Mathematical Physics Studies

Damien Calaque
Thomas Strobl *Editors*

Mathematical Aspects of Quantum Field Theories

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Damien Calaque · Thomas Strobl
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Editors
Damien Calaque
Université Montpellier 2
Montpellier
France

Thomas Strobl
Université Claude Bernard Lyon 1
Villeurbanne Cedex
France

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Foreword I

This volume contains the contributions to the 2012 les Houches winter school in mathematical physics, organized with the support of Labex Milyon and ESF ITGP. A common theme of the courses and talks at this winter school is classical and quantum field theory. This subject has a long history and has been since its inception a fertile ground for fruitful interactions between mathematics and physics. It now is a multifaceted research area attracting the interest of mathematicians and physicists alike.

The extended lecture notes of five courses of this school form the bulk of this volume and give an overview of the different points of views and recent results in field theory. The subject of Klaus Fredenhagen's lectures is algebraic quantum field theory, in the sense of Haag and Kastler, where the basic notion is that of local algebras of observables attached with portions of space-time. This approach was only recently extended to the case of curved space-times and these lecture notes, appearing here with material from lectures of Katarzyna Rejzner, offer a comprehensive description of the results, including perturbative renormalization, due to the lecturer and his collaborators. A similar emphasis on local observables, but in a different framework, is presented in the notes, written by Claudia Scheimbauer, of the lecture course of Kevin Costello on partially twisted supersymmetric gauge theories. A mathematical formulation of these theories in terms of factorization algebras of observables is presented, including perturbative quantization and the relation to Yangians and quantum enveloping algebras. Factorization algebras, first introduced by Beilinson and Drinfeld in the context of conformal field theory, have now a wide field of applications also outside quantum field theory, as illustrated in the lectures of Grégory Ginot, who gives an introduction to factorization algebras and factorization homology and presents applications to mapping spaces, higher Deligne conjectures, and string topology. The lectures of Alberto Cattaneo and Pavel Mnev give an introduction to semiclassical quantization of field theories with gauge symmetries. The approach of the lecturers, based on a collaboration with Nicolai Reshetikhin, is to systematically consider field theory on manifolds with boundary and study the behaviour when gluing along boundary components.

The aim, illustrated here in the case of abelian Chern–Simons theory, is to construct examples of quantum field theories on arbitrary manifolds by gluing them from simple pieces. This same idea appears in the lectures of Jørgen Andersen and Rinat Kashaev, who present their work on the construction of a topological quantum field theory in three-dimensions associated to the quantum Teichmüller theory. Here, the theory is constructed on a triangulated manifold combinatorially by gluing building blocks defined on tetrahedra.

Additionally to the lectures, there were several talks on field theories and their applications, some of which, along with other related contributions, are presented in this volume. The result is a collection of research works from researchers with different backgrounds in mathematics and physics, showing that quantum field theory has a broad range of applications, and will continue to be a source of inspiration for mathematics and mathematical physics.

Giovanni Felder
Anton Alekseev
Damien Calaque
Alberto Cattaneo
Maria Podkopaeva
Thomas Strobl
Andras Szenes

Foreword II

No development of modern science has had a more profound impact on human thinking than the advent of quantum theory. Wrenched out of centuries-old thought patterns, physicists of a generation ago found themselves compelled to embrace a new metaphysics. The distress, which this reorientation caused, continues to the present day. Basically, physicists have suffered a severe loss: their hold on reality. (Bryce S. DeWitt and Neill Graham: Resource Letter on the Interpretation of Quantum Mechanics, American Journal of Physics, July 1971)

In January/February of 2012, a Winter School on the topical subject of "Mathematical Aspects of Field Theories" took place at Les Houches—a wonderful village made available to the physics community as the location of conferences and schools by Cécile DeWitt-Morette, the wife of the late Bryce DeWitt, whom I have quoted above, and one of my favorite places on Earth. Damien Calaque and Thomas Strobl should be commended for having organized this School. They kindly invited me to participate in it; but, unfortunately, I had some minor health problems that prevented me from traveling to Les Houches.

Recently, the organizers asked me to write a short foreword for the proceedings of their School and they sent me various written contributions. Having taken a brief look at some of these writings, I feel the title of the School should really read "*highly* mathematical aspects of field theories". The material collected in this book looks fascinating. But I admit that I do not expect I would have understood much of what was discussed at the School. I imagine the organizers had invited me to participate in it with the thought that I would discuss some real-world applications of (quantum) field theory. I regret that I was unable to comply with their wishes. Let me emphasize, incidentally, that I do not associate any scholarly ambitions with this preface; it actually results from an experiment in "free association".

In comparison with classical field theory, Quantum Field Theory (QFT) is a rather recent endeavor, less than a 100 years old. Implicitly, it had first appeared in Planck's law for the spectral energy density of black body radiation and Einstein's theory of photons. It made its first explicit appearance (quantization of a harmonic string) in the famous "Dreimännerarbeit", in 1925, and got started in a more concrete sense in the work of Dirac on quantum electrodynamics (QED), soon afterward. Famous early contributions came from Jordan, Wigner, Heisenberg and Pauli (relativistic QFT), Kramers, Weisskopf (first attempts toward renormalizing QED), Wentzel, and others. Before and during World War II, E.C.G. Stückelberg de Breidenbach made prescient contributions to a manifestly relativistic formulation of QFT, the perturbative calculation of causal scattering amplitudes, and the renormalization of QED. He anticipated very sizable portions of the work of Feynman, for which the latter received the Nobel Prize, jointly with Schwinger and Tomonaga, in 1965.

The triumph of perturbative QED and, in particular, of calculating radiative corrections after performing suitable renormalizations (Lamb shift, "$g - 2$", etc.), convinced the physics community, at least temporarily, that relativistic QFT was a successful road toward understanding the physics of elementary particles. For QED, the comparison between theoretical calculations and experimental data was spectacularly successful, as is exceedingly well-known. After various lengthy and confusing detours, the discovery of the standard model (a $U(2) \times SU(3)$—gauge theory of electro-weak and strong interactions, from 1967 till the early 1970s) sealed the triumph of quantum field theory, and in particular of quantum gauge theory, as the most successful approach toward unifying special relativity with quantum mechanics, and thereby arriving at a satisfactory theory of elementary particles and of the fundamental interactions, with the exception of gravitation.

Indeed, relativistic quantum field theory is an attempt to combine the two most profound new theories of twentieth-century physics, Relativity Theory and Quantum Mechanics, in a mathematically consistent theoretical framework compatible with "Einstein causality" or "locality". In this attempt, space-time is most often chosen to be the Minkowski space of special relativity, i.e., it is viewed as a rigid screen on which Nature draws its imagery, and which is unaffected by material processes evolving in it. (Ambitious people, such as Bryce DeWitt, or Klaus Fredenhagen, who has a contribution to this book, and others, have formulated quantum field theory on fairly general Lorentzian manifolds with curvature. But the fact remains that the recoil of material processes on the structure of space-time is neglected or treated in a merely approximate, "self-consistent" fashion).

There cannot be any doubt that the use of a rigid classical model of space-time, such as Minkowski space, in a relativistic local quantum theory of matter is provisional and must ultimately lead to serious conceptual difficulties. Here are some reasons for this expectation: A relativistic local quantum theory on Minkowski space or other rigid models of classical Lorentzian space-times is of necessity a theory of systems with infinitely many degrees of freedom infinitely many of which can be localized in arbitrarily small cells of space-time—a property that appears to cause serious trouble as soon as gravity is not neglected anymore (conflict with General Relativity), besides giving rise to the infamous ultraviolet divergences well-known. In a more satisfactory, deeper unification of Relativity Theory and Quantum Theory—sometimes called "quantum gravity"—space is likely to appear as an *emergent structure* rather than as a fundamental one. In searches of "quantum gravity" it will presumably be advantageous to adopt Leibniz' views of space

(space as a structure encoding relations between events) rather than Newton's (a-priori character of space). A theory of this kind will have to be formulated without explicit reference to a particular model of space-time, and it is likely to prevent an infinity of degrees of freedom from accumulating in arbitrarily small cells of space-time, and hence might be expected to be ultraviolet finite. Many of us are tempted to believe that string theory will guide the way toward such a framework. Unfortunately, for the time being, we do not know of any conceptually clear, nonperturbative and manifestly background-independent formulations of string theory. However such a formulation will look like, it is likely that QFT—or, at least, *methods of QFT*—will be among its ingredients. (One might expect, for example, that two-dimensional superconformal field theory will be one of its important tools.) We are thus well advised to pursue studies of QFT and, in particular, of *mathematical aspects* of QFT—and, by the way, we should not avoid revisiting, from time to time, some of the puzzling mathematical and physical aspects of quantum theory. For, without arriving at a clearer understanding of quantum theory, we may never be able to unravel the mysteries at the root of a unification of the quantum theory of matter with a relativistic theory of space-time and gravitation!

Another source of some worry that something is not properly understood, yet, is the prominent role that *global* properties of rigid space-time play in deriving *local* consequences of relativistic QFT that ought to be valid under much more general assumptions on the nature of space-time. Here is an example: One of the spectacular predictions of relativistic local quantum theory (QFT) is the existence of *anti-matter*. This prediction, first explicitly proposed by Dirac, originally grew out from his work on a relativistic electron equation, the Dirac equation, and his idea of the "Dirac sea" as a cure to the problem of negative-energy states, and from an observation, due to Oppenheimer and Weyl, saying that the holes in Dirac's sea must have all the same properties as the electrons, except that their electric charge is opposite to that of electrons. But a general understanding of the necessary existence of anti-matter, in the guise of the CPT theorem, only came with Jost's work in axiomatic quantum field theory. Now, the strange fact is that Jost's proof of CPT makes use of some *global* properties of Minkowski space. But one would think that the existence of anti-matter is a *local* property of relativistic quantum theory, which will remain true for very general (and even for "dynamical") models of space time. Likewise, the general connection between the spin of quantum fields and their statistics is intimately connected to the property of the "vacuum state" to satisfy the Kubo-Martin-Schwinger (KMS) condition with respect to Lorentz boosts (as derived by Bisognano and Wichmann, using arguments slightly extending those of Jost), which refers to *global* symmetries of Minkowski space-time. And again, results of this type can be expected to remain valid for much more general "dynamical" models of space-time. So, quite clearly, there are most probably quite a few things we do not understand properly, yet, and we are well advised to continue studies of the *mathematical aspects of field theory*! (One might note, in passing, that, from today's perspective, Tomita-Takesaki modular theory—one of the miraculous developments in the theory of operator algebras—can be viewed as the general mathematical structure underlying Jost's CPT theorem, the Bisognano-Wichmann theorem, and the general analysis of equilibrium states in quantum statistical mechanics due to Haag, Hugenholtz and Winnink).

