Snapshots of Conformal Field Theory

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Abstract In snapshots, this exposition introduces conformal field theory, with a focus on those perspectives that are relevant for interpreting superconformal field theory by Calabi-Yau geometry. It includes a detailed discussion of the elliptic genus as an invariant which certain superconformal field theories share with the Calabi-Yau manifolds. K3 theories are (re)viewed as prime examples of superconformal field theories where geometric interpretations are known. A final snapshot addresses the K3-related Mathieu Moonshine phenomena, where a lead role is predicted for the chiral de Rham complex.

1 Introduction

Conformal quantum field theory (CFT) became popular in physics thanks to the work by Belavin, Polyakov and Zamolodchikov. In their seminal paper [7], on the one hand, they lay the mathematical foundations of axiomatic CFT, and on the other hand, they show the physical significance of CFT for surface phenomena in statistical physics by describing certain phase transitions of second order through CFT.

Another common source of conformal field theories is string theory, which is many theoreticians' favorite candidate for the unification of all interactions, including gravity. Here, particles are described by strings that move in some potentially complicated background geometry. The string dynamics are governed by a so-called non-linear sigma model, such that conformal invariance yields the string equations of motion. The quantum field theory living on the worldsheet of the string then is a CFT. This implies deep relations between CFT and geometry, which have already led to a number of intriguing insights in geometry, demanding for a more resilient bridge between mathematics and physics.

For example, in the early 90s mirror symmetry provided a first success story for the interaction between mathematics and physics in the context of CFT

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[14, 15, 57, 68]. However, a rigorous approach to those types of CFTs that are relevant for such deep insights in algebraic geometry was not available, at the time. As a result, the interaction between mathematics and physics in many cases amounted to a rather imbalanced division of work, where theoretical physicists provided the most amazing predictions and left them to the mathematicians for a proof, who in turn successfully detached their theories from their origins in physics.

With the advent of Monstrous Moonshine [9, 20, 42, 49, 88, 89], and with Borcherds' Fields Medal in 1998 "for his contributions to algebra, the theory of automorphic forms, and mathematical physics, including the introduction of vertex algebras and Borcherds' Lie algebras, the proof of the Conway-Norton moonshine conjecture and the discovery of a new class of automorphic infinite products" [71], the subject of conformal field theory, per se, began to become more popular in mathematics. Indeed, the comparatively new notion of vertex algebras provided a rigorous mathematical foundation to the most basic ingredients of conformal quantum field theory and thereby offered a viable approach to CFT for mathematicians. Nevertheless, the quest to fill the gap between abstract mathematical approaches to CFT and those types of models that are of interest in physics, and that are relevant for deeper insights in algebraic and enumerative geometry, has not yet been completed. The present work attempts to make a contribution to this quest.

Since this exposition can certainly only provide some snapshots of CFT, it has to follow a subjective selection and presentation of material. The guiding principle is the conviction that on the one hand, the foundation of the discussion has to be a mathematically rigorous definition of CFT, which is independent of string theory, while on the other hand, those predictions from CFT which affect the geometry of Calabi-Yau manifolds are among the most intriguing ones. To state and understand the latter, one needs to work with a mathematical formulation of CFT which allows to make contact with the non-linear sigma models in physics, thus sadly excluding a number of popular approaches to CFT. Moreover, the discussion is restricted to so-called two-dimensional Euclidean unitary CFTs.
In more detail, this work is structured as follows:

Section 2 provides a definition of some of the ingredients of CFT. The conformal vertex algebras serve as our point of entry in Sect. 2.1, since this part of CFT is probably the most natural for mathematicians. We proceed in Sect. 2.2 by listing the crucial ingredients that underlie a definition of superconformal field theory, along with additional required properties. The presentation makes no claim for completeness, but according to our declared conviction, we focus on those aspects that are relevant for the discussion of geometric interpretations as introduced later. This in particular restricts our attention to the so-called $N = (2, 2)$ superconformal field theories with space-time supersymmetry. A useful class of examples, which is well understood, is given by the toroidal $N = (2, 2)$ superconformal field theories presented in Sect. 2.3. We summarize the definition and properties of the chiral de Rham complex in Sect. 2.5, as an example of a sheaf of conformal vertex algebras on an arbitrary smooth algebraic variety, which thus provides a link between standard ingredients of CFT and geometric quantities. Since this link is not entirely understood to the very day, for clarification, our discussion rests on the special role of the elliptic genus

as an invariant that certain superconformal field theories share with the Calabi-Yau manifolds, as is discussed in some detail in Sect. 2.4.

The elliptic genus is also crucial for our definition of K3 theories in Sect. 3. This class of CFTs deserves some attention, as it provides the only examples of non-linear sigma models on Calabi-Yau manifolds other than tori, where at least there are precise predictions on the global form of the moduli space, implying some very explicit relations between quantities in geometry and CFT. We motivate the definition of K3 theories in detail, and we summarize some of the known properties of these theories. In particular, Proposition 2 recalls the dichotomy of $N = (2, 2)$ superconformal field theories at central charges $c = 6$, $\bar{c} = 6$ with space-time supersymmetry and integral $U(1)$-charges. Indeed, these theories fall into two classes, namely the toroidal and the K3 theories. Thus Proposition 2 is the conformal field theoretic counter part of the classification of Calabi-Yau 2-manifolds into complex two-tori, on the one hand, and K3 surfaces, on the other. Our proof [92, Sect. 7.1], which is little known, is summarized in the Appendix.

The final Sect. 4 is devoted to recent developments in the study of K3 theories, related to the mysterious phenomena known as Mathieu Moonshine. We recall the route to discovery of these phenomena, which also proceeds via the elliptic genus. We offer some ideas towards a geometric interpretation, arguing that one should expect the chiral de Rham complex to be crucial in unraveling the Mathieu Moonshine mysteries. The section closes with an open conjecture, which is related to Mathieu Moonshine, which however is formulated neither alluding to moonshine nor to CFT, and which therefore is hoped to be of independent interest.

2 Ingredients of Conformal Field Theory

The present section collects ingredients of conformal field theory (CFT), more precisely of *two-dimensional Euclidean unitary conformal field theory*. These adjectives translate into the properties of the underlying quantum field theory as follows: First, all fields are parametrized on a *two-dimensional worldsheet*, which comes equipped with a *Euclidean metric*. Second, the fields transform covariantly under *conformal maps* between such worldsheets. Furthermore, the space of states in such a CFT is equipped with a positive definite metric, with respect to which the infinitesimal conformal transformations act *unitarily*.

We begin by describing the simplest fields in our CFTs in terms of the so-called *vertex algebras* in Sect. 2.1. Next, Sect. 2.2 summarizes a definition of conformal field theory, with the toroidal conformal field theories presented as a class of examples in Sect. 2.3. In Sects. 2.4 and 2.5 the related notions of the elliptic genus and the chiral de Rham complex are discussed in the context of superconformal field theories. As such, the present section collects ingredients of CFT, with a focus on those ingredients that are under investigation to the very day.

2.1 Conformal and Superconformal Vertex Algebras

We begin by recalling the notion of *fields*, following [61]. The theory is built on the earlier results [9, 41, 67], see also [38] for a very readable exposition. This definition is most convenient, because it naturally implements the representation theory inherent to CFTs. As we shall see at the end of this section, for the chiral states of CFTs it also allows a straightforward definition of the n-point functions.

Definition 1 Consider a $\mathbb{C}$-vector space $\mathbb{H}$.

- $\mathbb{H}[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]]$ denotes the vector space of formal power series

$$v(z_1, \ldots, z_n) = \sum_{i_1, \ldots, i_n \in \mathbb{Z}} \widehat{v}_{i_1, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n}, \quad \widehat{v}_{i_1, \ldots, i_n} \in \mathbb{H}.$$

- For $A \in \text{End}_{\mathbb{C}}(\mathbb{H})[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]]$, and for $\alpha \in \mathbb{H}^* := \text{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{C})$ and $v \in \mathbb{H}$, we set

$$\langle \alpha, A(z_1, \ldots, z_n) v \rangle := \sum_{i_1, \ldots, i_n \in \mathbb{Z}} \langle \alpha, \widehat{A}_{i_1, \ldots, i_n} v \rangle z_1^{i_1} \cdots z_n^{i_n} \quad \in \mathbb{C}[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]],$$

  where on the right hand side, $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathbb{H}^*$ and $\mathbb{H}$.
- If $A(z) \in \text{End}_{\mathbb{C}}(\mathbb{H})[[z^{\pm 1}]]$ with $A(z) = \sum_n \widehat{A}_n z^n$, then ∂A denotes the formal derivative of A,

$$\partial A(z) = \sum_{n \in \mathbb{Z}} n \widehat{A}_n z^{n-1} \quad \in \text{End}_{\mathbb{C}}(\mathbb{H})[[z^{\pm 1}]].$$

- A formal power series $A(z) \in \text{End}_{\mathbb{C}}(\mathbb{H})[[z^{\pm 1}]]$ is called a FIELD on $\mathbb{H}$ if $A(z) = \sum_n \widehat{A}_n z^n$ obeys

$$\forall v \in \mathbb{H}: \quad \exists N \in \mathbb{Z} \text{ such that } \widehat{A}_n v = 0 \ \forall n < N.$$

The endomorphisms $\widehat{A}_n$ are called the MODES of the field A.

In other words, if A is a field on $\mathbb{H}$, then for every $v \in \mathbb{H}$ the expression $A(z)v = \sum_n (\widehat{A}_n v) z^n$ is a formal Laurent series with coefficients in $\mathbb{H}$ and with only finitely many non-zero contributions $(\widehat{A}_n v) z^n$ with $n < 0$. In the context of CFTs one can introduce a completion $\overline{\mathbb{H}}$ of $\mathbb{H}$ with respect to an appropriate topology and then for every $z \in \mathbb{C}^*$ view $A(z)$ as a linear operator from $\mathbb{H}$ to $\overline{\mathbb{H}}$, see for example [38, Sect. 1.2.1]. Accordingly, we call a field $A(z) = \sum_n \widehat{A}_n z^n$ *constant* if $\widehat{A}_n = 0$ for all $n \neq 0$. Similarly, if for $v \in \mathbb{H}$ we have $\widehat{A}_n v = 0$ for all $n < 0$, then we say that $A(z)v$ *is well-defined in* $z = 0$, and $A(z)v_{|z=0} = \widehat{A}_0 v$. Note that a field A according to Definition 1 can be viewed as an operator valued distribution, as usual in quantum field theory. Indeed, by means of the residue, $A(z)$ yields a linear map from complex

polynomials into $\mathbb{H}$. By definition, the space $\mathbb{H}$ carries a representation of the Lie algebra generated by the modes of every field on $\mathbb{H}$, with the Lie bracket that is inherited from $\mathrm{End}_\mathbb{C}(\mathbb{H})$, namely the commutator.

Let us now consider two fields A, B on $\mathbb{H}$. While the expressions $A(z)B(w)$ and $B(w)A(z)$ make sense as formal power series in $\mathrm{End}_\mathbb{C}(\mathbb{H})[[z^{\pm1}, w^{\pm1}]]$, a priori it is impossible to interpret them as fields. In general, we expect singular behavior for the coefficients when we insert $w = z$, and in fact the form of this singularity captures the most important aspects of CFT. Here, the notions of *locality* and *normal ordered products* come to aid:

Definition 2

1. Let ∂_w denote the formal derivative with respect to w in $\mathbb{C}[[z^{\pm1}, w^{\pm1}]]$. On $\mathbb{C}[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$, we define the $\mathbb{C}[[z, w]][z^{-1}, w^{-1}]$-linear operators $\iota_{z>w}$ and $\iota_{w>z}$ into $\mathbb{C}[[z^{\pm1}, w^{\pm1}]]$ with

$$\text{for } k \in \mathbb{N}: \quad \iota_{z>w}\left(k!(z-w)^{-k-1}\right) = \partial_w^k \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{w}{z}\right)^n,$$

$$\iota_{w>z}\left(k!(z-w)^{-k-1}\right) = -\partial_w^k \frac{1}{w} \sum_{n=0}^{\infty} \left(\frac{z}{w}\right)^n.$$

2. Fields A, B on $\mathbb{H}$ are called LOCAL WITH RESPECT TO EACH OTHER if there exist a so-called NORMAL ORDERED PRODUCT $:A(z)B(w): \in \mathrm{End}_\mathbb{C}(\mathbb{H})[[z^{\pm1}, w^{\pm1}]]$ and fields $X_0, \ldots, X_{N-1}$ and $:AB:$ on $\mathbb{H}$, such that for every $\alpha \in \mathbb{H}^*$ and $v \in \mathbb{H}$,

   - we have $\langle\alpha, :A(z)B(w): v\rangle \in \mathbb{C}[[z, w]][z^{-1}, w^{-1}]$,
   - in $\mathrm{End}_\mathbb{C}(\mathbb{H})[[z^{\pm1}]]$, we have $:AB:(z) = :A(z)B(w):_{|w=z}$,
   - in $\mathbb{C}[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$, we have

   $$\iota_{z>w}^{-1}\left(\langle\alpha, A(z)B(w)v\rangle\right) = \sum_{j=0}^{N-1} \frac{\langle\alpha, X_j(w)v\rangle}{(z-w)^{j+1}} + \langle\alpha, :A(z)B(w): v\rangle$$

   $$= \iota_{w>z}^{-1}\left(\langle\alpha, B(w)A(z)v\rangle\right).$$

   As a shorthand notation one writes the so-called OPERATOR PRODUCT EXPANSION (OPE)

   $$A(z)B(w) \sim \sum_{j=0}^{N-1} \frac{X_j(w)}{(z-w)^{j+1}}.$$

For the special fields that feature in CFTs, the formal power series in the above definition yield convergent functions in complex variables z and w on appropriate domains in $\mathbb{C}$. Then the operators $\iota_{z>w}$ and $\iota_{w>z}$ implement the Taylor expansions about $z = w$ in the domains $|z| > |w|$ and $|w| > |z|$, respectively. We therefore refer to these operators as *(formal) Taylor expansions*. The OPE thus captures the singular

behavior of the expressions $\langle \alpha, A(z)B(w)v \rangle$ when $z \sim w$, where locality of the fields A and B with respect to each other restricts the possible singularities to poles at $z = w$. For $[A(z), B(w)] := A(z)B(w) - B(w)A(z) \in \mathrm{End}_{\mathbb{C}}(\mathbb{H})[[z^{\pm 1}, w^{\pm 1}]]$, the observation that, in general, $\langle \alpha, [A(z), B(w)]v \rangle$ does not vanish, accounts for the fact that $(z - w)^{-1}$ and its derivatives have different Taylor expansions in the domains $|z| > |w|$ and $|w| > |z|$, respectively. Hence the modes of the fields X_j in the OPE encode the commutators $[\widehat{A}_n, \widehat{B}_m]$ of the modes of A and B.

