Thus, by $\langle \ddot{\mathrm{U}} \rangle$, we have $(\alpha \partial A, \alpha A) \in \ddot{\mathbb{H}}'$. By $\langle \ddot{\mathrm{Y}}1 \rangle$, we obtain $(X, Y) \in \ddot{\mathbb{H}}'$. □

The following theorem shows that four of the five completions that describe dyads allow for a characterization of the collection made of all homotopic pairs.

Theorem 3. *We have $\ddot{\mathbb{H}} = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{T}}, \ddot{\mathrm{U}}, \ddot{\mathrm{L}} \rangle$.*

Proof. Let $\ddot{\mathbb{H}}' = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{T}}, \ddot{\mathrm{U}}, \ddot{\mathrm{L}} \rangle$.

i) Setting $T = \emptyset$ in the definition of $\ddot{\mathrm{Y}}1$, we see that $\ddot{\mathbb{I}} \subseteq \ddot{\mathbb{H}}'$. We have $(X, Y) \in \ddot{\mathbb{H}}'$ whenever $X \overset{E}{\longmapsto} Y$ (Lemma 2 and Remark 1). Now, for any (R, S), (R, T), (S, T) in $\ddot{\mathbb{S}}$:
- If $(R, T) \in \ddot{\mathbb{H}}'$ and $S \overset{E}{\longmapsto} T$, then $(S, T) \in \ddot{\mathbb{H}}'$ and, by $\ddot{\mathrm{L}}$, we have $(R, S) \in \ddot{\mathbb{H}}'$;
- If $(S, T) \in \ddot{\mathbb{H}}'$ and $R \overset{E}{\longmapsto} S$, then $(R, S) \in \ddot{\mathbb{H}}'$ and, by $\ddot{\mathrm{T}}$, we have $(R, T) \in \ddot{\mathbb{H}}'$.
By induction, since $\ddot{\mathbb{H}} = \langle \ddot{\mathbb{I}}; \ddot{\mathrm{H}}2, \ddot{\mathrm{H}}4 \rangle$ (Prop. 7), it follows that $\ddot{\mathbb{H}} \subseteq \ddot{\mathbb{H}}'$.

ii) If $(X, Y) \in \ddot{\mathbb{C}}$, then it may be checked that Y collapses onto X. Thus $\ddot{\mathbb{C}} \subseteq \ddot{\mathbb{H}}$.

- Let $S, T \in \mathbb{S}$. Suppose $(S \cap T, T) \in \ddot{\mathbb{H}}$. Thus, there exists K independent from T such that $S \cap T \overset{*E}{\longmapsto} K$ and $T \overset{*E}{\longmapsto} K$ (Prop. 6). Then, there exists a copy $\lambda K \in Cop(K; T)$ such that $\lambda K \cap S = S \cap T$ (Prop. 5). Since $S \cap T \overset{*E}{\longmapsto} \lambda K$, we have $S \overset{*E}{\longmapsto} S \cup \lambda K$ (Prop. 1). We have also $\lambda K \cap (S \cup T) = T$. Thus, since $T \overset{*E}{\longmapsto} \lambda K$, we have $S \cup T \overset{*E}{\longmapsto} S \cup \lambda K$ (Prop. 1). Therefore $(S, S \cup T) \in \ddot{\mathbb{H}}$.

- Let $(R, S), (S, T), (R, T) \in \ddot{\mathbb{S}}$. Suppose $(R, S) \in \ddot{\mathbb{H}}$ and $(S, T) \in \ddot{\mathbb{H}}$. There exists K such that $S \overset{*E}{\longmapsto} K$, and $T \overset{*E}{\longmapsto} K$ (Prop. 6). By $\langle \ddot{\mathrm{H}}1 \rangle$, we have $(R, K) \in \ddot{\mathbb{H}}$. Thus, by $\langle \ddot{\mathrm{H}}2 \rangle$, we have $(R, T) \in \ddot{\mathbb{H}}$.