Serious study of mathematical aspects of quantum field theory is not a new development. After initially unsuccessful attempts of using QFT to construct a theory of nuclear forces and of mesons and failures to understand QED nonperturbatively, *axiomatic quantum field theory* was developed in the 50s and 60s of the past century; first in the guise of the Gårding-Wightman axioms for theories of local quantum fields, and then in the form of the Haag-Kastler axioms for nets of local observables. When the general structure of relativistic local quantum theory had been clarified, at least to some extent, it was felt to be important to construct models of quantum field theories satisfying all the Wightman- or the Haag-Kastler axioms. Thus, *constructive quantum field theory* was born in the middle of the 1960s. Axiomatic and constructive QFT gave rise to very fruitful developments in pure mathematics. In particular, complex analysis, functional analysis, functional integration and probability theory, "hard analysis", and group theory all drew inspiration from work on problems encountered in axiomatic and constructive quantum field theory—as they already had from the advent of quantum mechanics, two or three decades earlier.

Not only has pure mathematics profited from ideas, methods, and results of quantum field theory, but also theoretical physics: Various areas in theoretical physics other than particle physics, notably quantum-mechanical many-body theory and its applications to condensed matter physics, have been enriched by techniques first developed for the purposes of understanding relativistic QFT and applying it to particle physics and by corresponding advances in mathematics. Suffice it to recall, as a famous example, that Freeman Dyson made very successful applications of ideas and techniques he had discovered or learned in his work on QED to studies of the quantum theory of magnetism, etc. I hasten to add that concepts and ideas that had first appeared in statistical mechanics or condensed matter physics were subsequently successfully applied to QFT. Among the best examples is the observation that, in the Euclidian (imaginary-time) region, QFT looks like classical statistical mechanics and that certain Euclidian field theories can be viewed as gases of Brownian paths and loops (Symanzik's representation of scalar QFT's). This has made available certain methods developed in statistical mechanics, such as cluster expansions, lattice approximation, correlation inequalities, the Lee-Yang theorem, Kramers-Wannier duality, for purposes of quantum field theory. (Another frequently mentioned example is the Anderson-Higgs mechanism. However, one may argue with good reasons that essential features of this mechanism first appeared in Stückelberg's work on massive gauge fields, which was inspired by problems in particle physics).

It cannot be the purpose of a foreword like this one to give a detailed account of the history of the subject treated in the following chapters of a book—although, frankly, it would be tempting to dive more deeply into the history of (mathematical) quantum field theory at this point. However, this would require much more space than this preface can occupy and more time and care than I can afford. Suffice it, thus, to refer to some of the more mathematical developments in QFT and some of their applications in mathematics and physics by merely mentioning appropriate buzz words: Besides axiomatic and constructive quantum field theory, I would like to mention the perturbative renormalization group first discovered by Stückelberg and Petermann and by Gell-Mann and Low and then extended by Callan and Symanzik, which has profoundly changed the way we think about QFT, and has had spectacular applications in particle physics (ultraviolet asymptotic freedom—Politzer; Gross and Wilczek—as an explanation of Bjorken scaling, to mention one example) and in the theory of critical phenomena. I also wish to recall the development of systematic approaches to perturbative renormalization (BPHZ, analytic renormalization, Hepp's "axioms for renormalization", the Epstein-Glaser approach to renormalization, dimensional regularization and–renormalization, renormalization-group inspired renormalization à la Polchinski and Gallavotti, the use, by Connes and Kreimer, of the Hopf algebra of rooted trees and of a Riemann-Hilbert problem to set up perturbative renormalization theory, etc.), the Faddeev-Popov procedure of gauge fixing in general nonabelian gauge theories (extending ideas of Feynman and of DeWitt), the discovery of anomalies in gauge theories (Adler–-Bardeen–Bell–Jackiw, etc.). I continue by mentioning Wilson's form of the renormalization group and the development of conformal field theory; the discovery of supersymmetry; the discovery of BRST cohomology for gauge theories and of the BV formalism. I would like to draw attention to the advent of two-dimensional conformal field theory and of topological field theories, then to the discovery of connections between gravity theories on AdS spaces and conformal field theories on the boundary space-time (Maldacena correspondence), to end by mentioning the tantalizing role that ideas of integrability (Bethe-ansatz equations) have recently started to play in the analysis of supersymmetric gauge theories, etc. Most of these developments have had very interesting consequences in *mathematics*; but, of course, also—and, perhaps, more importantly—in *physics*. I feel it is important that one should not lose sight of one direction in the light of success in the other. I will now sketch some examples explaining what I mean.

Techniques that were originally inspired by problems in perturbative renormalization theory have recently had some fairly spectacular applications in algebraic geometry (e.g., multiple zeta values, polylogarithms, modular forms in QFT—work by Connes, Kreimer, Bloch, Marcolli, Brown, and others). Ideas and methods born from the study of gauge theories, of gauge fixing and gauge anomalies and of supersymmetry have had huge impact on developments in algebraic topology (moduli spaces of instantons on four-manifolds and Donaldson invariants, Seiberg-Witten invariants, Chas-Sullivan string topology, etc.). But, of course, for physicists, the work of 't Hooft and Veltman, Lee and Zinn-Justin, Becchi, Rouet and Stora, Batalin and Vilkovisky, and many others, demonstrating the renormalizability of the standard model of particle physics and other theories with infinite-dimensional local symmetries and leading to many concrete calculations of direct relevance to experiments at the LHC, is more important.

Another example concerns Wilson's form of the renormalization group and conformal field theory: Ideas that have grown out of the work of Wilson, Wegner, and others on the renormalization group and of somewhat related (earlier, but more narrow) work of Glimm and Jaffe on $\lambda\varphi^4$-theory in three dimensions have had enormously fruitful applications in the mathematics of functional integration, large-deviation theory, singular perturbation theory of operators, etc. Powerful new tools in analysis, such as multiscale analysis, and in probability theory have emerged therefrom. Conformal field theory has become a fundamental tool in string theory. Two-dimensional superconformal field theory has had important applications in algebraic topology (elliptic genera, chiral rings, ...). Ideas and methods from string theory and superconformal field theory have led to highly nontrivial results in enumerative geometry.—But let us not forget that Wilson's main aim had been to understand critical phenomena in the theory of continuous phase transitions and to calculate critical exponents, an enterprise that turned out to be extraordinarily successful, and that his ideas have changed the way we think of the role of quantum field theories in particle physics ("effective field theories"). And let us not forget that conformal field theories have played a crucial role in this enterprise as theories that encode renormalization group fixed points and often enable one to quantitatively determine critical exponents by identifying them with scaling dimensions of various fields of the theory.

It may be appropriate to also draw attention to Jones' theory of subfactors and of towers of von Neumann algebras. By studying the abstract problem of finding an invariant associated with the embedding of a subfactor in a von Neumann factor (initially for factors of type II_1—but extensions to factors of type III, due to Longo, followed soon), Vaughan Jones unravelled deep and very surprising connections between the theory of towers of von Neumann algebras, the theory of knots and links embedded in the three-dimensional sphere, unitary representations of the braid group, exactly solved models of two-dimensional classical statistical mechanics, representations of Lie algebras and quantum groups. His discoveries are highly original and, for me, have a magical touch. They represented the first substantial progress in knot theory in decades. According to Jones' own testimony, his knowledge of quantum theory apparently played an important role in his discoveries on subfactor theory; and, of course, the basic tools he used—the theory of operator algebras (Murray and von Neumann, and followers)—had grown out of studies of quantum theory. Jones' work was aimed at solving some deep problems in mathematics. But it turns out that some of the ideas and mathematical techniques he used had come up in algebraic quantum field theory, more precisely in the work of Doplicher, Haag and Roberts on the general theory of superselection sectors. Their work, which remains a rock of beauty in axiomatic QFT, concerned relativistic field theories in four-dimensional space-time. Constructive field theorists found quantum field theories in four dimensions to represent too serious a challenge, and had therefore constructed models of relativistic quantum fields on two- and three-dimensional Minkowski space. In the early 1970s, in studies of some two-dimensional quantum field models, Streater and Wilde and I had encountered the first very simple examples of the so-called "exchange algebras" of field operators, which, at the time, looked quite exotic. It turns out that exchange algebras of field operators can be characterized completely by so-called Yang-Baxter matrices, which determine representations of the braid group. But this was far from understood in the early 1970s. At this point, Jones was to become important for me. I write about him here not only because of his significance as an outstanding mathematician, but also because, in times when decency in the scientific community is somewhat endangered, he can serve as a role model of an intellectually honest and exceptionally generous scientist. As Joan Birman emphasized in her description of Jones' mathematical work at the 1990 International Congress of Mathematicians in Kyoto, where he received a Fields Medal, his style of working as a mathematician is informal and "encourages the free and open interchange of ideas". In 1987, during a sabbatical I spent at the IHES, I had the privilege to interact with Jones, and I greatly profited from his exceptional generosity and from "free and open interchange of ideas" with him. Thanks to these interactions, I learned that exchange algebras of field operators determine unitary representations of the braid group on n strands. This observation triggered the development of a general theory of braid (group) statistics, a form of quantum statistics only appearing in local quantum theories in two- and three-dimensional space-time. (This exotic form of quantum statistics had been missed by the pioneers of quantum mechanics). In the mid-1980s, my interest in braid statistics had been aroused by attempts, jointly with Marchetti, to understand Chern–Simons–Higgs theory in three dimensions and by tentative applications of such theories to the quantum Hall effect. Jones turned out to be the ideal advisor for understanding the relation between exchange algebras and braid statistics, and for how to proceed toward making further progress on all these problems.