The Definition 2 of the normal ordered product $:A(z)B(w):$ of two fields $A,\ B$ on $\mathbb{H}$ yields $:A(z)B(w): = A_+(z)B(w) + B(w)A_-(z)$ if $A(z) = A_+(z) + A_-(z)$, where $A_+(z) := \sum_{n \geq 0} \widehat{A}_n z^n$ and $A_-(z) := \sum_{n < 0} \widehat{A}_n z^n$. Hence our definition of normal ordered product amounts to a choice in decomposing $A(z) = A_+(z) + A_-(z)$ as stated, which accrues from the choice of decomposing the formal power series

$$\sum_{m=-\infty}^{\infty} z^m w^{-m-1} = \iota_{z>w}\left((z - w)^{-1}\right) - \iota_{w>z}\left((z - w)^{-1}\right) \quad \in \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]].$$

In the context of superconformal field theories, these notions are generalized to include *odd* fields; if both A and B are odd, then locality amounts to

$$\iota_{z>w}^{-1}\left(\langle \alpha, A(z)B(w)v \rangle\right) = \sum_{j=0}^{N-1} \frac{\langle \alpha, X_j(w)v \rangle}{(z - w)^{j+1}} + \langle \alpha, :A(z)B(w): v \rangle$$
$$= -\iota_{w>z}^{-1}\left(\langle \alpha, B(w)A(z)v \rangle\right),$$

abbreviated by the same OPE as in Definition 2, and the bracket $[\cdot, \cdot]$ in the above argument is replaced by a superbracket with $[A(z), B(w)] = A(z)B(w) + B(w)A(z)$ for odd fields $A,\ B$. The space $\mathbb{H}$, accordingly, furnishes a representation of the super-Lie algebra generated by the modes of the fields on $\mathbb{H}$.

The following list of examples provides some basic fields in the simplest CFTs:

Example 1 (*U(1)-current*)
We consider the complex Lie algebra $\mathcal{A}$ with $\mathbb{C}$-vector space basis $\{C;\ a_n,\ n \in \mathbb{Z}\}$, where C is a central element and the Lie bracket obeys

$$\forall m,\ n \in \mathbb{Z}: \quad [a_n, a_m] = m\delta_{n+m,0} \cdot \frac{C}{3}.$$

Choose some $c \in \mathbb{R}$ and let $\mathbb{H}$ denote the $\mathcal{A}$-module which under the $\mathcal{A}$-action is generated by a single non-zero vector Ω, with submodule of relations generated by

$$a_n\Omega = 0 \ \forall n \leq 0, \quad C\Omega = c\Omega.$$

The space $\mathbb{H}$ can be viewed as polynomial ring in the a_n with $n > 0$. One then checks that the so-called $U(1)$-*current*

$$J(z) := \sum_{n=-\infty}^{\infty} a_n z^{n-1}$$

is a well-defined field on $\mathbb{H}$ which obeys the OPE

$$J(z)J(w) \sim \frac{c/3}{(z-w)^2}.$$

In particular, J is local with respect to itself. Here and in the following, a constant field which acts by multiplication by $\Lambda \in \mathbb{C}$ on $\mathbb{H}$ is simply denoted by Λ.

Example 2 (*Virasoro field*)
For the $U(1)$-current J on the vector space $\mathbb{H}$ introduced in the previous example, assume $c \neq 0$ and let $T(z) := \frac{3}{2c} :JJ:(z)$. One checks that with $c^\bullet = 1$ this field on $\mathbb{H}$ obeys the OPE

$$T(z)T(w) \sim \frac{c^\bullet/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{1}$$

which for the modes of $T(z) = \sum_n L_n z^{n-2}$ translates into

$$\forall n, m \in \mathbb{Z}: \quad [L_n, L_m] = (m-n)L_{m+n} + \delta_{n+m,0}\, \tfrac{c^\bullet}{12}\, m(m^2-1). \tag{2}$$

The above Eq. (2) defines the *Virasoro algebra* at central charge $c^\bullet$, whose underlying vector space has $\mathbb{C}$-vector space basis $\{c^\bullet;\ L_n,\ n \in \mathbb{Z}\}$. This Lie algebra is the central extension by $\mathrm{span}_{\mathbb{C}}\{c^\bullet\}$ of the Lie algebra of infinitesimal conformal transformations of the punctured Euclidean plane $\mathbb{C}^*$.

Example 3 (*bc − βγ-system*)
Let $D \in \mathbb{N}$, and consider the super-Lie algebra $\mathcal{A}_D$ with $\mathbb{C}$-vector space basis $\{C;\ a_n^i, b_n^i, \varphi_n^i, \psi_n^i,\ n \in \mathbb{Z},\ i \in \{1, \ldots, D\}\}$, where the a_n^i, b_n^i and the central element C are even, while the φ_n^i, ψ_n^i are odd, and the only non-vanishing basic super-Lie brackets are

$$\forall m, n \in \mathbb{Z},\ i, j \in \{1, \ldots, D\}: \quad \begin{aligned} [a_n^i, b_m^j] &= \delta^{i,j}\delta_{n+m,0} \cdot C, \\ \{\psi_n^i, \varphi_m^j\} &= \delta^{i,j}\delta_{n+m,0} \cdot C. \end{aligned} \tag{3}$$

Here, $\{\cdot, \cdot\}$ denotes the super-Lie bracket between odd elements of $\mathcal{A}_D$, as is customary in the physics literature. Let $\mathbb{H}$ denote the $\mathcal{A}_D$-module which under the $\mathcal{A}_D$-action is generated by a single non-zero vector Ω, with submodule of relations generated by

$$\forall n \leq 0,\ m < 0,\ i, j \in \{1, \ldots, D\}: \quad \begin{aligned} a_n^i\Omega &= 0, \quad b_m^j\Omega = 0, \\ \psi_n^i\Omega &= 0, \quad \varphi_m^j\Omega = 0; \quad C\Omega = \Omega. \end{aligned}$$

Generalizing Examples 1 and 2 above, one checks that

$$a^i(z) := \sum_{n=-\infty}^{\infty} a_n^i z^{n-1}, \quad b^i(z) := \sum_{m=-\infty}^{\infty} b_m^i z^m,$$

$$\psi^i(z) := \sum_{n=-\infty}^{\infty} \psi_n^i z^{n-1}, \quad \varphi^i(z) := \sum_{m=-\infty}^{\infty} \varphi_m^i z^m, \quad i \in \{1, \ldots, D\}$$

defines pairwise local fields a^i, b^i, ψ^i, φ^i on $\mathbb{H}$. Moreover, one finds the OPEs

$$a^i(z)b^j(w) \sim \frac{\delta^{i,j}}{z-w}, \quad \varphi^i(z)\psi^j(w) \sim \frac{\delta^{i,j}}{z-w} \quad \forall\, i,\, j \in \{1, \ldots, D\},$$

while all other basic OPEs vanish, and the field

$$T^{\text{top}}(z) := \sum_{j=1}^{D} \left(:\partial b^j a^j:(z) + :\partial \varphi^j \psi^j:(z) \right) \tag{4}$$

is a Virasoro field obeying (1) at central charge $c^\bullet = 0$.

Example 4 (*Topological $N = 2$ superconformal algebra*)
With $\mathcal{A}_D$, $\mathbb{H}$, and the fields of the $bc - \beta\gamma$-system defined in the above Example 3, let

$$J(z) := \sum_{j=1}^{D} :\varphi^j \psi^j:(z), \quad Q(z) := \sum_{j=1}^{D} :a^j \varphi^j:(z), \quad G(z) := \sum_{j=1}^{D} :\psi^j \partial b^j:(z). \tag{5}$$

These fields obey the so-called *topological $N = 2$ superconformal algebra* at central charge $c = 3D$:

$$T^{\text{top}}(z)T^{\text{top}}(w) \sim \frac{2T^{\text{top}}(w)}{(z-w)^2} + \frac{\partial T^{\text{top}}(w)}{z-w}, \tag{6}$$

$$T^{\text{top}}(z)J(w) \sim -\frac{c/3}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}, \quad J(z)J(w) \sim \frac{c/3}{(z-w)^2},$$

$$T^{\text{top}}(z)Q(w) \sim \frac{Q(w)}{(z-w)^2} + \frac{\partial Q(w)}{z-w}, \quad Q(z)Q(w) \sim 0, \quad J(z)Q(w) \sim \frac{Q(w)}{z-w},$$

$$T^{\text{top}}(z)G(w) \sim \frac{2G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w}, \quad G(z)G(w) \sim 0, \quad J(z)G(w) \sim -\frac{G(w)}{z-w},$$

$$Q(z)G(w) \sim \frac{c/3}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T^{\text{top}}(w)}{z-w}. \tag{7}$$

Example 5 ($N = 2$ *superconformal algebra*)

Consider a $\mathbb{C}$-vector space $\mathbb{H}$ and pairwise local fields $T^{\text{top}}(z)$, $J(z)$, $Q(z)$, $G(z)$ on $\mathbb{H}$ which obey the topological $N = 2$ superconformal algebra (6)–(7) at central charge c. Now let

$$T(z) := T^{\text{top}}(z) - \frac{1}{2}\partial J(z), \quad G^+(z) := Q(z), \quad G^-(z) := G(z). \qquad (8)$$

Then $T(z)$ is another Virasoro field as in (1), but now with central charge $c^\bullet = c$, and the fields $T(z)$, $J(z)$, $G^+(z)$, $G^-(z)$ on $\mathbb{H}$ obey the so-called $N = 2$ *superconformal algebra* at central charge c,

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \qquad (9)$$

$$T(z)J(w) \sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}, \quad J(z)J(w) \sim \frac{c/3}{(z-w)^2},$$

$$T(z)G^\pm(w) \sim \frac{3/2\,G^\pm(w)}{(z-w)^2} + \frac{\partial G^\pm(w)}{z-w}, \quad J(z)G^\pm(w) \sim \pm\frac{G^\pm(w)}{z-w},$$

$$G^\pm(z)G^\mp(w) \sim \frac{c/3}{(z-w)^3} \pm \frac{J(w)}{(z-w)^2} + \frac{T(w) \pm \frac{1}{2}\partial J(w)}{z-w}, \quad G^\pm(z)G^\pm(w) \sim 0.$$

$$(10)$$

Equation (8) is referred to by the statement that the fields $T^{\text{top}}(z)$, $J(z)$, $Q(z)$, $G(z)$ are obtained from the fields $T(z)$, $J(z)$, $G^+(z)$, $G^-(z)$ by a *topological A-twist*. Analogously, fields $T^{\text{top}}(z)$, $-J(z)$, $Q(z)$, $G(z)$ which obey a topological $N = 2$ superconformal algebra at central charge c are obtained from fields $T(z)$, $J(z)$, $G^+(z)$, $G^-(z)$ which obey an $N = 2$ superconformal algebra at central charge c by a *topological B-twist* iff $T^{\text{top}}(z) = T(z) - \frac{1}{2}\partial J(z)$, $Q(z) = G^-(z)$, $G = G^+(z)$, see [35, 98]. On the level of the $N = 2$ superconformal algebras, the transition between topological A-twist and topological B-twist is induced by $(T, J, G^+, G^-) \mapsto (T, -J, G^-, G^+)$, an automorphism of the superconformal algebra. This automorphism is at the heart of *mirror symmetry* [68].

We are now ready to define one of the fundamental ingredients of CFT, namely the notion of *conformal vertex algebra*. The definition is taken from [38] and follows [9, 39, 61]:

Definition 3 A CONFORMAL VERTEX ALGEBRA AT CENTRAL CHARGE $c \in \mathbb{C}$ is given by the following data:

- A $\mathbb{Z}$-graded $\mathbb{C}$-vector space $W = \oplus_{m\in\mathbb{Z}} W_m$ called the SPACE OF STATES.
- A special vector $\Omega \in W_0$ called the VACUUM.
- A linear operator $L: W \to W$ called the TRANSLATION OPERATOR.
- A special vector $T \in W_2$ called the CONFORMAL VECTOR.

- A linear map

$$Y(\cdot, z) \colon W \longrightarrow \mathrm{End}(W)[[z^{\pm 1}]],$$

  called the STATE- FIELD CORRESPONDENCE, which assigns to every $A \in W$ a field $A(z) := Y(A, z)$ on W.

These data obey the following axioms:

- The VACUUM AXIOM: We have $\Omega(z) = 1$, and for every $A \in W$ and $A(z) = \sum_n \widehat{A}_n z^n$, we obtain $A(z)\Omega \in W[[z]]$, such that $A(z)\Omega$ is well-defined in $z = 0$ and

$$A(z)\Omega_{|z=0} = \widehat{A}_0 \Omega = A \in W.$$

  One says: The field $A(z)$ CREATES the state A from the vacuum.
- The TRANSLATION AXIOM:

$$L\Omega = 0 \quad \text{and} \quad \forall A \in W\colon \quad [L, A(z)] = \partial A(z).$$

- The LOCALITY AXIOM:
  All fields $A(z)$ with $A \in W$ are local with respect to each other.

  The (ungraded) vector space W with Ω, L, and the map Y is called a VERTEX ALGEBRA. In a CONFORMAL vertex algebra, in addition

- The field $T(z) = \sum_{n=-\infty}^{\infty} L_n z^{n-2}$ associated to the conformal vector T by the state-field correspondence is a Virasoro field obeying the OPE (1) with central charge $c^\bullet = c$.
- The translation operator L is given by $L = L_1$ and has degree 1.
- For all $m \in \mathbb{Z}$, $L_{0|W_m} = m$, and for $A \in W_m$, the field $A(z)$ has weight m, i.e. $A(z) = \sum_{n=-\infty}^{\infty} A_n z^{n-m}$ with $A_n \in \mathrm{End}(W)$ of degree n.

  In the context of superconformal field theories, the notion of conformal vertex algebras of Definition 3 is generalized to superconformal vertex algebras. For an $N = 2$ superconformal vertex algebra, the vector space W in the above Definition is graded by $\frac{1}{2}\mathbb{Z}$ instead of $\mathbb{Z}$, one needs to allow odd fields $A(z) = Y(A, z)$, which can have mode expansions in $z^{1/2} \cdot \mathrm{End}(W)[[z^{\pm 1}]]$, and one needs to generalize the notion of locality to such fields, as explained in the discussion of Definition 2. Finally, one needs to assume that there exist special states $J \in W_1$ and $G^{\pm} \in W_{3/2}$ such that the associated fields $J(z)$, $G^{\pm}(z)$ obey the $N = 2$ superconformal algebra (9)–(10).