- Let $(R, S), (S, T), (R, T) \in \ddot{\mathbb{S}}$. Suppose $(R, S) \in \ddot{\mathbb{H}}$ and $(R, T) \in \ddot{\mathbb{H}}$. By Prop. 6, there exists K independent from S such that $R \overset{*E}{\longmapsto} K$, and $S \overset{*E}{\longmapsto} K$. By Prop. 5, there exists $\lambda K \in Cop(K; S)$ such that $\lambda K \cap T = S$. Since $S \overset{*E}{\longmapsto} \lambda K$, we have $T \overset{*E}{\longmapsto} T \cup \lambda K$ (Prop. 1). By $\langle \ddot{\mathrm{H}}1 \rangle$, we have $(R, T \cup \lambda K) \in \ddot{\mathbb{H}}$. Since $R \overset{*E}{\longmapsto} \lambda K$, by $\langle \ddot{\mathrm{H}}3 \rangle$, we have $(\lambda K, T \cup \lambda K) \in \ddot{\mathbb{H}}$ and, by $\langle \ddot{\mathrm{H}}4 \rangle$, $(S, T \cup \lambda K) \in \ddot{\mathbb{H}}$. By $\langle \ddot{\mathrm{H}}2 \rangle$, we get $(S, T) \in \ddot{\mathbb{H}}$.

- Let $(R, S), (S, T), (R, T) \in \ddot{\mathbb{S}}$. Suppose $(R, T) \in \ddot{\mathbb{H}}$ and $(S, T) \in \ddot{\mathbb{H}}$. There exists K such that $S \overset{*E}{\longmapsto} K$, and $T \overset{*E}{\longmapsto} K$ (Prop. 6). By $\langle \ddot{\mathrm{H}}1 \rangle$, we have $(R, K) \in \ddot{\mathbb{H}}$. Thus, by $\langle \ddot{\mathrm{H}}2 \rangle$, we have $(R, S) \in \ddot{\mathbb{H}}$.

Thus, by induction, we have $\ddot{\mathbb{H}}' \subseteq \ddot{\mathbb{H}}$. □

By Th. 3, the only difference between the collection $\ddot{\mathbb{X}}$ of dyads and the collection $\ddot{\mathbb{H}}$ of homotopic pairs is the completion $\langle \ddot{\text{Y}}2 \rangle$. This difference may be illustrated by the following classical construction. Let P be the punctured Poincaré homology sphere. The complex P is not contractible since the fundamental group of P is not trivial, thus $(\emptyset, P) \notin \ddot{\mathbb{H}}$. Let α and β be two distinct vertices and let $S = \alpha P \cup \beta P$ be a suspension of P. Now the fundamental group of S is trivial and S is contractible. So we have $(\emptyset, S) \in \ddot{\mathbb{H}}$. Since $\ddot{\mathbb{H}} \subseteq \ddot{\mathbb{X}}$, we get $(\emptyset, S) \in \ddot{\mathbb{X}}$. But $(\emptyset, \alpha P) \in \ddot{\mathbb{X}}$ (Prop. 2 of [15]). Thus, by $\langle \ddot{\text{U}} \rangle$, it follows that $(\alpha P, S) \in \ddot{\mathbb{X}}$. By $\langle \ddot{\text{Y}}2 \rangle$, we deduce that $(\alpha P \cap \beta P, \beta P) \in \ddot{\mathbb{X}}$. We obtain $(P, \beta P) \in \ddot{\mathbb{X}}$. Since $(\emptyset, \beta P) \in \ddot{\mathbb{X}}$, by $\langle \ddot{\text{L}} \rangle$, we conclude that $(\emptyset, P) \in \ddot{\mathbb{X}}$, *i.e.*, that P is a dendrite (Th.ň2).

Remark 5. Let us consider the following completion on $\ddot{\mathbb{S}}$: For any $S, T \in \mathbb{S}$,

-> If $S \overset{E}{\longmapsto} T$, then $(S, T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\text{E}} \rangle$

Using Th. 3, we can check that we have $\ddot{\mathbb{H}} = \langle \ddot{\mathbb{I}}; \ddot{\text{E}}, \ddot{\text{T}}, \ddot{\text{U}}, \ddot{\text{L}} \rangle$.

