A Higher Stacky Perspective on Chern–Simons Theory

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Abstract The first part of this text is a gentle exposition of some basic constructions and results in the extended prequantum theory of Chern–Simons-type gauge field theories. We explain in some detail how the action functional of ordinary 3d Chern–Simons theory is naturally localized ("extended", "multi-tiered") to a map on the universal moduli stack of principal connections, a map that itself modulates a circle-principal 3-connection on that moduli stack, and how the iterated transgressions of this extended Lagrangian unify the action functional with its prequantum bundle and with the WZW-functional. In the second part we provide a brief review and outlook of the higher prequantum field theory of which this is a first example. This includes a higher geometric description of supersymmetric Chern–Simons theory, Wilson loops and other defects, generalized geometry, higher Spin-structures, anomaly cancellation, and various other aspects of quantum field theory.

1 Introduction

One of the fundamental examples of quantum field theory is 3-dimensional Chern–Simons gauge field theory as introduced in [88]. We give a pedagogical exposition of this from a new, natural, perspective of *higher geometry* formulated using *higher stacks* in *higher toposes* along the lines of [30] and references given there. Then we indicate how this opens the door to a more general understanding of *extended prequantum* (topological) field theory, constituting a pre-quantum analog of the extended quantum field theory as in [60], in the sense of higher geometric quantization [67].

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The aim of this text is twofold. On the one hand, we will attempt to dissipate the false belief that higher toposes are an esoteric discipline whose secret rites are reserved to initiates. To do this we will present a familiar example from differential topology, namely *Chern–Simons theory*, from the perspective of higher stacks, to show how this is a completely natural and powerful language in differential geometry. Furthermore, since any language is best appreciated by listening to it rather than by studying its grammar, in this presentation we will omit most of the rigorous definitions, leaving the reader the task to imagine and reconstruct them from the context. Clearly this does not mean that such definitions are not available: we refer the interested reader to [59] for the general theory of higher toposes and to [79] for general theory and applications of *differential cohesive* higher toposes that can express differential geometry, differential cohomology and prequantum gauge field theory; the reader interested in the formal mathematical aspects of the theory might enjoy looking at [81].

On the other hand, the purpose of this note is not purely pedagogical: we show how the stacky approach unifies in a natural way all the basic constructions in classical Chern–Simons theory (e.g., the action functional, the Wess-Zumino-Witten bundle gerbe, the symplectic structure on the moduli space of flat G-bundles as well as its prequantization), clarifies the relations of these with differential cohomology, and clearly points towards "higher Chern–Simons theories" and their higher and extended geometric prequantum theory. A brief survey and outlook of this more encompassing theory is given in the last sections. This is based on our series of articles including [28–31] and [74–76]. A set of lecture notes explaining this theory is [80].

We assume the reader has a basic knowledge of characteristic classes and of Chern–Simons theory. Friendly, complete and detailed introductions to these two topics can be found in [63] and [20, 32–34], respectively.

In this article we focus on the (extended) *geometric quantization* of Chern–Simons theory. Another important approach is the (extended) *perturbative quantization* based on path integrals in the BV-BRST formalism, as discussed notably in [1], based on the general program of extended perturbative BV-quantization laid out in [18, 19]. The BV-BRST formalism—a description of phase spaces/critical loci in higher ("derived") geometry—is also naturally formulated in terms of the higher cohesive geometry of higher stacks that we consider here, but further discussion of this point goes beyond the scope of this article. The interested reader can find more discussion in Sect. 1.2.15.2 and 3.10.8 of [79].

2 A Toy Example: 1-Dimensional $U(n)$-Chern–Simons Theory

Before describing the archetypical 3-dimensional Chern–Simons theory with a compact simply connected gauge group[1] from a stacky perspective, here we first look from this point of view at 1-dimensional Chern–Simons theory with gauge group

[1] We are using the term "gauge group" to refer to the structure group of the theory. This is not to be confused with the group of gauge transformations.

$U(n)$. Although this is a very simplified version, still it will show in an embryonic way all the features of the higher dimensional theory.[2] Moreover, a slight variant of this 1-dimensional CS theory shows up as a component of 3d Chern–Simons theory *with Wilson line defects*, this we discuss at the end of the exposition part in Sect. 3.4.5.

2.1 The Basic Definition

Let A be a $\mathfrak{u}_n$-valued differential 1-form on the circle S^1. Then $\frac{1}{2\pi i}\mathrm{tr}(A)$ is a real-valued 1-form, which we can integrate over S^1 to get a real number. This construction can be geometrically interpreted as a map

$$\{\text{trivialized } U(n)\text{-bundles with connections on } S^1\} \xrightarrow{\frac{1}{2\pi i}\int_{S^1}\mathrm{tr}} \mathbb{R}.$$

Since the Lie group $U(n)$ is connected, the classifying space $BU(n)$ of principal $U(n)$-bundles is simply connected, and so the set of homotopy classes of maps from S^1 to $BU(n)$ is trivial. By the characterizing property of the classifying space, this set is the set of isomorphism classes of principal $U(n)$-bundles on S^1, and so every principal $U(n)$-bundle over S^1 is trivializable. Using a chosen trivialization to pull-back the connection, we see that an arbitrary $U(n)$-principal bundle with connection (P, ∇) is (noncanonically) isomorphic to a trivialized bundle with connection, and so our picture enlarges to

$$\begin{array}{ccc} \{\text{trivialized } U(n)\text{-bundles with connections on } S^1\} & \xrightarrow{\frac{1}{2\pi i}\int_{S^1}\mathrm{tr}} & \mathbb{R} \\ \downarrow & & \\ \{U(n)\text{-bundles with connections on } S^1\}/\mathrm{iso} & & \end{array}$$

and it is tempting to fill the square by placing a suitable quotient of $\mathbb{R}$ in the right bottom corner. To see that this is indeed possible, we have to check what happens when we choose two different trivializations for the same bundle, i.e., we have to compute the quantity

$$\frac{1}{2\pi i}\int_{S^1} \mathrm{tr}(A') - \mathrm{tr}(A),$$

where A and A' are two 1-form incarnations of the same connection ∇ under different trivializations of the underlying bundle. What one finds is that this quantity is always an integer, thus giving a commutative diagram

[2] Even 1-dimensional Chern–Simons theory exhibits a rich structure once we pass to *derived* higher gauge groups as in [46]. This goes beyond the present exposition, but see Sect. 5.1 for an outlook and Sect. 5.7.10 of [79] for more details.

$$\begin{array}{ccc} \{\text{trivialized } U(n)\text{-bundles with connections on } S^1\} & \xrightarrow{\frac{1}{2\pi i}\int_{S^1}\mathrm{tr}} & \mathbb{R} \\ \downarrow & & \downarrow \\ \{U(n)\text{-bundles with connections on } S^1\}/\mathrm{iso} & \xrightarrow{\exp\int_{S^1}\mathrm{tr}} & U(1)\,. \end{array}$$

The bottom line in this diagram is the *1-dimensional Chern–Simons action for $U(n)$-gauge theory*. An elegant way of proving that $\frac{1}{2\pi i}\int_{S^1}\mathrm{tr}(A) - \mathrm{tr}(A')$ is always an integer is as follows. Once a trivialization has been chosen, one can extend a principal $U(n)$-bundle with connection (P, ∇) on S^1 to a trivialized principal $U(n)$-bundle with connection over the disk D^2. Denoting by the same symbol ∇ the extended connection and by A the 1-form representing it, then by Stokes' theorem we have

$$\frac{1}{2\pi i}\int_{S^1}\mathrm{tr}(A) = \frac{1}{2\pi i}\int_{\partial D^2}\mathrm{tr}(A) = \frac{1}{2\pi i}\int_{D^2} d\,\mathrm{tr}(A) = \frac{1}{2\pi i}\int_{D^2}\mathrm{tr}(F_\nabla),$$

where F_∇ is the curvature of ∇. If we choose two distinct trivializations, what we get are two trivialized principal $U(n)$-bundles with connection over D^2 together with an isomorphism of their boundary data. Using this isomorphism to glue together the two bundles, we get a (generally nontrivial) $U(n)$-bundle with connection $(\tilde{P}, \tilde{\nabla})$ on $S^2 = D^2 \coprod_{S^1} D^2$, the disjoint union of the upper and lower hemisphere glued along the equator, and

$$\frac{1}{2\pi i}\int_{S^1}\mathrm{tr}(A') - \mathrm{tr}(A) = \frac{1}{2\pi i}\int_{S^2}\mathrm{tr}(\tilde{\nabla}) = \langle c_1(\tilde{P}), [S^2]\rangle,$$

the first Chern number of the bundle $\tilde{P}$. Note how the generator c_1 of the second integral cohomology group $H^2(BU(n), \mathbb{Z}) \cong \mathbb{Z}$ has come into play.

Despite its elegance, the argument above has a serious drawback: it relies on the fact that S^1 is a boundary. And, although this is something obvious, still it is something nontrivial and indicates that generalizing 1-dimensional Chern–Simons theory to higher dimensional Chern–Simons theory along the above lines will force limiting the construction to those manifolds which are boundaries. For standard 3-dimensional Chern–Simons theory with a compact simply connected gauge group, this will actually be no limitation, since the oriented cobordism ring is trivial in dimension 3, but one sees that this is a much less trivial statement than saying that S^1 is a boundary. However, in any case, that would definitely not be true in general for higher dimensions, as well as for topological structures on manifolds beyond orientations.

2.2 A Lie Algebra Cohomology Approach

A way of avoiding the cobordism argument used in the previous section is to focus on the fact that

$$\frac{1}{2\pi i}\mathrm{tr} : \mathfrak{u}_n \to \mathbb{R}$$

is a Lie algebra morphism, i.e., it is a real-valued 1-cocycle on the Lie algebra $\mathfrak{u}_n$ of the group $U(n)$. A change of trivialization for a principal $U(n)$-bundle $P \to S^1$ is given by a gauge transformation $\boldsymbol{g} : S^1 \to U(n)$. If A is the $\mathfrak{u}_n$-valued 1-form corresponding to the connection ∇ in the first trivialization, the gauge-transformed 1-form A' is given by

$$A' = \boldsymbol{g}^{-1} A \boldsymbol{g} + \boldsymbol{g}^{-1} d\boldsymbol{g},$$

where $\boldsymbol{g}^{-1}d\boldsymbol{g} = \boldsymbol{g}^*\theta_{U(n)}$ is the pullback of the Maurer–Cartan form $\theta_{U(n)}$ of $U(n)$ via $\boldsymbol{g}$. Since $\frac{1}{2\pi i}\mathrm{tr}$ is an invariant polynomial (i.e., it is invariant under the adjoint action of $U(n)$ on $\mathfrak{u}_n$), it follows that

$$\frac{1}{2\pi i}\int_{S^1} \mathrm{tr}(A') - \mathrm{tr}(A) = \frac{1}{2\pi i}\int_{S^1} \boldsymbol{g}^*\mathrm{tr}(\theta_{U(n)}),$$

and our task is reduced to showing that the right-hand term is a "quantized" quantity, i.e., that it always assumes integer values. Since the Maurer–Cartan form satisfies the Maurer–Cartan equation

$$d\theta_{U(n)} + \tfrac{1}{2}[\theta_{U(n)}, \theta_{U(n)}] = 0,$$

we see that

$$d\mathrm{tr}(\theta_{U(n)}) = -\tfrac{1}{2}\mathrm{tr}\big([\theta_{U(n)}, \theta_{U(n)}]\big) = 0,$$

i.e., $\mathrm{tr}(\theta_{U(n)})$ is a closed 1-form on $U(n)$. As an immediate consequence,

$$\frac{1}{2\pi i}\int_{S^1} \boldsymbol{g}^*\mathrm{tr}(\theta_{U(n)}) = \langle \boldsymbol{g}^*[\tfrac{1}{2\pi i}\mathrm{tr}(\theta_{U(n)})], [S^1]\rangle$$

only depends on the homotopy class of $\boldsymbol{g} : S^1 \to U(n)$, and these homotopy classes are parametrized by the additive group $\mathbb{Z}$ of the integers. Notice how the generator $[\frac{1}{2\pi i}\mathrm{tr}(\theta_{U(n)})]$ of $H^1(U(n);\mathbb{Z})$ has appeared. This shows how this proof is related to the one in the previous section via the transgression isomorphism $H^1(U(n);\mathbb{Z}) \to H^2(BU(n);\mathbb{Z})$.

It is useful to read the transgression isomorphism in terms of differential forms by passing to real coefficients and pretending that $BU(n)$ is a finite dimensional smooth manifold. This can be made completely rigorous in various ways, e.g., by looking at $BU(n)$ as an inductive limit of finite dimensional Grasmannians. Then a connection on the universal $U(n)$-bundle $EU(n) \to BU(n)$ is described à la Ehresmann by a $\mathfrak{u}_n$-valued $U(n)$-equivariant 1-form A on $EU(n)$ which gives the Maurer–Cartan form when restricted to the fibers. The $\mathbb{R}$-valued 1-form $\frac{1}{2\pi i}\mathrm{tr}(A)$ restricted to the fibers gives the closed 1-form $\frac{1}{2\pi i}\mathrm{tr}(\theta_{U(n)})$ which is the generator of $H^1(U(n), \mathbb{R})$; the differential $d\frac{1}{2\pi i}\mathrm{tr}(A) = \frac{1}{2\pi i}\mathrm{tr}(F_A)$ is an exact 2-form on $EU(n)$ which is $U(n)$-invariant and so is the pullback of a closed 2-form on $BU(n)$ which, since it represents the first Chern class, is the generator of $H^2(U(n), \mathbb{R})$.

One sees that $\frac{1}{2\pi i}\mathrm{tr}$ plays a triple role in the above description, which might be initially confusing. To get a better understanding of what is going on, let us consider more generally an arbitrary compact connected Lie group G. Then the transgression isomorphism between $H^1(G, \mathbb{R})$ and $H^2(BG; \mathbb{R})$ is realized by a Chern–Simons element CS_1 for the Lie algebra $\mathfrak{g}$. This element is characterized by the following property: for $A \in \Omega^1(EG; \mathfrak{g})$ the connection 1-form of a principal G-connection on $EG \to BG$, we have the following transgression diagram

$$\langle F_A \rangle \xleftarrow{\ d\ } \mathrm{CS}_1(A) \xmapsto{A=\theta_G} \mu_1(\theta_G)\ ,$$

where on the left hand side $\langle - \rangle$ is a degree 2 invariant polynomial on $\mathfrak{g}$, and on the right hand side μ_1 is 1-cocycle on $\mathfrak{g}$. One says that CS_1 transgresses μ_1 to $\langle - \rangle$. Via the identification of $H^1(G; \mathbb{R})$ with the degree one Lie algebra cohomology $H^1_{\mathrm{Lie}}(\mathfrak{g}; \mathbb{R})$ and of $H^2(BG; \mathbb{R})$ with the vector space of degree 2 elements in the graded algebra $\mathrm{inv}(\mathfrak{g})$ (with elements of $\mathfrak{g}^*$ placed in degree 2), one sees that this indeed realizes the transgression isomorphism.

2.3 The First Chern Class as a Morphism of Stacks

Note that, by the end of the previous section, the base manifold S^1 has completely disappeared. This suggests that one should be able to describe 1-dimensional Chern–Simons theory with gauge group $U(n)$ more generally as a map

$$\{U(n)\text{-bundles with connections on } X\}/\mathrm{iso} \to \ ??,$$

where now X is an arbitrary manifold, and "??" is some natural target to be determined. To try to figure out what this natural target could be, let us look at something simpler and forget the connection. Then we know that the first Chern class gives a morphism of sets

$$c_1 : \{U(n)\text{-bundles on } X\}/\mathrm{iso} \to H^2(X; \mathbb{Z}).$$

Here the right hand side is much closer to the left hand side than it might appear at first sight. Indeed, the second integral cohomology group of X precisely classifies principal $U(1)$-bundles on X up to isomorphism, so that the first Chern class is actually a map

$$c_1 : \{U(n)\text{-bundles on } X\}/\text{iso} \to \{U(1)\text{-bundles over } X\}/\text{iso}.$$

Writing $\mathbf{B}U(n)(X)$ and $\mathbf{B}U(1)(X)$ for the groupoids of principal $U(n)$- and $U(1)$-bundles over X, respectively,[3] one can further rewrite c_1 as a function

$$c_1 : \pi_0\mathbf{B}U(n)(X) \to \pi_0\mathbf{B}U(1)(X)$$

between the connected components of these groupoids. This immediately leads one to suspect that c_1 could actually be $\pi_0(\mathbf{c}_1(X))$ for some morphism of groupoids $\mathbf{c}_1(X) : \mathbf{B}U(n)(X) \to \mathbf{B}U(1)(X)$. Moreover, naturality of the first Chern class suggests that, independently of X, there should actually be a morphism of stacks

$$\mathbf{c}_1 : \mathbf{B}U(n) \to \mathbf{B}U(1)$$

over the site of smooth manifolds.[4] Since a smooth manifold is built by patching together, in a smooth way, open balls of $\mathbb{R}^n$ for some n, this in turn is equivalent to saying that $\mathbf{c}_1 : \mathbf{B}U(n) \to \mathbf{B}U(1)$ is a morphism of stacks over the full sub-site of Cartesian spaces, where by definition a Cartesian space is a smooth manifold diffeomorphic to $\mathbb{R}^n$ for some n. To see that c_1 is indeed induced by a morphism of stacks, notice that $\mathbf{B}U(n)$ can be obtained by stackification from the simplicial presheaf which to a Cartesian space U associates the nerve of the action groupoid $*//C^\infty(U; U(n))$. This is nothing but saying, in a very compact way, that to give a principal $U(n)$-bundle on a smooth manifold X one picks a good open cover $\mathcal{U} = \{U_\alpha\}$ of X and local data given by smooth functions on the double intersections

$$g_{\alpha\beta} : U_{\alpha\beta} \to U(n)$$

such that $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ on the triple intersections $U_{\alpha\beta\gamma}$. The group homomorphism

$$\det : U(n) \to U(1)$$

maps local data $\{g_{\alpha\beta}\}$ for a principal $U(n)$ bundle to local data $\{h_{\alpha\beta} = \det(g_{\alpha\beta})\}$ for a principal $U(1)$-bundle and, by the basic properties of the first Chern class, one sees that

$$\mathbf{B}\det : \mathbf{B}U(n) \to \mathbf{B}U(1)$$

induces c_1 at the level of isomorphism classes, i.e., one can take $\mathbf{c}_1 = \mathbf{B}\det$.

Note that there is a canonical notion of *geometric realization* of stacks on smooth manifolds by topological spaces (see Sect. 4.3.4.1 of [79]). Under this realization the morphism of stacks $\mathbf{B}\det$ becomes a continuous function of classifying spaces $BU(n) \to K(\mathbb{Z}, 2)$ which represents the universal first Chern class.

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[3] That is, for the collections of all such bundles, with gauge transformations as morphisms.

[4] The reader unfamiliar with the language of higher stacks and simplicial presheaves in differential geometry can find an introduction in [31].

2.4 Adding Connections to the Picture

The above discussion suggests that what should really lie behind 1-dimensional Chern–Simons theory with gauge group $U(n)$ is a morphism of stacks

$$\hat{\mathbf{c}}_1 : \mathbf{B}U(n)_{\text{conn}} \to \mathbf{B}U(1)_{\text{conn}}$$

from the stack of $U(n)$-principal bundles with connection to the stack of $U(1)$-principal bundles with connection, lifting the first Chern class. This morphism is easily described, as follows. Local data for a $U(n)$-principal bundle with connection on a smooth manifold X are

- smooth $\mathfrak{u}_n$-valued 1-forms A_α on U_α;
- smooth functions $\boldsymbol{g}_{\alpha\beta} : U_{\alpha\beta} \to U(n)$,

such that

- $A_\beta = \boldsymbol{g}_{\alpha\beta}^{-1} A_\alpha \boldsymbol{g}_{\alpha\beta} + \boldsymbol{g}_{\alpha\beta}^{-1} d\boldsymbol{g}_{\alpha\beta}$ on $U_{\alpha\beta}$;
- $\boldsymbol{g}_{\alpha\beta}\boldsymbol{g}_{\beta\gamma}\boldsymbol{g}_{\gamma\alpha} = 1$ on $U_{\alpha\beta\gamma}$,

and this is equivalent to saying that $\mathbf{B}U(n)_{\text{conn}}$ is the stack of simplicial sets[5] which to a Cartesian space U assigns the nerve of the action groupoid

$$\Omega^1(U; \mathfrak{u}_n)//C^\infty(U; U(n)),$$

where the action is given by $\boldsymbol{g} : A \mapsto \boldsymbol{g}^{-1}A\boldsymbol{g} + \boldsymbol{g}^{-1}d\boldsymbol{g}$. To give a morphism $\hat{\mathbf{c}}_1 : \mathbf{B}U(n)_{\text{conn}} \to \mathbf{B}U(1)_{\text{conn}}$ we therefore just need to give a morphism of simplicial prestacks

$$\mathcal{N}(\Omega^1(-; \mathfrak{u}_n)//C^\infty(-; U(n))) \longrightarrow \mathcal{N}(\Omega^1(-; \mathfrak{u}_1)//C^\infty(-; U(1)))$$

lifting

$$\mathbf{B}\det : \mathcal{N}(*//C^\infty(-; U(n))) \longrightarrow \mathcal{N}(*//C^\infty(-; U(1))),$$

where $\mathcal{N}$ is the nerve of the indicated groupoid. In more explicit terms, we have to give a natural linear morphism

$$\varphi : \Omega^1(U; \mathfrak{u}_n) \to \Omega^1(U; \mathfrak{u}_1),$$

such that

$$\varphi(\boldsymbol{g}^{-1}A\boldsymbol{g} + \boldsymbol{g}^{-1}d\boldsymbol{g}) = \varphi(A) + \det(\boldsymbol{g})^{-1}d\det(\boldsymbol{g}),$$

and it is immediate to check that the linear map

$$\text{tr} : \mathfrak{u}_n \to \mathfrak{u}_1$$

[5] It is noteworthy that this indeed is a stack on the site CartSp. On the larger but equivalent site of all smooth manifolds it is just a prestack that needs to be further stackified.

does indeed induce such a morphism φ. In the end we get a commutative diagram of stacks

$$\begin{array}{ccc} \mathbf{B}U(n)_{\mathrm{conn}} & \xrightarrow{\hat{\mathbf{c}}_1} & \mathbf{B}U(1)_{\mathrm{conn}} \\ \downarrow & & \downarrow \\ \mathbf{B}U(n) & \xrightarrow{\mathbf{c}_1} & \mathbf{B}U(1) \end{array},$$

where the vertical arrows forget the connections.