  An important ingredient of CFT are the so-called n-point functions, which associate a function in n complex variables to every n-tuple of states in the CFT. These n-point functions are naturally related to the notion of vertex algebras, as we shall illustrate now. Assume that W is the $\mathbb{Z}$-graded vector space which underlies a conformal vertex algebra, with notations as in Definition 3. Furthermore, assume that W comes equipped with a positive definite scalar product $\langle \cdot, \cdot \rangle$, such that $W = \oplus_{m \in \mathbb{Z}} W_m$ is an orthogonal direct sum. Let $A(z)$, $B(w)$ denote the fields associated to A, $B \in W$ by the state-field correspondence, which are local

with respect to each other by the locality axiom. Hence by the very Definition 2 of locality, the formal power series $\langle \Omega, A(z)B(w)\Omega \rangle$ and $\langle \Omega, B(w)A(z)\Omega \rangle$ are obtained from the same series in $\mathbb{C}[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}]$ by means of the (formal) Taylor expansions $\iota_{z>w}$ and $\iota_{w>z}$, respectively. This series is denoted by $\langle A(z)B(w) \rangle \in \mathbb{C}[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}]$, such that

$$\iota_{z>w}\left(\langle A(z)B(w) \rangle\right) = \langle \Omega, A(z)B(w)\Omega \rangle, \quad \iota_{w>z}\left(\langle A(z)B(w) \rangle\right) = \langle \Omega, B(w)A(z)\Omega \rangle.$$

Then $\langle A(z)B(w) \rangle$ is an example of a 2-point function, and for $A_1, \ldots, A_n \in W$ one analogously defines the n-point functions $\langle A_1(z_1) \cdots A_n(z_n) \rangle$ by successive OPE. The additional properties of CFTs ensure that these n-point functions define meromorphic functions in complex variables $z_1, \ldots, z_n \in \mathbb{C}$, whose possible poles are restricted to the partial diagonals $z_i = z_j$, $i \neq j$.

2.2 Defining Conformal Field Theories

This section summarizes an axiomatic approach to conformal field theory. Instead of a full account, the focus lies on those ingredients of CFTs that are relevant for the remaining sections of this exposition. More details can be found e.g. in [91, 94]. We list the ingredients and defining properties of a two-dimensional Euclidean unitary conformal field theory at central charges c, $\overline{c}$:

Ingredient I. [The *space of states* $\mathbb{H}$]
The space $\mathbb{H}$ is a $\mathbb{C}$-vector space with positive definite scalar product $\langle \cdot, \cdot \rangle$ and with a compatible real structure $v \mapsto v^*$. Furthermore, there are two Virasoro fields $T(z)$, $\overline{T}(\overline{z})$ of central charges c, $\overline{c}$ on $\mathbb{H}$, see Eq. (1), where the OPE between T and $\overline{T}$ is trivial:
$$T(z)\overline{T}(\overline{z}) \sim 0.$$

The space of states of a CFT must have a number of additional properties:

Property A. The space of states $\mathbb{H}$ furnishes a *unitary* representation of the two commuting copies of a Virasoro algebra generated by the modes L_n, $\overline{L}_n$, $n \in \mathbb{Z}$, of the Virasoro fields $T(z)$ and $\overline{T}(\overline{z})$, which is *compatible with the real structure* of $\mathbb{H}$. The central elements c, $\overline{c}$ act by multiplication with fixed, real constants, also denoted c, $\overline{c} \in \mathbb{R}$. The operators L_0 and $\overline{L}_0$ are self-adjoint and positive semidefinite, and $\mathbb{H}$ decomposes into a direct sum of their simultaneous eigenspaces indexed by $R \subset \mathbb{R}^2$,

$$\mathbb{H} = \bigoplus_{(h,\overline{h})\in R} \mathbb{H}_{h,\overline{h}}, \quad \mathbb{H}_{h,\overline{h}} := \ker\left(L_0 - h \cdot \mathrm{id}\right) \cap \ker\left(\overline{L}_0 - \overline{h} \cdot \mathrm{id}\right).$$

By this we mean that R does not have accumulation points, and that every vector in $\mathbb{H}$ is a sum of contributions from finitely many different eigenspaces $\mathbb{H}_{h,\overline{h}}$. Moreover, every $\mathbb{H}_{h,\overline{h}}$ is finite dimensional.

Property A ensures that the space of states $\mathbb{H}$ of every conformal field theory furnishes a very well-behaved representation of two commuting copies of a Virasoro algebra. In addition, we need to assume that the *character* of this representation has favorable properties:

Property B. For $\tau \in \mathbb{C}$, $\Im(\tau) > 0$, let $q := \exp(2\pi i \tau)$; the *partition function*

$$Z(\tau) := \sum_{(h,\overline{h})\in R} \left(\dim_{\mathbb{C}} \mathbb{H}_{h,\overline{h}}\right) q^{h-c/24} \overline{q}^{\overline{h}-\overline{c}/24} = \mathrm{Tr}_{\mathbb{H}}\left(q^{L_0-c/24} \overline{q}^{\overline{L}_0-\overline{c}/24}\right)$$

is well defined for all values of τ in the complex upper halfplane, and it is invariant under modular transformations

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Since by Property B the partition function is modular invariant, it in particular is invariant under the translation $\tau \mapsto \tau + 1$ of the modular parameter. This implies that for every pair $(h, \overline{h}) \in R$ of eigenvalues of L_0 and $\overline{L}_0$, we have $h - \overline{h} \in \mathbb{Z}$. Hence the subspaces $W := \ker\left(\overline{L}_0\right)$ and $\overline{W} := \ker\left(L_0\right)$ are $\mathbb{Z}$-graded by L_0 and $\overline{L}_0$, respectively. To obtain a CFT, these subspaces are required to carry additional structure, which we are already familiar with:

Property C. The subspaces $W := \ker\left(\overline{L}_0\right)$ and $\overline{W} := \ker\left(L_0\right)$ of $\mathbb{H}$ carry the structure of conformal vertex algebras, see Definition 3, with $T(z)$ and $\overline{T}(\overline{z})$ the fields associated to the respective conformal vectors by the state-field correspondence. Moreover, the vacuum vector Ω of the conformal vertex algebra W agrees with the vacuum vector of $\overline{W}$, and Ω is a real unit vector yielding a basis of $W \cap \overline{W} = \mathbb{H}_{0,0}$.
The vertex algebras with underlying vector spaces W and $\overline{W}$ are called the *chiral algebras* of the CFT, and to simplify the terminology, we also refer to W and $\overline{W}$ as the chiral algebras.

As was discussed at the end of Sect. 2.1, in this setting there is a natural definition of n-point functions for the fields in the chiral algebras associated to W and $\overline{W}$. This definition, however, is not sufficient to capture the general n-point functions of conformal field theory. The notion is generalized along the following lines:

Ingredient II. [The system $\langle \cdots \rangle$ of n-point functions]
The space of states $\mathbb{H}$ is equipped with a system $\langle \cdots \rangle$ of *n-point functions*, that is, for every $n \in \mathbb{N}$ we have a map

$$\mathbb{H}^{\otimes n} \longrightarrow \mathrm{Maps}(\mathbb{C}^n \setminus \bigcup_{i \neq j} D_{i,j}, \mathbb{C}), \quad D_{i,j} := \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i = z_j \right\},$$

which is compatible with complex conjugation, and such that every function in the image is real analytic and allows an appropriate expansion about every partial diagonal $D_{i,j}$.

The following Property D, which along with Property E governs the behavior of the n-point functions, is immediate on the chiral algebras W and $\overline{W}$, by definition:

Property D. The n-point functions are *local*, that is, for every permutation $\sigma \in S_n$ and all $\phi_i \in \mathbb{H}$,

$$\langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle = \langle \phi_{\sigma(1)}(z_{\sigma(1)}) \cdots \phi_{\sigma(n)}(z_{\sigma(n)}) \rangle.$$

Consider an n-point function $\langle \phi(z_1) \cdots \phi(z_n) \rangle$ with $\phi = \phi_1 = \cdots = \phi_n \in \mathbb{H}$ as a function of one complex variable $z = z_k$, while all other z_l, $l \neq k$, are fixed. The closure of the domain of definition of this function is the *worldsheet* on which the n-point function is defined. Therefore, Ingredient II yields n-point functions whose worldsheet is the Riemann sphere $\overline{\mathbb{C}}$. As a basic feature of conformal field theory, the n-point functions are assumed to transform covariantly under conformal maps between worldsheets. In particular,

Property E. The n-point functions are *Poincaré covariant*, that is, for all isometries and all dilations f of the Euclidean plane $\mathbb{C}$, and for all $\phi_i \in \mathbb{H}_{h_i, \overline{h}_i}$,

$$\langle \phi_1(f(z_1)) \cdots \phi_n(f(z_n)) \rangle = \prod_{i=1}^{n} \left[\left(f'(z_i) \right)^{-h_i} \overline{f'(z_i)}^{-\overline{h}_i} \right] \langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle,$$

where $f'(z) = \partial_z f(z)$. Moreover, *infinitesimal translations* $\alpha L_1 + \overline{\alpha}\overline{L}_1$, $\alpha, \overline{\alpha} \in \mathbb{C}$, are represented by $\alpha\partial_z + \overline{\alpha}\partial_{\overline{z}}$, i.e. for arbitrary $\phi_i \in \mathbb{H}$,

$$\langle\phi_1(z_1)\cdots\phi_{n-1}(z_{n-1})(L_1\phi_n)(z_n)\rangle = \frac{\partial}{\partial z_n}\langle\phi_1(z_1)\cdots\phi_{n-1}(z_{n-1})\phi_n(z_n)\rangle,$$

$$\langle\phi_1(z_1)\cdots\phi_{n-1}(z_{n-1})(\overline{L}_1\phi_n)(z_n)\rangle = \frac{\partial}{\partial\overline{z}_n}\langle\phi_1(z_1)\cdots\phi_{n-1}(z_{n-1})\phi_n(z_n)\rangle.$$

The remaining requirements on the n-point functions, unfortunately, are rather more involved. Roughly, they firstly generalize Property E by ensuring that the representation of the two commuting copies of the Virasoro algebra on $\mathbb{H}$ (see Property A) induces an action by infinitesimal conformal transformations on the worldsheet. Furthermore, the operator product expansion of Definition 2 is generalized to induce the appropriate expansions of the n-point functions about partial diagonals, see Ingredient II. Finally, n-point functions must be defined on worldsheets with arbitrary genus. Since these additional properties are not needed explicitly in the remaining sections of the present exposition, here only the relevant keywords are listed in the final

Property F. The system $\langle\cdots\rangle$ of n-point functions is *conformally covariant*, and it *represents an operator product expansion* such that *reflection positivity* holds. Moreover, the *universality condition* holds, and sewing allows to define n-point functions on *worldsheets of arbitrary genus*.

As was mentioned at the beginning of this section, the ingredients of CFTs listed above yield *two-dimensional Euclidean unitary conformal field theories.* Indeed, these adjectives have been implemented in Properties A–F: According to the discussion that precedes Property E along with Property F, the worldsheets of our CFTs are *two-dimensional Euclidean* manifolds. *Conformality* is implemented by means of the two commuting copies of the Virasoro algebra, see the discussion of Eqs. (1) and (2), which act by infinitesimal conformal transformations on the worldsheets of the n-point functions by Properties E and F. On the space of states $\mathbb{H}$, Property A ensures that the representation of the infinitesimal conformal transformations is *unitary*.

Our approach to CFT is convenient, since it concretely implements the interplay between representation theory with the analytic properties of the n-point functions, which is characteristic of two-dimensional conformal quantum field theories. However, the relation to more general quantum field theories (QFTs) is not so evident. Let us briefly comment on this connection.

First, for the relevant QFTs we restrict to *Euclidean* quantum field theories according to a system of axioms that are based on the *Osterwalder-Schrader axioms* [78, 79], see [37, 82]. According to [78, 79], these axioms ensure that from such a QFT

one can construct a Hilbert space $\widetilde{\mathbb{H}}$ of states ϕ and associated fields Y_ϕ, where each field Y_ϕ yields a densely defined linear operator $Y_\phi(h)$ on $\widetilde{\mathbb{H}}$ for every test function h. Moreover, there is a special state Ω which plays the role of the vacuum as in our Property C.

The Osterwalder-Schrader axioms require the existence of correlation functions associated to every n-tuple of states in $\widetilde{\mathbb{H}}$ which resemble the n-point functions of CFT according to our Ingredient II. To obtain the fields of CFT from those of the general QFT, one needs to perform a procedure called *localization*. Within the Hilbert space $\widetilde{\mathbb{H}}$ one restricts to the subspace $\mathbb{H}$ which is generated by those states that are created by the localized field operators from the vacuum, generalizing the vacuum axiom of our Definition 3. The Osterwalder-Schrader axioms then ensure that locality (Property D), Poincaré covariance under isometries (Property E) and reflection positivity (Property F) hold for the n-point functions obtained from the correlation functions of the QFT.

According to [37], conformal covariance can be implemented by means of three additional axioms, ensuring the covariance of the n-point functions under dilations (Property E), the existence of the Virasoro fields (Ingredient I) and of an OPE (Properties C and F) with all the necessary features. See [82, Sect. 9.3] for an excellent account.

If a CFT is obtained from a conformally covariant QFT by localization, then one often says that the CFT is the *short distance limit* of the QFT. For details on this mathematical procedure see [37, 43, 90]. To the author's knowledge, it is unknown whether a CFT in the sense of our approach can always be viewed as a short distance limit of a full-fledged QFT.

With the above, we do not claim to provide a *minimal* axiomatic approach to CFT. For example, the requirement of Property F that n-point functions are well-defined on worldsheets of arbitrary genus implies modular invariance of the partition function, which was assumed separately in Property B. Indeed, the partition function $Z(\tau)$ is the 0-point function on a worldsheet torus with modulus τ, where conformal invariance implies that $Z(\tau)$ indeed solely depends on the complex structure represented by $\tau \in \mathbb{C}, \Im(\tau) > 0$. Property B is stated separately for clarity, and because modular invariance plays a crucial role in the discussion of the elliptic genus in Sect. 2.4 which is also essential for the remaining sections of this exposition, while we refrain from a detailed discussion of Property F.

Mathematical implications of modular invariance for CFTs were first pointed out by Cardy [16]. He observed that for those theories that had been studied by Belavin, Polyakov and Zamolodchikov in their seminal paper [7], and that describe physical phenomena in statistical physics, modular invariance of the partition function poses constraints on the operator content. These constraints can be useful for the classification of CFTs.

In special cases, modular invariance can be proven from first principles, assuming only that the n-point functions are well-defined on the Riemann sphere. In [74], Nahm argues that the assumption that the n-point functions on the torus define thermal states of the field algebra, which in turn is of type I, suffices to deduce modular invariance.

Under an assumption known as *Condition C* or *Condition C_2*, which amounts to certain quotients of the chiral algebras being finite dimensional, Zhu proves in [102] that modular invariance follows, as well. This covers a large class of examples of CFTs, among them the ones studied by Belavin, Polyakov and Zamolodchikov.