I do not engage in a discussion of how *topological Chern-Simons theory* entered the scene. Suffice it to say that, thanks to the work of Witten and others (including Chr. King and myself), it has come to play a very important role in the theory of invariants of knots and links and of general three-manifolds. This is undoubtedly an excellent example of a very successful application of ideas and techniques from quantum field theory to problems in pure mathematics.—Well, I would like to add that the observation that a class of three-dimensional topological field theories of which certain topological Chern-Simons theories are examples turn out to describe the large-scale physics of two-dimensional electron gases exhibiting the quantum Hall effect is, to say the least, equally remarkable and important. It has led to a classification of "universality classes" of such electron gases, To unravel this insight with some precision has kept me and some of my younger friends busy for quite a few years. A related, more recent development concerns the theory of "topological insulators", which are exciting new states of condensed matter with exotic gapless surface modes.

I would like to end this preface by mentioning an example of a discovery in pure mathematics that was arrived at independently by mathematical physicists working on comparatively concrete problems in algebraic quantum field theory and by a mathematician pursuing highly abstract mathematical concerns. Motivated by studies in the theory of superselection sectors in four-dimensional local quantum field theories, Doplicher and Roberts found a general solution of the Tannaka-Krein problem of reconstructing compact topological groups from certain C^*-tensor categories arising in quantum field theory. Motivated by purely mathematical considerations, Deligne independently proved related results. More recently, a partial duality theory between *braided* C^*-tensor categories and quantum groups has been developed, which, on one hand, was motivated by studies of quantum field theories with braid statistics in two- and three-dimensional space-time (Kerler and myself, Mack and Schomerus, and others) and, on the other hand, by rather abstract problems in the theory of quantum groups.

I hope these examples make it clear that the study of *mathematical aspects of quantum field theory* tends to generously pay off in that it may lead to the discovery of new mathematics. Even if, often, the new mathematics can also be found coming from a completely different starting point one cannot doubt the fruitfulness of exploiting quantum field theory for the purposes of pure mathematics. But, of course, one should always remember that the *original purpose* of relativistic quantum field theory has been to understand processes in particle physics at fairly high energies, and that applications of quantum field theory to problems from different areas in Physics have been spectacularly successful.

To conclude, I wish the readers enjoyable and fruitful encounters with the various essays collected in this book. I apologize for not commenting more specifically on these writings—but I do not think that this is my task here. Let me propose the following somewhat provocative variant of the remarks of DeWitt and Graham quoted at the beginning of this preface:

Few developments of modern science have had a more profound impact on mathematical thinking than the advent of quantum field theory. Wrenched out of old thought patterns, mathematicians have found themselves compelled to embrace a new way of formal reasoning. The distress which this reorientation has caused in some circles of the mathematical community continues to the present day. Basically, mathematicians using arguments inspired by quantum field theory have gained access to wonderful new insights, but suffered a severe loss: their hold on mathematical rigor.

Perhaps, this book will show that rigor need not be lost.

I dedicate this text to the memory of Edward Nelson, whose understanding of many “mathematical aspects of quantum field theory” was profound.

ETH Zürich Jürg Fröhlich

Preface

Despite its long history and its stunning experimental success, the mathematical foundation of perturbative quantum field theory (pQFT) is still a subject of ongoing research. This book aims at presenting some of the most recent developments in the field, and at reflecting the diversity of the approaches and the tools that have been invented and that are used. Some of the leading experts as well as newcomers in the field present their latest advances in the attempt for a better understanding of quantum, but also classical field theories.

The chosen material is, however, far from complete. As mentioned in the first foreword, the idea for this book grew out of a school in Les Houches on the subject, most lecturers agreeing to write a contribution. This then was complemented by selecting some of the customary young-participant-presentations to contribute, too, as well as by two, three additional invited articles. And, as mentioned in the second foreword, even though the book is aimed both at mathematicians and physicists, it is more oriented toward the mathematical developments. Here, it is maybe a pity that for example Nekrasov's lectures about the path integral on $N = 2$ supersymmetric gauge theories did not find entry into the present addition. But there are many more promising directions, which did not, at least one of which shall be mentioned below. Maybe this can be a reason to come back to the enterprise at a later point again, summarizing also those aspects, and possibly updating the ones which are contained in the present edition.

On this occasion, we use the opportunity to thank Jürg Fröhlich for his valuable physical insight arising from decades of own original work on the forefront of the subject and his complementary remarks on the physics that is involved in the mathematical descriptions, even if it is partially only in terms of some keywords due to lack of "space-time". In the winter school, there was in addition an inspiring opening lecture of another great person in the field, Ludwig Faddeev, who commented on his perspective on the still open one-million dollar Clay problem "Yang–Mills Existence and Mass Gap". His lecture had been published already in a similar form elsewhere, so that we briefly summarize it here only—also since it suits well to explain the problematic of the subject.

The task to win the prize may sound deceptively simple: essentially, one is asked to prove that Yang–Mills theory (for a semi-simple compact structure group G) *exists* (in its quantum version) and that there is a minimal mass for the spectrum of particles. Deceptively simple, since from the physical perspective there is absolutely *no* doubt that this theory exists (at least for $G = SU(2)$ or $G = SU(3)$); it is one of the corner stones of the standard model of elementary particles, verified experimentally to an incredible precision, as also emphasized by Fröhlich in his foreword. However, as pointed out by Faddeev in his lecture (but also independently by R. Jackiw), the problematic becomes already more evident if one notices that the underlying classical Yang–Mills theory is conformal, i.e., scale-invariant in four space-time dimensions. One way of seeing this is that the overall coupling constant does not carry any physical units in precisely this dimension. On the other hand, any mass of a particle to be specified in a physical theory needs to refer to some standard mass (like 1 kg). The definition of the theory does not carry any such a mass (or, equally, length) scale on the outset of the problem, i.e., in its classical formulation in terms of an action functional.

According to Faddeev, the remedy can lie only in the usually so unloved infinities encountered typically in interacting quantum field theories (QFTs). Those infinities that plagued the foundators of the theory, subsequently were handled with increasing success in a more or less well-founded theory of perturbative renormalization, but which still cost many contemporary students of theoretical and mathematical physics a large number of unpleasant hours; the latter fact is the case, since in particular in standard physics lectures on QFTs, often *the* experimentally verified *end is used to justify the* mathematical *means*, with a mathematical argumentation that either appears inconsistent or otherwise at least arbitrarily ad-hoc. On the other hand, the necessary regularization of the theory on the quantum level will introduce a length-scale, and in this way there can be hope that the resulting quantum Yang–Mills theory can yield a minimal mass in a well-defined way.

To formulate a mathematically well-defined and conceptually convincing regularization and renormalization scheme is one of the tasks of a mathematical approach to quantum field theory (QFT). But it goes even further: one wants the theory to satisfy a minimal number of axioms that seem to be enforced by compatibility with for example special relativity, running in part under the name of (Einstein) “locality” in this context. More precisely, basic considerations require a number of properties any “physically acceptable” QFT should satisfy. One version of such a set of axioms is the one formulated by Wightman. It contains for example (projective) equivariance of the quantum fields of the theory with respect to the action of the Poincare group (the isometry group of Minkowski space in four dimensions). Later, it was permitted also to trade in Euclidean four-space for the physical Minkowski space; the idea of the so-called Osterwalder–Schrader axioms being then that mathematically the theory is easier to define and the physical interpretation results in a second step by an appropriate analytic continuation, called Wick rotation in physics. The formulation of the Clay prize requires to define 4d quantum Yang–Mills theory with a rigor of *at least* such axioms.

The only problem here is that up to now there is not a *single* known interacting quantum field theory in four dimensions satisfying such a typical set of axioms; there are only examples of such theories in two or three space-time dimensions, which have, however, no physical significance and are (to be) considered as so-called “toy-models” only. As Max Kreuzer from the Technical University of Vienna used to say, torturing herewith some of the more mathematical-conceptually oriented students (all the more since the statement is true, at least from a physicist perspective): “The only theories satisfying the Wightman axioms are free theories.” A free theory is one that physically corresponds essentially to a single particle travelling alone through empty space not subject to any interactions and thus not subject to any experimental observations or tests. Clearly, this is highly dissatisfying, all the more, since the formulated axioms, in one or the other form, seem more or less unavoidable from a point of view of principles governing our *contemporary* understanding of quantum field theory.

At this point, we want to mention one of the unfortunate omissions of this volume, all the more since it contains a glimpse of hope for possibly finding an interacting QFT in four dimensions after all. The omission comes from a recent direction motivated by String Theory (but not only!) to consider QFTs on so-called noncommutative space-times. In fact, the idea is already quite old and pursues the goal that the “fuzzyness” of the underlying space resulting from non-commuting space(-time) coordinates could cure the problem of the UV-(or “high energy”/“small distance”) divergencies of QFTs mentioned already above. In the simplest setting, the commutator of the coordinates is a constant matrix Θ, corresponding to the deformation quantization of a constant Poisson tensor (in flat space). The resulting product of functions on space-time can then be described by the Moyal product of Θ. In this way, the classical action functional of the theory under investigation is replaced by one that is an infinite formal power series in Θ, reducing to the original functional for $\Theta = 0$. It is then *this* new functional to be used for the “quantization”, i.e., as a starting point of the construction of a pQFT.

Although first considerations indeed show improvement of the UV-behavior, it turns out that the problem is not solved in many cases (keyword “UV/IR”-mixing) and the original hype on the study of such theories seems to have decreased over the last years again. However, there is one proposal, the so-called Grosse–Wulkenhaar model, that resists many of the problems of other theories considered in this context and now even gives some hope to lead to a well-defined interacting QFT in four dimensions (although it is still too early to make this statement, there are at least several indications that look promising). One important issue to address at this point is that certainly the introduction of the tensor Θ on Minkowski or Eucildean space spoils its covariance. However, in a simultaneous limit sending Θ as well as the volume (made finite for an IR-regularization) to ∞, it was shown to lead to a covariant and local theory on Euclidean fourspace for this model. Reinterpreting thus this matrix Θ as another way of regularizing the theory, one is led to an apparently consistent, non-trivial quantum version of the φ^4-theory in four dimensions. The Wick rotation to Minkowski signature is a problem still under investigation on the day of this writing, while preliminary computer simulations in this direction seem promising.