7 Completions and Simple Homotopy

In the preceding section we have established a link between dyads and homotopic pairs. Here, we will clarify the relation between homotopic pairs and the more classical notion of simple homotopy (Th. 4). For that purpose, we introduce the following relation.

We denote by $\overset{H}{\sim}$ the binary relation on $\mathbb{S}$ such that, for all X, $Y \in \mathbb{S}$, we have $X \overset{H}{\sim} Y$ if and only if:
i) The complexes X and Y are disjoint; and
ii) There exists $K \in \mathbb{S}$ such that $(X, K) \in \ddot{\mathbb{H}}$ and $(Y, K) \in \ddot{\mathbb{H}}$.

For example, if $X \in \mathbb{S}$ and if $\lambda X \in Cop(X)$ is disjoint from X, then, by Prop. 3, we have $X \overset{*E}{\longmapsto} X \otimes \lambda X$ and $\lambda X \overset{*E}{\longmapsto} X \otimes \lambda X$. Thus, we have $X \overset{H}{\sim} \lambda X$.

Proposition 8. *Let $X, Y \in \mathbb{S}$ be disjoint complexes. We have $X \overset{H}{\sim} Y$ if and only if there exists $Z \in \mathbb{S}$ such that $X \overset{*E}{\longmapsto} Z$ and $Y \overset{*E}{\longmapsto} Z$.*

Proof. The "if" part is straightforward. Suppose there exists $K \in \mathbb{S}$ such that $(X, K) \in \ddot{\mathbb{H}}$ and $(Y, K) \in \ddot{\mathbb{H}}$. By Prop. 6, there exists Z' such that $X \overset{*E}{\longmapsto} Z'$ and $K \overset{*E}{\longmapsto} Z'$. By $\langle \ddot{\text{H}}1 \rangle$, we have $(Y, Z') \in \ddot{\mathbb{H}}$. Then, again by Prop. 6, there exists Z such that $Y \overset{*E}{\longmapsto} Z$ and $Z' \overset{*E}{\longmapsto} Z$. Thus, we have $X \overset{*E}{\longmapsto} Z$ and $Y \overset{*E}{\longmapsto} Z$. □

Lemma 3. *Let $X, Y \in \mathbb{S}$. If X and Y are simply homotopic, then there exists $\lambda Y \in Cop(Y)$ disjoint from X, and there exists $K \in \mathbb{S}$ such that $X \overset{*E}{\longmapsto} K$ and $\lambda Y \overset{*E}{\longmapsto} K$.*

Proof. Let X, $Y \in \mathbb{S}$.
i) If $X = Y$, then there exists $\lambda Y \in Cop(Y)$ disjoint from X. Let $K = X \otimes \lambda Y$.

By Prop. 3, the complex K satisfies the above condition.
ii) Suppose λY and K satisfy the above condition.

- Let X' such that $X \overset{E}{\longmapsto} X'$. Let $\mu K \in Cop(K)$ disjoint from K and X'. We have $\mu\lambda Y \preceq \mu K$ and $\mu\lambda Y$ is a copy of Y disjoint from X'. We set $K' = X' \cup (X \otimes \mu X) \cup \mu K$. We have $X' \overset{*E}{\longmapsto} X' \cup (X \otimes \mu X) \overset{*E}{\longmapsto} X' \cup (X \otimes \mu X) \cup \mu K = K'$, and $\mu\lambda Y \overset{*E}{\longmapsto} \mu K \overset{*E}{\longmapsto} (X \otimes \mu X) \cup \mu K \overset{*E}{\longmapsto} X' \cup (X \otimes \mu X) \cup \mu K = K'$.