An $N = (2, 2)$ *superconformal field theory* is a CFT as above, where the notion of locality is generalized according to what was said in the discussion of Definition 2, and the representations of the two commuting copies of a Virasoro algebra are extended to representations of $N = 2$ superconformal algebras, see Eqs. (9)–(10). As a first additional ingredient to these theories one therefore needs

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Ingredient III. [Compatible $\mathbb{Z}_2$-grading of the space of states]
The space of states $\mathbb{H}$ carries a $\mathbb{Z}_2$-grading $\mathbb{H} = \mathbb{H}_b \oplus \mathbb{H}_f$ into *worldsheet bosons* $\mathbb{H}_b$ (even) and *worldsheet fermions* $\mathbb{H}_f$ (odd), which is compatible with Properties A–F.
In more detail, for compatibility with Property A, the decomposition $\mathbb{H} = \mathbb{H}_b \oplus \mathbb{H}_f$ must be orthogonal and invariant under the action of the two commuting copies of the Virasoro algebra. In Property B, the trace defining the partition function is taken over the bosonic subspace $\mathbb{H}_b$, only. The chiral algebras introduced in Property C must contain $N = 2$ superconformal vertex algebras as introduced in the discussion of Definition 3, whose modes act unitarily on $\mathbb{H}$. The notion of locality in Property D is generalized to *semi-locality*, meaning that

$$\langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle = (-1)^I \langle \phi_{\sigma(1)}(z_{\sigma(1)}) \cdots \phi_{\sigma(n)}(z_{\sigma(n)}) \rangle$$

if $\sigma \in S_n$ and all $\phi_i \in \mathbb{H}$ have definite parity. Here, I is the number of inversions of odd states in σ, that is, the number of pairs (i, j) of indices with $i < j$ and $\sigma(i) > \sigma(j)$ and such that $\phi_i, \phi_j \in \mathbb{H}_f$. Properties E and F remain unchanged.

The fields in the chiral algebras of the CFT that furnish the two commuting copies of $N = 2$ superconformal vertex algebras according to Property III are generally denoted $T(z)$, $J(z)$, $G^+(z)$, $G^-(z)$ and $\overline{T}(\bar{z})$, $\overline{J}(\bar{z})$, $\overline{G}^+(\bar{z})$, $\overline{G}^-(\bar{z})$ with OPEs as in (9)–(10). The mode expansions for the even fields are denoted as

$$T(z) = \sum_n L_n z^{n-2}, \quad J(z) = \sum_n J_n z^{n-1}, \quad \overline{T}(\bar{z}) = \sum_n \overline{L}_n \bar{z}^{n-2}, \quad \overline{J}(\bar{z}) = \sum_n \overline{J}_n \bar{z}^{n-1},$$
$$(11)$$

in accord with Definition 3. As mentioned in the discussion after Definition 3, the odd fields $G^\pm(z)$ can have mode expansions either in $\mathrm{End}_{\mathbb{C}}(\mathbb{H})[[z^{\pm 1}]]$ or in $z^{1/2} \cdot \mathrm{End}_{\mathbb{C}}(\mathbb{H})[[z^{\pm 1}]]$, and analogously for $\overline{G}^\pm(\bar{z})$. This induces another $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading of the space of states $\mathbb{H}$,

$$\mathbb{H} = \mathbb{H}^{NS,NS} \oplus \mathbb{H}^{R,R} \oplus \mathbb{H}^{NS,R} \oplus \mathbb{H}^{R,NS}, \tag{12}$$

where

$$G^{\pm}(z) \in \mathrm{End}_{\mathbb{C}}(\mathbb{H}^{NS,\bullet})[[z^{\pm 1}]], \quad \overline{G}^{\pm}(\overline{z}) \in \mathrm{End}_{\mathbb{C}}(\mathbb{H}^{\bullet,NS})[[\overline{z}^{\pm 1}]]$$

in the so-called *Neveu-Schwarz-* or *NS-sector*, while

$$G^{\pm}(z) \in z^{1/2} \cdot \mathrm{End}_{\mathbb{C}}(\mathbb{H}^{R,\bullet})[[z^{\pm 1}]], \quad \overline{G}^{\pm}(\overline{z}) \in \overline{z}^{1/2} \cdot \mathrm{End}_{\mathbb{C}}(\mathbb{H}^{\bullet,R})[[\overline{z}^{\pm 1}]]$$

in the so-called *Ramond-* or *R-sector*. That is, on $\mathbb{H}^{S,\overline{S}}$ the fields $G^{\pm}(z)$ and $\overline{G}^{\pm}(\overline{z})$ have mode expansions according to the S and the $\overline{S}$ sector, respectively, with $S, \overline{S} \in \{R, NS\}$.

In what follows, we restrict our attention to so-called *non–chiral* $N = (2, 2)$ *superconformal field theories with space-time supersymmetry*:

Ingredient IV. [Space-time supersymmetry]

The space of states $\mathbb{H}$ carries another compatible $\mathbb{Z}_2$-grading by means of the properties of the odd fields $G^{\pm}(z)$ and $\overline{G}^{\pm}(\overline{z})$ of the $N = 2$ superconformal vertex algebra into

$$\mathbb{H} = \mathbb{H}^{NS} \oplus \mathbb{H}^{R}.$$

Here, the decomposition (12) reduces to $\mathbb{H}^{NS} := \mathbb{H}^{NS,NS}$ and $\mathbb{H}^{R} := \mathbb{H}^{R,R}$, while the sectors $\mathbb{H}^{NS,R}$ and $\mathbb{H}^{R,NS}$ are trivial.

Moreover, as representations of the two commuting $N = 2$ superconformal algebras of Ingredient III, $\mathbb{H}^{NS}$ and $\mathbb{H}^{R}$ are equivalent under an isomorphism $\Theta \colon \mathbb{H} \to \mathbb{H}$ which interchanges $\mathbb{H}^{NS}$ and $\mathbb{H}^{R}$ and which obeys

$$\begin{aligned}
[L_0, \Theta] &= \tfrac{c}{24}\Theta - \tfrac{1}{2}\Theta \circ J_0, \quad [J_0, \Theta] = -\tfrac{c}{6}\Theta, \\
[\overline{L}_0, \Theta] &= \tfrac{\overline{c}}{24}\Theta - \tfrac{1}{2}\Theta \circ \overline{J}_0, \quad [\overline{J}_0, \Theta] = -\tfrac{\overline{c}}{6}\Theta,
\end{aligned} \tag{13}$$

where $L_0, J_0, \overline{L}_0, \overline{J}_0$ are the zero-modes of the fields $T(z), J(z), \overline{T}(\overline{z}), \overline{J}(\overline{z})$ obtained from the mode expansions (11). The isomorphism Θ is induced by a field of the theory called *spectral flow*, and it is also known as *space-time supersymmetry*.

With the above notion of CFT, a number of examples are known, like minimal models, both the bosonic [7, 55] and the supersymmetric ones [13, 24, 81, 101]. In string theory, so-called non-linear sigma model constructions are believed to provide a map from certain manifolds to CFTs. While this construction is well understood for the simplest manifolds, namely for tori, the mathematical details in general are far from known.

2.3 Example: Toroidal Conformal Field Theories

For illustration and for later reference, this section very briefly presents the class of so-called *toroidal conformal field theories*. These theories are characterized by the existence of "sufficiently many $U(1)$-currents" as in Example 1 of Sect. 2.1.

We say that the chiral algebra $W = \oplus_m W_m$ of a CFT (see Property C, Sect. 2.2) *contains a* $\mathfrak{u}(1)^d$*-current algebra*, if W_1 contains an orthogonal system $\left(a_1^k \Omega, \ k \in \{1, \ldots, d\} \right)$ of states, which under the state-field correspondence of Definition 3 have associated fields $j^k(z)$, $k \in \{1, \ldots, d\}$, obeying the OPEs

$$\forall k, l \in \{1, \ldots, d\} : \qquad j^k(z) j^l(w) \sim \frac{\delta^{k,l}}{(z-w)^2}. \tag{14}$$

For (bosonic) CFTs we then have

Definition 4 A conformal field theory at central charges c, $\overline{c}$ is called TOROIDAL, if $c = \overline{c} = d$ with $d \in \mathbb{N}$, and if the chiral algebras $W = \oplus_m W_m$, $\overline{W} = \oplus_m \overline{W}_m$ of Property C, Sect. 2.2, each contain a $\mathfrak{u}(1)^d$-current algebra.

For our purposes, the *toroidal* $N = (2, 2)$ *superconformal field theories* are more relevant. They are characterized by the fact that their bosonic sector with space of states $\mathbb{H}_b$ contains a toroidal CFT at central charges $2D$, $2D$ in the sense of Definition 4, and in addition, they contain D left- and D right-moving so-called *Dirac fermions with coupled spin structures*. By this we mean first of all that the subspace $W_{1/2} \subset W$ of the vector space underlying the chiral algebra contains an orthogonal system $\left((\psi_\pm^k)_{1/2} \Omega, \ k \in \{1, \ldots, D\} \right)$ of states, which under the state-field correspondence of Definition 3 have associated (odd) fields $\psi_k^\pm(z)$, $k \in \{1, \ldots, D\}$, obeying the OPEs

$$\forall k, l \in \{1, \ldots, D\} : \qquad \psi_k^+(z) \psi_l^-(w) \sim \frac{\delta^{k,l}}{z-w}, \quad \psi_k^\pm(z) \psi_l^\pm(w) \sim 0, \tag{15}$$

and analogously for the subspace $\overline{W}_{1/2} \subset \overline{W}$ of the vector space underlying the second chiral algebra in Property C. In addition, all $\psi_k^\pm(z)$ are represented by formal power series in $\mathrm{End}_{\mathbb{C}}(\mathbb{H}^{NS})[[z^{\pm 1}]]$ on $\mathbb{H}^{NS}$, while on $\mathbb{H}^R$, they are represented in $z^{1/2} \cdot \mathrm{End}_{\mathbb{C}}(\mathbb{H}^R)[[z^{\pm 1}]]$, and analogously for the $\overline{\psi}_k^\pm(\overline{z})$. One shows that such a system of D left- and D right-moving Dirac fermions yields a well-defined CFT at central charges D, D (see e.g. [53, Sect. 8.2] or [91, Chap. 5]).

Definition 5 An $N = (2, 2)$ superconformal field theory at central charges c, $\overline{c}$ with space-time supersymmetry is toroidal, if $c = \overline{c} = 3D$ with $D \in \mathbb{N}$, and if this theory is the tensor product of a toroidal conformal field theory at central charges $2D$, $2D$ according to Definition 4, and a system of D left- and D right-moving Dirac fermions with coupled spin structures. Moreover, the fields $\psi_k^\pm(z), k \in \{1, \ldots, D\}$,

in (15) are the superpartners of the $U(1)$-currents $j^l(z)$, $l \in \{1, \ldots, 2D\}$ in (14), and analogously for the right-moving fields. By this we mean that for the fields $G^\pm(z)$, $\overline{G}^\pm(\overline{z})$ in the two commuting superconformal vertex algebras (9)–(10), and with notations as above, we have

$$\forall k \in \{1, \ldots, D\}: \quad G^\pm_{-1/2} a_1^k \Omega = (\psi^k_\pm)_{1/2}\Omega, \; G^\pm_{-1/2} a_1^{k+D} \Omega = \mp i(\psi^k_\pm)_{1/2}\Omega,$$
$$\overline{G}^\pm_{-1/2} \overline{a}_1^k \Omega = (\overline{\psi}^k_\pm)_{1/2}\Omega, \; \overline{G}^\pm_{-1/2} \overline{a}_1^{k+D} \Omega = \mp i(\overline{\psi}^k_\pm)_{1/2}\Omega.$$

The toroidal conformal and superconformal field theories have been very well understood by string theorists since the mid eighties [17, 76], and these theories have also been reformulated in terms of the vertex algebras presented in Sect. 2.1 [38, 61, 63]. This includes the interpretation of the toroidal conformal field theories as non-linear sigma models on tori, their deformations, and thus the structure of the moduli space of toroidal CFTs:

Theorem 1 ([76]) *The moduli space $\mathcal{M}^{\mathrm{tor}}_D$ of toroidal $N = (2, 2)$ superconformal field theories at central charges $c = \overline{c} = 3D$ with $D \in \mathbb{N}$ is a quotient of a $4D^2$-dimensional Grassmannian by an infinite discrete group,*

$$\mathcal{M}^{\mathrm{tor}}_D = O^+(2D, 2D; \mathbb{Z})\backslash\mathcal{T}^{2D,2D},$$
$$\textit{where } \mathcal{T}^{2D,2D} := O^+(2D, 2D; \mathbb{R})/SO(2D) \times O(2D).$$

Here, if $pq \neq 0$, then $O^+(p, q; \mathbb{R})$ denotes the group of those elements in $O(p, q; \mathbb{R}) = O(\mathbb{R}^{p,q})$ which preserve the orientation of maximal positive definite oriented subspaces in $\mathbb{R}^{p,q}$, and if $p \equiv q \bmod 8$, then $O^+(p, q; \mathbb{Z}) = O^+(p, q; \mathbb{R}) \cap O(\mathbb{Z}^{p,q})$ with $\mathbb{Z}^{p,q} \subset \mathbb{R}^{p,q}$ the standard even unimodular lattice of signature (p, q).

2.4 The Elliptic Genus

In this section, the conformal field theoretic elliptic genus is introduced and compared to the geometric elliptic genus that is known to topologists and geometers. This and the following section are completely expository with more details and proofs to be found in the literature as referenced.

Let us first consider an $N = (2, 2)$ superconformal field theory at central charges c, $\overline{c}$ with space-time supersymmetry according to the Ingredients I–IV of Sect. 2.2. For the zero-modes J_0, $\overline{J}_0$ of the fields $J(z)$, $\overline{J}(\overline{z})$ in the two commuting $N = 2$ superconformal algebras $T(z)$, $J(z)$, $G^+(z)$, $G^-(z)$, $\overline{T}(\overline{z})$, $\overline{J}(\overline{z})$, $\overline{G}^+(\overline{z})$, $\overline{G}^-(\overline{z})$ of Ingredient III according to (11) one finds: These linear operators are self-adjoint and simultaneously diagonalizable on the space of states $\mathbb{H} = \mathbb{H}^{NS} \oplus \mathbb{H}^R$. By Ingredient IV, the corresponding operator of spectral flow induces an equivalence of representations $\mathbb{H}^{NS} \cong \mathbb{H}^R$ of the two $N = 2$ superconformal algebras. This turns

out to imply that the linear operator $J_0 - \overline{J}_0$ has only integral eigenvalues, which are even on $\mathbb{H}_b$ and odd on $\mathbb{H}_f$, see e.g. [92, Sect. 3.1]. Hence $(-1)^{J_0 - \overline{J}_0}$ is an involution which yields the $\mathbb{Z}_2$-grading $\mathbb{H} = \mathbb{H}_b \oplus \mathbb{H}_f$ and an induced $\mathbb{Z}_2$-grading on $\mathbb{H}^R$. Following [30], this allows the definition of a supercharacter of the superconformal field theory, analogous to the partition function in Property B:

Definition 6 Consider an $N = (2, 2)$ superconformal field theory at central charges $c, \overline{c}$ with space-time supersymmetry. Set $q := \exp(2\pi i \tau)$ for $\tau \in \mathbb{C}$, $\Im(\tau) > 0$, and $y := \exp(2\pi i z)$ for $z \in \mathbb{C}$. Then

$$
\begin{aligned}
\mathcal{E}(\tau, z) &:= \mathrm{Str}_{\mathbb{H}^R} \left(y^{J_0} q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - \overline{c}/24} \right) \\
&= \mathrm{Tr}_{\mathbb{H}^R} \left((-1)^{J_0 - \overline{J}_0} y^{J_0} q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - \overline{c}/24} \right)
\end{aligned}
$$

is the CONFORMAL FIELD THEORETIC ELLIPTIC GENUS of the theory.