2.5 Degree 2 Differential Cohomology

If we now fix a base manifold X and look at isomorphism classes of principal $U(n)$-bundles (with connection) on X, we get a commutative diagram of sets

$$\begin{array}{ccc} \{U(n)\text{-bundles with connection on } X\}/\mathrm{iso} & \xrightarrow{\hat{c}} & \hat{H}^2(X;\mathbb{Z}) \\ \downarrow & & \downarrow \\ \{U(n)\text{-bundles on } X\}/\mathrm{iso} & \xrightarrow{c} & H^2(X;\mathbb{Z}) \end{array},$$

where $\hat{H}^2(X;\mathbb{Z})$ is the second differential cohomology group of X. This is defined as the degree 0 hypercohomology group of X with coefficients in the two-term Deligne complex, i.e., in the sheaf of complexes

$$C^\infty(-;U(1)) \xrightarrow{\frac{1}{2\pi i}d\log} \Omega^1(-;\mathbb{R}),$$

with $\Omega^1(-;\mathbb{R})$ in degree zero [8, 39]. That $\hat{H}^2(X;\mathbb{Z})$ classifies principal $U(1)$-bundles with connection is manifest by this description: via the Dold–Kan correspondence, the sheaf of complexes indicated above precisely gives a simplicial presheaf which produces $\mathbf{B}U(1)_{\mathrm{conn}}$ via stackification. Note that we have two natural morphisms of complexes of sheaves

$$\begin{array}{ccc} C^\infty(-;U(1)) & \xrightarrow{\frac{1}{2\pi i}d\log} & \Omega^1(-;\mathbb{R}) \\ \downarrow & & \downarrow \\ C^\infty(-;U(1)) & \longrightarrow & 0 \end{array} \quad \text{and} \quad \begin{array}{ccc} C^\infty(-;U(1)) & \xrightarrow{\frac{1}{2\pi i}d\log} & \Omega^1(-;\mathbb{R}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega^2(-;\mathbb{R})_{\mathrm{cl}} \end{array}.$$

The first one induces the forgetful morphism $\mathbf{B}U(1)_{\mathrm{conn}} \to \mathbf{B}U(1)$, while the second one induces the curvature morphism $F_{(-)} : \mathbf{B}U(1)_{\mathrm{conn}} \to \Omega^2(-;\mathbb{R})_{\mathrm{cl}}$ mapping a $U(1)$-bundle with connection to its curvature 2-form. From this one sees that degree 2 differential cohomology implements in a natural geometric way the simple idea of having an integral cohomology class together with a closed 2-form representing it in de Rham cohomology.

The last step that we need to recover the 1-dimensional Chern–Simons action functional from Sect. 2.1 is to give a natural morphism

$$\mathrm{hol} : \hat{H}^2(S^1;\mathbb{Z}) \to U(1)$$

so as to realize the 1-dimensional Chern–Simons action functional as the composition

$$\mathrm{CS}_1 : \{U(n)\text{-bundles with connection on } S^1\}/\mathrm{iso} \xrightarrow{\hat{c}} \hat{H}^2(S^1;\mathbb{Z}) \xrightarrow{\mathrm{hol}} U(1)\ .$$

As the notation “hol” suggests, this morphism is nothing but the holonomy morphism mapping a principal $U(1)$-bundle with connection on S^1 to its holonomy.

An enlightening perspective from which to look at this situation is in terms of fiber integration and moduli stacks of principal $U(1)$-bundles with connections over a base manifold X. Namely, for a fixed X we can consider the *mapping stack*

$$\mathbf{Maps}(X, \mathbf{B}U(1)_{\mathrm{conn}}),$$

which is presented by the simplicial presheaf that sends a Cartesian space U to the nerve of the groupoid of principal $U(1)$-bundles with connection on $U \times X$. In other words, $\mathbf{Maps}(X, \mathbf{B}U(1)_{\mathrm{conn}})$ is the *internal hom* space between X and $\mathbf{B}U(1)_{\mathrm{conn}}$ in the category of simplicial sheaves over the site of smooth manifolds. Then, if X is an oriented compact manifold of dimension one, the fiber integration formula from [44, 45] can be naturally interpreted as a morphism of simplicial sheaves

$$\mathrm{hol}_X : \mathbf{Maps}(X, \mathbf{B}U(1)_{\mathrm{conn}}) \to \underline{U}(1),$$

where on the right one has the sheaf of smooth functions with values in $U(1)$. Taking global sections over the point one gets the morphism of simplicial sets

$$\mathrm{hol}_X : \mathbf{H}(X, \mathbf{B}U(1)_{\mathrm{conn}}) \to U(1)_{\mathrm{discr}},$$

where on the right the Lie group $U(1)$ is seen as a 0-truncated simplicial object and where $\mathbf{H}(X, \mathbf{B}U(1)_{\mathrm{conn}})$ is a simplicial model for (the nerve of) the groupoid of principal $U(1)$-bundles with connection on X. Finally, passing to isomorphism classes/connected components one gets the morphism

$$\hat{H}^2(X;\mathbb{Z}) \to U(1).$$

This morphism can also be described in purely algebraic terms by noticing that for any 1-dimensional oriented compact manifold X the short exact sequence of complexes of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & C^\infty(-;U(1)) & \longrightarrow & C^\infty(-;U(1)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \frac{1}{2\pi i}d\log} & & \downarrow & & \\ 0 & \longrightarrow & \Omega^1(-;\mathbb{R}) & \longrightarrow & \Omega^1(-;\mathbb{R}) & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

induces an isomorphism

$$\Omega^1(X)/\Omega^1_{\mathrm{cl},\mathbb{Z}}(X) \xrightarrow{\sim} \hat{H}^2(X;\mathbb{Z})$$

in hypercohomology, where $\Omega^1(X)/\Omega^1_{\mathrm{cl},\mathbb{Z}}(X)$ is the group of differential 1-forms on X modulo those 1-forms which are closed and have integral periods. In terms of this isomorphism, the holonomy map is realized as the composition

$$\hat{H}^2(X;\mathbb{Z}) \xrightarrow{\sim} \Omega^1(X)/\Omega^1_{\mathrm{cl},\mathbb{Z}}(X) \xrightarrow{\exp(2\pi i \int_X -)} U(1).$$

2.6 The Brylinski–McLaughlin 2-Cocycle

It is natural to expect that the lift of the universal first Chern class c_1 to a morphism of stacks $\mathbf{c}_1 : \mathbf{B}U(n)_{\mathrm{conn}} \to \mathbf{B}U(1)_{\mathrm{conn}}$ is a particular case of a more general construction that holds for the generator c of the second integral cohomology group of an arbitrary compact connected Lie group G with $\pi_1(G) \cong \mathbb{Z}$. Namely, if $\langle - \rangle$ is the degree 2 invariant polynomial on $\mathfrak{g}[2]$ corresponding to the characteristic class c, then for any G-connection ∇ on a principal G-bundle $P \to X$ one has that $\langle F_\nabla \rangle$ is a closed 2-form on X representing the integral class c. This precisely suggests that (P, ∇) defines an element in degree 2 differential cohomology, giving a map

$$\{G\text{-bundles with connection on } X\}/\mathrm{iso} \to \hat{H}^2(X;\mathbb{Z}).$$

That this is indeed so can be seen following Brylinski and McLaughlin [12] (see [9] for an exposition an [10, 11] for related discussion). Let $\{A_\alpha, g_{\alpha\beta}\}$ the local data for a G-connection on $P \to X$, relative to a trivializing good open cover $\mathcal{U}$ of X. Then, since G is connected and the open sets $U_{\alpha\beta}$ are contractible, we can smoothly extend the transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ to functions $\hat{g}_{\alpha\beta} : [0,1] \times U_{\alpha\beta} \to G$ with $\hat{g}_{\alpha\beta}(0) = e$, the identity element of G, and $\hat{g}_{\alpha\beta}(1) = g_{\alpha\beta}$. Using the functions $\hat{g}_{\alpha\beta}$ one can interpolate from $A_\alpha\big|_{U_{\alpha\beta}}$ to $A_\beta|_{U_{\alpha\beta}}$ by defining the $\mathfrak{g}$-valued 1-form

$$\hat{A}_{\alpha\beta} = \hat{g}_{\alpha\beta}^{-1} A_\alpha|_{U_{\alpha\beta}} \hat{g}_{\alpha\beta} + \hat{g}_{\alpha\beta}^{-1} d\hat{g}_{\alpha\beta}$$

on $U_{\alpha\beta}$. Now pick a real-valued 1-cocycle μ_1 on the Lie algebra $\mathfrak{g}$ representing the cohomology class c and a Chern–Simons element CS_1 realizing the transgression from μ_1 to $\langle - \rangle$. Then the element

$$(\mathrm{CS}_1(A_\alpha), \quad \int_{\Delta^1} \mathrm{CS}_1(\hat{A}_{\alpha\beta}) \mod \mathbb{Z})$$

is a degree 2 cocycle in the Čech–Deligne total complex lifting the cohomology class $c \in H^2(BG, \mathbb{Z})$ to a differential cohomology class $\hat{c}$. Notice how modding out by $\mathbb{Z}$ in the integral $\int_{\Delta^1} \mathrm{CS}_1(\hat{A}_{\alpha\beta})$ precisely takes care of G being connected but not simply connected, with $H^1(G;\mathbb{Z}) \cong \pi_1(G) \cong \mathbb{Z}$. That is, choosing two different extensions $\hat{g}_{\alpha\beta}$ of $g_{\alpha\beta}$ will produce two different values for that integral, but their difference will lie in the rank 1 lattice of 1-dimensional periods of G, and with the correct normalization this will be a copy of $\mathbb{Z}$.

A close look at the construction of Brylinski and McLaughlin, see [31], reveals that it actually provides a refinement of the characteristic class $c \in H^2(BG; \mathbb{Z})$ to a commutative diagram of stacks

$$\begin{array}{ccc} \mathbf{B}G_{\mathrm{conn}} & \xrightarrow{\hat{\mathbf{c}}} & \mathbf{B}U(1)_{\mathrm{conn}} \\ \downarrow & & \downarrow \\ \mathbf{B}G & \xrightarrow{\mathbf{c}} & \mathbf{B}U(1) \end{array} .$$

2.7 The Presymplectic Form on $\mathbf{B}U(n)_{\mathrm{conn}}$

In geometric quantization it is customary to call *pre-quantization* of a symplectic manifold (M, ω) the datum of a $U(1)$-principal bundle with connection on M whose curvature form is ω.[6] Furthermore, it is shown that most of the good features of symplectic manifolds continue to hold under the weaker hypothesis that the 2-form ω is only closed; this leads to introducing the term *pre-symplectic manifold* to denote a smooth manifold equipped with a closed 2-form ω and to speak of *prequantum line bundles* for these. In terms of the morphisms of stacks described in the previous sections, a prequantization of a presymplectic manifold is a lift of the morphism $\omega : M \to \Omega^2(-\mathbb{R})_{\mathrm{cl}}$ to a map ∇ fitting into a commuting diagram

$$\begin{array}{ccc} & & \mathbf{B}U(1)_{\mathrm{conn}} \\ & \overset{\nabla}{\nearrow} & \downarrow{\scriptstyle F_{(-)}} \\ M & \xrightarrow{\omega} & \Omega^2(-;\mathbb{R})_{\mathrm{cl}} \end{array} ,$$

where the vertical arrow is the curvature morphism. From this perspective there is no reason to restrict M to being a manifold. By taking M to be the universal moduli stack $\mathbf{B}U(n)_{\mathrm{conn}}$, we see that the morphism $\hat{\mathbf{c}}_1$ can be naturally interpreted as giving a canonical prequantum line bundle over $\mathbf{B}U(n)_{\mathrm{conn}}$, whose curvature 2-form

$$\omega_{\mathbf{B}U(n)_{\mathrm{conn}}} : \mathbf{B}U(n)_{\mathrm{conn}} \xrightarrow{\hat{\mathbf{c}}_1} \mathbf{B}U(1)_{\mathrm{conn}} \xrightarrow{F} \Omega^2(-;\mathbb{R})_{\mathrm{cl}}$$

is the natural presymplectic 2-form on the stack $\mathbf{B}U(n)_{\mathrm{conn}}$: the invariant polynomial $\langle - \rangle$ viewed in the context of stacks. The datum of a principal $U(n)$-bundle with connection (P, ∇) on a manifold X is equivalent to the datum of a morphism $\varphi : X \to \mathbf{B}U(n)_{\mathrm{conn}}$, and the pullback $\varphi^*\omega_{\mathbf{B}U(n)_{\mathrm{conn}}}$ of the canonical 2-form on $\mathbf{B}U(n)_{\mathrm{conn}}$ is the curvature 2-form $\frac{1}{2\pi i}\mathrm{tr}(F_\nabla)$ on X. If (P, ∇) is a principal $U(n)$-bundle with connection over a compact closed oriented 1-dimensional manifold Σ_1

[6] See for instance [54] for an original reference on geometric quantization and see [67] for further pointers.

and the morphism $\varphi : \Sigma_1 \to \mathbf{B}U(n)_{\mathrm{conn}}$ defining it can be extended to a morphism $\tilde{\varphi} : \Sigma_2 \to \mathbf{B}U(n)_{\mathrm{conn}}$ for some 2-dimensional oriented manifold Σ_2 with $\partial\Sigma_2 = \Sigma_1$, then

$$CS_1(\nabla) = \exp \int_{\Sigma_2} \tilde{\varphi}^* \omega_{\mathbf{B}U(n)_{\mathrm{conn}}},$$

and the right hand side is independent of the extension $\tilde{\varphi}$. In other words,

$$CS_1(\nabla) = \exp \int_{\Sigma_2} \mathrm{tr}(F_{\tilde{\nabla}}),$$

for any extension $(\tilde{P}, \tilde{\nabla})$ of (P, ∇) to Σ_2. This way we recover the definition of the Chern–Simons action functional for $U(n)$-principal connections on S^1 given in Sect. 2.1.

More generally, the differential refinement $\hat{\mathbf{c}}$ of a characteristic class c of a compact connected Lie group G with $H^1(G; \mathbb{Z}) \cong \mathbb{Z}$, endows the stack $\mathbf{B}G_{\mathrm{conn}}$ with a canonical presymplectic structure with a prequantum line bundle given by $\hat{\mathbf{c}}$ itself, and the same considerations apply.

2.8 The Determinant as a Holonomy Map

We have so far met two natural maps with target the sheaf $\underline{U}(1)$ of smooth functions with values in the group $U(1)$. The first one was the determinant

$$\det : \underline{U}(n) \to \underline{U}(1),$$

and the second one was the holonomy map

$$\mathrm{hol}_X : \mathbf{Maps}(X; \mathbf{B}U(1)_{\mathrm{conn}}) \to \underline{U}(1),$$

defined on the moduli stack of principal $U(1)$-bundles with connection on a 1-dimensional compact oriented manifold X. To see how these two are related, take $X = S^1$ and notice that, by definition, a morphism from a smooth manifold M to the stack $\mathbf{Maps}(S^1; \mathbf{B}U(n)_{\mathrm{conn}})$ is the datum of a principal $U(n)$-bundle with connection over the product manifold $M \times S^1$. Taking the holonomy of the $U(n)$-connection along the fibers of $M \times S^1 \to M$ locally defines a smooth $U(n)$-valued function on M which is well defined up to conjugation. In other words, holonomy along S^1 defines a morphism from M to the stack $\underline{U}(n)//_{\mathrm{Ad}}\underline{U}(n)$, where Ad indicates the adjoint action. Since this construction is natural in M we have defined a natural $U(n)$-holonomy morphism

$$\mathrm{hol}^{U(n)} : \mathbf{Maps}(S^1; \mathbf{B}U(n)_{\mathrm{conn}}) \to \underline{U}(n)//_{\mathrm{Ad}}\underline{U}(n).$$

For $n = 1$, due to the fact that $U(1)$ is abelian, we also have a natural morphism $U(1)//_{\mathrm{Ad}}U(1) \to U(1)$, and the holonomy map hol_{S^1} factors as

$$\mathrm{hol}_{S^1} : \mathbf{Maps}(S^1; \mathbf{B}U(1)_{\mathrm{conn}}) \xrightarrow{\mathrm{hol}^{U(1)}} \underline{U}(1)//_{\mathrm{Ad}}\underline{U}(1) \to \underline{U}(1).$$

Therefore, by naturality of **Maps** we obtain the following commutative diagram

$$\begin{array}{ccccccc} & & \mathbf{Maps}(S^1;\mathbf{B}U(n)_{\mathrm{conn}}) & \xrightarrow{\mathbf{Maps}(S^1,\hat{\mathbf{c}}_1)} & \mathbf{Maps}(S^1;\mathbf{B}U(1)_{\mathrm{conn}}) & & \\ & & \downarrow{\scriptstyle \mathrm{hol}^{U(n)}} & & \downarrow{\scriptstyle \mathrm{hol}^{U(1)}} & & \\ \underline{U}(n) & \longrightarrow & \underline{U}(n)//_{\mathrm{Ad}}\underline{U}(n) & \xrightarrow{\det} & \underline{U}(1)//_{\mathrm{Ad}}\underline{U}(1) & \longrightarrow & \underline{U}(1)\,, \end{array}$$

where the leftmost bottom arrow is the natural quotient projection $\underline{U}(n) \to \underline{U}(n)//_{\mathrm{Ad}}$ $\underline{U}(n)$. In the language of [79] (3.9.6.4) one says that the determinant map is the "concretification" of the morphism $\mathbf{Maps}(S^1, \hat{\mathbf{c}}_1)$, we come back to this in Sect. 5.3. This construction immediately generalizes to the case of an arbitrary compact connected Lie group G with $H^1(G; \mathbb{Z}) \cong \mathbb{Z}$: the Lie group morphism $\rho : G \to U(1)$ integrating the Lie algebra cocycle μ_1 corresponding to the characteristic class $c \in H^2(BG; \mathbb{Z})$ is the concretification of $\mathbf{Maps}(S^1, \hat{\mathbf{c}})$.

2.9 Killing the First Chern Class: $SU(n)$-bundles

Recall from the theory of characteristic classes (see [63]) that the first Chern class is the obstruction to reducing the structure group of a principal $U(n)$-bundle to $SU(n)$. In the stacky perspective that we have been adopting so far this amounts to saying that the stack $\mathbf{B}SU(n)$ of principal $SU(n)$-bundles is the *homotopy fiber* of $\mathbf{c}_1$, hence the object fitting into the homotopy pullback diagram of stacks of the form

$$\begin{array}{ccc} \mathbf{B}SU(n) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}U(n) & \xrightarrow{\mathbf{c}_1} & \mathbf{B}U(1)\,. \end{array}$$

By the universal property of the homotopy pullback, this says that an $SU(n)$-principal bundle over a smooth manifold X is equivalently a $U(n)$-principal bundle P, together with a choice of trivialization of the associated determinant $U(1)$-principal bundle. Moreover, the whole groupoid of $SU(n)$-principal bundles on X is equivalent to the groupoid of $U(n)$-principal bundles on X equipped with a trivialization of their associated determinant bundle. To explicitly see this equivalence, let us write the local data for a morphism from a smooth manifold X to the homotopy pullback above. In terms of a fixed good open cover $\mathcal{U}$ of X, these are:

- smooth functions $\rho_\alpha : U_\alpha \to U(1)$;

- smooth functions $g_{\alpha\beta} : U_{\alpha\beta} \to U(n)$,

subject to the constraints

- $\det(g_{\alpha\beta})\rho_\beta = \rho_\alpha$ on $U_{\alpha\beta}$;
- $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ on $U_{\alpha\beta\gamma}$.

Morphisms between $\{\rho_\alpha, g_{\alpha\beta}\}$ and $\{\rho'_\alpha, g'_{\alpha\beta}\}$ are the gauge transformations locally given by $U(n)$-valued functions h_α on U_α such that $h_\alpha g_{\alpha\beta} = g'_{\alpha\beta}h_\beta$ and $\rho_\alpha \det(h_\alpha) = \rho'_\alpha$. The classical description of objects in $\mathbf{B}SU(n)$ corresponds to the gauge fixing $\rho_\alpha \equiv 1$; at the level of morphisms, imposing this gauge fixing constrains the gauge transformation h_α to satisfy $\det(h_\alpha) = 1$, i.e. to take values in $SU(n)$. From a categorical point of view, this amounts to saying that the embedding of the groupoid of $SU(n)$-principal bundles over X into the groupoid of morphisms from X to the homotopy fiber of $\mathbf{c}_1$ given by $\{g_{\alpha\beta}\} \mapsto \{1, g_{\alpha\beta}\}$ is fully faithful. It is also essentially surjective: use the embedding $U(1) \to U(n)$ given by $e^{it} \mapsto (e^{it}, 1, 1, \ldots, 1)$ to lift ρ_α^{-1} to a $U(n)$-valued function h_α with $\det(h_\alpha) = {\rho_\alpha}^{-1}$; then $\{h_\alpha\}$ is an isomorphism between $\{\rho_\alpha, g_{\alpha\beta}\}$ and $\{1, h_\alpha g_{\alpha\beta} {h_\beta}^{-1}\}$.

Similarly, the stack of $SU(n)$-principal bundles with $\mathfrak{su}_n$-connections is the homotopy pullback

$$
\begin{array}{ccc}
\mathbf{B}SU(n)_{\mathrm{conn}} & \longrightarrow & * \\
\downarrow & & \downarrow \\
\mathbf{B}U(n)_{\mathrm{conn}} & \xrightarrow{\hat{\mathbf{c}}_1} & \mathbf{B}U(1)_{\mathrm{conn}}
\end{array} .
$$

Details on this homotopy pullback description of $\mathbf{B}SU(n)_{\mathrm{conn}}$ can be found in [28].

In summary, what we have discussed means that the map $\hat{\mathbf{c}}_1$ between universal moduli stacks equivalently plays the following different roles:

1. it is a smooth and differential refinement of the universal first Chern class;
2. it induces a 1-dimensional Chern–Simons action functional by *transgression* to maps from the circle;
3. it represents the obstruction to lifting a smooth unitary structure to a smooth special unitary structure.

In the following we will consider higher analogs of $\hat{\mathbf{c}}_1$ and will see these different but equivalent roles of universal differential characteristic maps amplified further.