- Let X' such that $X' \overset{E}{\longmapsto} X$. Let $\mu K \in Cop(K)$ disjoint from K. We have $\mu\lambda Y \preceq \mu K$ and $\mu\lambda Y$ is a copy of Y disjoint from X'. We set $K' = (X' \otimes \mu X') \cup \mu K$. We have $X' \overset{*E}{\longmapsto} (X' \otimes \mu X') \overset{*E}{\longmapsto} (X' \otimes \mu X') \cup \mu K = K'$, and
$\mu\lambda Y \overset{*E}{\longmapsto} \mu K \overset{*E}{\longmapsto} (X' \otimes \mu X') \cup \mu K = K'$.
The proof is complete by induction on the number of elementary collapses and expansions that allow us to transform X into Y. □

Theorem 4. *Let $X, Y \in \mathbb{S}$ such that X and Y are disjoint. The complexes X and Y are simply homotopic if and only if $X \overset{H}{\sim} Y$.*

Proof. Let $X, Y \in \mathbb{S}$ be disjoint complexes.
i) If $X \overset{H}{\sim} Y$, then, by Prop. 8, X and Y are simply homotopic.
ii) Suppose X and Y are simply homotopic. Then, there exists $\lambda Y \in Cop(Y)$ disjoint from X, and there exists $K \in \mathbb{S}$ such that $X \overset{*E}{\longmapsto} K$ and $\lambda Y \overset{*E}{\longmapsto} K$ (Lemma 3). Furthermore, there exists $\mu K \in Cop(K; X)$ disjoint from Y (see Remark 4). Thus, $X \overset{*E}{\longmapsto} \mu K$ and $\mu\lambda Y \overset{*E}{\longmapsto} \mu K$. Let $K' = \mu K \cup (Y \otimes \mu\lambda Y)$. We have $X \overset{*E}{\longmapsto} \mu K \overset{*E}{\longmapsto} K'$ and $Y \overset{*E}{\longmapsto} Y \otimes \mu\lambda Y \overset{*E}{\longmapsto} K'$. It follows that $(X, K') \in \ddot{\mathbb{H}}$ and $(Y, K') \in \ddot{\mathbb{H}}$. Therefore $X \overset{H}{\sim} Y$. □

Since simple homotopy corresponds to a transitive relation, the following is a corollary of Th. 4.

Corollary 1. *Let $X, Y, Z \in \mathbb{S}$ be three mutually disjoint complexes. If $X \overset{H}{\sim} Y$ and $Y \overset{H}{\sim} Z$, then $X \overset{H}{\sim} Z$.*

If $X \in \mathbb{S}$ and if $\lambda X \in Cop(X)$, then X and λX are simply homotopic. Thus, we also have the following immediate consequence of Th. 4 and Prop. 8.

Corollary 2. *Let $X, Y \in \mathbb{S}$ and let $\lambda Y \in Cop(Y)$ disjoint from X. The complexes X and Y are simply homotopic if and only if there exists $K \in \mathbb{S}$ such that $X \overset{*E}{\longmapsto} K$ and $\lambda Y \overset{*E}{\longmapsto} K$.*

Remark 6. In [1], the following result was given (see Th. 4 of [1]):
Let $X, Y \in \mathbb{S}$. If X and Y are simply homotopic, then there exists $K \in \mathbb{S}$ and there exists a stellar sub-division $\widetilde{Y}$ of Y, such that $X \overset{*E}{\longmapsto} K$ and $\widetilde{Y} \overset{*E}{\longmapsto} K$.
Cor. 2 shows that we can have the same relationship between two homotopic complexes without involving sub-divisions, which change the structure of a complex. Only the notion of a copy is necessary. Observe that this result has been made possible thanks to the notion of a product, this construction allows us to have "more room" to perform homotopic transforms.

8 Conclusion

We proposed an extension of simple homotopy by considering homotopic pairs. The notion of a homotopic pair was formalized by means of completions. One of the main results of the paper (Th. 3) shows that a subset of the five completions that describe dyads allows for a complete characterization of homotopic pairs. Since dyads are linked to homology, we have a unified framework where a link between some notions relative to homotopy and some notions relative to homology may be expressed. It should be noted that such a link is not obvious in the classical framework [19].

In the future, we will further investigate the possibility to use completions for deriving results related to combinatorial topology.

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