Using known properties of the $N = 2$ superconformal algebra and of its irreducible unitary representations, one shows (see [23, 30, 100] for the original results and e.g. [92, Sect. 3.1] for a summary and proofs):

Proposition 1 *Consider the conformal field theoretic elliptic genus $\mathcal{E}(\tau, z)$ of an $N = (2, 2)$ superconformal field theory at central charges $c, \overline{c}$ with space-time supersymmetry.*
Then $\mathcal{E}(\tau, z)$ is holomorphic in τ and bounded when $\tau \to i\infty$.
It is invariant under smooth deformations of the underlying superconformal field theory to any other space-time supersymmetric $N = (2, 2)$ superconformal field theory with the same central charges.
Moreover, $\mathcal{E}(\tau, z)$ transforms covariantly under modular transformations,

$$
\mathcal{E}(\tau + 1, z) = \mathcal{E}(\tau, z), \quad \mathcal{E}(-1/\tau, z/\tau) = e^{2\pi i \frac{c}{6} \cdot \frac{z^2}{\tau}} \mathcal{E}(\tau, z).
$$

If in addition $c = \overline{c} \in 3\mathbb{N}$, and all eigenvalues of J_0 and $\overline{J}_0$ in the Ramond sector lie in $\frac{c}{6} + \mathbb{Z}$, then

$$
\mathcal{E}(\tau, z + 1) = (-1)^{\frac{c}{3}} \mathcal{E}(\tau, z), \quad \mathcal{E}(\tau, z + \tau) = q^{-\frac{c}{6}} y^{-\frac{c}{3}} \mathcal{E}(\tau, z).
$$

In other words, $\mathcal{E}(\tau, z)$ is a WEAK JACOBI FORM (with a character, if $c/3$ is odd) of WEIGHT 0 and INDEX $c/6$.

Note that the additional assumptions on the central charges and the eigenvalues of J_0 and $\overline{J}_0$ in the last statement of Proposition 1 are expected to hold for superconformal field theories that are obtained by a non-linear sigma model construction from some Calabi-Yau D-manifold.

On the other hand, following Hirzebruch's seminal work on multiplicative sequences and their genera [59], the elliptic genus is known to topologists as a ring

homomorphism from the cobordism ring of smooth oriented compact manifolds into a ring of modular functions [60, 66]. For simplicity we assume that our underlying manifold X is a Calabi-Yau D-manifold. Then its associated geometric elliptic genus $\mathcal{E}_X(\tau, z)$ can be viewed as a modular function obeying the transformation properties of Proposition 1 with $c = 3D$ and interpolating between the standard topological invariants of X, namely its *Euler characteristic* $\chi(X)$, its *signature* $\sigma(X)$, and its *topological Euler characteristic* $\chi(\mathcal{O}_X)$.

To understand this in more detail, first recall the definition of the topological invariants mentioned above: For $y \in \mathbb{C}$ the Hirzebruch χ_y-genus [59] is defined by

$$\chi_y(X) := \sum_{p,q=0}^{D} (-1)^q y^p h^{p,q}(X),$$

where the $h^{p,q}(X)$ are the Hodge numbers of X. Then

$$\chi(X) := \chi_{-1}(X), \quad \sigma(X) := \chi_{+1}(X), \quad \chi(\mathcal{O}_X) := \chi_0(X). \tag{16}$$

Note that by the usual symmetries among the Hodge numbers $h^{p,q}(X)$ of a complex Kähler manifold X, the signature $\sigma(X) = \sum_{p,q}(-1)^q h^{p,q}(X)$ vanishes if the complex dimension D of X is odd; we have thus trivially extended the usual definition of the signature on oriented compact manifolds whose real dimension is divisible by 4 to all compact complex Kähler manifolds.

To motivate a standard formula for the specific elliptic genus which is of relevance to us, see Definition 7, we draw the analogy to the interpretation of the topological invariants (16) in terms of the Atiyah-Singer Index Theorem [5]. For any complex vector bundle E on X and a formal variable x, we introduce the shorthand notations

$$\Lambda_x E := \bigoplus_p x^p \Lambda^p E, \quad S_x E := \bigoplus_p x^p S^p E,$$

where $\Lambda^p E$, $S^p E$ denote the exterior and the symmetric powers of E, respectively, along with the *Chern character* on such formal power series in x whose coefficients are complex vector bundles F_p:

$$\mathrm{ch}(\bigoplus_p x^p F_p) := \sum_p x^p \mathrm{ch}(F_p).$$

Then by the Hirzebruch-Riemann-Roch formula [58], which can be viewed as a special case of the Atiyah-Singer Index Theorem, one finds

$$\chi_y(X) = \int_X \mathrm{Td}(X)\mathrm{ch}(\Lambda_y T^*), \tag{17}$$

where $\mathrm{Td}(X)$ denotes the *Todd genus* and $T := T^{1,0}X$ is the holomorphic tangent bundle of X. Generalizing the expression in Eq. (17) and following [60, 65, 97] we now have

Definition 7 Let X denote a compact complex D-manifold with holomorphic tangent bundle $T := T^{1,0}X$. Set

$$\mathbb{E}_{q,-y} := y^{-D/2} \bigotimes_{n=1}^{\infty} \left(\Lambda_{-yq^{n-1}}T^* \otimes \Lambda_{-y^{-1}q^n}T \otimes S_{q^n}T^* \otimes S_{q^n}T \right),$$

viewed as a formal power series with variables $y^{\pm 1/2}$, q, whose coefficients are holomorphic vector bundles on X.

Analogously to Definition 1, the integral $\int_X$ is extended linearly to the vector space of formal power series whose coefficients are characteristic classes on X. Then with $q := \exp(2\pi i\tau)$ and $y := \exp(2\pi iz)$, the holomorphic Euler characteristic of $\mathbb{E}_{q,-y}$,

$$\mathcal{E}_X(\tau, z) := \int\limits_X \mathrm{Td}(X)\mathrm{ch}(\mathbb{E}_{q,-y}) \quad \in \quad y^{-D/2} \cdot \mathbb{Z}[[y^{\pm 1}, q]],$$

is the (geometric) ELLIPTIC GENUS of X.

By [60, 65, 97], the elliptic genus $\mathcal{E}_X(\tau, z)$ in fact yields a well-defined function in $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$ and in $z \in \mathbb{C}$. If X is a Calabi-Yau D-manifold, then $\mathcal{E}_X(\tau, z)$ is a weak Jacobi form (with a character, if D is odd) of weight 0 and index $D/2$ [11]. In other words, with $c := 3D$ the elliptic genus $\mathcal{E}_X(\tau, z)$ obeys the transformation properties stated for $\mathcal{E}(\tau, z)$ in Proposition 1, and it is bounded when $\tau \to i\infty$. One checks that by definition, the elliptic genus indeed is a topological invariant which interpolates between the standard topological invariants of Eq. (16), namely $\mathcal{E}_X(\tau, z) \overset{\tau \to i\infty}{\longrightarrow} y^{-D/2}\chi_{-y}(X)$ and

$$\mathcal{E}_X(\tau, z = 0) = \chi(X), \quad \mathcal{E}_X(\tau, z = 1/2) = (-1)^{D/2}\sigma(X) + \mathcal{O}(q),$$
$$q^{D/4}\mathcal{E}_X(\tau, z = (\tau + 1)/2) = (-1)^{D/2}\chi(\mathcal{O}_X) + \mathcal{O}(q). \tag{18}$$

According to Witten [96, 97], the expression for the elliptic genus $\mathcal{E}_X(\tau, z)$ in Definition 7 can be interpreted as a regularized version of a $U(1)$-equivariant index of a Dirac-like operator on the loop space of X, see also [66]. This explains the notation chosen in Definition 7, and it also motivates why one expects that for CFTs which are obtained by a non-linear sigma model construction from some Calabi-Yau D-manifold X, the conformal field theoretic elliptic genus of Definition 6 agrees with the geometric elliptic genus of X as in Definition 7. Note that the resulting equation

$$\mathcal{E}_X(\tau, z) = \mathrm{Str}_{\mathbb{H}^R} \left(y^{J_0}q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24} \right) \tag{19}$$

would furnish a natural generalization of the *McKean-Singer Formula* [72]. While non-linear sigma model constructions are not understood sufficiently well to even attempt a general proof of this equation, there is some evidence for its truth. On the one hand, as was pointed out in Sect. 2.3, $N = (2, 2)$ superconformal field theories obtained from a non-linear sigma model on a complex torus are very well understood. One confirms that their conformal field theoretic elliptic genus vanishes, as does the geometric elliptic genus of a complex torus. Equation (19) is also compatible with the construction of symmetric powers of the manifold X [25]. Moreover, compatibility of the elliptic genus with orbifold constructions was proved in [12, 40]. Further evidence in favor of the expectation (19) arises from a discussion of the chiral de Rham complex, see Sect. 2.5.

2.5 The Chiral de Rham Complex

As was pointed out above, non-linear sigma model constructions of $N = (2, 2)$ superconformal field theories are in general not very well understood. Therefore, a direct proof of the expected equality (19) is out of reach. However, instead of a full-fledged superconformal field theory, in [70] the authors construct a *sheaf of superconformal vertex algebras*, known as the *chiral de Rham complex* Ω_X^{ch}, on any complex manifold X. The chiral de Rham complex of X is expected to be closely related to the non-linear sigma model on X, as we shall discuss in the present section.

Let us begin by summarizing the construction of the chiral de Rham complex Ω_X^{ch} for a complex D-dimensional manifold X, see [8, 56, 69, 70]. First, to any coordinate neighborhood $U \subset X$ with holomorphic coordinates $(z^1, \ldots, z^D)$ one associates a $bc - \beta\gamma$ system $\Omega_X^{\mathrm{ch}}(U)$ as in Example 3, see Sect. 2.1. Here, the even fields a^j, b^j are interpreted as arising from quantizing the local sections $\partial/\partial z^j$, z^j of the sheaf of polyvector fields on X, while the odd fields ϕ^j, ψ^j correspond to the local sections dz^j, $\partial/\partial(dz^j)$ of the sheaf of differential operators on the de Rham algebra of differential forms. Indeed, by (3) the map

$$(\partial/\partial z^j, \; z^j, \; dz^j, \; \partial/\partial(dz^j)) \longmapsto (a_0^j, \; b_0^j, \; \phi_0^j, \; \psi_0^j)$$

induces a super-Lie algebra homomorphism.

According to [70], coordinate transforms on X induce corresponding transformation rules for the fields a^j, b^j, ϕ^j, ψ^j which are compatible with the structure of the $bc - \beta\gamma$-system as discussed in Example 3. This allows to glue the $\Omega_X^{\mathrm{ch}}(U)$ accordingly, and by localization, one indeed obtains a well-defined sheaf of vertex algebras over X, with a (non-associative) action of $\mathcal{O}_X$ on it.

A key result of [70] is the fact that under appropriate assumptions on X, there are well-defined global sections of the sheaf $\mathrm{End}_{\mathbb{C}}(\Omega_X^{\mathrm{ch}})[[z^{\pm 1}]]$, which are locally given by the fields (4), (5) of the topological $N = 2$ superconformal algebra (6)–(7) discussed in Example 4 of Sect. 2.1:

Theorem 2 ([70]) *Let X denote a compact complex manifold of dimension D. As discussed above, there is an associated sheaf Ω_X^{ch} of vertex algebras on X. On every holomorphic coordinate chart $U \subset X$, let $T^{\mathrm{top}}(z)$, $J(z)$, $Q(z)$, $G(z)$ denote the local sections in $\mathrm{End}_{\mathbb{C}}(\Omega_X^{\mathrm{ch}}(U))[[z^{\pm 1}]]$ defined by (4), (5), with mode expansions*

$$T^{\mathrm{top}}(z) = \sum_n L_n^{\mathrm{top}} z^{n-2}, \quad J(z) = \sum_n J_n z^{n-1},$$

$$Q(z) = \sum_n Q_n z^{n-1}, \quad G(z) = \sum_n G_n z^{n-2},$$

respectively. Then the following holds:

1. *The linear operators $F := J_0$ and $d_{\mathrm{dR}}^{\mathrm{ch}} := -Q_0$ are globally well-defined. Moreover, F defines a $\mathbb{Z}$-grading on Ω_X^{ch}, while $(d_{\mathrm{dR}}^{\mathrm{ch}})^2 = 0$, such that*

$$\forall p \in \mathbb{Z}: \quad \Omega_X^{\mathrm{ch},p}(U) := \left\{ \Phi \in \Omega_X^{\mathrm{ch}}(U) \mid F\Phi = p\Phi \right\}$$

   yields a complex $(\Omega_X^{\mathrm{ch},\bullet}, d_{\mathrm{dR}}^{\mathrm{ch}})$, which is called the CHIRAL DE RHAM COMPLEX.
2. *The map $(z^j, dz^j) \mapsto (b_0^j, \phi_0^j)$ induces a quasi-isomorphism from the usual de Rham complex to the chiral de Rham complex of X.*
3. *The local fields $T^{\mathrm{top}}(z)$ given in (4) define a global field on the chiral de Rham complex, by which we mean a global section of the sheaf $\mathrm{End}_{\mathbb{C}}(\Omega_X^{\mathrm{ch}})[[z^{\pm 1}]]$. The chiral de Rham complex therefore is bigraded by F and L_0^{top}.*
4. *If X is a Calabi-Yau manifold, then the local fields $J(z)$, $Q(z)$, $G(z)$ given in (5) also define global fields on the chiral de Rham complex.*

As mentioned above, the sheaf Ω_X^{ch} is not quasi-coherent. However, it has a filtration which is compatible with the bigrading of Theorem 2 and such that the corresponding graded object yields a quasi-coherent sheaf isomorphic to (the sheaf of sections of) $(-y)^{D/2}\mathbb{E}_{q,y}$ as in Definition 7. This is used extensively in [10, 11] to study the Čech cohomology $H^*(X, \Omega_X^{ch})$. Note that this means classical Čech cohomology, ignoring the differential $d_{\mathrm{dR}}^{\mathrm{ch}}$ of the chiral de Rham complex. The authors find:

Theorem 3 ([10, 11]) *Consider a Calabi-Yau D-manifold X, and the Čech cohomology $H^*(X, \Omega_X^{ch})$ of its chiral de Rham complex Ω_X^{ch}. Equip it with the induced bigrading by the operators $F = J_0$ and L_0^{top} of Theorem 2 and the $\mathbb{Z}_2$-grading by $(-1)^{F+q}$ on $H^q(X, \Omega_X^{ch})$. Then $H^*(X, \Omega_X^{ch})$ carries a natural structure of a topological $N = 2$ superconformal vertex algebra [10, Proposition 3.7 and Definition 4.1]. Moreover [11], the GRADED EULER CHARACTERISTIC of the chiral de Rham complex, that is, the supertrace of the operator $y^{-D/2} \cdot (y^{J_0} q^{L_0^{\mathrm{top}}})$ on $H^*(X, \Omega_X^{ch})$, yields the elliptic genus $\mathcal{E}_X(\tau, z)$ of Definition 7.*

Thus Theorem 3 indicates a possible relationship between the chiral de Rham complex Ω_X^{ch} of a Calabi-Yau D-manifold X and a non-linear sigma model on X,

since it recovers the (geometric!) elliptic genus $\mathcal{E}_X(\tau, z)$ by means of a supertrace which at least in spirit agrees with the expression on the right hand side of Eq. (19). Note that $L_0^{\text{top}} = L_0 - \frac{1}{2}J_0$ by (8) [using Definition 1 and (11)]. Therefore, using the fact that the elliptic genus is holomorphic, along with the spectral flow (13), the conformal field theoretic elliptic genus of Definition 6 can be expressed as

$$\mathcal{E}(\tau, z) = \text{Str}_{\mathbb{H}^R}\left(y^{J_0}q^{L_0 - c/24}\right)$$
$$= y^{-c/6}\text{Str}_{\mathbb{H}^{NS}}\left((yq^{-1/2})^{J_0}q^{L_0}\right) = y^{-c/6}\text{Str}_{\mathbb{H}^{NS}}\left(y^{J_0}q^{L_0^{\text{top}}}\right).$$

Hence recalling $c = 3D$ for a non-linear sigma model on a Calabi-Yau D-manifold, one is led to conjecture that one might be able to identify an appropriate cohomology of $\mathbb{H}^{NS}$ with $H^*(X, \Omega_X^{ch})$.