The remarks of the introduction up to here aimed at a complementary argumentation to the one of Fröhlich of why one would wish to have a mathematically well-founded theory of quantum fields describing known physics at high energies. Even on the level of perturbation theory, i.e., in terms of formal power series, the situation concerning physically relevant theories in this context is far from satisfactory. Theories of physical relevance are in some sense of quite a different nature than those of relevant mathematical impact: while the first ones are characterized by so-called "propagating degrees of freedom", the latter ones are mostly of "topological" nature. Essentially or at least in a first approximation, the difference lies in the dimension of the (generic part of the) moduli space of (classical) solutions to the Euler–Lagrange equations of the theory modulo its gauge symmetries. For physically relevant theories, this needs to be an infinite-dimensional space, reflecting the fact that physically observable excitations describing elementary particles can be generated locally everywhere in space-time, while for topological models this space is usually finite-dimensional. On the quantum level, the latter type of theories are then called topological quantum field theories (TQFTs).

One of the most famous examples of a TQFT, if not the most famous one among mathematicians, is the Chern–Simons theory. The major breakthrough was made by Witten, who observed that one could recover link and three-manifold invariants *via* the path integral quantization of the Chern–Simons classical action functional. The so-called A- and B-models are other famous examples of TQFTs. They are related by mirror symmetry to one another, a notion originating from physical intuition, relating seemingly different, but in the end equivalent quantum string theories. Mirror symmetry and its relation to enumerative and algebraic geometry became a major research area of pure mathematics by itself in the mean time.

A generalization of the A- and B-model is the Poisson sigma model (PSM), celebrating its twentieth anniversary this year. It was discovered in the context of toy models of coupled gravity and Yang–Mills theories defined on two-dimensional space-time manifolds Σ (Ikeda and Schaller–Strobl). Already at this very beginning it was realized that the quantization of the PSM is intimately related to the quantization of the target Poisson manifold—applying a particular non-perturbative quantization scheme to this theory, the integrality condition of geometric quantization pops up for the symplectic leaves of the Poisson target (cf also Alekseev–Schaller–Strobl). However, only in an unparalleld work of Kontsevich it was observed that already the perturbative quantization of the PSM on a trivial world-sheet topology solves the by then longstanding problem of deformation quantization of Poisson manifolds, leading him to his famous formality theorem (several steps of this procedure were retraced in a series of works by Cattaneo-Felder).

This is a good example of the use of (T)QFTs in mathematics: one trades in the apparently simpler problem of quantization of a Poisson structure on $\mathbb{R}^n$ for the quantization of a *field theory* the target of which is this Poisson manifold $M = \mathbb{R}^n$. This now is an infinite dimensional space, the functional being defined over vector bundle morphisms from $T\Sigma$ to T^*M. Moreover, one needs to factor out an infinite-dimensional gauge group, the quotient yielding in general a complicated, singular, but in this case finite-dimensional space. However, it turns out that the application of *standard* techniques developed in the context of perturbative QFTs with gauge symmetries leads to formulas relevant to the finite dimensional target space that otherwise proved resistant over decades for being invented directly!

The flow is expected to also go into the other direction, however, i.e., one expects to learn from TQFTs and related mathematics for how to sharpen our approaches for the construction of physically more relevant QFTs. It is in this spirit instance that Tamarkin wrote a 100 pages paper only about the renormalization of the PSM—in a standard physics approach the perturbative renormalization of such a topological model would be dealt with in at most a few paragraphs. The functorial approach to TQFTs, as developed also at the examples of topological strings (like the A- and B-model), led to an axiomatic definition of them in terms of the Atyah–Segal axioms. In the lectures of Fredenhagen about the formulation of pQFTs on curved space-times of Lorentzian signature one finds a reformulation of standard QFT axioms closely related to such a functorial perspective.

For the present, as mentioned rather mathematically oriented volume on QFT (cf. also the foreword of Fröhlich), this is maybe one of the main perspectives from our editors' side to its contributions: the hope that, *on the long run*, topological models and mathematics in general can have something to say about (also physically relevant) QFTs. It is thus not so surprising that one out of in total four parts to this book is devoted to mathematics around the Chern–Simons theory. Subsequent to Witten's work, Reshetikhin and Turaev proposed a rigorous mathematical construction of a (nonperturbative!) quantization of the Chern–Simons theory in terms of quantum groups and modular tensor categories. And despite this great achievement, there are many questions that remain open in the context of Chern–Simons theory, both of computational and theoretical nature.

In the context of the PSM, on the other hand, one seems still quite far from a nonperturbative quantization. So, this model is not yet really defined as a TQFT—in the sense of the Atiyah–Segal axioms, although there is no serious doubt that such a formulation should exist. However, already now the PSM teaches us at least two more lessons related to the present volume: First, as found by Cattaneo and Felder, the reduced phase space of the PSM, i.e. its Weinstein symplectic quotient, when smooth, carries the structure of a symplectic groupoid (cf. also the contribution of I. Contreras to this volume). And this groupoid is precisely the one that integrates the Lie algebroid T^*M associated to the target Poisson manifold M, a construction suggesting the one needed for the integration problem of general Lie algebroids to Lie groupoids, finally solved by Crainic and Fernandes (in the sense of necessary and sufficient conditions for a smooth integration to exist). This is only one of the examples for a renewed interest in geometrical questions related to field theories already on the *classical* side. Such an understanding of the classical theory is also important in order to identify the difficulties specific to the quantum side when trying to provide rigorous constructions of QFTs. One of the four parts to this book is thus devoted to merely classical or semi-classical investigations of field theories.

Second, the PSM can be viewed as a Chern–Simons theory for the Lie algebroid T^*M: while the integrand of the Chern–Simons theory for an ordinary Lie algebra arises as a transgression of the Pontryagin class, "$tr(F \wedge F) = \mathrm{d}(CS)$", likewise the integrand of the PSM relates to a characteristic 3-form class, "$F^i \wedge F_i = \mathrm{d}(PSM)$" where here F corresponds to the obstruction of the vector bundle morphism $T\Sigma \rightarrow T^*M$ to be a Lie algebroid morphism—it has a 1-form part F^i (from the base map) in addition to a standard 2-form part for curvatures. In fact, there is a topological sigma model that reduces to the PSM in two dimensions and includes the Chern–Simons theory in three, and this is the so-called AKSZ sigma model (after Alexandrov–Kontsevich–Schwarz–Zaboronski; cf. the contribution of Bonavolontà–Kotov as well as the introduction of one of us to this book); and even the relation to higher characteristic classes extends to those (Kotov–Strobl, cf. also Fiorenza–Rogers–Schreiber as well as the contribution of Fiorenza–Sati–Schreiber to this volume). In general, there is a—to our mind useful—trend to higher structures in theories of relevance to mathematical physics and this is also reflected partially in the present book.

One of the, from a mathematical point-of-view, most well-understood classes of QFTs which are *not* topological consists of 2-dimensional conformal field theories (CFTs). In this context the axiomatization of the operator product expansion has led to the notion of vertex (operator) and chiral algebras, which are now widely used both in mathematics and physics. There have been several attempts to generalize these and base the axiomatics of perturbative QFT and the renormalization procedure on the operator product expansion: Kontsevich (unpublished), Hollands, and Costello–Gwilliam (see e.g., the contribution of Costello–Scheimbauer to this volume) in the Euclidean context, Fredenhagen et al in the Lorentzian context (cf. the contribution of Fredenhager–Rejzner to this volume). All these approaches share two things: the appearance of a pattern resembling the one of little disk operads, which axiomatizes the physical concept of "locality," and the use of techniques from deformation quantization.

The concept of locality in $2d$ conformal field theory can also be formulated by *defining* a CFT as a functor from a suitable category of cobordisms to vector spaces (cf. Atiyah–Segal) satisfying certain properties. The foundational work of Beilinson–Drinfeld on chiral algebras exhibits a close relation between these two approaches to the concept of locality: namely, *chiral homology* associates a CFT *à la* Atiyah–Segal with any (conformal) vertex algebra. Recently, Lurie defined a topological analog of chiral homology, known as *factorization homology*: it assigns a TQFT to any algebra over the little n-disk operad, or to any E_n-algebra (cf. contributions of Markarian and Tanaka to this volume for an approach to Chern–Simons theory using factorization homology), which can be proven to be *fully extended* (Scheimbauer). Fully extended TQFTs are known, after the *cobordism hypothesis* (Lurie, Baez–Dolan), to be the "most local" TQFT (cf. also the contibutions of Fiorenza–Sati–Schreiber and Cattaneo–Mnev–Reshetikhin to this volume). Factorization/chiral homology can actually be defined for any factorization algebra (Costello–Gwilliam); a new concept that encompasses the ones of E_n-, vertex and chiral algebras, and whose definition was designed to encode the general algebraic structure of local observables of an arbitrary field theory. It has applications that range from conjectures on renormalization of lattice models (cf. the introduction of one of us to this book) to algebraic topology (cf. Ginot's contribution to the last part of this volume).

We now give a very brief overview on the contents of the book, which starts with an introductory chapter that emphasizes the importance of derived and homotopical (or higher) structures in the mathematical treatment of TQFTs.

Summary of Part I

The first Part is about local aspects of perturbative quantum field theory, with an emphasis on the axiomatization of the algebra behind the operator product expansion and the ideas coming from deformation quantization techniques.

It begins with a Chapter, by Fredenhager–Rejzner, summarizing the approach that was developed for the Lorentzian signature and applicable to also curved (globally hyperbolic) space-times, applying a quantization procedure to QFT by adapting deformation quantization to its setting. It then continues with a contribution, by Costello–Scheimbauer, on partially twisted supersymmetric four-dimensional gauge theories that are studied using the foundational work of Costello and Costello–Gwilliam. The last chapter, written by Wendland, is a short review of Conformal Field Theory, summarizing in particular recent progress made in that field and its relation to the geometry of $K3$ surfaces and *Mathieu moonshine*.

Summary of Part II

The second Part focuses on Chern–Simons (CS) gauge theories.

It begins with a Chapter of Andersen–Kashaev on a construction of $SL(2,\mathbb{C})$ quantum CS theory by means of Teichmüller theory and the quantum dilogarithm of Faddeev. This is followed by a Chapter of Fiorenza–Sati–Schreiber, exhibiting higher structures in a systematic way in the context of an extended prequantum theory of CS-type gauge field theories. The subsequent Chapter consists of two contributions, one by Markarian and one by Tanaka, and deals with the relation between three-dimensional CS theory and factorization homology. Part II is completed by a review of Thuillier about the use of Deligne–Beilinson cohomology for an alternative or deepened understanding of abelian $U(1)$ CS theory.