3 The Archetypical Example: 3d Chern–Simons Theory

We now pass from the toy example of 1-dimensional Chern–Simons theory to the archetypical example of 3-dimensional Chern–Simons theory, and in fact to its extended (or "multi-tiered") geometric prequantization.

While this is a big step as far as the content of the theory goes, a pleasant consequence of the higher geometric formulation of the 1d theory above is that *conceptually* essentially nothing new happens when we move from 1-dimensional theory to 3-dimensional theory (and further). For the 3d theory we only need to restrict our attention to simply connected compact simple Lie groups, so as to have $\pi_3(G) \cong \mathbb{Z}$ as the first nontrivial homotopy group, and to move from stacks to higher stacks, or more precisely, to 3-stacks. (For non-simply connected groups one needs a little bit more structure, as we briefly indicate in Sect. 4.)

3.1 Higher $U(1)$-bundles with Connections and Differential Cohomology

The basic 3-stack naturally appearing in ordinary 3d Chern–Simons theory is the 3-stack $\mathbf{B}^3U(1)_{\mathrm{conn}}$ of principal $U(1)$-3-bundles with connection (also known as $U(1)$-bundle-2-gerbes with connection). It is convenient to introduce in general the n-stack $\mathbf{B}^nU(1)_{\mathrm{conn}}$ and to describe its relation to differential cohomology.

By definition, $\mathbf{B}^nU(1)_{\mathrm{conn}}$ is the n-stack obtained by stackifying the prestack on Cartesian spaces which corresponds, via the Dold–Kan correspondence, to the $(n+1)$-term Deligne complex

$$\underline{U}(1)[n]^{\infty}_{D} = \left(\underline{U}(1) \xrightarrow{\frac{1}{2\pi i}d\log} \Omega^1(-;\mathbb{R}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(-;\mathbb{R})\right),$$

where $\underline{U}(1)$ is the sheaf of smooth functions with values in $U(1)$, and with $\Omega^n(-;\mathbb{R})$ in degree zero. It is immediate from the definition that the equivalence classes of $U(1)$-n-bundles with connection on a smooth manifold X are classified by the $(n+1)$-st differential cohomology group of X,

$$\hat{H}^{n+1}(X;\mathbb{Z}) \cong \mathbb{H}^0(X;\underline{U}(1)[n]^{\infty}_{D}) \cong \pi_0\mathbf{H}(X;\mathbf{B}^nU(1)_{\mathrm{conn}}),$$

where in the middle we have degree zero hypercohomology of X with coefficients in $\underline{U}(1)[n]^{\infty}_{D}$. Similarly, the n-stack of $U(1)$-n-bundles (without connection) $\mathbf{B}^nU(1)$ is obtained via Dold–Kan and stackification from the sheaf of chain complexes

$$\underline{U}(1)[n] = \left(\underline{U}(1) \to 0 \to \cdots \to 0\right),$$

with $C^{\infty}(-;U(1))$ in degree n. Equivalence classes of $U(1)$-n-bundles on X are in natural bijection with

$$H^{n+1}(X;\mathbb{Z}) \cong H^n(X;\underline{U}(1)) \cong \mathbb{H}^0(X;\underline{U}(1)[n]) \cong \pi_0\mathbf{H}(X;\mathbf{B}^nU(1)).$$

The obvious morphism of chain complexes of sheaves $\underline{U}(1)[n]^\infty_D \to \underline{U}(1)[n]$ induces the "forget the connection" morphism $\mathbf{B}^n U(1)_{\text{conn}} \to \mathbf{B}^n U(1)$ and, at the level of equivalence classes, the natural morphism

$$\hat{H}^{n+1}(X;\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z})$$

from differential cohomology to integral cohomology. If we denote by $\Omega^{n+1}(-;\mathbb{R})_{\text{cl}}$ the sheaf (a 0-stack) of closed n-forms, then the morphism of complexes $\underline{U}(1)[n]^\infty_D \to \Omega^{n+1}(-;\mathbb{R})_{\text{cl}}$ given by

$$\begin{array}{ccccccc} \underline{U}(1) & \xrightarrow{\frac{1}{2\pi i}d\log} & \Omega^1(-;\mathbb{R}) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^n(-;\mathbb{R}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle d} \\ 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & \Omega^{n+1}(-;\mathbb{R})_{\text{cl}} \end{array}$$

induces the morphism of stacks $\mathbf{B}^n U(1)_{\text{conn}} \xrightarrow{F_{(-)}} \Omega^{n+1}(-;\mathbb{R})_{\text{cl}}$ mapping a circle n-bundle ($(n-1)$-bundle gerbe) with connection to the curvature $(n+1)$-form of its connection. At the level of differential cohomology, this is the morphism

$$\hat{H}^{n+1}(X;\mathbb{Z}) \to \Omega^{n+1}(X;\mathbb{R})_{\text{cl}}.$$

The last n-stack we need to introduce to complete this sketchy picture of differential cohomology formulated on universal moduli stacks is the n-stack $\flat\mathbf{B}^{n+1}\mathbb{R}$ associated with the chain complex of sheaves

$$\flat\underline{\mathbb{R}}[n+1]^\infty = \left(\Omega^1(-;\mathbb{R}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(-;\mathbb{R}) \xrightarrow{d} \Omega^{n+1}(-;\mathbb{R})_{\text{cl}}\right),$$

with $\Omega^{n+1}(-;\mathbb{R})_{\text{cl}}$ in degree zero. The obvious morphism of complexes of sheaves $\Omega^{n+1}(-;\mathbb{R})_{\text{cl}} \to \flat\underline{\mathbb{R}}[n+1]^\infty$ induces a morphism of stacks $\Omega^{n+1}(-;\mathbb{R})_{\text{cl}} \to \flat\mathbf{B}^{n+1}\mathbb{R}$. Moreover one can show (see, e.g., [31, 79]) that there is a "universal curvature characteristic" morphism curv : $\mathbf{B}^n U(1) \to \flat\mathbf{B}^{n+1}\mathbb{R}$ and a homotopy pullback diagram

$$\begin{array}{ccc} \mathbf{B}^n U(1)_{\text{conn}} & \xrightarrow{F} & \Omega^{n+1}(-;\mathbb{R})_{\text{cl}} \\ \downarrow & & \downarrow \\ \mathbf{B}^n U(1) & \xrightarrow{\text{curv}} & \flat\mathbf{B}^{n+1}\mathbb{R}\ , \end{array}$$

of higher moduli stacks in $\mathbf{H}$, which induces in cohomology the commutative diagram

$$\begin{array}{ccc} \hat{H}^{n+1}(X;\mathbb{Z}) & \xrightarrow{F} & \Omega^{n+1}(X;\mathbb{R})_{\text{cl}} \\ \downarrow & & \downarrow \\ H^{n+1}(X;\mathbb{Z}) & \longrightarrow & H^{n+1}_{\text{dR}}(X;\mathbb{R})\ . \end{array}$$

This generalizes to any degree $n \geq 1$ what we remarked in Sect. 2.5 for the degree 2 case: differential cohomology encodes in a systematic and geometric way the simple idea of having an integral cohomology class together with a closed differential form representing it in de Rham cohomology. For $n = 0$ we have $\hat{H}^1(X;\mathbb{Z}) \equiv H^0(X;\underline{U}(1)) = C^\infty(X;U(1))$ and the map $\hat{H}^1(X;\mathbb{Z}) \to H^1(X;\mathbb{Z})$ is the morphism induced in cohomology by the short exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{R}} \to \underline{U}(1) \to 1.$$

At the level of stacks, this corresponds to the morphism

$$U(1) \to \mathbf{B}\mathbb{Z}$$

induced by the canonical principal $\mathbb{Z}$-bundle $\mathbb{R} \to U(1)$.

3.2 Compact Simple and Simply Connected Lie Groups

From a cohomological point of view, a compact simple and simply connected Lie group G is the degree 3 analogue of the group $U(n)$ considered in our 1-dimensional toy model. That is, the homotopy (hence the homology) of G is trivial up to degree 3, and $\pi_3(G) \cong H^3(G;\mathbb{Z}) \cong \mathbb{Z}$, by the Hurewicz isomorphism. Passing from G to its classifying space BG we find $H^4(BG;\mathbb{Z}) \cong \mathbb{Z}$, so that the fourth integral cohomology group of BG is generated by a fundamental characteristic class $c \in H^4(BG;\mathbb{Z})$. All other elements in $H^4(BG;\mathbb{Z})$ are of the form kc for some integer k, usually called the "level" in the physics literature. For P a G-principal bundle over a smooth manifold X, we will write $c(P)$ for the cohomology class $f^*c \in H^4(X,\mathbb{Z})$, where $f : X \to BG$ is any classifying map for P. This way we realize c as a map

$$c : \{\text{principal } G\text{-bundles on } X\}/\text{iso} \to H^4(X;\mathbb{Z}).$$

Moving to real coefficients, the fundamental characteristic class c is represented, via the isomorphism $H^4(BG;\mathbb{R}) \cong H^3(G;\mathbb{R}) \cong H^3_{\text{Lie}}(\mathfrak{g},\mathbb{R})$ by the canonical 3-cocycle μ_3 on the Lie algebra $\mathfrak{g}$ of G, i.e., up to normalization, by the 3-cocycle $\langle[-,-],-\rangle$, where $\langle -,-\rangle$ is the Killing form of $\mathfrak{g}$ and $[-,-]$ is the Lie bracket. On the other hand, via the Chern-Weil isomorphism

$$H^*(BG;\mathbb{R}) \cong \text{inv}(\mathfrak{g}[2]),$$

the characteristic class c corresponds to the Killing form, seen as a degree four invariant polynomial on $\mathfrak{g}$ (with elements of $\mathfrak{g}^*$ placed in degree 2). The transgression between μ_3 and $\langle -,-\rangle$ is witnessed by the canonical degree 3 Chern–Simons element CS_3 of $\mathfrak{g}$. That is, for a $\mathfrak{g}$-valued 1-form A on some manifold, let

$$\text{CS}_3(A) = \langle A, dA\rangle + \tfrac{1}{3}\langle A,[A,A]\rangle.$$

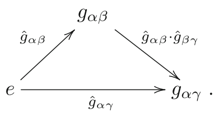
Then, for $A \in \Omega^1(EG; \mathfrak{g})$ the connection 1-form of a principal G-connection on $EG \to BG$, we have the following transgression diagram

$$\langle F_A, F_A \rangle \xleftarrow{\ d\ } \mathrm{CS}_3(A) \xmapsto{A=\theta_G} \mu_3(\theta_G, \theta_G, \theta_G)\ ,$$

where θ_G is the Maurer–Cartan form of G (i.e., the restriction of A to the fibers of $EG \to BG$) and $F_A = dA + \frac{1}{2}[A, A]$ is the curvature 2-form of A. Notice how both the invariance of the Killing form and the Maurer–Cartan equation $d\theta_G + \frac{1}{2}[\theta_G, \theta_G] = 0$ play a rôle in the above transgression diagram.

3.3 The Differential Refinement of Degree 4 Characteristic Classes

The description of the Brylinski–McLaughlin 2-cocycle from Sect. 2.6 has an evident generalization to degree four. Indeed, let $\{A_\alpha, g_{\alpha\beta}\}$ be the local data for a G-connection ∇ on $P \to X$, relative to a trivializing good open cover $\mathcal{U}$ of X, with G a compact simple and simply connected Lie group. Then, since G is connected and the open sets $U_{\alpha\beta}$ are contractible, we can smoothly extend the transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ to functions $\hat{g}_{\alpha\beta} : [0, 1] \times U_{\alpha\beta} \to G$ with $\hat{g}_{\alpha\beta}(0) = e$, the identity element of G, and $\hat{g}_{\alpha\beta}(1) = g_{\alpha\beta}$, and using the functions $\hat{g}_{\alpha\beta}$ one can interpolate from $A_\alpha\big|_{U_{\alpha\beta}}$ to $A_\beta\big|_{U_{\alpha\beta}}$ as in Sect. 2.6, defining a $\mathfrak{g}$-valued 1-form $\hat{A}_{\alpha\beta} = \hat{g}_{\alpha\beta}^{-1} A_\alpha\big|_{U_{\alpha\beta}} \hat{g}_{\alpha\beta} + \hat{g}_{\alpha\beta}^{-1} d\hat{g}_{\alpha\beta}$. On the triple intersection $U_{\alpha\beta\gamma}$ we have the paths in G



Since G is simply connected we can find smooth functions

$$\hat{g}_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \times \Delta^2 \to G$$

filling these 2-simplices, and we can use these to extend the interpolation between $\hat{A}_{\alpha\beta}$, $\hat{A}_{\beta\gamma}$ and $\hat{A}_{\gamma\alpha}$ over the 2-simplex. Let us denote this interpolation by $\hat{A}_{\alpha\beta\gamma}$. Finally, since G is 2-connected, on the quadruple intersections we can find smooth functions

$$\hat{g}_{\alpha\beta\gamma\delta} : U_{\alpha\beta\gamma\delta} \times \Delta^3 \to G$$

cobounding the union of the 2-simplices corresponding to the $\hat{g}_{\alpha\beta\gamma}$'s on the triple intersections. We can again use the $\hat{g}_{\alpha\beta\gamma\delta}$'s to interpolate between the $\hat{A}_{\alpha\beta\gamma}$'s over the 3-simplex. Finally, one considers the degree zero Čech–Deligne cochain with coefficients in $\underline{U}(1)[3]^\infty_D$

$$\left(\mathrm{CS}_3(A_\alpha), \int_{\Delta^1} \mathrm{CS}_3(\hat{A}_{\alpha\beta}), \int_{\Delta^2} \mathrm{CS}_3(\hat{A}_{\alpha\beta\gamma}), \int_{\Delta^3} \mathrm{CS}_3(\hat{A}_{\alpha\beta\gamma\delta}) \mod \mathbb{Z} \right). \quad (1)$$

Brylinski and McLaughlin [12] show (see also [9] for an exposition and [10, 11] for related discussion) that this is indeed a degree zero Čech–Deligne cocycle, and thus defines an element in $\hat{H}^4(X;\mathbb{Z})$. Moreover, they show that this cohomology class only depends on the isomorphism class of (P, ∇), inducing therefore a well-defined map

$$\hat{c} : \{G\text{-bundles with connection on } X\}/\mathrm{iso} \to \hat{H}^4(X;\mathbb{Z}).$$

Notice how modding out by $\mathbb{Z}$ in the rightmost integral in the above cochain precisely takes care of $\pi_3(G) \cong H^3(G;\mathbb{Z}) \cong \mathbb{Z}$. Notice also that, by construction,

$$\int_{\Delta^3} \mathrm{CS}_3(\hat{A}_{\alpha\beta\gamma\delta}) = \int_{\Delta^3} \hat{\boldsymbol{g}}_{\alpha\beta\gamma\delta}{}^* \, \mu_3(\theta_G \wedge \theta_G \wedge \theta_G),$$

where θ_G is the Maurer–Cartan form of G. Hence the Brylinski–McLaughlin cocycle lifts the degree 3 cocycle with coefficients in $\underline{U}(1)$

$$\int_{\Delta^3} \hat{\boldsymbol{g}}_{\alpha\beta\gamma\delta}{}^* \, \mu_3(\theta_G \wedge \theta_G \wedge \theta_G) \mod \mathbb{Z},$$

which represents the characteristic class $c(P)$ in $H^3(X;\underline{U}(1)) \cong H^4(X;\mathbb{Z})$. As a result, the differential characteristic class $\hat{c}$ lifts the characteristic class c, i.e., we have a natural commutative diagram

$$\begin{array}{ccc} \{G\text{-bundles with connection on } X\}/\mathrm{iso} & \xrightarrow{\hat{c}} & \hat{H}^4(X;\mathbb{Z}) \\ \downarrow & & \downarrow \\ \{G\text{-bundles on } X\}/\mathrm{iso} & \xrightarrow{c} & H^4(X;\mathbb{Z}) . \end{array}$$

By looking at the Brylinski–McLaughlin construction through the eyes of simplicial integration of ∞-Lie algebras one sees [31] that the above commutative diagram is naturally enhanced to a commutative diagram of stacks

$$\begin{array}{ccc} \mathbf{B}G_{\mathrm{conn}} & \xrightarrow{\hat{\mathbf{c}}} & \mathbf{B}^3U(1)_{\mathrm{conn}} \\ \downarrow & & \downarrow \\ \mathbf{B}G & \xrightarrow{\mathbf{c}} & \mathbf{B}^3U(1) . \end{array}$$

As we are going to show, the morphism $\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3U(1)_{\mathrm{conn}}$ that refines the characteristic class c to a morphism of stacks is the morphism secretly governing all basic features of level 1 three-dimensional Chern–Simons theory with gauge group G. Similarly, for any $k \in \mathbb{Z}$, one has a morphism of stacks

$$k\hat{\mathbf{c}} : \mathbf{B}G_{\text{conn}} \to \mathbf{B}^3 U(1)_{\text{conn}}$$

governing level k 3d Chern–Simons theory with gauge group G. Indeed, this map may be regarded as the very *Lagrangian* of 3d Chern–Simons theory *extended* (“localized”, “multi-tiered”) to codimension 3. We discuss this next.

3.4 Prequantum n-bundles on Moduli Stacks of G-connections on a Fixed Manifold

We discuss now how the differential refinement $\hat{\mathbf{c}}$ of the universal characteristic map c constructed above serves as the *extended* Lagrangian for 3d Chern–Simons theory in that its *transgression* to mapping stacks out of k-dimensional manifolds yields all the “geometric prequantum” data of Chern–Simons theory in the corresponding dimension, in the sense of geometric quantization. For the purpose of this exposition we use terms such as “prequantum n-bundle” freely without formal definition. We expect the reader can naturally see at least vaguely the higher prequantum picture alluded to here. A more formal survey of these notions is in Sect. 5.4.

If X is a compact oriented manifold without boundary, then there is a fiber integration in differential cohomology lifting fiber integration in integral cohomology [48]:

$$\begin{array}{ccc} \hat{H}^{n+\dim X}(X \times Y; \mathbb{Z}) & \xrightarrow{\int_X} & \hat{H}^n(Y; \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^{n+\dim X}(X \times Y; \mathbb{Z}) & \xrightarrow{\int_X} & H^n(Y; \mathbb{Z}) \, . \end{array}$$

In [44] Gomi and Terashima describe an explicit lift of this to the level of Čech–Deligne cocycles; see also [25]. One observes [30] that such a lift has a natural interpretation as a morphism of moduli stacks

$$\mathrm{hol}_X : \mathbf{Maps}(X, \mathbf{B}^{n+\dim X} U(1)_{\text{conn}}) \to \mathbf{B}^n U(1)_{\text{conn}}$$

from the $(n+\dim X)$-stack of moduli of $U(1)$-$(n+\dim X)$-bundles with connection over X to the n-stack of $U(1)$-n-bundles with connection (Sect. 2.4 of [30]). Therefore, if Σ_k is a compact oriented manifold of dimension k with $0 \le k \le 3$, we have a composition

$$\mathbf{Maps}(\Sigma_k, \mathbf{B}G_{\text{conn}}) \xrightarrow{\mathbf{Maps}(\Sigma_k, \hat{\mathbf{c}})} \mathbf{Maps}(\Sigma_k, \mathbf{B}^3 U(1)_{\text{conn}}) \xrightarrow{\mathrm{hol}_{\Sigma_k}} \mathbf{B}^{3-k} U(1)_{\text{conn}} .$$

This is the canonical $U(1)$-$(3-k)$-bundle with connection over the moduli space of principal G-bundles with connection over Σ_k induced by $\hat{\mathbf{c}}$: the *transgression* of $\hat{\mathbf{c}}$ to the mapping space. Composing on the right with the curvature morphism we get the underlying canonical closed $(4-k)$-form

$$\mathbf{Maps}(\Sigma_k, \mathbf{B}G_{\mathrm{conn}}) \to \Omega^{4-k}(-;\mathbb{R})_{\mathrm{cl}}$$

on this moduli space. In other words, the moduli stack of principal G-bundles with connection over Σ_k carries a canonical *pre*-$(3-k)$*-plectic structure* (the higher order generalization of a symplectic structure, [67]) and, moreover, this is equipped with a canonical geometric prequantization: the above $U(1)$-$(3-k)$-bundle with connection.

Let us now investigate in more detail the cases $k = 0, 1, 2, 3$.

3.4.1 $k = 0$: The Universal Chern–Simons 3-Connection $\hat{\mathbf{c}}$

The connected 0-manifold Σ_0 is the point and, by definition of **Maps**, one has a canonical identification

$$\mathbf{Maps}(*, \mathbf{S}) \cong \mathbf{S}$$

for any (higher) stack **S**. Hence the morphism

$$\mathbf{Maps}(*, \mathbf{B}G_{\mathrm{conn}}) \xrightarrow{\mathbf{Maps}(*,\hat{\mathbf{c}})} \mathbf{Maps}(*, \mathbf{B}^3U(1)_{\mathrm{conn}})$$

is nothing but the universal differential characteristic map $\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3U(1)_{\mathrm{conn}}$ that refines the universal characteristic class c. This map modulates a circle 3-bundle with connection (bundle 2-gerbe) on the universal moduli stack of G-principal connections. For $\nabla : X \longrightarrow \mathbf{B}G_{\mathrm{conn}}$ any given G-principal connection on some X, the pullback

$$\hat{\mathbf{c}}(\nabla) : \; X \xrightarrow{\nabla} \mathbf{B}G_{\mathrm{conn}} \xrightarrow{\hat{\mathbf{c}}} \mathbf{B}^3U(1)_{\mathrm{conn}}$$

is a 3-bundle (bundle 2-gerbe) on X which is sometimes in the literature called the *Chern–Simons 2-gerbe* of the given connection ∇. Accordingly, $\hat{\mathbf{c}}$ modulates the *universal* Chern–Simons bundle 2-gerbe with universal 3-connection. From the point of view of higher geometric quantization, this is the *prequantum 3-bundle* of extended prequantum Chern–Simons theory.

This means that the prequantum $U(1)$-$(3-k)$-bundles associated with k- dimensional manifolds are all determined by the prequantum $U(1)$-3-bundle associated with the point, in agreement with the formulation of fully extended topological field theories [36]. We will denote by the symbol $\omega^{(4)}_{\mathbf{B}G_{\mathrm{conn}}}$ the pre-3-plectic 4-form induced on $\mathbf{B}G_{\mathrm{conn}}$ by the curvature morphism.