The details of such an identification are still more subtle, however. Indeed, by construction, the chiral de Rham complex depends only on the complex structure of X, while the non-linear sigma model, in addition, depends on the complexified Kähler structure of X. It is therefore natural to expect the vertex algebra of Theorem 3 to yield a truncated version of the non-linear sigma model by means of the topological twists mentioned in Example 5 of Sect. 2.1. Since the crucial bundle $\mathbb{E}_{q,-y}$ of Definition 7 resembles an infinite-dimensional Fock space, while the traditional topological A- and B-twists yield finite dimensional spaces of states, the so-called *half-twisted sigma model* according to Witten [99] is the most natural candidate. It still cannot yield the vertex algebra of Theorem 3, since it depends both on the complex and on the complexified Kähler structure of X. Moreover, the Čech resolution, which is implicit in $H^*(X, \Omega_X^{ch})$, does not resemble the standard features of non-linear sigma models on X. According to Kapustin, however, a large volume limit of Witten's half twisted sigma model on X yields the cohomology of Ω_X^{ch} with respect to yet another resolution of the complex, the so-called *Dolbeault resolution* [62] .

3 Conformal Field Theory on K3

As emphasized repeatedly, non-linear sigma model constructions are in general not well understood, except for the toroidal conformal field theories presented in Sect. 2.3. Recall however that there are only two topologically distinct types of Calabi-Yau 2-manifolds, namely the complex 2-tori and the K3 surfaces (see e.g. [6, Chap. VIII] for an excellent introduction to the geometry of K3 surfaces). By the Kummer construction, one obtains an example of a K3 surface by means of a $\mathbb{Z}_2$-orbifold procedure from every complex 2-torus. On the other hand, $\mathbb{Z}_2$-orbifolds of the toroidal CFTs are also reasonably well understood. One therefore expects to be able to construct examples of CFTs which allow a non-linear sigma model interpretation on some K3 surface. Compared to CFTs on higher-dimensional Calabi-Yau D-manifolds, those on K3 surfaces indeed provide a borderline case, in the sense

that much more is known about these so-called *K3 theories*. Most importantly, we can give a mathematical definition of such theories without ever mentioning non-linear sigma model constructions. The current section presents this definition and summarizes some of the known properties of K3 theories.

To motivate the mathematical definition of K3 theories, let us recall the conformal field theoretic elliptic genus of Sect. 2.4. Here we assume that we are given an $N = (2, 2)$ superconformal field theory that obeys the following conditions, which are necessary for the CFT to allow a non-linear sigma model interpretation on some Calabi-Yau 2-manifold: The theory is superconformal at central charges $c = 6$, $\overline{c} = 6$ with space-time supersymmetry, and such that all eigenvalues of J_0 and $\overline{J}_0$ are integral. This latter condition is equivalent to the assumption that in addition to the spectral flow operator of Ingredient IV in Sect. 2.2, the theory possesses a quartet of *two-fold left- and right-handed spectral flow operators* $\Theta^{\pm}$, $\overline{\Theta}^{\pm}$. By this we mean that these operators act analogously to $\Theta^{\pm 2}$ on the space of states, with Θ as in (13), namely

$$[L_0, \Theta^{\pm}] = \tfrac{c}{6}\Theta^{\pm} \mp \Theta^{\pm} \circ J_0, \ \ [J_0, \Theta^{\pm}] = \mp\tfrac{c}{3}\Theta^{\pm},$$
$$[\overline{L}_0, \overline{\Theta}^{\pm}] = \tfrac{\overline{c}}{6}\overline{\Theta}^{\pm} \mp \overline{\Theta}^{\pm} \circ \overline{J}_0, \ \ [\overline{J}_0, \overline{\Theta}^{\pm}] = \mp\tfrac{\overline{c}}{3}\overline{\Theta}^{\pm},$$

but with all other commutators vanishing. The fields associated to $\Theta^{\pm}\Omega$, $\overline{\Theta}^{\pm}\Omega$ by the state-field correspondence (Definition 3) are denoted $J^{\pm}(z)$ and $\overline{J}^{\pm}(\overline{z})$, respectively. By Proposition 1, the conformal field theoretic elliptic genus $\mathcal{E}(\tau, z)$ of such a CFT is a weak Jacobi form of weight 0 and index 1. However, the space of such Jacobi forms is one-dimensional, as follows from the methods introduced in [36] (see [11] or [92, Theorem. 3.1.12] for direct proofs). According to the discussion that follows Definition 7, the (geometric) elliptic genus $\mathcal{E}_{K3}(\tau, z)$ of a K3 surface is a weak Jacobi form of weight 0 and index 1 as well, which by (18) is non-zero, since $\mathcal{E}_{K3}(\tau, z = 0) = \chi(K3) = 24$. The precise form of the function $\mathcal{E}_{K3}(\tau, z)$ is well-known, and we obtain

$$\mathcal{E}(\tau, z) = a \cdot \mathcal{E}_{K3}(\tau, z) = a \cdot \left(2y + 20 + 2y^{-1} + \mathcal{O}(q)\right) \tag{20}$$

for some constant a. In fact,

Proposition 2 ([92, Sect. 7.1]) *Consider an $N = (2, 2)$ superconformal field theory at central charges $c = 6$, $\overline{c} = 6$ with space-time supersymmetry and such that all the eigenvalues of J_0 and of $\overline{J}_0$ are integral.*

1. *The elliptic genus of this CFT either vanishes, or it agrees with the geometric elliptic genus $\mathcal{E}_{K3}(\tau, z)$ of a K3 surface.*
2. *The conformal field theoretic elliptic genus vanishes if and only if the theory is a toroidal $N = (2, 2)$ superconformal field theory according to Definition 5.*

This result is mentioned in [75] and proved in [92, Sect. 7.1], where the proof however contains a few typos. The sketch of a corrected proof is banned to the

Appendix, since it uses a number of properties of superconformal field theories with space-time supersymmetry which are well-known to the experts, but which we have not derived in this exposition.

While as mentioned before, the toroidal $N = (2, 2)$ superconformal field theories are well understood, it is also not hard to find examples of theories whose conformal field theoretic elliptic genus is $\mathcal{E}_{K3}(\tau, z)$, see [30]. In particular, the authors of [30] prove that the standard $\mathbb{Z}_2$-orbifold of every toroidal $N = (2, 2)$ superconformal field theory at central charges $c = 6$, $\bar{c} = 6$ yields such an example. By the above this is in accord with the expectations based on the Kummer construction, hence our

Definition 8 A superconformal field theory is called a K3 THEORY, if the following conditions hold: The CFT is an $N = (2, 2)$ superconformal field theory at central charges $c = 6$, $\bar{c} = 6$ with space-time supersymmetry, all the eigenvalues of J_0 and of $\overline{J}_0$ are integral, and the conformal field theoretic elliptic genus of the theory is

$$\mathcal{E}(\tau, z) = \mathcal{E}_{K3}(\tau, z).$$

Possibly, every K3 theory allows a non-linear sigma model interpretation on some K3 surface, however a proof is far out of reach. Nevertheless, under standard assumptions on the deformation theory of such theories it is possible to determine the form of every connected component of the moduli space of K3 theories. Namely, one assumes that all deformations by so-called marginal operators are integrable for these theories, an assumption which can be justified in string theory and which is demonstrated to all orders of perturbation theory in [26]. Then, based on the previous results [18, 83], one obtains

Theorem 4 ([3, 75]) *With the notations introduced in Theorem 1, let $\mathcal{T}^{4,20}$ denote the Grassmannian of maximal positive definite oriented subspaces of $\mathbb{R}^{4,20}$,*

$$\mathcal{T}^{4,20} := O^+(4, 20; \mathbb{R})/SO(4) \times O(20).$$

By $\mathcal{T}_0^{4,20} \subset \mathcal{T}^{4,20}$ we denote the set of all those maximal positive definite oriented subspaces $x \subset \mathbb{R}^{4,20}$ which have the property that $x^\perp$ does not contain any roots, that is, all $\alpha \in x^\perp \cap \mathbb{Z}^{4,20}$ obey $\langle \alpha, \alpha \rangle \neq -2$.

If the above-mentioned assumptions on deformations of K3 theories hold, namely that all deformations by so-called marginal operators are integrable, then each connected component $\mathcal{M}_s^{K3}$ of the moduli space of K3 theories has the following form:

$$\mathcal{M}_s^{K3} = O^+(4, 20; \mathbb{Z}) \backslash \mathcal{T}_0^{4,20}.$$

This result reinforces the expectation that one connected component $\mathcal{M}_\sigma^{K3}$ of the moduli space of K3 theories can be identified with the space of non-linear sigma models on K3 surfaces, since in addition, we have

Proposition 3 ([3]) *The partial completion $\mathcal{T}^{4,20}$ of the smooth universal covering space $\mathcal{T}_0^{4,20}$ of $\mathcal{M}_s^{K3}$ can be isometrically identified with the* PARAMETER SPACE OF

NON- LINEAR SIGMA MODELS ON $K3$. *Namely, denoting by X the diffeomorphism type of a K3 surface, $\mathcal{T}^{4,20}$ is a cover of the space of triples (Σ, V, B) where Σ denotes a* HYPERKÄHLER STRUCTURE *on X, $V \in \mathbb{R}^+$ is interpreted as the* VOLUME *of X, and B is the de Rham cohomology class of a real closed two-form on X, a so-called* B- FIELD.

If a K3 theory in $\mathcal{M}_s^{K3}$ lifts to a point in $\mathcal{T}^{4,20}$ which is mapped to the triple (Σ, V, B), then (Σ, V, B) is called a GEOMETRIC INTERPRETATION *of the K3 theory.*

In [75, 93] it is shown that the expectation that non-linear sigma models on K3 yield K3 theories indeed is compatible with orbifold constructions, more precisely with every orbifold construction of a K3 surface from a complex two-torus by means of a discrete subgroup of SU(2). As mentioned above, one might conversely expect that every K3 theory with geometric interpretation (Σ, V, B) can be constructed as a non-linear sigma model on a K3 surface, specified by the data (Σ, V, B) – at least the existence of a non-linear sigma model interpretation has not been disproved for any K3 theory, so far.

The statement of Proposition 3 makes use of the fact that every K3 surface is a hyperkähler manifold. The analogous statement for K3 theories is the observation that the two commuting copies of $N = 2$ superconformal algebras (9)–(10) are each extended to an $N = 4$ *superconformal algebra* in these theories. This is a direct consequence of our Definition 8 of K3 theories. Indeed, as mentioned at the beginning of this section, the assumption of space-time supersymmetry together with the integrality of the eigenvalues of J_0 and $\overline{J}_0$ imply that the fields $J^{\pm}(z)$, $\overline{J}^{\pm}(\overline{z})$ corresponding to two-fold left- and right-handed spectral flow are fields of the CFT. One checks that at central charges $c = 6$, $\overline{c} = 6$, these fields create states in the subspaces W_1 and $\overline{W}_1$ of the vector spaces underlying the chiral algebras of Property C (see the vacuum axiom in Definition 3), whose J_0- (respectively $\overline{J}_0$-) eigenvalues are ± 2. Moreover, with the $U(1)$-currents $J(z)$, $\overline{J}(\overline{z})$ of the two commuting copies of $N = 2$ superconformal vertex algebras, the fields $J^{\pm}(z)$, $\overline{J}^{\pm}(\overline{z})$ generate two commuting copies of a so-called $\mathfrak{su}(2)_1$-*current algebra*, which in turn is known to extend the $N = 2$ superconformal algebra to an $N = 4$ superconformal algebra [1].

The characters of the irreducible unitary representations of the $N = 4$ superconformal algebra at arbitrary central charges have been determined in [31–34, 84]. Their transformation properties under modular transforms in general are *not* modular, in contrast to the situation at lower supersymmetry, where an infinite class of characters of irreducible unitary representations does enjoy modularity. Instead, the $N = 4$ characters exhibit a so-called *Mock modular* behavior, see e.g. [22] for a recent account. Since in the context of non-linear sigma models, $N = 4$ supersymmetry is linked to the geometric concept of hyperkähler manifolds [2], this seems to point towards a connection between Mock modularity and hyperkähler geometry. The nature of this connection however, to date, is completely mysterious.

4 The Elliptic Genus of K3

Recall that the elliptic genus $\mathcal{E}_{K3}(\tau, z)$ of K3 plays center stage in our Definition 8 of K3 theories. Though this function is explicitly known and well understood, recent years have uncovered a number of mysteries around it. In the present section, some of these mysteries are discussed. This involves more open than solved problems, and as a reminder, the titles of all the following subsections are questions instead of statements.