Summary of Part III

The third Part of this book is devoted to a classical or at most semi-classical analysis of field theories.

It begins with a Chapter of Cattaneo–Mnev–Reshetikhin, introducing some very recent work on the treatment of constraints and boundary conditions in classical field theories, with an emphasis on the BV and BFV formalism. The subsequent contribution, written by Kotov–Bonavolontà, deals with the BV-BRST formalism in the context of AKSZ sigma models, improving previous local results to a global level. The following Chapter of Li-Bland–Ševera provides a beautiful treatment of the (quasi-)Hamiltonian and Poisson geometry of various moduli spaces of flat connections on quilted surfaces, which are relevant in classical Chern–Simons and WZW theories. The final Chapter of this Part aims at understanding the construction of the sympletic groupoid associated to the PSM from the axiomatics of Frobenius algebras.

Summary of Part IV

The fourth Part consists of a single Chapter written by Ginot. It provides a detailed account of the mathematical foundations of Factorization Algebras and Factorization Homology, making extensive use of higher homotopical structures, thus closing the circle opened in the introductory Chapter.

We would like to conclude this preface with a quotation from the Clay Institute's official description of the "Yang–Mills existence and Mass gap" problem as formulated by Arthur Jaffe and Edward Witten:

> *… one does not yet have a mathematically complete example of a quantum gauge theory in four-dimensional space-time, nor even a precise definition of quantum gauge theory in four dimensions. Will this change in the 21st century? We hope so*!

We wholeheartedly share this wish, and hope in turn that some of the mathematical concepts presented in this book will help to better understand, one day, quantum field theories in four dimensions.

France

Damien Calaque
Thomas Strobl

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Contents

Contributors

Jørgen Ellegaard Andersen Center for Quantum Geometry of Moduli Spaces, University of Aarhus, Aarhus, Denmark

Giuseppe Bonavolontà University of Luxembourg, Luxembourg City, Luxembourg

Damien Calaque Université Montpellier 2, Montpellier Cedex 5, France

Alberto S. Cattaneo Institut Für Mathematik, Universität Zürich, Zürich, Switzerland

Ivan Contreras Department of Mathematics, University of California, Berkeley, CA, USA

Kevin Costello Department of Mathematics, Northwestern University, Evanston, IL, USA

Domenico Fiorenza Università degli Studi di Roma "La Sapienza", Roma, Italy

Klaus Fredenhagen II. Institut für Theoretische Physik, Universität Hamburg, Hamburg, Germany

Grégory Ginot Institut de Mathématiques de Jussieu - Paris Rive gauche, Université Pierre et Marie Curie - Sorbonne Universités, Paris Cedex 05, France

Rinat Kashaev University of Geneva, Genève 4, Switzerland

Alexei Kotov Department of Mathematics and Statistics, Faculty of Science and Technology, University of Tromsø, Tromsø, Norway

David Li-Bland Department of Mathematics, University of California, Berkeley, USA

Nikita Markarian Department of Mathematics, National Research University Higher School of Economics, Moscow, Russia

Pavel Mnev St. Petersburg Department of V.A. Steklov Institute of Mathematics of the Russian Academy of Sciences, St. Petersburg, Russia; Chebyshev Laboratory, St. Petersburg State University, St. Petersburg, Russia

Katarzyna Rejzner Department of Mathematics, University of York, York, UK

Nicolai Reshetikhin Department of Mathematics, University of California, Berkeley, CA, USA; KdV Institute for Mathematics, University of Amsterdam, Amsterdam, The Netherlands; ITMO University, Saint Petersburg, Russia

Hisham Sati University of Pittsburgh, Pittsburgh, PA, USA

Claudia Scheimbauer Department of Mathematics, ETH Zürich, Zürich, Switzerland

Urs Schreiber Radboud University Nijmegen, Nijmegen, The Netherlands

Pavol Ševera Department of Mathematics, Université de Genève, Geneva, Switzerland; on leave from Department of Theoretical Physics, FMFI UK, Bratislava, Slovakia

Hiro Lee Tanaka Department of Mathematics, Harvard University, Cambridge, MA, USA

Frank Thuillier LAPTh, Université de Savoie, CNRS, Annecy-le-Vieux Cedex, France

Katrin Wendland Mathematics Institute, Freiburg University, Freiburg im Breisgau, Germany

A Derived and Homotopical View on Field Theories

Damien Calaque

Homological technics have been widely used in physics for a very long time. It seems that their first appearance in quantum field theory goes back to the so-called *Faddaev-Popov ghosts* [16], which have later been mathematically identified as Chevalley generators. In more geometric terms one would nowadays justify their appearance as follows: the quotient space of the phase space by symmetries of the Lagrangian $\mathcal{L}$ might be singular and one shall rather deal with the *quotient stack* instead.

The usefulness of (higher) stacks in quantum field theory is argued in Chap. 6. Let me anyway emphasize that the quotient stack carries some relevant information (such as finite gauge symmetries) that can't be encoded by simply adding new fields.

Another crucial step is the introduction of anti-fields and anti-ghosts. A geometric explanation for anti-fields is that the quantities one wants to compute localize on the critical points of $\mathcal{L}$, which might be degenerate or non-isolated. A smart idea is to consider *the derived critical locus* of $\mathcal{L}$ instead, which one defines as the derived intersection of the graph of $\mathrm{d}_{dR}\mathcal{L}$ with the zero section inside the cotangent of the phase space. A derived intersection can be concretely computed by first applying a (homological) perturbation to one of the two factors and then taking the intersection: anti-fields then simply appear as Koszul generators.

The derived critical locus inherits a (-1)-shifted symplectic structure (see below) which is at the heart of the anti-bracket formalism (a.k.a. BV formalism) [5]. The symmetries of the Lagragian act in a Hamiltonian way on the derived critical locus, and anti-ghosts appear when one is taking the derived zeroes of the moments.

We refer to [25] for related considerations and a wonderful exposition of the homological nature of the BV formalism.

All this seems to be nowadays well-known, but we would like to emphasize two points:

- the usual homological approach to higher structures (see e.g. Chaps. 3 and 10) does not distinguish clearly the "derived" and "stacky" directions, while the rapidly emerging field of derived geometry takes care of it.

D. Calaque (✉)
I3M, Université Montpellier 2, 34095 Montpellier Cedex 5, France
e-mail: damien.calaque@um2.fr

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- one **has to** make use of derived geometry in order to get symplectic structures: the use of non-derived stacks in Chap. 6 systematically destroys the non-degeneracy of Hamiltonian structures.

The second point is very much related to what happens with symplectic structures on moduli spaces (which are deeply studied in Chap. 11). For instance, the moduli stack of flat G-bundles (G being a compact Lie group) on a closed oriented surface **does not** carry any symplectic structure for very simple degree reasons: its tangent complex sits in cohomological degrees -1 and 0. It is only when restricted to a specific locus that the natural pre-sympectic form becomes non-degenerate. But there is a natural **derived stack** of flat G-bundles on a closed surface which is symplectic (its tangent complex sits in degrees -1, 0 and 1).

In this introductory chapter we provide an informal and partial discussion of the usefulness of derived and homotopical technics in field theories.

We begin with a description of field theories of AKSZ type [2] in the framework of derived (algebraic) geometry. The derived geometric approach makes very transparent the fact that this class of theories fits into the axiomatic framework of Atiyah–Segal [3, 26]. We refer to Chap. 9 for a detailed discussion of the compatibility between the BV formalism and the Atiyah–Segal framework.

We then discuss two mathematical formulations of the physical concept of locality: *factorization algebras* and *fully extended field theories*. We put a lot of emphasis on topological field theories and say a few words about conformal field theories. We also mention how these two approaches are related.

We finally end this Chapter with the example of 3d Chern–Simons theory with a finite gauge group and sketch how one could recover the results of [17, Sect. 4] from this approach.

1 Classical Fields and the AKSZ-PTVV Construction

Classical fields are usually described mathematically as sections of (infinite dimensional) fiber bundles. A large class of theories, called *σ-models*, actually describe fields as maps. In the seminal paper [2] the authors introduce the notion of *Q-manifolds*, that allow one to deal with many theories as σ-models. Moreover, the so-called *AKSZ-construction* make them fit into the framework of the *BV formalism* [5] (a.k.a. *anti-bracket formalism*).

A mathematical treatment of perturbative quantum field theory within the framework of the BV quantization (not only for AKSZ theories) can be found in [13] (see also Chap. 9 for examples).

1.1 Transgression

At the heart of the AKSZ formalism [2] and its modern reformulation in [22] (known as PTVV formalism, which is formulated in the language of *derived geometry*[1]) one finds the so-called *transgression procedure*. Let X, Y be generalized spaces (Q-manifolds in the AKSZ formalism, derived stacks in the PTVV formalism). Let ω be a symplectic form of cohomological degree n on Y and assume that X carries an integration theory of cohomological degree d. Then the formula

$$\int_X ev^*\omega\,,$$

where $ev : X \times \mathbf{Map}(X, Y) \longrightarrow Y$ is the evaluation map, defines a symplectic form of cohomological degree $n - d$ on the mapping space $\mathbf{Map}(X, Y)$.

1.1.1 AKSZ versus PTVV: Integration Theory

There are subtle but important differences between the AKSZ and the PTVV formalisms.

In the case of the AKSZ formalism, the integration theory one is referring to is nothing but the *Berezin integration* [7]. Here are three examples of Q-manifolds carrying an integration theory of cohomological degree d in this sense:

1. $(V[1], 0)$, where V is vector space of dimension d.
2. $\Sigma_{dR} := \big(T[1]\Sigma, \mathrm{d}_{dR}\big)$, where Σ is a compact oriented differentiable manifold of dimension d.
3. $\Sigma_{Dol} := \big(T^{0,1}[1]\Sigma, \overline{\partial}\big)$, where Σ is a compact complex manifold of dimension d equipped with a nowhere vanishing top degree holomorphic form η.

Within the PTVV formalism an integration theory of degree d on a derived stack X is a chain map $[X] : \mathbf{R}\Gamma(\mathcal{O}_\Sigma) \longrightarrow \mathbf{k}[-d]$, where $\mathbf{R}\Gamma(\mathcal{O}_\Sigma)$ denotes the complex of derived global functions on Σ, which satisfies a suitable non-degeneracy condition (the definition of non-degeneracy mimics the abstract formulation of Poincaré duality). Any integration theory of cohomological degree d on a Q-manifold in the AKSZ sense induces an integration theory on its associated derived stack in the PTVV sense. But:

different Q-manifolds might have equivalent associated derived stacks.