3.4.2 $k = 1$: The Wess-Zumino-Witten Bundle Gerbe

We now come to the transgression of the extended Chern–Simons Lagrangian to the closed connected 1-manifold, the circle $\Sigma_1 = S^1$. Here we find a higher analog of the construction described in Sect. 2.8. Notice that, on the one hand, we can think of the mapping stack $\mathbf{Maps}(\Sigma_1, \mathbf{B}G_{\mathrm{conn}}) \simeq \mathbf{Maps}(S^1, \mathbf{B}G_{\mathrm{conn}})$ as a kind of moduli stack of G-connections on the circle—up to a slight subtlety, which we explain in more detail below in Sect. 5.3. On the other hand, we can think of that mapping stack as the *free loop space* of the universal moduli stack $\mathbf{B}G_{\mathrm{conn}}$.

The subtlety here is related to the differential refinement, so it is instructive to first discard the differential refinement and consider just the smooth characteristic map $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^3U(1)$ which underlies the extended Chern–Simons Lagrangian and which modulates the universal circle 3-bundle on $\mathbf{B}G$ (without connection). Now, for every pointed stack $* \to \mathbf{S}$ we have the corresponding (categorical) *loop space* $\Omega\mathbf{S} := * \times_{\mathbf{S}} *$, which is the homotopy pullback of the point inclusion along itself. Applied to the moduli stack $\mathbf{B}G$ this recovers the Lie group G, identified with the sheaf (i.e., the 0-stack) of smooth functions with target G: $\Omega\mathbf{B}G \simeq \underline{G}$. This kind of looping/delooping equivalence is familiar from the homotopy theory of classifying spaces; but notice that since we are working with smooth (higher) stacks, the loop space $\Omega\mathbf{B}G$ also knows the smooth structure of the group G, i.e. it knows G as a Lie group. Similarly, we have

$$\Omega\mathbf{B}^3U(1) \simeq \mathbf{B}^2U(1)$$

and so forth in higher degrees. Since the looping operation is functorial, we may also apply it to the characteristic map $\mathbf{c}$ itself to obtain a map

$$\Omega\mathbf{c} : \underline{G} \to \mathbf{B}^2U(1)$$

which modulates a $\mathbf{B}U(1)$-principal 2-bundle on the Lie group G. This is also known as the *WZW-bundle gerbe*; see [41, 83]. The reason, as discussed there and as we will see in a moment, is that this is the 2-bundle that underlies the 2-connection with surface holonomy over a worldsheet given by the Wess-Zumino-Witten action functional. However, notice first that there is more structure implied here: for any pointed stack $\mathbf{S}$ there is a natural equivalence $\Omega\mathbf{S} \simeq \mathbf{Maps}_*(\Pi(S^1), \mathbf{S})$, between the loop space object $\Omega\mathbf{S}$ and the moduli stack of *pointed maps* from the categorical circle $\Pi(S^1) \simeq \mathbf{B}\mathbb{Z}$ to $\mathbf{S}$. Here Π denotes the *path ∞-groupoid* of a given (higher) stack.[7] On the other hand, if we do not fix the base point then we obtain the *free loop space object* $\mathcal{L}\mathbf{S} \simeq \mathbf{Maps}(\Pi(S^1), \mathbf{S})$. Since a map $\Pi(\Sigma) \to \mathbf{B}G$ is equivalently a map $\Sigma \to \flat\mathbf{B}G$, i.e., a flat G-principal connection on Σ, the free loop space $\mathcal{L}\mathbf{B}G$ is equivalently the moduli stack of flat G-principal connections on S^1. We will come back to this perspective in Sect. 5.3. The homotopies that do not fix the base point act by conjugation on loops, hence we have, for any smooth (higher) group, that

$$\mathcal{L}\mathbf{B}G \simeq \underline{G}//_{\mathrm{Ad}}\underline{G}$$

is the (homotopy) quotient of the adjoint action of G on itself; see [64] for details on homotopy actions of smooth higher groups. For G a Lie group this is the familiar

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[7] The existence and functoriality of the path ∞-groupoids is one of the features characterizing the higher topos of higher smooth stacks as being *cohesive*, see [79].

adjoint action quotient stack. But the expression holds fully generally. Notably, we also have

$$\mathcal{L}\mathbf{B}^3U(1) \simeq \mathbf{B}^2U(1)//_{\mathrm{Ad}}\mathbf{B}^2U(1)$$

and so forth in higher degrees. However, in this case, since the smooth 3-group $\mathbf{B}^2U(1)$ is abelian (it is a groupal E_∞-algebra) the adjoint action splits off in a direct factor and we have a projection

$$\mathcal{L}\mathbf{B}^3U(1) \simeq \mathbf{B}^2U(1) \times (*//\mathbf{B}^2U(1)) \xrightarrow{p_1} \mathbf{B}^2U(1) \ .$$

In summary, this means that the map $\Omega\mathbf{c}$ modulating the WZW 2-bundle over G descends to the adjoint quotient to the map

$$p_1 \circ \mathcal{L}\mathbf{c} : \underline{G}//_{\mathrm{Ad}}\underline{G} \to \mathbf{B}^2U(1),$$

and this means that the WZW 2-bundle is canonically equipped with the structure of an ad_G-*equivariant* bundle gerbe, a crucial feature of the WZW bundle gerbe [41, 42].

We emphasize that the derivation here is fully general and holds for any smooth (higher) group G and any smooth characteristic map $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^nU(1)$. Each such pair induces a WZW-type $(n-1)$-bundle on the smooth (higher) group G modulated by $\Omega\mathbf{c}$ and equipped with G-equivariant structure exhibited by $p_1 \circ \mathcal{L}\mathbf{c}$. We discuss such higher examples of higher Chern–Simons-type theories with their higher WZW-type functionals further below in Sect. 4.

We now turn to the differential refinement of this situation. In analogy to the above construction, but taking care of the connection data in the extended Lagrangian $\hat{\mathbf{c}}$, we find a homotopy commutative diagram in $\mathbf{H}$ of the form

$$\begin{array}{ccccccc} & & \mathbf{Maps}(S^1;\mathbf{B}G_{\mathrm{conn}}) & \xrightarrow{\mathbf{Maps}(S^1,\hat{\mathbf{c}})} & \mathbf{Maps}(S^1;\mathbf{B}^3U(1)_{\mathrm{conn}}) & & \\ & & \downarrow{\scriptstyle \mathrm{hol}} & & \downarrow{\scriptstyle \mathrm{hol}} & & \\ \underline{G} & \longrightarrow & \underline{G}//_{\mathrm{Ad}}\underline{G} & \xrightarrow{\mathbf{wzw}} & \mathbf{B}^2U(1)_{\mathrm{conn}}//_{\mathrm{Ad}}\mathbf{B}^2U(1)_{\mathrm{conn}} & \longrightarrow & \mathbf{B}^2U(1)_{\mathrm{conn}} \ , \end{array}$$

where the vertical maps are obtained by forming holonomies of (higher) connections along the circle. The lower horizontal row is the differential refinement of $\Omega\mathbf{c}$: it modulates the Wess-Zumino-Witten $U(1)$-bundle gerbe with connection

$$\mathbf{wzw} : \underline{G} \to \mathbf{B}^2U(1)_{\mathrm{conn}}.$$

That $\mathbf{wzw}$ is indeed the correct differential refinement can be seen, for instance, by interpreting the construction by Carey et al. [15] in terms of the above diagram. That is, choosing a basepoint x_0 in S^1 one obtains a canonical lift of the leftmost vertical arrow:

$$\begin{array}{ccc} & & \mathbf{Maps}(S^1;\mathbf{B}G_{\mathrm{conn}}) \\ & \nearrow^{(P_{x_0},\nabla_{x_0})} & \downarrow{\scriptstyle \mathrm{hol}} \\ \underline{G} & \longrightarrow & \underline{G}//_{\mathrm{Ad}}\underline{G} \ , \end{array}$$

where $(P_{x_0}\nabla_{x_0})$ is the principal G-bundle with connection on the product $G \times S^1$ characterized by the property that the holonomy of ∇_{x_0} along $\{\boldsymbol{g}\} \times S^1$ with starting point $(\boldsymbol{g}, x_0)$ is the element $\boldsymbol{g}$ of G. Correspondingly, we have a homotopy commutative diagram

$$\begin{array}{ccccccc} & & \mathbf{Maps}(S^1;\mathbf{B}G_{\mathrm{conn}}) & \xrightarrow{\mathbf{Maps}(S^1,\hat{\mathbf{c}})} & \mathbf{Maps}(S^1;\mathbf{B}^3U(1)_{\mathrm{conn}}) & & \\ & \nearrow^{(P_{x_0},\nabla_{x_0})} & \downarrow{\scriptstyle \mathrm{hol}} & & \downarrow{\scriptstyle \mathrm{hol}} & \searrow^{\mathrm{hol}_{S^1}} & \\ \underline{G} & \longrightarrow & \underline{G}/\!/_{\mathrm{Ad}}\underline{G} & \xrightarrow{\mathbf{wzw}} & \mathbf{B}^2U(1)_{\mathrm{conn}}/\!/_{\mathrm{Ad}}\mathbf{B}^2U(1)_{\mathrm{conn}} & \longrightarrow & \mathbf{B}^2U(1)_{\mathrm{conn}}\,. \end{array}$$

Then Proposition 3.4 from [15] identifies the upper path (hence also the lower path) from $\underline{G}$ to $\mathbf{B}^2U(1)_{\mathrm{conn}}$ with the Wess-Zumino-Witten bundle gerbe.

Passing to equivalence classes of global sections, we see that **wzw** induces, for any smooth manifold X, a natural map $C^\infty(X; G) \to \hat{H}^2(X; \mathbb{Z})$. In particular, if $X = \Sigma_2$ is a compact Riemann surface, we can further integrate over X to get

$$\boldsymbol{wzw} : C^\infty(\Sigma_2; G) \to \hat{H}^2(X; \mathbb{Z}) \xrightarrow{\int_{\Sigma_2}} U(1).$$

This is the *topological term* in the Wess-Zumino-Witten model; see [14, 38, 40]. Notice how the fact that **wzw** factors through $\underline{G}/\!/_{\mathrm{Ad}}\underline{G}$ gives the conjugation invariance of the Wess-Zumino-Witten bundle gerbe, hence of the topological term in the Wess-Zumino-Witten model.

3.4.3 $k = 2$: The Symplectic Structure on the Moduli Space of Flat Connections on Riemann Surfaces

For Σ_2 a compact Riemann surface, the transgression of the extended Lagrangian $\hat{\mathbf{c}}$ yields a map

$$\mathbf{Maps}(\Sigma_2; \mathbf{B}G_{\mathrm{conn}}) \xrightarrow{\mathbf{Maps}(\Sigma_2,\hat{\mathbf{c}})} \mathbf{Maps}(\Sigma_2; \mathbf{B}^3U(1)_{\mathrm{conn}}) \xrightarrow{\mathrm{hol}_{\Sigma_2}} \mathbf{B}U(1)_{\mathrm{conn}},$$

modulating a circle-bundle with connection on the moduli space of gauge fields on Σ_2. The underlying curvature of this connection is the map obtained by composing this with

$$\mathbf{B}U(1)_{\mathrm{conn}} \xrightarrow{F_{(-)}} \Omega^2(-;\mathbb{R})_{\mathrm{cl}} \quad ,$$

which gives the canonical presymplectic 2-form

$$\omega : \ \mathbf{Maps}(\Sigma_2; \mathbf{B}G_{\mathrm{conn}}) \longrightarrow \Omega^2(-;\mathbb{R})_{\mathrm{cl}}$$

on the moduli stack of principal G-bundles with connection on Σ_2. Equivalently, this is the transgression of the invariant polynomial

$$\langle - \rangle : \ \mathbf{B}G_{\mathrm{conn}} \longrightarrow \Omega^4_{\mathrm{cl}}$$

to the mapping stack out of Σ_2. The restriction of this 2-form to the moduli stack $\mathbf{Maps}(\Sigma_2; \flat\mathbf{B}G_{\mathrm{conn}})$ of flat principal G-bundles on Σ_2 induces a canonical symplectic structure on the moduli space

$$\mathrm{Hom}(\pi_1(\Sigma_2), G)/_{\mathrm{Ad}}G$$

of flat G-bundles on Σ_2. Such a symplectic structure seems to have been first made explicit in [3] and then identified as the phase space structure of Chern–Simons theory in [88]. Observing that differential forms on the moduli stack, and hence de Rham cocycles $\mathbf{B}G \to \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$, may equivalently be expressed by simplicial forms on the bar complex of G, one recognizes in the above transgression construction a stacky refinement of the construction of [87].

To see more explicitly what this form ω is, consider any test manifold $U \in \mathrm{CartSp}$. Over this the map of stacks ω is a function which sends a G-principal connection $A \in \Omega^1(U \times \Sigma_2)$ (using that every G-principal bundle over $U \times \Sigma_2$ is trivializable) to the 2-form

$$\int_{\Sigma_2} \langle F_A \wedge F_A \rangle \in \Omega^2(U).$$

Now if A represents a field in the phase space, hence an element in the concretification of the mapping stack, then it has no "leg"[8] along U, and so it is a 1-form on Σ_2 that depends smoothly on the parameter U: it is a U-parameterized *variation* of such a 1-form. Accordingly, its curvature 2-form splits as

$$F_A = F_A^{\Sigma_2} + d_U A,$$

where $F_A^{\Sigma_2} := d_{\Sigma_2} A + \frac{1}{2}[A \wedge A]$ is the U-parameterized collection of curvature forms on Σ_2. The other term is the *variational differential* of the U-collection of forms. Since the fiber integration map $\int_{\Sigma_2} : \Omega^4(U \times \Sigma_2) \to \Omega^2(U)$ picks out the component of $\langle F_A \wedge F_A \rangle$ with two legs along Σ_2 and two along U, integrating over the former we have that

$$\omega|_U = \int_{\Sigma_2} \langle F_A \wedge F_A \rangle = \int_{\Sigma_2} \langle d_U A \wedge d_U A \rangle \in \Omega^2_{\mathrm{cl}}(U).$$

In particular if we consider, without loss of generality, $(U = \mathbb{R}^2)$-parameterized variations and expand

$$d_U A = (\delta_1 A) du^1 + (\delta_2 A) du^2 \in \Omega^2(\Sigma_2 \times U),$$

then

[8] That is, when written in local coordinates (u, σ) on $U \times \Sigma_2$, then $A = A_i(u, \sigma) du^i + A_j(u, \sigma) d\sigma^j$ reduces to the second summand.

$$\omega|_U = \int_{\Sigma_2} \langle \delta_1 A, \delta_2 A \rangle .$$

In this form the symplectic structure appears, for instance, in prop. 3.17 of [32] (in [88] this corresponds to (3.2)).

In summary, this means that the circle bundle with connection obtained by transgression of the extended Lagrangian $\hat{\mathbf{c}}$ is a *geometric prequantization* of the phase space of 3d Chern–Simons theory. Observe that traditionally prequantization involves an arbitrary *choice*: the choice of prequantum bundle with connection whose curvature is the given symplectic form. Here we see that in *extended* prequantization this choice is eliminated, or at least reduced: while there may be many differential cocycles lifting a given curvature form, only few of them arise by transgression from a higher differential cocycles in top codimension. In other words, the restrictive choice of the single geometric prequantization of the invariant polynomial $\langle -, - \rangle : \mathbf{B}G_{\mathrm{conn}} \to \Omega^4_{\mathrm{cl}}$ by $\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$ down in top codimension induces canonical choices of prequantization over all Σ_k in all lower codimensions $(n - k)$.

3.4.4 $k = 3$: The Chern–Simons Action Functional

Finally, for Σ_3 a compact oriented 3-manifold without boundary, transgression of the extended Lagrangian $\hat{\mathbf{c}}$ produces the morphism

$$\mathbf{Maps}(\Sigma_3; \mathbf{B}G_{\mathrm{conn}}) \xrightarrow{\mathbf{Maps}(\Sigma_3, \hat{\mathbf{c}})} \mathbf{Maps}(\Sigma_3; \mathbf{B}^3 U(1)_{\mathrm{conn}}) \xrightarrow{\mathrm{hol}_{\Sigma_3}} \underline{U}(1).$$

Since the morphisms in $\mathbf{Maps}(\Sigma_3; \mathbf{B}G_{\mathrm{conn}})$ are *gauge transformations* between field configurations, while $\underline{U}(1)$ has no non-trivial morphisms, this map necessarily gives a *gauge invariant* $U(1)$-valued function on field configurations. Indeed, evaluating over the point and passing to isomorphism classes (hence to gauge equivalence classes), this induces the *Chern–Simons action functional*

$$S_{\hat{\mathbf{c}}} : \{G\text{-bundles with connection on } \Sigma_3\}/\mathrm{iso} \to U(1).$$

It follows from the description of $\hat{\mathbf{c}}$ given in Sect. 3.3 that if the principal G-bundle $P \to \Sigma_3$ is trivializable then

$$S_{\hat{\mathbf{c}}}(P, \nabla) = \exp 2\pi i \int_{\Sigma_3} \mathrm{CS}_3(A),$$

where $A \in \Omega^1(\Sigma_3, \mathfrak{g})$ is the $\mathfrak{g}$-valued 1-form on Σ_3 representing the connection ∇ in a chosen trivialization of P. This is actually always the case, but notice two things: first, in the stacky description one does not need to know a priori that every principal

G-bundle on a 3-manifold is trivializable; second, the independence of $S_{\hat{\mathbf{c}}}(P, \nabla)$ on the trivialization chosen is automatic from the fact that $S_{\hat{\mathbf{c}}}$ is a morphism of stacks read at the level of equivalence classes.

Furthermore, if (P, ∇) can be extended to a principal G-bundle with connection $(\tilde{P}, \tilde{\nabla})$ over a compact 4-manifold Σ_4 bounding Σ_3, one has

$$S_{\hat{\mathbf{c}}}(P, \nabla) = \exp 2\pi i \int_{\Sigma_4} \tilde{\varphi}^* \omega^{(4)}_{\mathbf{B}G_{\mathrm{conn}}} = \exp 2\pi i \int_{\Sigma_4} \langle F_{\tilde{\nabla}}, F_{\tilde{\nabla}} \rangle,$$

where $\tilde{\varphi} : \Sigma_4 \to \mathbf{B}G_{\mathrm{conn}}$ is the morphism corresponding to the extended bundle $(\tilde{P}, \tilde{\nabla})$. Notice that the right hand side is independent of the extension chosen. Again, this is always the case, so one can actually take the above equation as a definition of the Chern–Simons action functional, see, e.g., [32, 33]. However, notice how in the stacky approach we do not need a priori to know that the oriented cobordism ring is trivial in dimension 3. Even more remarkably, the stacky point of view tells us that there would be a natural and well-defined 3d Chern–Simons action functional even if the oriented cobordism ring were nontrivial in dimension 3 or even if not every G-principal bundle on a 3-manifold were trivializable. An instance of checking that a nontrivial higher cobordism group vanishes can be found in [57], allowing for the application of the construction of Hopkins–Singer [48].

3.4.5 The Chern–Simons Action Functional with Wilson Loops

To conclude our exposition of the examples of 1d and 3d Chern–Simons theory in higher geometry, we now briefly discuss how both unify into the theory of 3d Chern–Simons gauge fields with Wilson line defects. Namely, for every embedded knot

$$\iota : S^1 \hookrightarrow \Sigma_3$$

in the closed 3d worldvolume and every complex linear representation $R : G \to \mathrm{Aut}(V)$ one can consider the *Wilson loop observable* $W_{\iota,R}$ mapping a gauge field $A : \Sigma \to \mathbf{B}G_{\mathrm{conn}}$, to the corresponding "Wilson loop holonomy"

$$W_{\iota,R} : A \mapsto \mathrm{tr}_R(\mathrm{hol}(\iota^* A)) \in \mathbb{C}.$$

This is the trace, in the given representation, of the parallel transport defined by the connection A around the loop ι (for any choice of base point). It is an old observation[9] that this Wilson loop $W(C, A, R)$ is itself the *partition function* of a 1-dimensional topological σ-model quantum field theory that describes the topological sector of a particle charged under the nonabelian background gauge field A. In Sect. 3.3 of [88] it was therefore emphasized that Chern–Simons theory with Wilson loops should really

[9] This can be traced back to [4]; a nice modern review can be found in Sect. 4 of [6].

be thought of as given by a single Lagrangian which is the sum of the 3d Chern–Simons Lagrangian for the gauge field as above, plus that for this topologically charged particle.

We now briefly indicate how this picture is naturally captured by higher geometry and refined to a single *extended* Lagrangian for coupled 1d and 3d Chern–Simons theory, given by maps on higher moduli stacks. In doing this, we will also see how the ingredients of Kirillov's orbit method and the Borel-Weil-Bott theorem find a natural rephrasing in the context of smooth differential moduli stacks. The key observation is that for $\langle\lambda, -\rangle$ an integral weight for our simple, connected, simply connected and compact Lie group G, the contraction of $\mathfrak{g}$-valued differential forms with λ extends to a morphism of smooth moduli stacks of the form

$$\langle\lambda, -\rangle \ : \ \Omega^1(-, \mathfrak{g})//\underline{T}_\lambda \to \mathbf{B}U(1)_{\text{conn}},$$

where $T_\lambda \hookrightarrow G$ is the maximal torus of G which is the stabilizer subgroup of $\langle\lambda, -\rangle$ under the coadjoint action of G on $\mathfrak{g}^*$. Indeed, this is just the classical statement that exponentiation of $\langle\lambda, -\rangle$ induces an isomorphism between the integral weight lattice $\Gamma_{\text{wt}}(\lambda)$ relative to the maximal torus T_λ and the $\mathbb{Z}$-module $\text{Hom}_{\text{Grp}}(T_\lambda, U(1))$ and that under this isomorphism a gauge transformation of a $\mathfrak{g}$-valued 1-form A turns into that of the $\mathfrak{u}(1)$-valued 1-form $\langle\lambda, A\rangle$.

Comparison with the discussion in Sect. 2 shows that this is the extended Lagrangian of a 1-dimensional Chern–Simons theory. In fact it is just a slight variant of the trace-theory discussed there: if we realize $\mathfrak{g}$ as a matrix Lie algebra and write $\langle\alpha, \beta\rangle = \text{tr}(\alpha \cdot \beta)$ as the matrix trace, then the above Chern–Simons 1-form is given by the "λ-shifted trace"

$$\text{CS}_\lambda(A) := \text{tr}(\lambda \cdot A) \in \Omega^1(-; \mathbb{R}).$$

Then, clearly, while the "plain" trace is invariant under the adjoint action of all of G, the λ-shifted trace is invariant only under the subgroup T_λ of G that fixes λ.