4.1 A Non-geometric Decomposition of the Elliptic Genus?

As was mentioned at the end of Sect. 3, our very Definition 8 ensures that every K3 theory enjoys $N = (4, 4)$ supersymmetry. The current section summarizes how this induces a decomposition of the function $\mathcal{E}_{K3}(\tau, z)$, which is a priori not motivated geometrically and which turns out to bear some intriguing surprises.

In what follows, assume that we are given a K3 theory according to Definition 8 with space of states $\mathbb{H} = \mathbb{H}^{NS} \oplus \mathbb{H}^{R}$. Both $\mathbb{H}^{NS}$ and $\mathbb{H}^{R}$ can be decomposed into direct sums of irreducible unitary representations with respect to the $N = (4, 4)$ superconformal symmetry. According to [31, 32], there are three types of irreducible unitary representations of the $N = 4$ superconformal algebra at central charge $c = 6$, namely the *vacuum representation*, the *massless matter representation*, and finally the *massive matter representations* which form a one-parameter family indexed by $h \in \mathbb{R}_{>0}$. For later convenience we focus on the Ramond-sector $\mathbb{H}^{R}$ of our theory and denote the respective irreducible unitary representations by $\mathcal{H}_0$, $\mathcal{H}_{\mathrm{mm}}$, $\mathcal{H}_h$ ($h \in \mathbb{R}_{>0}$). This notation alludes to the properties of the corresponding representations in the Neveu-Schwarz sector $\mathbb{H}^{NS}$, which are related to the representations in $\mathbb{H}^{R}$ by spectral flow Θ according to (13). Indeed, the vacuum representation in the NS-sector has the vacuum Ω as its ground state. The massive matter representations are characterized by the spontaneous breaking of supersymmetry at every mass level, including the ground state [95].

Setting $y = \exp(2\pi i z)$ and $q = \exp(2\pi i \tau)$ for $z, \tau \in \mathbb{C}$ with $\Im(\tau) > 0$ as before and using $c/24 = 1/4$, the characters of the irreducible unitary $N = 4$ representations that are relevant to our discussion are denoted by

$$\chi_a(\tau, z) := \mathrm{Str}_{\mathcal{H}_a}\left(y^{J_0} q^{L_0 - 1/4}\right) = \mathrm{Tr}_{\mathcal{H}_a}\left((-1)^{J_0} y^{J_0} q^{L_0 - 1/4}\right), \quad a \in \mathbb{R}_{\geq 0} \cup \{\mathrm{mm}\}.$$

These functions have been determined explicitly in [32]. For our purposes, only the following properties are relevant,

$$\chi_0(\tau, z = 0) = -2, \qquad\qquad \chi_{\mathrm{mm}}(\tau, z = 0) = 1,$$
$$\forall h > 0: \quad \chi_h(\tau, z) = q^h \widehat{\chi}(\tau, z) \qquad \text{with} \quad \widehat{\chi}(\tau, z) = \chi_0(\tau, z) + 2\chi_{\mathrm{mm}}(\tau, z),$$
$$\text{hence} \quad \chi_h(\tau, z = 0) = \widehat{\chi}(\tau, z = 0) = 0. \tag{21}$$

The constant $\chi_a(\tau, z = 0)$ yields the so-called *Witten index* [95] of the respective representation.

The most general ansatz for a decomposition of $\mathbb{H}^R$ into irreducible representations of the two commuting $N = 4$ superconformal algebras therefore reads

$$\mathbb{H}^R = \bigoplus_{a,\,\overline{a} \in \mathbb{R}_{\geq 0} \cup \{\mathrm{mm}\}} m_{a,\overline{a}} \mathcal{H}_a \otimes \overline{\mathcal{H}_{\overline{a}}}$$

with appropriate non-negative integers $m_{a,\overline{a}}$. Then

$$\mathrm{Tr}_{\mathbb{H}^R}\left((-1)^{J_0 - \overline{J}_0} y^{J_0} \overline{y}^{\overline{J}_0} q^{L_0 - 1/4} \overline{q}^{\overline{L}_0 - 1/4} \right) = \sum_{a,\,\overline{a} \in \mathbb{R}_{\geq 0} \cup \{\mathrm{mm}\}} m_{a,\overline{a}} \cdot \chi_a(\tau, z) \cdot \overline{\chi_{\overline{a}}(\tau, z)},$$

together with Definition 6 yields the conformal field theoretic elliptic genus of our CFT as

$$\mathcal{E}(\tau, z) = \sum_{a,\,\overline{a} \in \mathbb{R}_{\geq 0} \cup \{\mathrm{mm}\}} m_{a,\overline{a}} \cdot \chi_a(\tau, z) \cdot \overline{\chi_{\overline{a}}(\tau, z = 0)}. \tag{22}$$

This expression simplifies dramatically on insertion of (21). In addition, the known properties of K3 theories impose a number of constraints on the coefficients $m_{a,\overline{a}}$. First, since under spectral flow, $\mathcal{H}_0$ is mapped to the irreducible representation of the $N = 4$ superconformal algebra whose ground state is the vacuum Ω, the uniqueness of the vacuum (see Property C) implies $m_{0,0} = 1$. Moreover, from the proof of Proposition 2 (see the Appendix) or from the known explicit form (20) of $\mathcal{E}(\tau, z)$, we deduce that in every K3 theory, $m_{0,\mathrm{mm}} = m_{\mathrm{mm},0} = 0$. Finally, according to the discussion of Property B in Sect. 2.2, $\mathbb{H}_{h,\overline{h}} \cap \mathbb{H}_b$ can only be non-trivial if $h - \overline{h} \in \mathbb{Z}$, which on $\mathbb{H}_{h,\overline{h}} \cap \mathbb{H}_f \cap \mathbb{H}^{NS}$ generalizes to $h - \overline{h} \in \frac{1}{2} + \mathbb{Z}$. Since the groundstates of $\mathcal{H}_0$, $\mathcal{H}_{\mathrm{mm}}$, $\mathcal{H}_h$ under spectral flow yield states with L_0-eigenvalues 0, $\frac{1}{2}$, h, and J_0-eigenvalues 0, ± 1, 0, respectively [32], this implies that $m_{0,\overline{h}}$, $m_{\mathrm{mm},\overline{h}}$, $m_{h,0}$, $m_{h,\mathrm{mm}}$ with $h, \overline{h} > 0$ can only be non-zero if $h, \overline{h} \in \mathbb{N}$. In conclusion, we obtain a refined ansatz for the $N = (4, 4)$ decomposition of $\mathbb{H}^R$,

$$\mathbb{H}^R = \mathcal{H}_0 \otimes \overline{\mathcal{H}_0} \oplus h^{1,1} \mathcal{H}_{\mathrm{mm}} \otimes \overline{\mathcal{H}_{\mathrm{mm}}} \oplus \bigoplus_{h,\,\overline{h} \in \mathbb{R}_{>0}} k_{h,\overline{h}} \mathcal{H}_h \otimes \overline{\mathcal{H}_{\overline{h}}}$$
$$\oplus \bigoplus_{n=1}^{\infty} \left[f_n \mathcal{H}_n \otimes \overline{\mathcal{H}_0} \oplus \overline{f_n} \mathcal{H}_0 \otimes \overline{\mathcal{H}_n} \right] \tag{23}$$
$$\oplus \bigoplus_{n=1}^{\infty} \left[g_n \mathcal{H}_n \otimes \overline{\mathcal{H}_{\mathrm{mm}}} \oplus \overline{g_n} \mathcal{H}_{\mathrm{mm}} \otimes \overline{\mathcal{H}_n} \right].$$

Here, all the coefficients $h^{1,1}$, $k_{h,\overline{h}}$, f_n, $\overline{f}_n$, g_n, $\overline{g}_n$ are non-negative integers, whose precise values depend on the specific K3 theory under inspection.

By (22), and inserting (21) and the refined ansatz (23), we obtain

$$\mathcal{E}(\tau, z) = -2\chi_0(\tau, z) + h^{1,1}\chi_{\mathrm{mm}}(\tau, z) + \sum_{n=1}^{\infty}[-2f_n + g_n]\,\chi_n(\tau, z)$$

$$= -2\chi_0(\tau, z) + h^{1,1}\chi_{\mathrm{mm}}(\tau, z) + e(\tau)\,\widehat{\chi}(\tau, z) \text{ with } e(\tau) := \sum_{n=1}^{\infty}[g_n - 2f_n]\,q^n.$$

Now recall from Definition 8 that $\mathcal{E}(\tau, z) = \mathcal{E}_{\mathrm{K3}}(\tau, z)$, where by the discussion preceding (20) we have $\mathcal{E}_{\mathrm{K3}}(\tau, z = 0) = 24$. Using (21), this implies $h^{1,1} = 20$. Since the geometric elliptic genus $\mathcal{E}_{\mathrm{K3}}(\tau, z)$ is a topological invariant of all K3 surfaces, we conclude

Proposition 4 *The elliptic genus $\mathcal{E}_{\mathrm{K3}}(\tau, z)$ of K3 decomposes into the characters of irreducible unitary representations of the $N = 4$ superconformal algebra in the Ramond sector according to*

$$\mathcal{E}_{\mathrm{K3}}(\tau, z) = -2\chi_0(\tau, z) + 20\chi_{\mathrm{mm}}(\tau, z) + e(\tau)\,\widehat{\chi}(\tau, z),$$

$$\text{where } e(\tau) := \sum_{n=1}^{\infty}[g_n - 2f_n]\,q^n,$$

and the coefficients g_n, f_n give the respective multiplicities of representations in the decomposition (23). While the values of g_n, f_n vary within the moduli space of K3 theories, the coefficients $g_n - 2f_n$ of $e(\tau)$ are invariant.

A decomposition of $\mathcal{E}_{\mathrm{K3}}(\tau, z)$ in the spirit of Proposition 4 was already given in [30]. In [77] and independently in [92, Conjecture 7.2.2] it was conjectured that all coefficients of the function $e(\tau)$ are non-negative, for the following reason: Recall that under spectral flow, the irreducible representation $\mathcal{H}_0$ is mapped to the representation of the $N = 4$ superconformal algebra whose ground state is the vacuum Ω. Therefore, in (23), the coefficients f_n determine those contributions to the subspace $W_n \subset W$ of the vector space underlying the chiral algebra of Property C that do *not* belong to the vacuum representation under the $N = (4, 4)$ supersymmetry. For any fixed value of $n \in \mathbb{N}$ with $n > 0$, we generically expect no such additional contributions to W_n. In other words, we expect that generically $f_n = 0$ and thus that the nth coefficient of $e(\tau)$ agrees with $g_n \geq 0$. Since these coefficients are invariant on the moduli space of K3 theories, they should always be non-negative.

The conjectured positivity of the coefficients $g_n - 2f_n$ is proved in [27, 29] in the context of an intriguing observation. Namely, in [29], Eguchi, Ooguri and Tachikawa observe that each of these coefficients seems to give the dimension of a representation of a certain sporadic group, namely of the *Mathieu group* M_{24}. For small values of n,

they find dimensions of irreducible representations, while at higher order, more work is required to arrive at a well-defined conjecture. The quest for understanding this observation, which is often referred to as *Mathieu Moonshine*, has sparked enormous interest in the mathematical physics community. Building on results of [19, 28, 44, 45], the observation has been recently verified by Gannon in the following form:

Theorem 5 ([50]) *There are virtual representations of the Mathieu group M_{24} on spaces $\mathcal{R}_0$, $\mathcal{R}_{mm}$, and true representations on spaces $\mathcal{R}_n$, $n \in \mathbb{N}_{>0}$, such that*

$$\mathcal{R} := \mathcal{H}_0 \otimes \mathcal{R}_0 \oplus \mathcal{H}_{mm} \otimes \mathcal{R}_{mm} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_n \otimes \mathcal{R}_n$$

has the following properties: With the $N = 4$ superconformal algebra acting non-trivially only on the first factor in each summand of $\mathcal{R}$, and the Mathieu group M_{24} acting non-trivially only on the second factor, one obtains functions

$$\forall g \in M_{24}: \quad \mathcal{E}_g(\tau, z) := Tr_{\mathcal{R}} \left(g y^{J_0} q^{L_0 - 1/4} \right)$$

which under modular transformations generate a collection of M_{24}-TWISTED ELLIPTIC GENERA *of K3. In particular, $\mathcal{E}_{id}(\tau, z) = \mathcal{E}_{K3}(\tau, z)$.*

4.2 A Geometric Mathieu Moonshine Phenomenon?

While Theorem 5 beautifully specifies a well-defined formulation of the Mathieu Moonshine observation and proves it, the proof does not offer any insight into the role of the Mathieu group M_{24} in the context of K3 theories. The present section summarizes some ideas for a possible interpretation that is based in geometry.

Indeed, the relevance of the group M_{24} for the geometry of K3 surfaces had been discovered much earlier by Mukai:

Theorem 6 ([73]) *Let G denote a finite group of* SYMPLECTIC AUTOMORPHISMS *of a K3 surface X. By this we mean that X denotes a K3 surface whose complex structure has been fixed, and that G is a finite group of biholomorphic maps on X whose induced action on the holomorphic volume form is trivial.*

Then G is a subgroup of the Mathieu group M_{24}. More precisely, G is a subgroup of one out of a list of 11 subgroups of $M_{23} \subset M_{24}$, the largest of which has order 960.

Hence although M_{24} does play a crucial role in describing symplectic automorphisms of K3 surfaces, Theorem 6 cannot immediately explain Mathieu Moonshine. Indeed, Mathieu Moonshine suggests that there is an action of the entire group M_{24} on some mathematical object which underlies the elliptic genus of K3, while Theorem 6 implies that no K3 surface allows M_{24} as its symplectic automorphism group.

Namely, the theorem states that the maximal order of a symplectic automorphism group of any K3 surface is 960, which is smaller by orders of magnitude than the order 244.823.040 of M_{24}.

Since the *non-geometric* decomposition of $\mathcal{E}_{K3}(\tau, z)$ by means of $N = 4$ supersymmetry presented in Sect. 4.1 led to the discovery of Mathieu Moonshine, one may suspect that rather than the properties of K3 surfaces, the properties of K3 theories should explain the Mathieu Moonshine phenomena. However, symmetry groups of K3 theories, in general, need not be subgroups of M_{24}, as apparently was first noted independently by the authors of [29, 85]. Conversely, no K3 theory can have M_{24} as its symmetry group, as follows from [46], where Gaberdiel, Hohenegger and Volpato generalize a very enlightening second proof of Theorem 6 due to Kondo [64] to a classification result for symmetries of K3 theories.