This is an important point. Derived stacks are model-independent: it doesn't matter how a derived stack is constructed. In the physics language one could view derived stacks as reduced phase space while Q-manifolds carry some information about

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[1] We refer to [27] and references therein for an introduction to derived geometry.

the original phase space (e.g. the moduli stack of flat G-bundles, compared the Q-manifold of all G-connections).

Note that there is a stack with an integration theory that can't be described using Q-manifolds. Let Σ be a Poincaré duality d-space; there is stack Σ_B classifying local systems on Σ (it can be explicitely described using a combinatorial presentation of Σ, such as a triangulation or a cellular structure). Derived global functions on Σ_B are cochains on Σ and thus the fundamental class $[\Sigma]$ determines an integration theory of degree d on Σ_B.

1.1.2 AKSZ versus PTVV: Symplectic Structures

The model independence of derived stacks forces all definitions to be homotopy invariant and as such the required properties can't be strictly satisfied (i.e. they might only hold up to coherent homotopies). This is particularly visible when it comes to closed forms. Roughly speaking, the complex of forms on a derived stack (or a Q-manifold) has two "graduations": the weight (k-forms have weight k) and the cohomological degree. Similarly the differential has two components: the internal differential d_{int} (the Lie derivative with respect to the cohomological vector field Q) and the de Rham differential d_{dR}. In the PTVV formalism a k-form of degree n is a weight k d_{int}-cocycle ω_0 of cohomological degree n, and

being a closed form is an additional structure.

Namely, a closed k-form of degree n consists in a sequence $(\omega_0, \omega_1, \dots)$ where

- ω_0 is a k-form of degree n.
- ω_i has weight $k + i$ and cohomological degree n.
- $\mathrm{d}_{dR}(\omega_i) \pm \mathrm{d}_{int}(\omega_{i+1}) = 0$.

Somehow we are considering forms which are closed up to homotopy, while the AKSZ formalism only considers closed forms which are strictly closed.

Something similar happens for the non-degeneracy property when one defines symplectic structures. An *n-symplectic structure* is the data of a closed 2-form of degree n such that its underlying 2-form of degree n is *non-degenerate* (recall that in the AKSZ formalism the underlying form coincides with the closed one): non-degenerate means that the morphism it induces between the tangent and the cotangent complexes is a quasi-isomorphism (while it is required to be an isomorphism in the AKSZ formalism).

Remark 1 The AKSZ formalism also makes an extensive use of infinite dimensional differential geometry, while derived geometry is designed so that many derived mapping stacks are still locally representable by finite dimensional objects (it is often the case that the reduced phase space is a finite dimensional object even though the original phase space is not).

Example 1 Here are some nice examples of symplectic structures in the derived setting:

- if G is a compact Lie group then $BG = [*/G]$ carries a 2-symplectic structure (see [22]).
- if G is any Lie group then $[\mathfrak{g}^*/G] = T^*[1](BG)$ carries a 1-symplectic structure (see [8]).
- if G is a compact Lie group then $[G/G] = \mathbf{Map}(S^1_B, BG)$ carries a 1-symplectic structure (see [8, 23]).
- the derived critical locus of a function carries a (-1)-symplectic structure (see [22]).

1.2 Transgression with Boundary

In [10, 11] (see also Chap. 9) the AKSZ construction is extended to the case when the source of the σ-model has a boundary and the authors use it to produce field theories that satisfy the axiomatics of Atiyah–Segal [3, 26]. The analogous construction also exists for the PTVV formalism (see [8]).

1.2.1 AKSZ versus PTVV: Lagrangian Structures

Let $X \xrightarrow{f} Y$ be a morphism of generalized spaces and assume we have an n-symplectic structure ω on Y. As usual in derived geometry (and more generally in homotopy theory), being Lagrangian is not a property but rather an additional structure. Namely, a *Lagrangian structure* on f is a homotopy γ (inside the space of closed 2-forms of degree n on X) between $f^*\omega$ and 0 such that the underlying path γ_0 between $f^*\omega_0$ and 0 is non-degenerate. In more explicit terms:

- $\gamma = (\gamma_0, \gamma_1, \ldots)$ is such that $f^*\omega_0 = \mathrm{d}_{int}(\gamma_0)$ and
$$f^*\omega_i = \mathrm{d}_{int}(\gamma_i) \pm \mathrm{d}_{dR}(\gamma_{i-1})\,.$$
- the identity satisfied by γ_0 ensures that the map $\mathbb{T}_X \longrightarrow f^*\mathbb{L}_Y[n]$ given by $f^*\omega_0$ lifts to $\mathbb{T}_X \longrightarrow \mathbb{L}_f[n+1]$, where $\mathbb{L}_f$ is the relative cotangent complex. The non-degeneracy condition says that it is a quasi-isomorphism.

Usual Lagrangian subspaces are Lagrangian in the above sense, but any kind of map can carry a Lagrangian structure. There are actually Lagrangian structures arising in a quite surprising way:

Example 2 (*See* [8, 9, 23]). (a) A Lagrangian structure on the morphism $X \longrightarrow *_{(n+1)}$, where $*_{(n+1)}$ is the point equipped with its canonical $(n+1)$-symplectic structure, is the same as an n-symplectic structure on X.

(b) A moment map $\mu : X \longrightarrow \mathfrak{g}^*$ induces a Lagrangian structure on the map $[\mu] : [X/G] \longrightarrow [\mathfrak{g}^*/G]$.
(c) A Lie group valued moment map (in the sense of [1]) $\mu : X \longrightarrow G$, where G is a compact Lie group, induces a Lagrangian structure on the map $[\mu] : [X/G] \longrightarrow [G/G]$.

1.2.2 Relative Integration Theory

A *relative integration theory* (a.k.a. non-degenerate boundary structure or relative orientation, see [8]) on a morphism $X \xrightarrow{f} Y$ is the data of an integration theory $[X]$ on X together with a homotopy η between $f_*[X]$ and 0 that is non-degenerate.[2]

Example 3 There are two important examples of relative integration theories on a morphism. Consider a compact oriented $(d+1)$-manifold Σ with oriented boundary $\partial\Sigma$. Then the morphisms $(\partial\Sigma)_{dR} \longrightarrow \Sigma_{dR}$ and $(\partial\Sigma)_B \longrightarrow \Sigma_B$ both carry a relative integration theory.

Let $X \xrightarrow{f} Y$ be a morphism together with a relative integration theory $([X], \eta)$, and let Z be equipped with an n-symplectic structure ω. It is shown in [8] that

$$\int_\eta ev^*\omega$$

defines a Lagrangian structure on the pull-back morphism $\mathbf{Map}(Y, Z) \longrightarrow \mathbf{Map}(X, Z)$.

1.2.3 Field Theories from Transgression with Boundary

Given a generalized space Y together with an n-symplectic structure, the process of transgression with boundary allows one to produce a functor $\mathbf{Map}(-, Y)$ from a category with

- objects being generalized spaces with an integration theory,
- morphisms from X_1 to X_2 being cospans $X_1 \coprod \overline{X_2} \to X_{12}$ equipped with a relative integration theory,[3]
- composition being given by gluing: $X_{12} \circ X_{23} := X_{12} \coprod_{X_2} X_{23}$.

[2] We won't detail what non-degeneracy means here, but simply say that its definition again mimics the main abstract feature of relative Poincaré duality.

[3] Here an below, $\overline{??}$ means that we consider the opposite integration theory or the opposite symplectic structure on ?? (it should be clear from the context).

to a category with

- objects being generalized spaces with a shifted symplectic structure,
- morphisms from Z_1 to Z_2 being Lagrangian correspondences $Z_{12} \to Z_1 \times \overline{Z_2}$,
- composition being given by fiber products: $Z_{12} \circ Z_{23} := Z_{12} \underset{Z_2}{\times} Z_{23}$.

If we restrict objects of the source category to those of the form described in Example 3, then we precisely get a topological field theory taking values in a category of Lagrangian correspondences. Note that usually, categories of Lagrangian correspondences are ill-defined (as some compositions might not be well-behaved), but working in the homotopy setting and considering derived fiber products resolves this problem.

Remark 2 The gadget one actually has to work with is called an $(\infty, 1)$*-category*, and one shall emphasize that *categroids* (which appear in the main references for Chaps. 5 and 12) are often shadows of an underlying $(\infty, 1)$-category (in other words, even though some compositions might seem to be ill-defined, they actually happen to be well-defined *up to homotopy*).

1.3 Examples

We now provide examples of classical topological field theories that can be treated using the above approach, even though some superconformal field theories (as described in Chap. 4) can be obtained as well.

1.3.1 Classical Chern–Simons Theory

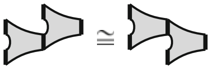
Classical Chern–Simons theory can be recovered if one starts with $Y = BG$ for a compact Lie group G. Details can be found in [23]. One can also include all kinds of boundary conditions (Lagrangian morphisms) or domain-walls (Lagrangian correspondences), which allow to recover all the symplectic moduli spaces of flat connections over quilted surfaces that are obtained *via* the quasi-Hamiltonian formalism in Chap. 11.

1.3.2 Moore–Tachikawa Theory

There is a $2d$ TFT that have been sketched by Moore and Tachikawa [21], of which the target category is a certain category of holomorphic symplectic varieties. This category is a particular case of our category of Lagrangian correspondences (see [9]) and it is very likely that their TFT can be obtained from mapping spaces.

1.3.3 Poisson σ-model

Let (X, π) be a Poisson manifold and consider its π-twisted 1-shifted cotangent $Y := T_X[-1]_\pi$. The derived stack Y, resp. the zero section morphism $X \longrightarrow Y$, can be shown to carry a 1-symplectic structure, resp. a Lagrangian structure. One can show that the mapping stack from $(\,)_B$ to Y with boundary condition in X, which happens to be the derived self-intersection $\mathcal{G} := X \times_Y^h X$ of X into Y, is 0-symplectic (see [8, 9, 27] for general statements about symplectic structures on relative derived mapping stacks). The cobordism with boundary is sent to a Lagrangian correspondence between $\mathcal{G} \times \mathcal{G}$ and $\mathcal{G}$, which turns $\mathcal{G}$ into an algebra object within the $(\infty, 1)$-category of Lagrangian correspondences. For instance, associativity of composition is given by the following diffeomorphism:



In [12] Contreras and Scheimbauer show that $\mathcal{G}$ is actually a Calabi-Yau algebra (in the sense of [19]), which clarifies the mysterious axioms of a relational symplectic groupoid of Chap. 12.