Notice that the domain of $\langle\lambda, -\rangle$ naturally sits inside $\mathbf{B}G_{\text{conn}}$ by the canonical map

$$\Omega^1(-, \mathfrak{g})//\underline{T}_\lambda \to \Omega^1(-, \mathfrak{g})//\underline{G} \simeq \mathbf{B}G_{\text{conn}}.$$

One sees that the homotopy fiber of this map is the *coadjoint orbit* $\mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$ of $\langle\lambda, -\rangle$, equipped with the map of stacks

$$\theta : \mathcal{O}_\lambda \simeq \underline{G}//\underline{T}_\lambda \to \Omega^1(-, \mathfrak{g})//\underline{T}_\lambda$$

which over a test manifold U sends $\boldsymbol{g} \in C^\infty(U, G)$ to the pullback $\boldsymbol{g}^*\theta_G$ of the Maurer–Cartan form. Composing this with the above extended Lagrangian $\langle\lambda, -\rangle$ yields a map

$$\langle\lambda, \theta\rangle : \ \mathcal{O}_\lambda \xrightarrow{\theta} \Omega^1(-, \mathfrak{g})//\underline{T}_\lambda \xrightarrow{\langle\lambda, -\rangle} \mathbf{B}U(1)_{\text{conn}}$$

which modulates a canonical $U(1)$-principal bundle with connection on the coadjoint orbit. One finds that this is the canonical prequantum bundle used in the orbit method [53]. In particular its curvature is the canonical symplectic form on the coadjoint orbit.

So far this shows how the ingredients of the orbit method are incarnated in smooth moduli stacks. This now immediately induces Chern–Simons theory with Wilson loops by considering the map $\Omega^1(-,\mathfrak{g})//\underline{T}_\lambda \to \mathbf{B}G_{\mathrm{conn}}$ itself as the target[10] for a field theory defined on knot inclusions $\iota : S^1 \hookrightarrow \Sigma_3$. This means that a field configuration is a diagram of smooth stacks of the form

$$\begin{array}{ccc} S^1 & \xrightarrow{(\iota^* A)^g} & \Omega^1(-,\mathfrak{g})//\underline{T}_\lambda \\ {\scriptstyle \iota}\downarrow & \swarrow\!\!\!\swarrow_{g} & \downarrow \\ \Sigma_3 & \xrightarrow[A]{} & \mathbf{B}G_{\mathrm{conn}} \end{array},$$

i.e., that a field configuration consists of

- a gauge field A in the "bulk" Σ_3;
- a G-valued function $\boldsymbol{g}$ on the embedded knot

such that the restriction of the ambient gauge field A to the knot is equivalent, via the gauge transformation $\boldsymbol{g}$, to a $\mathfrak{g}$-valued connection on S^1 whose local $\mathfrak{g}$-valued 1-forms are related each other by local gauge transformations taking values in the torus T_λ. Moreover, a gauge transformation between two such field configurations $(A, \boldsymbol{g})$ and $(A', \boldsymbol{g}')$ is a pair (t_{Σ_3}, t_{S^1}) consisting of a G-gauge transformation t_{Σ_3} on Σ_3 and a T_λ-gauge transformation t_{S^1} on S^1, intertwining the gauge transformations $\boldsymbol{g}$ and $\boldsymbol{g}'$. In particular if the bulk gauge field on Σ_3 is held fixed, i.e., if $A = A'$, then t_{S^1} satisfies the equation $\boldsymbol{g}' = \boldsymbol{g}\, t_{S^1}$. This means that the Wilson-line components of gauge-equivalence classes of field configurations are naturally identified with smooth functions $S^1 \to G/T_\lambda$, i.e., with smooth functions on the Wilson loop with values in the coadjoint orbit. This is essentially a rephrasing of the above statement that G/T_λ is the homotopy fiber of the inclusion of the moduli stack of Wilson line field configurations into the moduli stack of bulk field configurations.

We may postcompose the two horizontal maps in this square with our two extended Lagrangians, that for 1d and that for 3d Chern–Simons theory, to get the diagram

$$\begin{array}{ccccc} S^1 & \xrightarrow{(\iota^* A)^g} & \Omega^1(-,\mathfrak{g})//T & \xrightarrow{\langle\lambda,-\rangle} & \mathbf{B}U(1)_{\mathrm{conn}} \\ {\scriptstyle \iota}\downarrow & \swarrow\!\!\!\swarrow_{g} & \downarrow & & \\ \Sigma_3 & \xrightarrow{A} & \mathbf{B}G_{\mathrm{conn}} & \xrightarrow{\hat{\mathbf{c}}} & \mathbf{B}^3U(1)_{\mathrm{conn}} \end{array}.$$

Therefore, writing $\mathbf{Fields}_{\mathrm{CS+W}}\left(S^1 \overset{\iota}{\hookrightarrow} \Sigma_3\right)$ for the moduli stack of field configurations for Chern–Simons theory with Wilson lines, we find two action functionals as the composite top and left morphisms in the diagram

---

[10] This means that here we are secretly moving from the topos of (higher) stacks on smooth manifolds to its *arrow topos*, see Sect. 5.2.

$$\begin{array}{ccccc} \mathbf{Fields}_{\mathrm{CS+W}}\left(S^1 \overset{\iota}{\hookrightarrow} \Sigma_3\right) & \longrightarrow & \mathbf{Maps}(\Sigma_3, \mathbf{B}G_{\mathrm{conn}}) & \xrightarrow{\mathrm{hol}_{\Sigma_3}\mathbf{Maps}(\Sigma_3,\hat{\mathbf{c}})} & \underline{U}(1) \\ \downarrow & & \downarrow & & \\ \mathbf{Maps}(S^1, \Omega^1(-,\mathfrak{g})//T_\lambda) & \longrightarrow & \mathbf{Maps}(S^1, \mathbf{B}G_{\mathrm{con}}) & & \\ \downarrow{\scriptstyle \mathrm{hol}_{S^1}\mathbf{Maps}(S^1,\langle\lambda,-\rangle)} & & & & \\ \underline{U}(1) & & & & \end{array}$$

in $\mathbf{H}$, where the top left square is the homotopy pullback that characterizes maps in $\mathbf{H}^{(\Delta^1)}$ in terms of maps in $\mathbf{H}$. The product of these is the action functional

$$\begin{array}{ccc} \mathbf{Fields}_{\mathrm{CS+W}}\left(S^1 \overset{\iota}{\hookrightarrow} \Sigma_3\right) & \longrightarrow & \mathbf{Maps}(\Sigma_3, \mathbf{B}^3U(1)_{\mathrm{conn}}) \times \mathbf{Maps}(S^1, \mathbf{B}U(1)_{\mathrm{conn}}) \\ & & \downarrow \\ & & \underline{U}(1) \times \underline{U}(1) \xrightarrow{\quad\cdot\quad} \underline{U}(1)\,. \end{array}$$

where the rightmost arrow is the multiplication in $U(1)$. Evaluated on a field configuration with components $(A, \boldsymbol{g})$ as just discussed, this is

$$\exp\left(2\pi i \left(\int_{\Sigma_3} \mathrm{CS}_3(A) + \int_{S^1} \langle \lambda, (\iota^* A)^{\boldsymbol{g}} \rangle \right)\right).$$

This is indeed the action functional for Chern–Simons theory with Wilson loop ι in the representation R corresponding to the integral weight $\langle \lambda, - \rangle$ by the Borel-Weil-Bott theorem, as reviewed for instance in Sect. 4 of [6].

Apart from being an elegant and concise repackaging of this well-known action functional and the quantization conditions that go into it, the above reformulation in terms of stacks immediately leads to prequantum line bundles in Chern–Simons theory with Wilson loops. Namely, by considering the codimension 1 case, one finds the symplectic structure and the canonical prequantization for the moduli stack of field configurations on surfaces with specified singularities at specified punctures [88]. Moreover, this is just the first example in a general mechanism of (extended) action functionals with defect and/or boundary insertions. Another example of the same mechanism is the gauge coupling action functional of the open string. This we discuss in Sect. 5.4.2.

4 Extension to More General Examples

The way we presented the two examples of the previous sections indicates that they are clearly just the beginning of a rather general pattern of extended prequantized higher gauge theories of Chern–Simons type: for every smooth higher group G with universal differential higher moduli stack $\mathbf{B}G_{\text{conn}}$ (and in fact for any higher moduli stack at all, as further discussed in Sect. 5.1) every differentially refined universal characteristic map of stacks

$$\mathbf{L} : \mathbf{B}G_{\text{conn}} \longrightarrow \mathbf{B}^n U(1)_{\text{conn}}$$

constitutes an extended Lagrangian—hence, by iterated transgression, the action functional, prequantum theory and WZW-type action functional—of an n-dimensional Chern–Simons type gauge field theory with (higher) gauge group G. Moreover, just moving from higher stacks on the site of smooth manifolds to higher stacks on the site of smooth supermanifolds one has an immediate and natural generalization to super-Chern–Simons theories. Here we briefly survey some examples of interest, which were introduced in detail in [76] and [30]. Further examples and further details can be found in Sect. 5.7 of [79].

4.1 String Connections and Twisted String structures

Notice how we have moved from the 1d Chern–Simons theory of Sect. 2 to the 3d Chern-Simon theory of Sect. 3 by replacing the connected but not 1-connected compact Lie group $U(n)$ with a compact 2-connected but not 3-connected Lie group G. The natural further step towards a higher dimensional Chern–Simons theory would then be to consider a compact Lie group which is (at least) 3-connected. Unfortunately, there exists no such Lie group: if G is compact and simply connected then its third homotopy group will be nontrivial, see e.g. [62]. However, a solution to this problem does exist if we move from compact Lie groups to the more general context of smooth higher groups, i.e. if we focus on the stacks of principal bundles rather than on their gauge groups. As a basic example, think of how we obtained the stacks $\mathbf{B}SU(n)$ and $\mathbf{B}SU(n)_{\text{conn}}$ out of $\mathbf{B}U(n)$ and $\mathbf{B}U(n)_{\text{conn}}$ in Sect. 2.9. There we first obtained these stacks as homotopy fibers of the morphisms of stacks

$$\mathbf{c}_1 : \mathbf{B}U(n) \to \mathbf{B}U(1); \quad \hat{\mathbf{c}}_1 : \mathbf{B}U(n)_{\text{conn}} \to \mathbf{B}U(1)_{\text{conn}}$$

refining the first Chern class. Then, in a second step, we identified these homotopy fibers with the stack of principal bundles (with and without connection) for a certain compact Lie group, which turned out to be $SU(n)$. However, the homotopy fiber definition would have been meaningful even in case we would have been unable to show that there was a compact Lie group behind it, or even in case there would have been no such. This may seem too far a generalization, but actually Milnor's theorem [61] would have assured us in any case that there existed a *topological* group $SU(n)$ whose classifying space is homotopy equivalent to the topological realization of the homotopy fiber $\mathbf{B}SU(n)$, that is, equivalently, to the homotopy fiber of the topological realization of the morphism $\mathbf{c}_1$. This is nothing but the topological characteristic map

$$c_1 : BU(n) \to BU(1) \simeq K(\mathbb{Z}, 2)$$

defining the first Chern class. In other words, one defines the space $BSU(n)$ as the homotopy pullback

$$\begin{array}{ccc} BSU(n) & \longrightarrow & * \\ \downarrow & & \downarrow \\ BU(n) & \xrightarrow{c_1} & K(\mathbb{Z},2) \end{array} ;$$

the based loop space $\Omega BSU(n)$ has a natural structure of topological group "up to homotopy", and Milnor's theorem precisely tells us that we can strictify it, i.e. we can find a topological group $SU(n)$ (unique up to homotopy) such that $SU(n) \simeq \Omega BSU(n)$. Moreover, $BSU(n)$, defined as a homotopy fiber, will be a classifying space for this "homotopy-$SU(n)$" group. From this perspective, we see that having a model for the homotopy-$SU(n)$ which is a compact Lie group is surely something nice to have, but that we would have nevertheless been able to speak in a rigorous and well-defined way of the groupoid of smooth $SU(n)$-bundles over a smooth manifold X even in case such a compact Lie model did not exist. The same considerations apply to the stack of principal $SU(n)$-bundles with connections.

These considerations may look redundant, since one is well aware that there is indeed a compact Lie group $SU(n)$ with all the required features. However, this way of reasoning becomes prominent and indeed essential when we move to higher characteristic classes. The fundamental example is probably the following. For $n \geq 3$ the spin group Spin(n) is compact and simply connected with $\pi_3(\mathrm{Spin}(n)) \cong \mathbb{Z}$. The generator of $H^4(B\mathrm{Spin}(n); \mathbb{Z})$ is the first factional Pontrjagin class $\frac{1}{2}p_1$, which can be equivalently seen as a characteristic map

$$\tfrac{1}{2}p_1 \colon B\mathrm{Spin}(n) \to K(\mathbb{Z}; 4).$$

The String group String(n) is then defined as the topological group whose classifying space is the homotopy fiber of $\frac{1}{2}p_1$, i.e., the homotopy pullback

$$\begin{array}{ccc} B\mathrm{String}(n) & \longrightarrow & * \\ \downarrow & & \downarrow \\ B\mathrm{Spin}(n) & \xrightarrow{\frac{1}{2}p_1} & K(\mathbb{Z},4) \end{array} ;$$

this defines String(n) uniquely up to homotopy. The topological group String(n) is 6-connected with $\pi_7(\mathrm{String}(n)) \cong \mathbb{Z}$. The generator of $H^8(B\mathrm{String}(n); \mathbb{Z})$ is the second fractional Pontrjagin class $\frac{1}{6}p_2$, see [75]. One can then define the 3-stack of smooth String(n)-principal bundles as the homotopy pullback

$$\begin{array}{ccc} \mathbf{B}\mathrm{String}(n) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}\mathrm{Spin}(n) & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3U(1) \end{array} ,$$

where $\frac{1}{2}\mathbf{p}_1$ is the morphism of stacks whose topological realization is $\frac{1}{2}p_1$. In other words, a String(n)-principal bundle over a smooth manifold X is the datum of a Spin(n)-principal bundle over X together with a trivialization of the associated $B^2U(1)$-principal 3-bundle. The characteristic map

$$\tfrac{1}{6}p_2 : B\mathrm{String}(n) \to K(\mathbb{Z}; 8)$$

is the topological realization of a morphism of stacks

$$\tfrac{1}{6}\mathbf{p}_2 : \mathbf{B}\mathrm{String}(n) \to \mathbf{B}^7 U(1),$$

see [31, 76]. Similarly, one can define the 3-stack of smooth String bundles with connections as the homotopy pullback

$$\begin{array}{ccc} \mathbf{B}\mathrm{String}(n)_{\mathrm{conn}} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}\mathrm{Spin}(n)_{\mathrm{conn}} & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1} & \mathbf{B}^3 U(1)_{\mathrm{conn}} \end{array},$$

where $\frac{1}{2}\hat{\mathbf{p}}_1$ is the lift of $\frac{1}{2}\mathbf{p}_1$ to the stack of Spin(n)-bundles with connections. Again, this means that a String(n)-bundle with connection over a smooth manifold X is the datum of a Spin(n)-bundle with connection over X together with a trivialization of the associated $U(1)$-3-bundle with connection. The morphism $\frac{1}{6}\mathbf{p}_2$ lifts to a morphism

$$\tfrac{1}{6}\hat{\mathbf{p}}_2 : \mathbf{B}\mathrm{String}(n)_{\mathrm{conn}} \to \mathbf{B}^7 U(1)_{\mathrm{conn}},$$

see [31], and this defines a 7d Chern–Simons theory with gauge group the String(n)-group.

In the physics literature one usually considers also a more flexible notion of String connection, in which one requires that the underlying $U(1)$-3-bundle of a Spin(n)-bundle with connection is trivialized, but does not require the underlying 3-connection to be trivialized. In terms of stacks, this corresponds to considering the homotopy pullback

$$\begin{array}{ccc} \mathbf{B}\mathrm{String}(n)_{\mathrm{conn}'} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}\mathrm{Spin}(n)_{\mathrm{conn}} & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3 U(1) \end{array};$$

see, e.g., [84]. Furthermore, it is customary to consider not only the case where the underlying $U(1)$-3-bundle (with or without connection) is trivial, but also the case when it is equivalent to a fixed *background* $U(1)$-3-bundle (again, eventually with connection). Notably, the connection 3-form of this fixed background is the C-field of the M-theory literature (cf. [70, 71]). The moduli stacks of Spin(n)-bundles on a smooth manifold X with possibly nontrivial fixed $U(1)$-3-bundle background are called [76] moduli stacks of *twisted* String bundles on X. A particular interesting case is when the twist is independent of X, hence is itself given by a universal characteristic class, hence by a twisting morphism

$$\mathbf{c} : \ \mathbf{S} \longrightarrow \mathbf{B}^3 U(1) \ ,$$

where $\mathbf{S}$ is some (higher moduli) stack. In this case, indeed, one can define the stack $\mathbf{B}\mathrm{String}(n)^{\mathbf{c}}$ of $\mathbf{c}$-twisted String(n)-structures as the homotopy pullback

$$\begin{array}{ccc} \mathbf{B}\mathrm{String}(n)^{\mathbf{c}} & \longrightarrow & \mathbf{S} \\ \downarrow & & \downarrow^{\mathbf{c}} \\ \mathbf{B}\mathrm{Spin}(n) & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1} & \mathbf{B}^3U(1) \end{array} ,$$

and similarly for the stack of **c**-twisted String(n)-connections. This is a higher analog of Spin^c-structures, whose universal moduli stack sits in the analogous homotopy pullback diagram

$$\begin{array}{ccc} \mathbf{B}\mathrm{Spin}^c(n) & \longrightarrow & \mathbf{B}U(1) \\ \downarrow & & \downarrow^{\mathbf{c}_1 \bmod 2} \\ \mathbf{B}SO(n) & \xrightarrow{\mathbf{w}_1} & \mathbf{B}^2\mathbb{Z}_2 \end{array} .$$

(For more on higher Spin^c-structures see also [72, 73] and Sect. 5.2 of [79].). By a little abuse of terminology, when the twisting morphism **a** is the refinement of a characteristic class for a compact simply connected simple Lie group G to a morphism of stacks $\mathbf{a} : \mathbf{B}G \to \mathbf{B}^3U(1)$, one may speak of *G-twisted structures* rather than of **a**-twisted structures.

By the discussion in Sect. 3 the differential twisting maps $\frac{1}{2}\hat{\mathbf{p}}_1$ and $\hat{\mathbf{a}}$ appearing here are at the same time extended Lagrangians of Chern–Simons theories. Together with the nature of homotopy pullback, it follows [31] that a field $\phi : X \to \mathbf{B}\mathrm{String}^{\mathbf{a}}{}_{\mathrm{conn}}$ consists of a pair of gauge fields and a homotopy between their Chern–Simons data, namely of

1. a Spin-connection $\nabla_{\mathfrak{so}}$;
2. a G-connection $\nabla_{\mathfrak{g}}$;
3. a twisted 2-form connection B whose curvature 3-form H is locally given by $H = dB + \mathrm{CS}(\nabla_{\mathfrak{so}}) - \mathrm{CS}(\nabla_{\mathfrak{g}})$.

These are the data for (Green–Schwarz-) anomaly-free background gauge fields (gravity, gauge field, Kalb–Ramond field) for the heterotic string [76]. A further refinement of this construction yields the universal moduli stack for the *supergravity C-field* configurations in terms of E_8-twisted String connections [29]. Here the presence of the differential characteristic maps $\hat{\mathbf{c}}$ induces the Chern–Simons gauge-coupling piece of the supergravity 2-brane (the *M2-brane*) action functional.

4.2 Cup-Product Chern–Simons Theories

In Sect. 3 we had restricted attention to 3d Chern–Simons theory with simply connected gauge groups. Another important special case of 3d Chern–Simons theory is that with gauge group the circle group $U(1)$, which is of course not simply connected. In this case the universal characteristic map that controls the theory is the differential refinement of the *cup product class* $c_1 \cup c_1$. Here we briefly indicate this case and the analogous higher dimensional Chern–Simons theories obtained from cup products of higher classes and from higher order cup products.