Because by the above, the symmetries of K3 theories seem not to explain Mathieu Moonshine, in a series of papers [85–87] it has been argued that possibly, the action of M_{24} arises as a *combined action* of all finite symplectic symmetry groups of K3 surfaces. This idea can be motivated by the mathematical properties of the elliptic genus which were presented in Sects. 2.4 and 2.5:

By Theorem 3, the geometric elliptic genus $\mathcal{E}_{K3}(\tau, z)$ is recovered from the chiral de Rham complex Ω_X^{ch} of a K3 surface X as its graded Euler characteristic, that is, as the supertrace of the appropriate operator on the Čech cohomology $H^*(X, \Omega_X^{\mathrm{ch}})$. In accord with [56, (2.1.3)], one can expect that every symplectic automorphism of a K3 surface X induces an action on $H^*(X, \Omega_X^{\mathrm{ch}})$. Therefore, the Čech cohomology $H^*(K3, \Omega_{K3}^{\mathrm{ch}})$ of the chiral de Rham complex appears to be an excellent candidate for the desired mathematical object which both underlies the elliptic genus, and which carries actions of all finite symplectic automorphism groups of K3 surfaces, thus combining them to the action of a possibly larger group.

Note that according to Theorem 3, there exists a natural structure of a vertex algebra on $H^*(K3, \Omega_{K3}^{\mathrm{ch}})$. This additional structure on the mathematical object which underlies the elliptic genus is in complete accord with the implications of Theorem 5. Indeed, it was already conjectured in [47, 48], that Mathieu Moonshine is governed by some vertex algebra which carries an M_{24}-action, whose properties would immediately induce the modular transformation properties of the twisted elliptic genera of Theorem 5. As was argued in the discussion of Theorem 3, $H^*(X, \Omega_X^{\mathrm{ch}})$ is moreover expected to be related to a non-linear sigma model on X, at least in a large volume limit, providing the desired link to K3 theories. Indeed, the required compatibility with a large volume limit might also explain the restriction to those symmetries of K3 theories which can be induced by some symplectic automorphism of a K3 surface, as seems to be the case for the generators of M_{24} in Mathieu Moonshine.

Unfortunately, despite all its convincing properties promoting it to an excellent candidate to resolve Mathieu Moonshine, the vertex algebra structure of $H^*(X, \Omega_X^{\mathrm{ch}})$ is notoriously hard to calculate, as are the precise properties of general non-linear sigma models on K3, even in a large volume limit. Therefore, these ideas remain conjectural, so far. Sadly, known alternative constructions for vertex algebras that underlie the elliptic genus and that are easier to calculate seem not to explain Mathieu Moonshine [21].

A possible mechanism of combining symplectic automorphism groups of distinct K3 surfaces to larger groups is presented in [85, 87], and the following result can be seen as evidence in favor of these ideas:

Proposition 5 ([86]) *Consider the "smallest massive" representation of M_{24} that occurs in Theorem 5, that is, the representation on $\mathcal{R}_1$.*

The space $\mathcal{R}_1$ is isomorphic to a certain vector space V^{CFT} of states which is common to all K3 theories that are obtained by a standard $\mathbb{Z}_2$-orbifold construction from a toroidal $N = (2, 2)$ superconformal field theory. Moreover, on V^{CFT}, the combined action of all finite symplectic automorphism groups of Kummer surfaces induces a faithful action of the maximal subgroup $\mathbb{Z}_2^4 \rtimes A_8$ of order 322.560 in M_{24}. The resulting representation on V^{CFT} is equivalent to the representation of $\mathbb{Z}_2^4 \rtimes A_8$ on $\mathcal{R}_1$ which is induced by restriction from M_{24} to this subgroup.

This result is the first piece of evidence in the literature for any trace of an M_{24}-action on a space of states of a K3 theory. It is remarkable that the CFT techniques produce precisely the representation of a maximal subgroup of M_{24} which is predicted by Mathieu Moonshine according to the idea of "combining symplectic automorphism groups". Note that the group $\mathbb{Z}_2^4 \rtimes A_8$ is not a subgroup of M_{23}, indicating that indeed M_{24} rather than M_{23} should be expected to be responsible for Mathieu Moonshine, despite Theorem 6, by which all finite symplectic automorphism groups of K3 surfaces are subgroups of M_{23}. This preference for M_{24} to M_{23} is in accord with the findings of [50].

Encouraged by Proposition 5, one may hope that in a large volume limit, V^{CFT} can be identified with a subspace of $H^*(\text{K3}, \Omega_{\text{K3}}^{\text{ch}})$, inducing an equivalence of vertex algebras. Furthermore, by combining the action of $\mathbb{Z}_2^4 \rtimes A_8$ with the action of finite symplectic automorphism groups of K3 surfaces which are not Kummer, one may hope to generate an action of the entire group M_{24}. Finally, one may hope that this result generalizes to the remaining representations on $\mathcal{R}_n$, $n > 1$, found in Theorem 5. In conclusion, there is certainly much work left.

4.3 A Geometric Decomposition of the Elliptic Genus?

Even if the ideas presented in Sect. 4.2 prove successful, then so far, they give no indication for the reason for M_{24} -of all groups- to arise from the combined action of finite symplectic automorphism groups of K3 surfaces. Circumventing this intrinsic problem, the current section presents a simpler conjecture which can be formulated independently of Mathieu Moonshine. If true, however, it could serve as a step towards understanding Mathieu Moonshine.

Taking the idea seriously that there should be a purely geometric explanation for Mathieu Moonshine, one main obstacle to unraveling its mysteries is the lack of geometric interpretation for the non-geometric decomposition of $\mathcal{E}_{\text{K3}}(\tau, z)$ stated in Proposition 4, which is at the heart of the discovery of Mathieu Moonshine. Recall that the derivation of Proposition 4 rests on the identification (19) of the geometric elliptic

genus of a Calabi-Yau D-manifold X with the conformal field theoretic elliptic genus of a CFT that is obtained from X by a non-linear sigma model construction. We have incorporated this identification into our Definition 8, and it is the motivation for decomposing the geometric elliptic genus of K3 into the characters of irreducible unitary representations of the $N = 4$ superconformal algebra at central charge $c = 6$. While the conformal field theoretic elliptic genus by Definition 6 is obtained as a trace over the space of states $\mathbb{H}^R$, the geometric elliptic genus by Definition 7 is an analytic trace over a formal power series $\mathbb{E}_{q,-y}$ whose coefficients are holomorphic vector bundles on our K3 surface. The decomposition of the space of states $\mathbb{H}^R$ of every K3 theory by $N = (4, 4)$ supersymmetry which was performed in Sect. 4.1 to derive Proposition 4 should accordingly be counterfeited by a decomposition of $\mathbb{E}_{q,-y}$. We thus expect

Conjecture 1 *Let X denote a K3 surface with holomorphic tangent bundle $T :=$ $T^{1,0}X$, and consider $\mathbb{E}_{q,-y}$ as in Definition 7. Furthermore, let $e(\tau)$ denote the function defined in Proposition 4. Then there are polynomials p_n, $n \in \mathbb{N}_{>0}$, such that*

$$\mathbb{E}_{q,-y} = -\mathcal{O}_X \cdot \chi_0(\tau, z) + T \cdot \chi_{\mathrm{mm}}(\tau, z) + \sum_{n=1}^{\infty} p_n(T) \cdot q^n \widehat{\chi}(\tau, z),$$

$$and \quad e(\tau) = \sum_{n=1}^{\infty} \left(\int_X Td(X) p_n(T) \right) \cdot q^n,$$

where $p_k(T) = \sum_{n=0}^{N_k} a_n T^{\otimes n}$ if $p_k(x) = \sum_{n=0}^{N_k} a_n x^n$, and where $T^{\otimes 0} = \mathcal{O}_X$ is understood.

If (19) is interpreted as a generalization of the McKean-Singer Formula, as indicated in the discussion of that equation, then Conjecture 1 can be viewed as a generalization of a *local index theorem* [4, 51, 52, 80]. Note that the conjecture is formulated without even alluding to Mathieu Moonshine, so it may be of independent interest. If true, then for each $n \in \mathbb{N}_{>0}$, every finite symplectic automorphism group of a K3 surface X naturally acts on $p_n(T)$, and one may hope that this will yield insight into the descent of this action to the representation of M_{24} on $\mathcal{R}_n$ which was found in Theorem 5.

Appendix—Proof of Proposition 2 in Sect. 3

The entire proof of Proposition 2 rests on the study of the $+1$-eigenspace of the
linear operator $\overline{J}_0$ on the subspace $\overline{W}_{1/2}$ of the vector space $\overline{W}$ underlying the chiral
algebra. First, one shows that this eigenspace is either trivial or two-dimensional,
and from this one deduces claim 1. of the proposition. One direction of claim 2. is
checked by direct calculation, using the defining properties of toroidal $N = (2, 2)$
superconformal field theories. To obtain the converse, one shows that $\mathcal{E}(\tau, z) \equiv 0$
implies that an antiholomorphic counterpart of the conformal field theoretic elliptic
genus must vanish as well, from which claim 2 is shown to follow.

1. Assume that the space $\overline{W}_{1/2}$ contains an eigenvector of $\overline{J}_0$ with eigenvalue $+1$.
   We denote the field associated to this state by $\overline{\psi}_1^+(\overline{z})$. The properties of the
   real structure on the space of states $\mathbb{H}$ of our CFT imply that there is a com-
   plex conjugate state with $\overline{J}_0$-eigenvalue -1 whose associated field we denote
   by $\overline{\psi}_1^-(\overline{z})$. The properties of unitary irreducible representations of the Vira-
   soro algebra imply that these fields form a *Dirac fermion* [see the discussion
   around (15)], and that therefore $\overline{J}_1(\overline{z}) := \frac{1}{2} :\overline{\psi}_1^+\overline{\psi}_1^-:(\overline{z})$ is a $U(1)$-current as in
   Example 1 in Sect. 2.1. By a procedure known as *GKO-construction* [54], one
   obtains $\overline{J}(\overline{z}) = \overline{J}_1(\overline{z}) + \overline{J}_2(\overline{z})$ for the field $\overline{J}(\overline{z})$ in the $N = 2$ superconfor-
   mal algebra (9)–(10), and $\overline{J}_k(\overline{z}) = i\partial\overline{H}_k(\overline{z})$ with $\overline{\psi}_1^\pm(\overline{z}) =: e^{\pm i\overline{H}_1}:(\overline{z})$. The
   fields of twofold right-handed spectral flow, which by assumption are fields of
   the theory, are moreover given by $\overline{J}^\pm(\overline{z}) =: e^{\pm i(\overline{H}_1 + \overline{H}_2)}:(\overline{z})$. Their OPEs with
   the $\overline{\psi}_1^\pm(\overline{z})$ yield an additional Dirac-fermion, with fields $\overline{\psi}_2^\pm(\overline{z}) =:: e^{\pm i\overline{H}_2}:(\overline{z})$ in
   the CFT. This proves that the ± 1-eigenspaces of $\overline{J}_0$ on $\overline{W}_{1/2}$ each are precisely
   two-dimensional, since by the same argument no further Dirac fermions can be
   fields of the theory. Note that by definition, the corresponding states belong to
   the sector $\mathbb{H}_f \cap \mathbb{H}^{NS} \subset \mathbb{H}$ of the space of states of our theory.
   In summary, the $+1$-eigenspace of the linear operator $\overline{J}_0$ on $\overline{W}_{1/2}$ is either trivial
   or two-dimensional.

   We now study the leading order contributions in the conformal field theoretic
   elliptic genus $\mathcal{E}(\tau, z)$ of our theory. From (20) and by the very Definition 6 we
   deduce that $2ay^{-1}$ counts states in the subspace $V \subset \mathbb{H}^R$ where $L_0, \overline{L}_0$ both take
   eigenvalue $\frac{c}{24} = \frac{1}{4} = \frac{\overline{c}}{24}$ and J_0 takes eigenvalue -1. More precisely,

$$2a = \mathrm{Tr}_V\left((-1)^{J_0 - \bar{J}_0}\right) = -\mathrm{Tr}_V\left((-1)^{\bar{J}_0}\right).$$

As follows from properties of the so-called *chiral ring*, see e.g. [92, Sect. 3.1.1], a basis of V is obtained by spectral flow Θ (see Ingredient IV in Sect. 2.2) from (i) the vacuum Ω, (ii) the state whose corresponding field is $\bar{J}^+(\bar{z})$, and (iii) a basis of the $+1$-eigenspace of the linear operator $\bar{J}_0$ on $\overline{W}_{1/2}$. Since according to (13), the eigenvalues of $\bar{J}_0$ after spectral flow to V are (i) -1, (ii) $+1$, (iii) 0, the above trace vanishes if the $+1$-eigenspace of the linear operator $\bar{J}_0$ on $\overline{W}_{1/2}$ is two-dimensional, implying $2a = 0$, and if this eigenspace is trivial, then we obtain $2a = 2$.

In conclusion, the conformal field theoretic elliptic genus of our theory either vanishes, in which case the $+1$-eigenspace of the linear operator $\bar{J}_0$ on $\overline{W}_{1/2}$ is created by two Dirac fermions, or $\mathcal{E}(\tau, z) = \mathcal{E}_{K3}(\tau, z)$.   □

2. a. Using the details of toroidal $N = (2, 2)$ superconformal field theories that are summarized in Sect. 2.3, one checks by a direct calculation that the conformal field theoretic elliptic genus of all such theories vanishes.   □

   b. To show the converse, first observe that in our discussion of $N = (2, 2)$ superconformal field theories, the two commuting copies of a superconformal algebra are mostly treated on an equal level. However, the Definition 6 breaks this symmetry, and

$$\overline{\mathcal{E}}(\bar{\tau}, \bar{z}) := \mathrm{Tr}_{\mathbb{H}^R}\left((-1)^{J_0 - \bar{J}_0} \bar{y}^{\bar{J}_0} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24}\right)$$

   should define an equally important antiholomorphic counterpart of the conformal field theoretic elliptic genus. In our case by the same reasoning as for $\mathcal{E}(\tau, z)$, it must yield zero or $\overline{\mathcal{E}}_{K3}(\bar{\tau}, \bar{z})$. Note that Proposition 1 implies that

$$\mathcal{E}(\tau, z = 0) = \overline{\mathcal{E}}(\bar{\tau}, \bar{z} = 0)$$

   is a constant, which in fact is known as the *Witten index* [95–97]. In particular, by (18) we have $\mathcal{E}_{K3}(\tau, z = 0) = 24$, hence $\mathcal{E}(\tau, z) \equiv 0$ implies $\overline{\mathcal{E}}(\bar{\tau}, \bar{z}) \equiv 0$. It remains to be shown that our theory is a toroidal theory according to Definition 5 in this case.

   But Step 1. of our proof then implies that the $+1$-eigenspace of the linear operator $\bar{J}_0$ on $\overline{W}_{1/2}$ is created by two Dirac fermions and that the analogous statement holds for the $+1$-eigenspace of the linear operator J_0 on $W_{1/2}$. Hence we have Dirac fermions $\psi_k^{\pm}(z)$ and $\overline{\psi}_k^{\pm}(\bar{z})$, $k \in \{1, 2\}$, with OPEs as in (15). Compatibility with supersymmetry then implies that the superpartners of these fields yield the two $\mathfrak{u}(1)^4$-current algebras, as is required in order to identify our theory as a toroidal one.   □

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