2 Mathematical Formulations of Locality

The AKSZ-PTVV theories are expected to be local, in the sense that one can compute everything from local data that one would later glue. In this section we briefly sketch two mathematical approaches to the concept of locality.

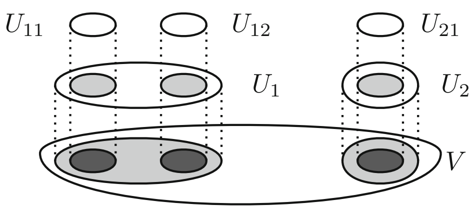
2.1 Factorization Algebras

A *factorization algebra* E over a topological space X consists of

- the data of a vector space (or a cochain complex) E_U for every open subset $U \subset X$.
- the data of a linear map (or a chain map) $\bigotimes_{i \in I} E_{U_i} \longrightarrow E_V$ for every inclusion $\coprod_{i \in I} U_i \subset V$ of pairwise disjoint open subsets.

satisfying the following properties:

- associativity, that can more or less be depicted as follows:



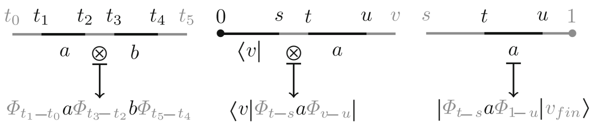
- gluing (one can reconstruct E_U from a nice open cover $\mathcal{U}$ of U and $E_{\mathcal{U}}$).

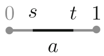
Remark 3 The gluing property is typically a locality property.

We refer to Chaps. 3 and 13 for precise definitions.

Example 4 (*Topological quantum mechanics, see* [14]). Let A be an associative algebra (e.g. $A = \mathbf{End}(V)$) and let $(\Phi_t)_t$ be a 1-parameter group of automorphisms of A (e.g. $\Phi_t = e^{-\frac{it}{\hbar}H}$ is the time evolution). We also give ourselves a right A-module M_r (e.g. V^*) and a left A-module M_ℓ (e.g. V), together with initial and final states $v_{init} \in M_r$ and $v_{fin} \in M_\ell$. From these data one can describe a factorization algebra E on the closed interval $X = [0, 1]$.

- on open intervals of X we set: $E_{[0,s[} = M_r$, $E_{]t,u[} = A$ and $E_{]v,1]} = M_\ell$.
- here are examples of the factorization product:



- one can show that $E_{[0,1]} = M_r \underset{A}{\otimes} M_\ell$ ($\mathbb{C}$ in our example).
- we finally interpret 

$\longmapsto \langle v_{init}|\Phi_s a \Phi_{1-t}|v_{fin}\rangle$ as an expectation value.

2.1.1 Factorization Algebras in the BV Formalism

Producing factorization algebras from the local observables in the BV formalism is the main achievement of Costello–Gwilliam (see [14], and also Chap. 3). At the classical level they consider observables with compact support in order to get factorization algebras. It seems that for topological and conformal AKSZ (or PTVV) theories one can consider mapping spaces with compact support in order to get a factorization algebra structure on classical local observables. In particular it is expected that the transgression procedure (both for symplectic and Lagrangian structures) still makes sense locally and glues well.

The main difficult part in Costello–Gwilliam work is of course the quantization of these classical theories. One has to consider effective field theories in the sense of [13] and renormalize (when possible). In the $2d$ conformal case one gets in the end a structure which is very similar to the one of a vertex algebra (see Chap. 3 for a precise statement and Chap. 4 for the definition of a vertex algebra and its use in conformal field theory). In the topological case one obtains in the end a *locally constant* factorization algebra: on $\mathbb{R}^n$ this boils down to the datum of an algebra over the little disks operad.

Renormalization is actually trivial in the topological case, even though it is not so obvious in Costello's framework. We propose here a different approach to the quantization of classical topological BV theories. The first step is to discretize the theory one is working with, so that one can easily write a factorization algebra of classical discrete local observables that carries a bracket of degree 1. The main point is that on a finite region the algebra of local observable is finitely generated, so that BV quantization can be performed very easily (there is no need to apply any kind of energy cut-off as we have only finitely many states).

The final and very hard step is to make the mesh of the discretization tend to zero. There is some magic that happens for topological theories:

there is no need to make the mesh tend to zero.

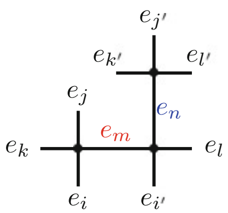
The reason is that, even though the factorization algebra of local observables is not locally constant, it becomes *locally constant at a sufficiently large scale* (the scale depending on the size of the mesh).

Remark 4 Understanding the renormalization procedure for lattice field theories in terms of factorization algebras could lead to a non-perturbative alternative to the constructions of QFTs proposed in [13, 14]. At the moment we[4] can only recover the Weyl algebra from a discrete $1d$ model. The next step would be to understand the renormalization of discrete models in 2 dimensions (with an emphasis on conformal ones).

2.1.2 Locally Constant Factorization Algebras from Discrete Models

One can prove that any factorization algebra that is locally constant above a given scale gives rise to a locally constant factorization algebra that coincides with the original one above that scale. The idea is very simple: discard the badly behaved part (the one below the given scale) and replace it by a rescaled copy of what happens at large scale... note that implementing this idea actually requires the use of the higher categorical machinery.

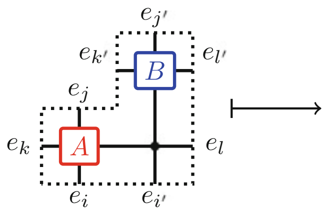
Let us provide a potential application of this quite intuitive idea to lattice models. We will formulate things in dimension 2 but it works in arbitrary dimension. Let H, V be vector spaces of states (horizontal and vertical) and let $R \in GL(H \otimes V)$ be an interaction matrix: $R^{jl}_{ik} = \exp\left(-\frac{1}{kT}\epsilon^{jl}_{ik}\right)$. Computing a state sum is nothing but tensor calculus:



$$\longmapsto \quad R^{jm}_{ik} R^{nl}_{i'm} R^{j'l'}_{nk'}$$

One can define a factorization algebra $\mathcal{F}_R$ which associates the space of its boundary states to a given open region of $\mathbb{R}^2$, and for which the factorization product can be depicted in the following way:

[4] This is a joint project with Giovanni Felder.



$$A \otimes B \mapsto \left(A^{jm}_{ik} R^{nl}_{i'm} B^{j'\ '}_{nk'}\right)^{jj'll'}_{ii'kk'}$$

Note that the lattice $\mathbb{Z}^2$ acts on global sections $\mathcal{F}_R(\mathbb{R}^2)$ of $\mathcal{F}_R$.

Conjecture 1 (Kontsevich). *$C^\bullet\left(\mathbb{Z}^2, \mathcal{F}_R(\mathbb{R}^2)\right)$ has an action of the (chains on the) little disks operads in dimension 2.*

The idea to prove this conjecture is to define a new factorization algebra $\widetilde{\mathcal{F}}_R$, very similar to $\mathcal{F}_R$ but carrying an additional discretized de Rham differential,[5] such that

- $\widetilde{\mathcal{F}}_R$ is locally constant at scale > 2.
- $\widetilde{\mathcal{F}}_R(\mathbb{R}^2) = C^\bullet\left(\mathbb{Z}^2, \mathcal{F}_R(\mathbb{R}^2)\right)$.

This would imply Kontsevich's conjecture.

2.2 *Fully Extended Field Theories*

The axiomatics of fully extended field theories is a higher categorical analog of Atiyah–Segal axiomatics. Roughly speaking, it is a symmetric monoidal functor from a symmetric monoidal higher category of cobordisms to another symmetric monoidal higher category. Higher categories of cobordism can be informally described as follows (we refer to [19] for precise definitions in the topological setting):

- objects are 0 dimensional manifolds of a certain type.
- 1-morphisms are 1-cobordisms between these.
- 2-morphisms are 2-cobordisms,
- ...

It is only for topological field theories that the above has been formalized in a mathematically precise way (see [4, 19]). The *cobordism hypothesis* (which is now a Theorem thanks to the work of Lurie) states that fully extended topological field theories are completely determined by their value on the point. One can see this as a very strong locality property (everything can be reconstructued from the point!). Objects that are images of the point under fully extended TFTs are called *fully dualizable*: being fully dualizable is a very strong finiteness requirement.

We refer to [18] for a very nice review of the cobordism hypothesis (note that the cobordism hypothesis appears implicitly in Chaps. 6 and 9).

[5] Roughly, $\mathcal{F}_R$ carries a discrete flat connection and $\widetilde{\mathcal{F}}_R$ is the factorization algebra of derived flat sections of $\mathcal{F}_R$.

2.2.1 Examples of Fully Extended TFTs

In dimension 1, fully dualizable objects are genuine dualizable objects (e.g. finite dimensional vector spaces).

Classical field theories of AKSZ-PTVV type are fully extended. This has been announced (without proof) in [11] and [8, 9]. The target category to work with is a suitable category of iterated Lagrangian correspondences, that is currently the subject of ongoing investigations.

It is expected that modular tensor categories are fully dualizable in the 4-category of braided monoidal categories, leading to a large class of fully extended $4d$ TFTs.

2.2.2 Chiral and Factorization Homologies

Locality in $2d$ conformal field theory can be formalized either using modular functors or vertex algebras. Chiral homology, that was invented by Beilinson–Drinfeld [6], allows one to produce a modular functor out of a (quasi-conformal) vertex algebra.

Factorization homology (a.k.a. topological chiral homology) achieves the same goal in the topological setting. If A is an algebra over the little disks operad and M is a framed manifold then factorization homology of M with coefficients in A, denoted

$$\int_M A\,,$$

is defined as the "integral", over all open balls in M, of the value of A on them. Lurie proved [19, 20] that factorization homology is indeed a TFT, and conjectured that it is fully extended. Chapter 7 presents perturbative Chern–Simons theory in dimension 3 as a by-product of factorization homology.