The cup product $\cup$ in integral cohomology can be lifted to a cup product $\hat{\cup}$ in differential cohomology, i.e., for any smooth manifold X we have a natural commutative diagram

$$\begin{array}{ccc} \hat{H}^p(X;\mathbb{Z}) \otimes \hat{H}^q(X;\mathbb{Z}) & \xrightarrow{\hat{\cup}} & \hat{H}^{p+q}(X;\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^p(X;\mathbb{Z}) \otimes H^q(X;\mathbb{Z}) & \xrightarrow{\cup} & H^{p+q}(X;\mathbb{Z}) \end{array},$$

for any $p, q \geq 0$. Moreover, this cup product is induced by a cup product defined at the level of Čech–Deligne cocycles, the so called *Beilinson-Drinfeld cup product*, see [8]. This, in turn, may be seen [30] to come from a morphism of higher universal moduli stacks

$$\widehat{\cup} : \mathbf{B}^{n_1}U(1)_{\text{conn}} \times \mathbf{B}^{n_2}U(1)_{\text{conn}} \to \mathbf{B}^{n_1+n_2+1}U(1)_{\text{conn}}.$$

Moreover, since the Beilinson–Deligne cup product is associative up to homotopy, this induces a well-defined morphism

$$\mathbf{B}^{n_1}U(1)_{\text{conn}} \times \mathbf{B}^{n_2}U(1)_{\text{conn}} \times \cdots \times \mathbf{B}^{n_{k+1}}U(1)_{\text{conn}} \to \mathbf{B}^{n_1+\cdots+n_{k+1}+k}U(1)_{\text{conn}}.$$

In particular, for $n_1 = \cdots = n_{k+1} = 3$, one finds a cup product morphism

$$\left(\mathbf{B}^3U(1)_{\text{conn}}\right)^{k+1} \to \mathbf{B}^{4k+3}U(1)_{\text{conn}}.$$

Furthermore, one sees from the explicit expression of the Beilinson–Deligne cup product that, on a local chart U_α, if the 3-form datum of a connection on a $U(1)$-3-bundle is the 3-form C_α, then the $(4k+3)$-form local datum for the corresponding connection on the associated $U(1)$-$(4k+3)$-bundle is

$$C_\alpha \wedge \underbrace{dC_\alpha \wedge \cdots \wedge dC_\alpha}_{k \text{ times}}.$$

Now let G be a compact and simply connected simple Lie group and let $\hat{\mathbf{c}} : \mathbf{B}G_{\text{conn}} \to \mathbf{B}^3U(1)_{\text{conn}}$ be the morphism of stacks underlying the fundamental characteristic class $c \in H^4(BG, \mathbb{Z})$. Then we can consider the $(k+1)$-fold product of $\hat{\mathbf{c}}$ with itself:

$$\hat{\mathbf{c}} \mathbin{\hat{\cup}} \hat{\mathbf{c}} \mathbin{\hat{\cup}} \cdots \mathbin{\hat{\cup}} \hat{\mathbf{c}} : \mathbf{B}G_{\text{conn}} \xrightarrow{(\hat{\mathbf{c}},\ldots,\hat{\mathbf{c}})} \left(\mathbf{B}^3U(1)_{\text{conn}}\right)^{k+1} \xrightarrow{\hat{\cup}} \mathbf{B}^{4k+3}U(1)_{\text{conn}}.$$

If X is a compact oriented smooth manifold, fiber integration along X gives the morphism

$$\mathbf{Maps}(X, \mathbf{B}G_{\text{conn}}) \to \mathbf{Maps}(X, \mathbf{B}^{4k+3}U(1)_{\text{conn}}) \xrightarrow{\text{hol}_X} \mathbf{B}^{4k+3-\dim X}U(1)_{\text{conn}}.$$

In particular, if $\dim X = 4k+3$, by evaluating over the point and taking equivalence classes we get a canonical morphism

$$\{G\text{-bundles with connections on } X\}/\text{iso} \to U(1).$$

This is the action functional of the $(k+1)$-fold *cup product Chern–Simons theory* induced by the $(k+1)$-fold cup product of c with itself [30]. This way one obtains, for every $k \geq 0$, a $(4k+3)$-dimensional theory starting with a 3d Chern–Simons theory. Moreover, in the special case that the principal G-bundle on X is topologically trivial, this action functional has a particularly simple expression: it is given by

$$\exp 2\pi i \int_X \mathrm{CS}_3(A) \wedge \langle F_A, F_A \rangle \wedge \cdots \wedge \langle F_A, F_A \rangle,$$

where $A \in \Omega^1(X; \mathfrak{g})$ is the $\mathfrak{g}$-valued 1-form on X representing the connection in the chosen trivialization of the G-bundle. But notice that in this more general situation now not *every* gauge field configuration will have an underlying trivializable (higher) bundle anymore, the way it was true for the 3d Chern–Simons theory of a simply connected Lie group in Sect. 3.

More generally, one can consider an arbitrary smooth (higher) group G, e.g. $U(n) \times \mathrm{Spin}(m) \times \mathrm{String}(l)$, together with $k+1$ characteristic maps $\hat{\mathbf{c}}_i : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^{n_i} U(1)_{\mathrm{conn}}$ and one can form the $(k+1)$-fold product

$$\hat{\mathbf{c}}_1 \hat{\cup} \cdots \hat{\cup}\, \hat{\mathbf{c}}_{k+1} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^{n_1 + \cdots n_{k+1} + k} U(1)_{\mathrm{conn}},$$

inducing a $(n_1 + \cdots n_{k+1} + k)$-dimensional Chern–Simons-type theory. For instance, if G_1 and G_2 are two compact simply connected simple Lie groups, then we have a 7d cup product Chern–Simons theory associated with the cup product $\hat{\mathbf{c}}_1 \hat{\cup}\, \hat{\mathbf{c}}_2$. If (P_1, ∇_1) and (P_2, ∇_2) are a pair of topologically trivial principal G_1- and G_2-bundles with connections over a 7-dimensional oriented compact manifold without boundary X, the action functional of this Chern–Simons theory on this pair is given by

$$\exp 2\pi i \int_X \mathrm{CS}_3(A_1) \wedge \langle F_{A_2}, F_{A_2} \rangle = \exp 2\pi i \int_X \mathrm{CS}_3(A_2) \wedge \langle F_{A_1}, F_{A_1} \rangle,$$

where A_i is the connection 1-forms of ∇_i, for $i = 1, 2$. Notice how in general a G_i-principal bundle on a 7-dimensional manifold is not topologically trivial, but still we have a well defined cup-product Chern–Simons action $S_{\hat{\mathbf{c}}_1 \hat{\cup} \hat{\mathbf{c}}_2}$. In the topologically nontrivial situation, however, there will not be such a simple global expression for the action.

Let us briefly mention a few representative important examples from string theory and M-theory which admit a natural interpretation as cup-product Chern–Simons theories, the details of which can be found in [30]. For all examples presented below we write the Chern–Simons action for the topologically trivial sector.

- *Abelian higher dimensional CS theory and self-dual higher gauge theory.* For every $k \in \mathbb{N}$ the differential cup product yields the extended Lagrangian

$$\mathbf{L} : \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}} \longrightarrow \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}} \times \mathbf{B}^{2k+1}U(1)_{\mathrm{conn}} \xrightarrow{\hat{\cup}} \mathbf{B}^{4k+3}U(1)_{\mathrm{conn}}$$

for a $4k+3$-dimensional Chern–Simons theory of $(2k+1)$-form connections on higher circle bundles (higher bundle gerbes). Over a 3-dimensional manifold Σ the corresponding action functional applied to gauge fields A whose underlying bundle is trivial is given by

$$\exp 2\pi i \int_{\Sigma} \mathrm{CS}_1(A) \cup d\mathrm{CS}_1(A) = \exp 2\pi i \int_{\Sigma} A \wedge F_A,$$

where $F_A = dA$ is the curvature of a $U(1)$-connection A. Similarly, the transgression of $\mathbf{L}$ to codimension 1 over a manifold Σ of dimension $4k+2$ yields the prequantization of a symplectic form on $(2k+1)$-form connections which, by a derivation analogous to that in Sect. 3.4.3, is given by

$$\omega(\delta A_1, \delta A_1) = \int_{\Sigma} \delta A_1 \wedge \delta A_1.$$

A complex polarization of this symplectic structure is given by a choice of conformal metric on Σ and the corresponding canonical coordinates are complex Hodge self-dual forms on Σ. This yields the famous holographic relation between higher abelian Chern–Simons theory and self-dual higher abelian gauge theory in one dimension lower.

- *The M5-brane self-dual theory.* In particular, for $k=1$ it was argued in [89] that the 7-dimensional Chern–Simons theory which we refine to an extended prequantum theory by the extended Lagrangian

$$\mathbf{L} : \mathbf{B}^{3}U(1)_{\mathrm{conn}} \longrightarrow \mathbf{B}^{3}U(1)_{\mathrm{conn}} \times \mathbf{B}^{3}U(1)_{\mathrm{conn}} \xrightarrow{\hat{\cup}} \mathbf{B}^{7}U(1)_{\mathrm{conn}}$$

describes, in this holographic manner, the quantum theory of the self-dual 2-form in the 6-dimensional worldvolume theory of a single M5-brane. Since moreover in [90] it was argued that this abelian 7-dimensional Chern–Simons theory is to be thought of as the abelian piece in the Chern–Simons term of 11-dimensional supergravity compactified on a 4-sphere, and since this term in general receives non-abelian corrections from "flux quantization" (see [29] for a review of these and for discussion in the present context of higher moduli stacks), we discussed in [28] the appropriate non-abelian refinement of this 7d Chern–Simons term, which contains also cup product terms of the form $\hat{\mathbf{a}}_1 \hat{\cup} \hat{\mathbf{a}}_2$ as well we the term $\frac{1}{6}\widehat{\mathbf{p}}_2$ from Sect. 4.1.
- *Five-dimensional and eleven-dimensional supergravity.* The topological part of the five-dimensional supergravity action is $\exp 2\pi i \int_{Y^5} A \wedge F_A \wedge F_A$, where A is a $U(1)$-connection. Writing the action as $\exp 2\pi i \int_{Y^5} \mathrm{CS}_1(A) \cup d\mathrm{CS}_1(A) \cup d\mathrm{CS}_1(A)$, one sees this is a 3-fold Chern–Simons theory. Next, in eleven dimensions, the C-field C_3 with can be viewed as a 3-connection on a 2-gerbe with 4-curvature G_4. By identifying the C-field with the Chern–Simons 3-form $\mathrm{CS}_3(A)$

of a $U(1)$-3-connection A, the topological action $\exp 2\pi i \int_{Y11} C_3 \wedge G_4 \wedge G_4$, is seen to be of the form $\exp 2\pi i \int_{Y11} \mathrm{CS}_3(A) \cup d\mathrm{CS}_3(A) \cup d\mathrm{CS}_3(A)$. This realizes the 11d supergravity C-field action as the action for a 3-tier cup-product abelian Chern–Simons theory induced by a morphism of 3-stacks [29].

4.3 Super-Chern–Simons Theories

The (higher) topos $\mathbf{H}$ of (higher) stacks on the smooth site of manifolds which we have been considering for most of this paper has an important property common to various similar toposes such as that on supermanifolds: it satisfies a small set of axioms called (differential) *cohesion*, see [79]. Moreover, essentially every construction described in the above sections makes sense in an arbitrary cohesive topos. For constructions like homotopy pullbacks, mapping spaces, adjoint actions etc., this is true for every topos, while the differential cohesion in addition guarantees the existence of differential geometric structures such as de Rham coefficients, connections, differential cohomology, etc. This setting allows to transport all considerations based on the cohesion axioms across various kinds of geometries. Notably, one can speak of higher *supergeometry*, and hence of fermionic quantum fields, simply by declaring the site of definition to be that of supermanifolds: indeed, the higher topos of (higher) stacks on supermanifolds is differentially cohesive ([79], Sect. 4.6). This leads to a natural notion of *super-Chern–Simons theories*.

In order to introduce these notions, we need a digression on higher complex line bundles. Namely, we have been using the n-stacks $\mathbf{B}^n U(1)$, but without any substantial change in the theory we could also use the n-stacks $\mathbf{B}^n \mathbb{C}^\times$ with the multiplicative group $U(1)$ of norm 1 complex numbers replaced by the full multiplicative group of non-zero complex numbers. Since we have a fiber sequence

$$\mathbb{R}_{>0} \to \mathbb{C}^\times \to U(1)$$

with topologically contractible fiber, under geometric realization $|-|$ the canonical map $\mathbf{B}^n U(1) \to \mathbf{B}^n \mathbb{C}^\times$ becomes an equivalence. Nevertheless, some constructions are more naturally expressed in terms of $U(1)$-principal n-bundles, while others are more naturally expressed in terms of $\mathbb{C}^\times$-principal n-bundles, and so it is useful to be able to switch from one description to the other. For $n = 1$ this is the familiar fact that the classifying space of principal $U(1)$-bundles is homotopy equivalent to the classifying space of complex line bundles. For $n = 2$ we still have a noteworthy (higher) linear algebra interpretation: $\mathbf{B}^2\mathbb{C}^\times$ is naturally identified with the 2-stack $2\mathbf{Line}_\mathbb{C}$ of *complex line 2-bundles*. Namely, for R a commutative ring (or more generally an E_∞-ring), one considers the 2-category of R-algebras, bimodules and bimodule homomorphisms (e.g. [22]). We may think of this as the 2-category of *2-vector spaces* over R (appendix A of [78], Sect. 4.4 of [82], Sect. 7 of [36]). Notice that this 2-category is naturally braided monoidal. We then write

$$2\mathrm{Line}_R \hookrightarrow 2\mathrm{Vect}_R$$

for the full sub-2-groupoid on those objects which are invertible under this tensor product: the *2-lines* over R. This is the *Picard 2-groupoid* over R, and with the inherited monoidal structure it is a 3-group, the *Brauer 3-group* of R. Its homotopy groups have a familiar algebraic interpretation:

- $\pi_0(2\mathrm{Line}_R)$ is the *Brauer group* of R;
- $\pi_1(2\mathrm{Line}_R)$ is the ordinary *Picard group* of R (of ordinary R-lines);
- $\pi_2(2\mathrm{Line}_R) \simeq R^\times$ is the *group of units*.

(This is the generalization to $n = 2$ of the familiar Picard 1-groupoid $1\mathrm{Line}_R$ of invertible R-modules.) Since the construction is natural in R and naturality respects 2-lines, by taking R to be a sheaf of k-algebras, with k a fixed field, one defines the 2-stacks $2\mathbf{Vect}_k$ of *k-2-vector bundles* and $2\mathbf{Line}_k$ of 2-line bundles over k. If k is algebraically closed, then there is, up to equivalence, only a single 2-line and only a single invertible bimodule, hence $2\mathbf{Line}_k \simeq \mathbf{B}^2 k^\times$. In particular, we have that

$$2\mathbf{Line}_{\mathbb{C}} \simeq \mathbf{B}^2\mathbb{C}^\times .$$

The background B-field of the bosonic string has a natural interpretation as a section of the differential refinement $\mathbf{B}^2\mathbb{C}^\times_{\mathrm{conn}}$ of the 2-stack $\mathbf{B}^2\mathbb{C}^\times$. Hence, by the above discussion, it is identified with a 2-connection on a complex 2-line bundle. However, a careful analysis, due to [23] and made more explicit in [35], shows that for the superstring the background B-field is more refined. Expressed in the language of higher stacks the statement is that the superstring B-field is a connection on a complex *super*-2-line bundle. This means that one has to move from the (higher) topos of (higher) stacks on the site of smooth manifolds to that of stacks on the site of smooth supermanifolds (Sect. 4.6 of [79]). The 2-stack of complex 2-line bundles is then replaced by the 2-stack $2\mathbf{sLine}_{\mathbb{C}}$ of super-2-line bundles, whose global points are complex Azumaya superalgebras. Of these there are, up to equivalence, not just one but two: the canonical super 2-line and its “superpartner” [85]. Moreover, there are now, up to equivalence, two different invertible 2-linear maps from each of these super-lines to itself. In summary, the homotopy sheaves of the super 2-stack of super line 2-bundles are

- $\pi_0(2\mathbf{sLine}_{\mathbb{C}}) \simeq \mathbb{Z}_2$,
- $\pi_1(2\mathbf{sLine}_{\mathbb{C}}) \simeq \mathbb{Z}_2$,
- $\pi_2(2\mathbf{sLine}_{\mathbb{C}}) \simeq \mathbb{C}^\times$.

Since the homotopy groups of the group $\mathbb{C}^\times$ are $\pi_0(\mathbb{C}^\times) = 0$ and $\pi_1(\mathbb{C}^\times) = \mathbb{Z}$, it follows that the geometric realization of this 2-stack has homotopy groups

- $\pi_0(|2\mathbf{sLine}_{\mathbb{C}}|) \simeq \mathbb{Z}_2$,
- $\pi_1(|2\mathbf{sLine}_{\mathbb{C}}|) \simeq \mathbb{Z}_2$,
- $\pi_2(|2\mathbf{sLine}_{\mathbb{C}}|) \simeq 0$,
- $\pi_3(|2\mathbf{sLine}_{\mathbb{C}}|) \simeq \mathbb{Z}$.

These are precisely the correct coefficients for the twists of complex K-theory [24], witnessing the fact that the B-field background of the superstring twists the Chan-Paton bundles on the D-branes [23, 35].

The braided monoidal structure of the 2-category of complex super-vector spaces induces on $2\mathbf{sLine}_{\mathbb{C}}$ the structure of a *braided 3-group*. Therefore, one has a naturally defined 3-stack $\mathbf{B}(2\mathbf{sLine}_{\mathbb{C}})_{\mathrm{conn}}$ which is the supergeometric refinement of the coefficient object $\mathbf{B}^3\mathbb{C}^{\times}_{\mathrm{conn}}$ for the extended Lagrangian of bosonic 3-dimensional Chern–Simons theory. Therefore, for G a super-Lie group a super-Chern–Simons theory, inducing a super-WZW action functional on G, is naturally given by an extended Lagrangian which is a map of higher moduli stacks of the form

$$\mathbf{L} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}(2\mathbf{sLine}_{\mathbb{C}})_{\mathrm{conn}}.$$

Notice that, by the canonical inclusion $\mathbf{B}^3\mathbb{C}^{\times}_{\mathrm{conn}} \to \mathbf{B}(2\mathbf{sLine}_{\mathbb{C}})_{\mathrm{conn}}$, every bosonic extended Lagrangian of 3d Chern–Simons type induces such a supergeometric theory with trivial super-grading part.

5 Outlook: Higher Prequantum Theory

The discussion in Sects. 2 and 3 of low dimensional Chern–Simons theories and the survey on higher dimensional Chern–Simons theories in Sect. 4, formulated and extended in terms of higher stacks, is a first indication of a fairly comprehensive theory of higher and extended prequantum gauge field theory that is naturally incarnated in a suitable context of higher stacks. In this last section we give a brief glimpse of some further aspects. Additional, more comprehensive expositions and further pointers are collected for instance in [79, 80].

5.1 *σ-models*

The Chern–Simons theories presented in the previous sections are manifestly special examples of the following general construction: one has a universal (higher) stack **Fields** of field configurations for a certain field theory, equipped with an *extended* Lagrangian, namely with a map of higher stacks

$$\mathbf{L} \; : \; \mathbf{Fields} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$$

to the n-stack of $U(1)$-principal n-bundles with connections. The Lagrangian $\mathbf{L}$ induces Lagrangian data in arbitrary codimension: for every closed oriented worldvolume Σ_k of dimension $k \leq n$ there is a *transgressed* Lagrangian

$$\mathbf{Maps}(\Sigma_k; \mathbf{Fields}) \xrightarrow{\mathbf{Maps}(\Sigma_k;\mathbf{L})} \mathbf{Maps}(\Sigma_k; \mathbf{B}^n U(1)_{\mathrm{conn}}) \xrightarrow{\mathrm{hol}_{\Sigma_k}} \mathbf{B}^{n-k}U(1)_{\mathrm{conn}}$$

defining the (off-shell) prequantum $U(1)$-$(n-k)$-bundle of the given field theory. In particular, the curvature forms of these bundles induce the canonical pre-$(n-k)$-plectic structure on the moduli stack of field configurations on Σ_k.

In codimension 0, i.e., for $k = n$ one has the morphism of stacks

$$\exp(2\pi i \int_{\Sigma_n} -) : \mathbf{Maps}(\Sigma_n; \mathbf{Fields}) \to \underline{U}(1)$$

and so taking global sections over the point and passing to equivalence classes one finds the *action functional*

$$\exp(2\pi i \int_{\Sigma_n} -) : \{\text{Field configurations}\}/\text{equiv} \to U(1).$$

Notice how the stacky origin of the action functional automatically implies that its value only depends on the gauge equivalence class of a given field configuration. Moreover, the action functional of an extended Lagrangian field theory as above is manifestly a σ-model action functional: the target "space" is the universal moduli stack of field configurations itself. Furthermore, the composition

$$\omega : \mathbf{Fields} \xrightarrow{\mathbf{L}} \mathbf{B}^n U(1)_{\text{conn}} \xrightarrow{F_{(-)}} \Omega^{n+1}(-; \mathbb{R})_{\text{cl}}$$

shows that the stack of field configurations is naturally equipped with a pre-n-plectic structure [67], which means that actions of extended Lagrangian field theories in the above sense are examples of σ-models with (pre)-n-plectic targets. For *binary* dependence of the n-plectic form on the fields this includes the AKSZ σ-models [2, 16–19, 26, 50, 51, 55, 56, 69]. For instance, from this perspective, the action functional of classical 3d Chern–Simons theory is the σ-model action functional with target the stack $\mathbf{B}G_{\text{conn}}$ equipped with the pre-3-plectic form $\langle -, - \rangle : \mathbf{B}G_{\text{conn}} \to \Omega^4_{\text{cl}}$ (the Killing form invariant polynomial) as discussed in 3. If we consider binary invariant polynomials in *derived* geometry, hence on objects with components also in negative degree, then also closed bosonic string field theory as in [91] is an example (see 5.7.10 of [79]) as are constructions such as [21]. Examples of n-plectic structures of higher arity on moduli stacks of higher gauge fields are in [28, 30].

More generally, we have transgression of the extended Lagrangian over manifolds Σ_k with boundary $\partial\Sigma_k$. Again by inspection of the constructions in [44] in terms of Deligne complexes, one finds that under the Dold–Kan correspondence these induce the corresponding constructions on higher moduli stacks: the *higher parallel transport* of $\mathbf{L}$ over Σ_k yields a *section* of the $(n-k+1)$-bundle which is modulated over the boundary by $\mathbf{Maps}(\partial\Sigma_k, \mathbf{B}G_{\text{conn}}) \to \mathbf{B}^{n-k+1}U(1)_{\text{conn}}$. This is the incarnation at the prequantum level of the *propagator* of the full extended TQFT in the sense of [60] over Σ_k, as indicated in [58]. Further discussion of this full prequantum field theory obtained this way is well beyond the scope of the present article. However, below in Sect. 5.4 we indicate how familiar *anomaly cancellation* constructions in open string theory naturally arise as examples of such transgression of extended Lagrangians over worldvolumes with boundary.

5.2 Fields in Slices: Twisted Differential Structures

Our discussion of σ-model-type actions in the previous section might seem to suggest that all the fields that one encounters in field theory have moduli that form (higher) stacks on the site of smooth manifolds. However, this is actually not the case and one need not look too far in order to find a counterexample: the field of gravity in general relativity is a (pseudo-)Riemannian metric on spacetime, and there is no such thing as a stack of (pseudo-)Riemannian metrics on the smooth site. This is nothing but the elementary fact that a (pseudo-)Riemannian metric cannot be pulled back along an arbitrary smooth morphism between manifolds, but only along local diffeomorphisms. Translated into the language of stacks, this tells us that (pseudo-)Riemannian metrics is a stack on the étale site of smooth manifolds, but not on the smooth site.[11] Yet we can still look at (pseudo-)Riemannian metrics on a smooth n-dimensional manifold X from the perspective of the topos $\mathbf{H}$ of stacks over the smooth site, and indeed this is the more comprehensive point of view. Namely, working in $\mathbf{H}$ also means to work with all its *slice toposes* (or *over-toposes*) $\mathbf{H}/_{\mathbf{S}}$ over the various objects $\mathbf{S}$ in $\mathbf{H}$. For the field of gravity this means working in the slice $\mathbf{H}/_{\mathbf{B}GL(n;\mathbb{R})}$ over the stack $\mathbf{B}GL(n;\mathbb{R})$.[12]

Once again, this seemingly frightening terminology is just a concise and rigorous way of expressing a familiar fact from Riemannian geometry: endowing a smooth n-manifold X with a pseudo-Riemannian metric of signature $(p, n-p)$ is equivalent to performing a reduction of the structure group of the tangent bundle of X to $O(p, n-p)$. Indeed, one can look at the tangent bundle as a morphism $\tau_X : X \to \mathbf{B}GL(n;\mathbb{R})$.

Example: Orthogonal Structures.

The above reduction is then the datum of a homotopy lift of τ_X

$$\begin{array}{ccc} & & \mathbf{B}O(n) \\ & {}^{o_X}\nearrow \;\; \Downarrow e & \downarrow \\ X & \xrightarrow{\tau_X} & \mathbf{B}\mathrm{GL}(n;\mathbb{R}) \end{array},$$

where the vertical arrow

$$\mathbf{OrthStruc}_n : \ \mathbf{B}O(n) \longrightarrow \mathbf{B}\mathrm{GL}(n;\mathbb{R})$$

is induced by the inclusion of groups $O(n) \hookrightarrow GL(n;\mathbb{R})$. Such a commutative diagram is precisely a map

$$(o_X, e) : \ \tau_X \longrightarrow \mathbf{OrthStruc}_n$$

in the slice $\mathbf{H}/_{\mathbf{B}GL(n;\mathbb{R})}$. The homotopy e appearing in the above diagram is precisely the *vielbein field* (frame field) which exhibits the reduction, hence which induces the

[11] See [13] for a comprehensive treatment of the étale site of smooth manifolds and of the higher topos of higher stacks over it.

[12] More detailed discussion of how (quantum) fields generally are maps in slices of cohesive toposes has been given in the lecture notes [80] and in Sects. 1.2.16, 5.4 of [79].

Riemannian metric. So the moduli stack of Riemannian metrics in n dimensions is $\mathbf{OrthStruc}_n$, not as an object of the ambient cohesive topos $\mathbf{H}$, but of the slice $\mathbf{H}_{/\mathbf{B}\mathrm{GL}(n)}$. Indeed, a map between manifolds regarded in this slice, namely a map $(\phi, \eta) : \tau_Y \to \tau_X$, is equivalently a smooth map $\phi : Y \to X$ in $\mathbf{H}$, but equipped with an equivalence $\eta : \phi^*\tau_X \to \tau_Y$. This precisely exhibits ϕ as a local diffeomorphism. In this way the slicing formalism automatically knows along which kinds of maps metrics may be pulled back.

Example: (Exceptional) Generalized Geometry.

If we replace in the above example the map $\mathbf{OrthStruc}_n$ with inclusions of other maximal compact subgroups, we similarly obtain the moduli stacks for *generalized* geometry (metric and B-field) as appearing in type II superstring backgrounds (see, e.g., [47]), given by

$$\mathbf{typeII} : \ \mathbf{B}(O(n) \times O(n)) \longrightarrow \mathbf{B}O(n,n) \quad \in \mathbf{H}_{/\mathbf{B}O(n,n)}$$

and of *exceptional generalized geometry* appearing in compactifications of 11-dimensional supergravity [49], given by

$$\mathbf{ExcSugra}_n : \ \mathbf{B}K_n \longrightarrow \mathbf{B}E_{n(n)} \quad \in \mathbf{H}_{/\mathbf{B}E_{n(n)}}.$$

For instance, a manifold X in type II-geometry is represented by $\tau_X^{\mathrm{gen}} : X \to \mathbf{B}O(n,n)$ in the slice $\mathbf{H}_{/\mathbf{B}O(n,n)}$, which is the map modulating what is called the *generalized tangent bundle*, and a field of generalized type II gravity is a map $(o_X^{\mathrm{gen}}, e) : \tau_X^{\mathrm{gen}} \to \mathbf{typeII}$ to the moduli stack in the slice. One checks that the homotopy e is now precisely what is called the *generalized vielbein field* in type II geometry. We read off the kind of maps along which such fields may be pulled back: a map $(\phi, \eta) : \tau_Y^{\mathrm{gen}} \to \tau_X^{\mathrm{gen}}$ is a *generalized* local diffeomorphism: a smooth map $\phi : Y \to X$ equipped with an equivalence of generalized tangent bundles $\eta : \phi^*\tau_X^{\mathrm{gen}} \to \tau_Y^{\mathrm{gen}}$. A directly analogous discussion applies to the exceptional generalized geometry.

Furthermore, various topological structures are generalized fields in this sense, and become fields in the more traditional sense after differential refinement.

Example: Spin Structures.

The map $\mathbf{SpinStruc} : \mathbf{B}\mathrm{Spin} \to \mathbf{B}\mathrm{GL}$ is, when regarded as an object of $\mathbf{H}_{/\mathbf{B}\mathrm{GL}}$, the moduli stack of spin structures. Its differential refinement $\mathbf{SpinStruc}_{\mathrm{conn}} : \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \to \mathbf{B}\mathrm{GL}_{\mathrm{conn}}$ is such that a domain object $\tau_X^{\nabla} \in \mathbf{H}_{/\mathbf{GL}_{\mathrm{conn}}}$ is given by an affine connection, and a map $(\nabla_{\mathrm{Spin}}, e) : \tau_X^{\nabla} \to \mathbf{SpinStruc}_{\mathrm{conn}}$ is precisely a *Spin connection* and a Lorentz frame/vielbein which identifies ∇ with the corresponding Levi-Civita connection.

This example is the first in a whole tower of *higher Spin structure* fields [74–76], each of which is directly related to a corresponding higher Chern–Simons theory. The next higher example in this tower is the following.

Example: Heterotic Fields.

For $n \geq 3$, let **Heterotic** be the map

$$\mathbf{Heterotic}: \ \mathbf{B}\mathrm{Spin}(n) \xrightarrow{(p, \frac{1}{2}\mathbf{p}_1)} \mathbf{B}\mathrm{GL}(n;\mathbb{R}) \times \mathbf{B}^3 U(1)$$

regarded as an object in the slice $\mathbf{H}_{/\mathbf{B}\mathrm{GL}(n;\mathbb{R})\times\mathbf{B}^3U(1)}$. Here p is the morphism induced by

$$\mathrm{Spin}(n) \to O(n) \hookrightarrow GL(n;\mathbb{R})$$

while $\frac{1}{2}\mathbf{p}_1 : \mathbf{B}\mathrm{Spin}(n) \to \mathbf{B}^3U(1)$ is the morphism of stacks underlying the first fractional Pontrjagin class which we met in Sect. 4.1. To regard a smooth manifold X as an object in the slice $\mathbf{H}_{/\mathbf{B}\mathrm{GL}(n;\mathbb{R})\times\mathbf{B}^3U(1)}$ means to equip it with a $U(1)$-3-bundle $\mathbf{a}_X : X \to \mathbf{B}^3U(1)$ in addition to the tangent bundle $\tau_X : X \to \mathbf{B}GL(n;\mathbb{R})$. A Green–Schwarz anomaly-free background field configuration in heterotic string theory is (the differential refinement of) a map $(s_X, \phi) : (\tau_X, \mathbf{a}_X) \to \mathbf{Heterotic}$, i.e., a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{s_X} & \mathbf{B}\mathrm{Spin} \\ & {\scriptstyle (\tau_X, \mathbf{a}_X)} \searrow \ \swarrow_{\phi} \ \swarrow {\scriptstyle \mathbf{Heterotic}} & \\ & \mathbf{B}\mathrm{GL}(n) \times \mathbf{B}^3U(1) \,. & \end{array}$$

The 3-bundle $\mathbf{a}_X$ serves as a twist: when $\mathbf{a}_X$ is trivial then we are in presence of a String structure on X; so it is customary to refer to (s_X, ϕ) as to an $\mathbf{a}_X$*-twisted String structure* on X, in the sense of [76, 86]. The Green–Schwarz anomaly cancellation condition is then imposed by requiring that $\mathbf{a}_X$ (or rather its differential refinement) factors as

$$X \longrightarrow \mathbf{B}SU \xrightarrow{\mathbf{c}_2} \mathbf{B}^3U(1) \ \ ,$$

where $\mathbf{c}_2(E)$ is the morphism of stacks underlying the second Chern class. Notice that this says that the extended Lagrangians of Spin- and SU-Chern–Simons theory in 3-dimensions, as discussed above, at the same time serve as the twists that control the higher background gauge field structure in heterotic supergravity backgrounds.

Example: Dual Heterotic Fields.

Similarly, the morphism

$$\mathbf{DualHeterotic}: \ \mathbf{B}\mathrm{String}(n) \xrightarrow{(p, \frac{1}{6}\mathbf{p}_2)} \mathbf{B}\mathrm{GL}(n;\mathbb{R}) \times \mathbf{B}^7 U(1)$$

governs field configurations for the dual heterotic string. These examples, in their differentially refined version, have been discussed in [76]. The last example above is governed by the extended Lagrangian of the 7-dimensional Chern–Simons-type higher gauge field theory of String-2-connections. This has been discussed in [28].

There are many more examples of (quantum) fields modulated by objects in slices of a cohesive higher topos. To close this brief discussion, notice that the twisted String structure example has an evident analog in one lower degree: a central extension of Lie groups $A \to \hat{G} \to G$ induces a long fiber sequence

$$\underline{A} \longrightarrow \underline{\hat{G}} \longrightarrow \underline{G} \longrightarrow \mathbf{B}A \longrightarrow \mathbf{B}\hat{G} \longrightarrow \mathbf{B}G \xrightarrow{\mathbf{c}} \mathbf{B}^2A$$

in $\mathbf{H}$, where $\mathbf{c}$ is the group 2-cocycle that classifies the extension. If we regard this as a coefficient object in the slice $\mathbf{H}_{/\mathbf{B}^2 A}$, then regarding a manifold X in this slice means to equip it with an $(\mathbf{B}A)$-principal 2-bundle (an A-bundle gerbe) modulated by a map $\tau_X^A : X \to \mathbf{B}^2 A$; and a field $(\phi, \eta) : \tau_X^A \to \mathbf{c}$ is equivalently a G-principal bundle $P \to X$ equipped with an equivalence $\eta : \mathbf{c}(E) \simeq \tau_X^A$ with the 2-bundle which obstructs its lift to a $\hat{G}$-principal bundle (the "lifting gerbe"). The differential refinement of this setup similarly yields G-gauge fields equipped with such an equivalence. A concrete example for this is discussed below in Sect. 5.4.

This special case of fields in a slice is called a *twisted (differential) $\hat{G}$-structure* in [76] and a *relative field* in [37]. In more generality, the terminology *twisted (differential)* $\mathbf{c}$-*structures* is used in [76] to denote spaces of fields of the form $\mathbf{H}/_{\mathbf{S}}(\sigma_X, \mathbf{c})$ for some slice topos $\mathbf{H}/_{\mathbf{S}}$ and some coefficient object (or "twisting object") $\mathbf{c}$; see also the exposition in [80]. In fact in full generality (quantum) fields in slice toposes are equivalent to cocycles in (generalized and parameterized and possibly non-abelian and differential) *twisted cohomology*. The constructions on which the above discussion is built is given in some generality in [64].

In many examples of twisted (differential) structures/fields in slices the twist is constrained to have a certain factorization. For instance the twist of the (differential) String-structure in a heterotic background is constrained to be the (differential) second Chern-class of a (differential) $E_8 \times E_8$-cocycle, as mentioned above; or for instance the gauging of the 1d Chern–Simons fields on a knot in a 3d Chern–Simons theory bulk is constrained to be the restriction of the bulk gauge field, as discussed in Sect. 3.4.5. Another example is the twist of the Chan-Paton bundles on D-branes, discussed below in Sect. 5.4, which is constrained to be the restriction of the ambient Kalb–Ramond field to the D-brane. In all these cases the fields may be thought of as being maps in the slice topos that arise from maps in the *arrow topos* $\mathbf{H}^{\Delta^1}$. A moduli stack here is a map of moduli stacks

$$\mathbf{Fields}_{\text{bulk+def}} : \mathbf{Fields}_{\text{def}} \longrightarrow \mathbf{Fields}_{\text{bulk}}$$

in $\mathbf{H}$; and a domain on which such fields may be defined is an object $\Sigma_{\text{bulk}} \in \mathbf{H}$ equipped with a map (often, but not necessarily, an inclusion) $\Sigma_{\text{def}} \to \Sigma_{\text{bulk}}$, and a field configuration is a square of the form

$$\begin{array}{ccc} \Sigma_{\text{def}} & \xrightarrow{\phi_{\text{def}}} & \mathbf{Fields}_{\text{def}} \\ \downarrow & \swarrow_{\simeq} & \downarrow{\scriptstyle \mathbf{Fields}} \\ \Sigma_{\text{bulk}} & \xrightarrow[\phi_{\text{bulk}}]{} & \mathbf{Fields}_{\text{bulk}} \end{array}$$

in $\mathbf{H}$. If we now fix ϕ_{bulk} then $(\phi_{\text{bulk}})|_{\Sigma_{\text{def}}}$ serves as the twist, in the above sense, for ϕ_{def}. If $\mathbf{Fields}_{\text{def}}$ is trivial (the point/terminal object), then such a field is a cocycle in *relative cohomology*: a cocycle ϕ_{bulk} on Σ_{bulk} equipped with a trivialization $(\phi_{\text{bulk}})|_{\Sigma_{\text{def}}}$ of its restriction to Σ_{def}.

The fields in Chern–Simons theory with Wilson loops displayed in Sect. 3.4.5 clearly constitute an example of this phenomenon. Another example is the field content of type II string theory on a 10-dimensional spacetime X with D-brane $Q \hookrightarrow X$, for which the above diagram reads

$$
\begin{array}{ccc}
Q & \longrightarrow & \mathbf{B}\mathrm{PU}_{\mathrm{conn}} \\
\big\downarrow & \swArrow & \big\downarrow{\scriptstyle \mathbf{dd}_{\mathrm{conn}}} \\
X & \xrightarrow{\;B\;} & \mathbf{B}^2 U(1)_{\mathrm{conn}}
\end{array} ,
$$

discussed further below in Sect. 5.4. In [29] we discussed how the supergravity C-field over an 11-dimensional Hořava-Witten background with 10-dimensional boundary $X \hookrightarrow Y$ is similarly a relative cocyle, with the coefficients controlled, once more, by the extended Chern–Simons Lagrangian

$$\hat{\mathbf{c}} : \ \mathbf{B}(E_8 \times E_8)_{\mathrm{conn}} \longrightarrow \mathbf{B}^3 U(1)_{\mathrm{conn}} \ ,$$

now regarded in $\mathbf{H}^{(\Delta^1)}$.

5.3 Differential Moduli Stacks

In the exposition in Sects. 2 and 3 above we referred, for ease of discussion, to the mapping stacks of the form $\mathbf{Maps}(\Sigma_k, \mathbf{B}G_{\mathrm{conn}})$ as moduli stacks of G-gauge fields on Σ_k. From a more refined perspective this is not quite true. While certainly the global points of these mapping stacks are equivalently the G-gauge field configurations on Σ_k, for U a parameter space, the U-parameterized collections in the mapping stack are not quite those of the intended moduli stack: for the former these are gauge fields and gauge transformations on $U \times \Sigma_k$, while for the latter these are genuine cohesively U-parameterized collections of gauge fields on Σ_k.

In the exposition above we saw this difference briefly in Sect. 3.4.3, where we constrained a 1-form $A \in \Omega^1(U \times \Sigma, \mathfrak{g})$ (a U-plot of the mapping stack) to vanish on vector fields tangent to U; this makes it a smooth function on U with values in connections on Σ. More precisely, for G a Lie group and Σ a smooth manifold, let

$$G\mathbf{Conn}(\Sigma) \in \mathbf{H}$$

be the stack which assigns to any $U \in \mathrm{CartSp}$ the groupoid of smoothly U-parameterized collections of smooth G-principal connections on Σ, and of smoothly U-parameterized collections of smooth gauge transformations between these connections. This is the actual moduli stack of G-connections. In this form, but over a different site of definition, it appears for instance in geometric Langlands duality. In physics this stack is best known in the guise of its infinitesimal approximation: the corresponding Lie algebroid is dually the (off-shell) *BRST-complex* of the gauge theory, and the BRST ghosts are the cotangents to the morphisms in $G\mathbf{Conn}(\Sigma)$ at the identity.

Notice that while the mapping stack is itself not quite the right answer, there is a canonical map that comes to the rescue

$$\mathbf{Maps}(\Sigma, \mathbf{B}G_{\mathrm{conn}}) \longrightarrow G\mathbf{Conn}(\Sigma) \ .$$

We call this the *concretification* map. We secretly already saw an example of this in Sect. 3.4.2, where this was the map $\mathbf{Maps}(S^1, \mathbf{B}G_{\mathrm{conn}}) \longrightarrow \underline{G}//_{\mathrm{Ad}}\underline{G}$.

In more complicated examples, such as for higher groups G and base spaces Σ which are not plain manifolds, it is in general less evident what $G\mathbf{Conn}(\Sigma)$ should be. But if the ambient higher topos is cohesive, then there is a general abstract procedure that produces the differential moduli stack. This is discussed in Sects. 3.9.6.4 and 4.4.15.3 of [79] and in [65].

5.4 Prequantum Geometry in Higher Codimension

We had indicated in Sect. 3.4 how a single extended Lagrangian, given by a map of universal higher moduli stacks $\mathbf{L} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$, induces, by transgression, circle $(n-k)$-bundles with connection

$$\mathrm{hol}_{\Sigma_k}\mathbf{Maps}(\Sigma_k, \mathbf{L}) \ : \ \mathbf{Maps}(\Sigma_k, \mathbf{B}G_{\mathrm{conn}}) \longrightarrow \mathbf{B}^{n-k}U(1)_{\mathrm{conn}}$$

on moduli stacks of field configurations over each closed k-manifold Σ_k. In codimension 1, hence for $k = n-1$, this reproduces the ordinary *prequantum circle bundle* of the n-dimensional Chern–Simons type theory, as discussed in Sect. 3.4.3. The space of sections of the associated line bundle is the space of *prequantum states* of the theory. This becomes the space of genuine quantum states after choosing a *polarization* (i.e., a decomposition of the moduli space of fields into *canonical coordinates* and *canonical momenta*) and restricting to polarized sections (i.e., those depending only on the canonical coordinates). But moreover, for each Σ_k we may regard $\mathrm{hol}_{\Sigma_k}\mathbf{Maps}(\Sigma_k, \mathbf{L})$ as a *higher prequantum bundle* of the theory in higher codimension hence consider its prequantum geometry in higher codimension.

We discuss now some generalities of such a higher geometric prequantum theory and then show how this perspective sheds a useful light on the gauge coupling of the open string, as part of the transgression of prequantum 2-states of Chern–Simons theory in codimension 2 to prequantum states in codimension 1.

5.4.1 Higher Prequantum States and Prequantum Operators

We indicate here the basic concepts of higher extended prequantum theory and how they reproduce traditional prequantum theory.[13]

Consider a (pre)-n-plectic form, given by a map

$$\omega : X \longrightarrow \Omega^{n+1}(-; \mathbb{R})_{\mathrm{cl}}$$

[13] A discussion of this and the following can be found in Sects. 3.9.13 and 4.4.19 of [79]; see also [27].

in $\mathbf{H}$. A *n-plectomorphism* of (X, ω) is an auto-equivalence of ω regarded as an object in the slice $\mathbf{H}_{/\Omega^{n+1}_{\mathrm{cl}}}$, hence a diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & X \\ & {\scriptstyle\omega}\searrow \quad \swarrow{\scriptstyle\omega} & \\ & \Omega^{n+1}(-;\mathbb{R})_{\mathrm{cl}} & \end{array} .$$

A *prequantization* of (X, ω) is a choice of prequantum line bundle, hence a choice of lift ∇ in

$$\begin{array}{ccc} & & \mathbf{B}^n U(1)_{\mathrm{conn}} \\ & {\scriptstyle\nabla}\nearrow & \downarrow{\scriptstyle F_{(-)}} \\ X & \xrightarrow{\omega} & \Omega^{n+1}_{\mathrm{cl}} \end{array} ,$$

modulating a circle n-bundle with connection on X. We write $\mathbf{c}(\nabla) : X \xrightarrow{\nabla} \mathbf{B}^n U(1)_{\mathrm{conn}} \to \mathbf{B}^n U(1)$ for the underlying $(\mathbf{B}^{n-1}U(1))$-principal n-bundle. An autoequivalence

$$\hat{O} : \nabla \xrightarrow{\simeq} \nabla$$

of the prequantum n-bundle regarded as an object in the slice $\mathbf{H}_{/\mathbf{B}^n U(1)_{\mathrm{conn}}}$, hence a diagram in $\mathbf{H}$ of the form

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & X \\ & {\scriptstyle\nabla}\searrow \;\; \swArrow_{O} \;\; \swarrow{\scriptstyle\nabla} & \\ & \mathbf{B}^n U(1)_{\mathrm{conn}} & \end{array}$$

is an (exponentiated) *prequantum operator* or *quantomorphism* or *regular contact transformation* of the prequantum geometry (X, ∇). These form an ∞-group in $\mathbf{H}$. The L_∞-algebra of this *quantomorphism ∞-group* is the higher *Poisson bracket* Lie algebra of the system. If X is equipped with group structure then the quantomorphisms covering the action of X on itself form the *Heisenberg ∞-group*. The homotopy labeled O in the above diagram is the *Hamiltonian* of the prequantum operator. The image of the quantomorphisms in the symplectomorphisms (given by composition the above diagram with the curvature morphism $F_{(-)} : \mathbf{B}^n U(1)_{\mathrm{conn}} \to \Omega^{n+1}_{\mathrm{cl}}$) is the group of *Hamiltonian n-plectomorphisms*. A lift of an ∞-group action $G \to \mathbf{Aut}(X)$ on X from automorphisms of X (diffeomorphism) to quantomorphisms is a *Hamiltonian action*, infinitesimally (and dually) a *momentum map*.

To define higher prequantum states we fix a representation (V, ρ) of the circle n-group $\mathbf{B}^{n-1}U(1)$. By the general results in [64] this is equivalent to fixing a homotopy fiber sequence of the form

$$\begin{array}{ccc} \underline{V} & \longrightarrow & \underline{V}/\!/\mathbf{B}^{n-1}U(1) \\ & & \downarrow{\scriptstyle\rho} \\ & & \mathbf{B}^n U(1) \end{array}$$

in $\mathbf{H}$. The vertical morphism here is the *universal ρ-associated V-fiber ∞-bundle* and characterizes ρ itself. Given such, a section of the V-fiber bundle which is ρ-associated to $\mathbf{c}(\nabla)$ is equivalently a map

$$\Psi : \mathbf{c}(\nabla) \longrightarrow \rho$$

in the slice $\mathbf{H}_{/\mathbf{B}^n U(1)}$. This is a higher *prequantum state* of the prequantum geometry (X, ∇). Since every prequantum operator $\hat{O}$ as above in particular is an auto-equivalence of the underlying prequantum bundle $\hat{O} : \mathbf{c}(\nabla) \xrightarrow{\simeq} \mathbf{c}(\nabla)$ it canonically acts on prequantum states given by maps as above simply by precomposition

$$\Psi \mapsto \hat{O} \circ \Psi.$$

Notice also that from the perspective of Sect. 5.2 all this has an equivalent interpretation in terms of twisted cohomology: a preqantum state is a cocycle in twisted V-cohomology, with the twist being the prequantum bundle. And a prequantum operator/quantomorphism is equivalently a twist automorphism (or "generalized local diffeomorphism").

For instance if $n = 1$ then ω is an ordinary (pre)symplectic form and ∇ is the connection on a circle bundle. In this case the above notions of prequantum operators, quantomorphism group, Heisenberg group and Poisson bracket Lie algebra reproduce exactly all the traditional notions if X is a smooth manifold, and generalize them to the case that X is for instance an orbifold or even itself a higher moduli stack, as we have seen. The canonical representation of the circle group $U(1)$ on the complex numbers yields a homotopy fiber sequence

$$\begin{array}{ccc} \underline{\mathbb{C}} & \longrightarrow & \underline{\mathbb{C}}/\!/\underline{U}(1) \\ & & \downarrow{\scriptstyle \rho} \\ & & \mathbf{B}U(1) \end{array},$$

where $\underline{\mathbb{C}}/\!/\underline{U}(1)$ is the stack corresponding to the ordinary action groupoid of the action of $U(1)$ on $\mathbb{C}$, and where the vertical map is the canonical functor forgetting the data of the local $\mathbb{C}$-valued functions. This is the *universal complex line bundle* associated to the universal $U(1)$-principal bundle. One readily checks that a prequantum state $\Psi : \mathbf{c}(\nabla) \to \rho$, hence a diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & \underline{\mathbb{C}}/\!/\underline{U}(1) \\ & {\scriptstyle \mathbf{c}(\nabla)}\searrow \quad \swarrow{\scriptstyle \rho} & \\ & \mathbf{B}U(1) & \end{array}$$

in $\mathbf{H}$ is indeed equivalently a section of the complex line bundle canonically associated to $\mathbf{c}(\nabla)$ and that under this equivalence the pasting composite

$$\begin{array}{ccccc} X & \xrightarrow{\simeq} & X & \longrightarrow & \underline{\mathbb{C}}/\!/\underline{U}(1) \\ & {\scriptstyle \mathbf{c}(\nabla)}\searrow \; \overset{O}{\swarrow} & \downarrow{\scriptstyle \mathbf{c}(\nabla)} & \swarrow{\scriptstyle \rho} & \\ & & \mathbf{B}U(1) & & \end{array}$$

is the result of the traditional formula for the action of the prequantum operator $\hat{O}$ on Ψ.

Instead of forgetting the connection on the prequantum bundle in the above composite, one can equivalently equip the prequantum state with a differential refinement, namely with its *covariant derivative* and then exhibit the prequantum operator action directly. Explicitly, let $\mathbb{C}//U(1)_{\mathrm{conn}}$ denote the quotient stack $(\underline{\mathbb{C}} \times \Omega^1(-,\mathbb{R}))//\underline{U}(1)$, with $U(1)$ acting diagonally. This sits in a homotopy fiber sequence

$$\begin{array}{ccc} \underline{\mathbb{C}} & \longrightarrow & \underline{\mathbb{C}}//\underline{U}(1)_{\mathrm{conn}} \\ & & \downarrow{\scriptstyle \rho_{\mathrm{conn}}} \\ & & \mathbf{B}U(1)_{\mathrm{conn}} \end{array}$$

which may be thought of as the differential refinement of the above fiber sequence $\underline{\mathbb{C}} \to \underline{\mathbb{C}}//\underline{U}(1) \to \mathbf{B}U(1)$. (Compare this to Sect. 3.4.5, where we had similarly seen the differential refinement of the fiber sequence $\underline{G}/\underline{T}_\lambda \to \mathbf{B}T_\lambda \to \mathbf{B}G$, which analogously characterizes the canonical action of G on the coset space G/T_λ.) Prequantum states are now equivalently maps

$$\widehat{\boldsymbol{\Psi}} : \nabla \longrightarrow \rho_{\mathrm{conn}}$$

in $\mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}}$. This formulation realizes a section of an associated line bundle equivalently as a connection on what is sometimes called a groupoid bundle. As such, $\widehat{\boldsymbol{\Psi}}$ has not just a 2-form curvature (which is that of the prequantum bundle) but also a 1-form curvature: this is the covariant derivative $\nabla\sigma$ of the section.

Such a relation between sections of higher associated bundles and higher covariant derivatives holds more generally. In the next degree for $n = 2$ one finds that the quantomorphism 2-group is the Lie 2-group which integrates the *Poisson bracket Lie 2-algebra* of the underlying 2-plectic geometry as introduced in [67]. In the next section we look at an example for $n = 2$ in more detail and show how it interplays with the above example under transgression.

The above higher prequantum theory becomes a genuine quantum theory after a suitable higher analog of a choice of *polarization*. In particular, for $\mathbf{L} : X \to \mathbf{B}^n U(1)_{\mathrm{conn}}$ an extended Lagrangian of an n-dimensional quantum field theory as discussed in all our examples here, and for Σ_k any closed manifold, the polarized prequantum states of the transgressed prequantum bundle $\mathrm{hol}_{\Sigma_k}\mathbf{Maps}(\Sigma_k, \mathbf{L})$ should form the $(n-k)$-vector spaces of higher quantum states in codimension k. These states would be assigned to Σ_k by the *extended quantum field theory*, in the sense of [60], obtained from the extended Lagrangian $\mathbf{L}$ by extended geometric quantization. There is an equivalent reformulation of this last step for $n = 1$ given simply by the push-forward of the prequantum line bundle in K-theory (see Sect. 6.8 of [43]) and so one would expect that accordingly the last step of higher geometric quantization involves similarly a push-forward of the associated V-fiber ∞-bundles above in some higher generalized cohomology theory. But this remains to be investigated.

5.4.2 Example: The Anomaly-Free Gauge Coupling of the Open String

As an example of these general phenomena, we close by briefly indicating how the higher prequantum states of 3d Chern–Simons theory in codimension 2 reproduce the *twisted Chan-Paton gauge bundles* of open string backgrounds, and how their transgression to codimension 1 reproduces the cancellation of the Freed-Witten-Kapustin anomaly of the open string.

By the above, the Wess-Zumino-Witten gerbe $\mathbf{wzw} : G \to \mathbf{B}^2 U(1)_{\mathrm{conn}}$ as discussed in Sect. 3.4.2 may be regarded as the *prequantum 2-bundle* of Chern–Simons theory in codimension 2 over the circle. Equivalently, if we consider the WZW σ-model for the string on G and take the limiting TQFT case obtained by sending the kinetic term to 0 while keeping only the gauge coupling term in the action, then it is the extended Lagrangian of the string σ-model: its transgression to the mapping space out of a *closed* worldvolume Σ_2 of the string is the topological piece of the exponentiated WZW σ-model action. For Σ_2 with boundary the situation is more interesting, and this we discuss now.

The *Heisenberg 2-group* of the prequantum geometry $(G, \mathbf{wzw})$ is[14] the *String 2-group* (see the appendix of [28] for a review), the smooth 2-group $\mathrm{String}(G)$ which is, up to equivalence, the loop space object of the homotopy fiber of the smooth universal class $\mathbf{c}$

$$\mathbf{B}\mathrm{String}(G) \longrightarrow \mathbf{B}G \xrightarrow{\mathbf{c}} \mathbf{B}^3 U(1).$$

The canonical representation of the 2-group $BU(1)$ is on the complex K-theory spectrum, whose smooth (stacky) refinement is given by $\mathbf{B}U := \varinjlim_n \mathbf{B}U(n)$ in $\mathbf{H}$ (see Sect. 5.4.3 of [79] for more details). On any component for fixed n the action of the smooth 2-group $\mathbf{B}U(1)$ is exhibited by the long homotopy fiber sequence

$$U(1) \longrightarrow U(n) \to \mathrm{PU}(n) \longrightarrow \mathbf{B}U(1) \longrightarrow \mathbf{B}U(n) \longrightarrow \mathbf{B}\mathrm{PU}(n) \xrightarrow{\mathbf{dd}_n} \mathbf{B}^2 U(1)$$

in $\mathbf{H}$, in that $\mathbf{dd}_n$ is the universal $(\mathbf{B}U(n))$-fiber 2-bundle which is associated by this action to the universal $(\mathbf{B}U(1))$-2-bundle.[15] Using the general higher representation theory in $\mathbf{H}$ as developed in [64], a local section of the $(\mathbf{B}U(n))$-fiber prequantum 2-bundle which is $\mathbf{dd}_n$-associated to the prequantum 2-bundle $\mathbf{wzw}$, hence a local prequantum 2-state, is, equivalently, a map

$$\Psi : \mathbf{wzw}|_Q \longrightarrow \mathbf{dd}_n$$

in the slice $\mathbf{H}_{/\mathbf{B}^2 U(1)}$, where $\iota_Q : Q \hookrightarrow G$ is some subspace. Equivalently (compare with the general discussion in Sect. 5.2), this is a map

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[14] This follows for instance as the Lie integration of the result in [5] that the Heisenberg Lie 2-algebra here is the $\mathfrak{string}(\mathfrak{g})$ Lie 2-algebra; see also [27].

[15] The notion of $(\mathbf{B}U(n))$-fiber 2-bundle is equivalently that of nonabelian $U(n)$-*gerbes* in the original sense of Giraud, see [64]. Notice that for $n = 1$ this is more general than then notion of $U(1)$-bundle gerbe: a G-gerbe has structure 2-group $\mathbf{Aut}(\mathbf{B}G)$, but a $U(1)$-bundle gerbe has structure 2-group only in the left inclusion of the fiber sequence $\mathbf{B}U(1) \hookrightarrow \mathbf{Aut}(\mathbf{B}U(1)) \to \mathbb{Z}_2$.

$$(\Psi, \mathbf{wzw}) : \iota_Q \longrightarrow \mathbf{dd}_n$$

in $\mathbf{H}^{(\Delta^1)}$, hence a diagram in $\mathbf{H}$ of the form

$$\begin{array}{ccc} Q & \xrightarrow{\Psi} & \mathbf{B}\mathrm{PU}(n) \\ {\scriptstyle \iota_Q}\big\downarrow & \swarrow & \big\downarrow{\scriptstyle \mathbf{dd}_n} \\ G & \xrightarrow[\mathbf{wzw}]{} & \mathbf{B}^2 U(1) \,. \end{array}$$

One finds (Sect. 5.4.3 of [79]) that this equivalently modulates a unitary bundle on Q which is *twisted* by the restriction of **wzw** to Q as in twisted K-theory (such a twisted bundle is also called a *gerbe module* if **wzw** is thought of in terms of bundle gerbes [7]). So

$$\mathbf{dd}_n \in \mathbf{H}_{/\mathbf{B}^2 U(1)}$$

is the moduli stack for twisted rank-n unitary bundles. As with the other moduli stacks before, one finds a differential refinement of this moduli stack, which we write

$$(\mathbf{dd}_n)_{\mathrm{conn}} : (\mathbf{B}U(n)//\mathbf{B}U(1))_{\mathrm{conn}} \to \mathbf{B}^2 U(1)_{\mathrm{conn}},$$

and which modulates twisted unitary bundles with twisted connections (bundle gerbe modules with connection). Hence a differentially refined state is a map $\widehat{\Psi} : \mathbf{wzw}|_Q \to (\mathbf{dd}_n)_{\mathrm{conn}}$ in $\mathbf{H}_{/\mathbf{B}^2 U(1)_{\mathrm{conn}}}$; and this is precisely a twisted gauge field on a D-brane Q on which open strings in G may end. Hence these are the *prequantum 2-states* of Chern–Simons theory in codimension 2. Precursors of this perspective of Chan-Paton bundles over D-branes as extended prequantum 2-states can be found in [68, 77].

Notice that by the above discussion, together the discussion in Sect. 5.2, an equivalence

$$\hat{O} : \mathbf{wzw} \xrightarrow{\simeq} \mathbf{wzw}$$

in $\mathbf{H}_{/\mathbf{B}^2 U(1)_{\mathrm{conn}}}$ has two different, but equivalent, important interpretations:

1. it is an element of the *quantomorphism 2-group* (i.e. the possibly non-linear generalization of the Heisenberg 2-group) of 2-prequantum operators;
2. it is a twist automorphism analogous to the generalized diffeomorphisms for the fields in gravity.

Moreover, such a transformation is locally a structure well familiar from the literature on D-branes: it is locally (on some cover) given by a transformation of the B-field of the form $B \mapsto B + d_{\mathrm{dR}} a$ for a local 1-form a (this is the *Hamiltonian 1-form* in the interpretation of this transformation in higher prequantum geometry) and its prequantum operator action on prequantum 2-states, hence on Chan-Paton gauge fields

$$\widehat{\Psi} : \mathbf{wzw} \longrightarrow (\mathbf{dd}_n)$$

(by precomposition) is given by shifting the connection on a twisted Chan-Paton bundle (locally) by this local 1-form a. This local gauge transformation data

$$B \mapsto B + da, \quad A \mapsto A + a,$$

is familiar from string theory and D-brane gauge theory (see e.g. [66]). The 2-prequantum operator action $\Psi \mapsto \hat{O}\Psi$ which we see here is the fully globalized refinement of this transformation.

Surface Transport and the Twisted Bundle Part of Freed-Witten-Kapustin Anomalies.

The map $\widehat{\Psi} : (\iota_Q, \mathbf{wzw}) \to (\mathbf{dd}_n)_{\mathrm{conn}}$ above is the gauge-coupling part of the extended Lagrangian of the *open* string on G in the presence of a D-brane $Q \hookrightarrow G$. We indicate what this means and how it works. Note that for all of the following the target space G and background gauge field **wzw** could be replaced by any target space with any circle 2-bundle with connection on it.

The object ι_Q in $\mathbf{H}^{(\Delta^1)}$ is the target space for the open string. The worldvolume of that string is a smooth compact manifold Σ with boundary inclusion $\iota_{\partial\Sigma} : \partial\Sigma \to \Sigma$, also regarded as an object in $\mathbf{H}^{(\Delta^1)}$. A field configuration of the string σ-model is then a map

$$\phi : \iota_\Sigma \to \iota_Q$$

in $\mathbf{H}^{(\Delta^1)}$, hence a diagram

$$\begin{array}{ccc} \partial\Sigma & \longrightarrow & Q \\ {\scriptstyle \iota_{\partial\Sigma}} \big\downarrow & & \big\downarrow {\scriptstyle \iota_Q} \\ \Sigma & \xrightarrow{\ \phi\ } & G \end{array}$$

in $\mathbf{H}$, hence a smooth function $\phi : \Sigma \to G$ subject to the constraint that the boundary of Σ lands on the D-brane Q. Postcomposition with the background gauge field $\widehat{\Psi}$ yields the diagram

$$\begin{array}{ccccc} \partial\Sigma & \longrightarrow & Q & \xrightarrow{\ \widehat{\Psi}\ } & (\mathbf{B}U(n)/\!/U(1))_{\mathrm{conn}} \\ {\scriptstyle \iota_{\partial\Sigma}} \big\downarrow & & \big\downarrow {\scriptstyle \iota_Q} & & \\ \Sigma & \xrightarrow{\ \phi\ } & G & \xrightarrow[\ \mathbf{wzw}\ ]{} & \mathbf{B}^2U(1)_{\mathrm{conn}}\ . \end{array}$$

Comparison with the situation of Chern–Simons theory with Wilson lines in Sect. 3.4.5 shows that the total action functional for the open string should be the product of the fiber integration of the top composite morphism with that of the bottom composite morphisms. Hence that functional is the product of the surface parallel transport of the **wzw** B-field over Σ with the line holonomy of the twisted Chan-Paton bundle over $\partial\Sigma$.

This is indeed again true, but for more subtle reasons this time, since the fiber integrations here are *twisted*. For the surface parallel transport we mentioned this already at the end of Sect. 5.1: since Σ has a boundary, parallel transport over Σ

does not yield a function on the mapping space out of Σ, but rather a section of the line bundle on the mapping space out of $\partial\Sigma$, pulled back to this larger mapping space.

Furthermore, the connection on a twisted unitary bundle does not quite have a well-defined traced holonomy in $\mathbb{C}$, but rather a well defined traced holonomy up to a coherent twist. More precisely, the transgression of the WZW 2-connection to maps out of the circle as in Sect. 3.4 fits into a diagram of moduli stacks in $\mathbf{H}$ of the form

$$\begin{array}{ccc} \mathbf{Maps}(S^1, (\mathbf{B}U(n)/\!/\mathbf{B}U(1))_{\mathrm{conn}}) & \xrightarrow{\mathrm{tr\,hol}_{S^1}} & \underline{\mathbb{C}}/\!/\underline{U}(1)_{\mathrm{conn}} \\ \big\downarrow{\scriptstyle \mathbf{Maps}(S^1,(\mathbf{dd}_n)_{\mathrm{conn}})} & & \big\downarrow \\ \mathbf{Maps}(S^1,\mathbf{B}^2U(1)_{\mathrm{conn}}) & \xrightarrow{\mathrm{hol}_{S^1}} & \mathbf{B}U(1)_{\mathrm{conn}}\,. \end{array}$$

This is a transgression-compatibility of the form that we have already seen in Sect. 3.4.2.

In summary, we obtain the transgression of the extended Lagrangian of the open string in the background of B-field and Chan-Paton bundles as the following pasting diagram of moduli stacks in $\mathbf{H}$ (all squares are filled with homotopy 2-cells, which are notationally suppressed for readability)

$$\begin{array}{ccccc} \mathbf{Fields}_{\mathrm{OpenString}}(\iota_{\partial\Sigma}) & \longrightarrow & \mathbf{Maps}(\Sigma,G) & \xrightarrow{\exp(2\pi i\int_\Sigma[\Sigma,\mathbf{wzw}])} & \underline{\mathbb{C}}/\!/\underline{U}(1)_{\mathrm{conn}} \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathbf{Maps}(\iota_{\partial\Sigma},G)} & & \big\downarrow \\ \mathbf{Maps}(S^1,Q) & \xrightarrow{\mathbf{Maps}(S^1,\iota_Q)} & \mathbf{Maps}(S^1,G) & & \\ \big\downarrow{\scriptstyle \mathbf{Maps}(S^1,\widehat{\Psi})} & & & \searrow{\scriptstyle \mathbf{Maps}(S^1,\mathbf{wzw})} & \\ \mathbf{Maps}(S^1,(\mathbf{B}U(n)/\!/\mathbf{B}U(1))_{\mathrm{conn}}) & \xrightarrow{\mathbf{Maps}(S^1,(\mathbf{dd}_n)_{\mathrm{conn}})} & & & \mathbf{Maps}(S^1,\mathbf{B}^2U(1)_{\mathrm{conn}}) \\ \big\downarrow{\scriptstyle \mathrm{tr\,hol}_{S^1}} & & & & \searrow{\scriptstyle \mathrm{hol}_{S^1}} \\ \underline{\mathbb{C}}/\!/\underline{U}(1)_{\mathrm{conn}} & \longrightarrow & & & \mathbf{B}U(1)_{\mathrm{conn}} \end{array}$$

Here

- the top left square is the homotopy pullback square that computes the mapping stack $\mathbf{Maps}(\iota_{\partial\Sigma},\iota_Q)$ in $\mathbf{H}^{(\Delta^1)}$, which here is simply the smooth space of string configurations $\Sigma \to G$ which are such that the string boundary lands on the D-brane Q;
- the top right square is the twisted fiber integration of the **wzw** background 2-bundle with connection: this exhibits the parallel transport of the 2-form connection over the worldvolume Σ with boundary S^1 as a section of the pullback of the transgression line bundle on loop space to the space of maps out of Σ;
- the bottom square is the above compatibility between the twisted traced holonomy of twisted unitary bundles and the transgression of their twisting 2-bundles.

The total diagram obtained this way exhibits a difference between two section of a single complex line bundle on $\mathbf{Fields}_{\mathrm{OpenString}}(\iota_{\partial\Sigma})$ (at least one of them non-vanishing), hence a map

$$\exp\left(2\pi i \int_{\Sigma} [\Sigma, \mathbf{wzw}]\right) \cdot \operatorname{tr}\operatorname{hol}_{S^1}([S^1, \widehat{\Psi}]) : \mathbf{Fields}_{\mathrm{OpenString}}(\iota_{\partial\Sigma}) \longrightarrow \underline{\mathbb{C}}.$$

This is the well-defined action functional of the open string with endpoints on the D-brane $Q \hookrightarrow G$, charged under the background **wzw** B-field and under the twisted Chan-Paton gauge bundle $\widehat{\Psi}$.

Unwinding the definitions, one finds that this phenomenon is precisely the twisted-bundle-part, due to Kapustin [52], of the Freed-Witten anomaly cancellation for open strings on D-branes, hence is the Freed-Witten-Kapustin anomaly cancellation mechanism either for the open bosonic string or else for the open type II superstring on Spin^c-branes. Notice how in the traditional discussion the existence of twisted bundles on the D-brane is identified just as *some* construction that happens to cancel the B-field anomaly. Here, in the perspective of extended quantization, we see that this choice follows uniquely from the general theory of extended prequantization, once we recognize that $\mathbf{dd}_n$ above is (the universal associated 2-bundle induced by) the canonical representation of the circle 2-group $\mathbf{B}U(1)$, just as in one codimension up $\mathbb{C}$ is the canonical representation of the circle 1-group $U(1)$.

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