The fact that factorization homology is a fully extended TFT was recently proved in [24].

2.2.3 Chern–Simons Theory with a Finite Gauge Group

Let G be a finite group.

Remark 5 The cotangent complex of BG reduces to $\{0\}$, so that BG is trivially n-symplectic for any $n \in \mathbb{Z}$. Therefore symplectic structures won't play a significant rôle for this specific example. But they are essential when one deals with non-discrete compact Lie groups.

We have a $3d$ fully extended TFT with values in a higher category of iterated correspondences that is given by $\mathbf{Map}(-, BG)$.

It is very unlikely that the category of correspondences can provide numerical invariants. In order to get that we have to "linearize" our field theory.

Let us sketch how to do this in dimensions 1-2-3:

- we replace $\mathbf{Map}\big(S^1_B, BG\big) = [G/G]$ by its category of quasi-coherent sheaves $QCoh([G/G])$, which is nothing but the category $Rep\big(D(G)\big)$ of (complexes of) representation of the Drinfeld double of G.
- the correspondence given by $\mathbf{Map}(\Sigma, BG)$ for a 2-cobordism Σ can be used to produce a convolution functor $QCoh([G/G]^k) \longrightarrow QCoh([G/G]^l)$.
- mapping spaces from $3d$ manifolds produce natural transformations of functors.

Remark 6 One can even associate the monoidal category $QCoh(BG) = Rep(G)$ to the point. It is important to notice that not every object is fully dualizable in the 3-category of monoidal categories. But $Rep(G)$ surely is,[6] so that we have a nice and linear enough fully extended TFT.

It would be interesting to get back this fully extended Chern–Simons TFT with finite gauge group by means of factorization homology. In order to do so one shall construct a locally constant factorization algebra on $\mathbb{R}^3$ that is locally constant. We would suggest to use a discrete model.

Remark 7 Observe that $Rep(G)$ is a fusion category and is thus, after [15], a fully dualizable object in the symmetric monoidal 3-category of monoidal categories. It thus produces a fully extended $3d$ TFT. The fact that the partition function of this TFT can be computed *via* a state sum (see [28]) is a strong evidence in favor of our suggestion.

One must say that already for Yang-Mills theory in dimension 2 it is an interesting task to produce an E_2-algebra from the data of a Hopf algebra with an integral, by means of a discrete model.

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[6] This is very much related to the fact that the symplectic structure on BG is zero. In the case of compact groups, $Rep(G)$ isn't finite enough and must be deformed in order to get a rigid enough object... this is where the quantum group comes from.

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Part I
Locality in Perturbative QFTs

One of the most well-understood class of quantum field theories from a mathematical point of view consists of two-dimensional conformal field theories (CFT). In this context the axiomatization of the operator product expansion has led to the notion of vertex (operator) and chiral algebras, which are now widely used both in mathematics and physics. There have been several attempts to generalize these and base the axiomatics of perturbative QFT and the renormalization procedure on the operator product expansion: Kontsevich (unpublished), Hollands, and Costello-Gwilliam in the Euclidean context, Fredenhagen et al. in the Lorentzian context. All these approaches share two things: the appearance of a pattern ressembling that of little disk operads, and the use of techniques from deformation quantization.

Part I begins with "Perturbative Algebraic Quantum Field Theory" (written by Klaus Fredenhagen and Katarzyna Rejzner) summarizing the approach that was developped for the Lorentzian signature. It then continues with "Lectures on Mathematical Aspects of (Twisted) Supersymmetric Gauge Theories" (written by Kevin Costello and Claudia Scheimbauer) about Costello's approach to supersymmetric four-dimensional gauge theories, in the Euclidean context, where the notion of factorization algebra appears to be proeminent.

Part I ends with "Snapshots of Conformal Field Theory" (written by Katrin Wendland) which reviews some recent developments in (super)conformal field theory. It reports in particular some very exciting considerations about the geometry of K3 surfaces and the *Mathieu Moonshine*.

Perturbative Algebraic Quantum Field Theory

Klaus Fredenhagen and Katarzyna Rejzner

Abstract These notes are based on the course given by Klaus Fredenhagen at the Les Houches Winter School in Mathematical Physics (January 29–February 3, 2012) and the course *QFT for mathematicians* given by Katarzyna Rejzner in Hamburg for the Research Training Group 1670 (February 6–11, 2012). Both courses were meant as an introduction to modern approach to perturbative quantum field theory and are aimed both at mathematicians and physicists.

1 Introduction

Quantum field theory (QFT) is at present the by far most successful description of fundamental physics. Elementary physics is to a large extent explained by a specific quantum field theory, the so-called Standard Model. All the essential structures of the standard model are nowadays experimentally verified. Outside of particle physics, quantum field theoretical concepts have been successfully applied also to condensed matter physics.

In spite of its great achievements, quantum field theory also suffers from several longstanding open problems. The most serious problem is the incorporation of gravity. For some time, many people believed that such an incorporation would require a radical change in the foundation of the theory, and one favored theories with rather different structures as e.g. string theory or loop quantum gravity. But up to now these alternative theories did not really solve the problem; moreover there are several indications that QFT might be more relevant to quantum gravity than originally expected.

K. Fredenhagen (✉)
II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany
e-mail: klaus.fredenhagen@desy.de

K. Rejzner
Department of Mathematics, University of York, Heslington, YO10 5DD York, UK
e-mail: kasia.rejzner@york.ac.uk

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Another great problem of QFT is the difficulty of constructing interesting examples. In nonrelativistic quantum mechanics the construction of a selfadjoint Hamiltonian is possible for most cases of interest, in QFT, however the situation is much worse. Models under mathematical control are

- free theories
- superrenormalizable models in 2 and 3 dimensions
- conformal field theories in 2 dimensions
- topological theories in 3 dimensions
- integrable theories in 2 dimensions

but no single interacting theory in 4 dimensions, in particular neither the standard model nor any of its subtheories like QCD or QED. Instead one has to evaluate the theory in uncontrolled approximations, mainly using formal perturbation theory, and, in the case of QCD, lattice gauge theories.

If one attempts to incorporate gravity, an additional difficulty is the apparent nonlocality of quantum physics which is in conflict with the geometrical interpretation of gravity in Einstein's theory. Even worse, the traditional treatment of QFT is based on several additional nonlocal concepts, including

- vacuum (defined as the state of lowest energy)
- particles (defined as irreducible representations of the Poincaré group)
- S-matrix (relies on the notion of particles)
- path integral (involves nonlocal correlations)
- euclidean space (does not exist for generic Lorentzian spacetime)

There exists, however, a formulation of QFT which is based entirely on local concepts. This is Algebraic QFT (AQFT), or, synonymously, Local Quantum Physics [20]. AQFT relies on the algebraic formulation of quantum theory in the sense of the original approach by Born, Heisenberg and Jordan and formalized in terms of C^*-algebras by I. Segal. The step from quantum mechanics to QFT is performed by incorporating the principle of locality in terms of local algebras of observables. This is the algebraic approach to field theory proposed by Haag and Kastler [18]. By the Haag-Ruelle scattering theory the Haag-Kastler framework on Minkowski space, together with some mild assumptions on the energy momentum spectrum, already implies the existence of scattering states of particles and of the S-matrix.

It required some time before this framework could be generalized to generic Lorentzian spacetimes. A direct approach was performed by Dimock [12], but the framework he proposed did not contain an appropriate notion of covariance. Such a notion, termed *local covariance* was introduced more recently in a programmatic paper by Brunetti, Verch and one of us (K.F.) [9] motivated by the attempt to define the renormalized perturbation series of QFT on curved backgrounds [7, 21, 22]. It amounts to an assignment of algebras of observable to generic spacetimes, subject to a certain coherence condition formulated in the language of category theory. In Sect. 3 we will describe the framework in detail.

The framework of locally covariant field theory is a plausible system of axioms for a generally covariant field theory. Before we enter the problem of constructing examples of quantum field theory satisfying these axioms we describe the corresponding structure in classical field theory (Sect. 4). Main ingredient is the so-called Peierls bracket by which the classical algebra of observables becomes a Poisson algebra.

Quantization can be done in the sense of formal deformation quantization, i.e. in terms of formal power series in $\hbar$ at least for free field theories, and one obtains an abstract algebra resembling the algebra of Wick polynomials on Fock space (Sect. 5). Interactions can then be introduced by the use of a second product in this algebra, namely the time ordered product. Disregarding for a while the notorious UV divergences of QFT we show how interacting theories can be constructed in terms of the free theory (Sect. 6).

In the final part of these lectures (Sect. 7) we treat the UV divergences and their removal by renormalization. Here again the standard methods are nonlocal and loose their applicability on curved spacetimes. Fortunately, there exists a method which is intrinsically local, namely causal perturbation theory. Causal perturbation theory was originally proposed by Stückelberg and Bogoliubov and rigorously elaborated by Epstein and Glaser [16] for theories on Minkowski space. The method was generalized by Brunetti and one of us (K.F) [7] to globally hyperbolic spacetimes and was then combined with the principle of local covariance by Hollands and Wald [21, 22]. The latter authors were able to show that renormalization can be done in agreement with the principle of local covariance. The UV divergences show up in ambiguities in the definition of the time ordered product. These ambiguities are characterized by a group [10, 13, 23], namely the renormalization group as originally introduced by Petermann and Stückelberg [38].

2 Algebraic Quantum Mechanics

Quantum mechanics in its original formulation in the Dreimännerarbeit by Born, Heisenberg and Jordan is based on an identification of observables with elements of a noncommutative involutive complex algebra with unit.

Definition 1 An involutive complex algebra $\mathfrak{A}$ is an algebra over the field of complex numbers, together with a map, $^* : \mathfrak{A} \to \mathfrak{A}$, called an involution. The image of an element A of $\mathfrak{A}$ under the involution is written A^*. Involution is required to have the following properties:

1. for all $A, B \in \mathfrak{A}$: $(A + B)^* = A^* + B^*$, $(AB)^* = B^* A^*$,
2. for every $\lambda \in \mathbb{C}$ and every $A \in \mathfrak{A}$: $(\lambda A)^* = \overline{\lambda} A^*$,
3. for all $A \in \mathfrak{A}$: $(A^*)^* = A$.

In quantum mechanics such an abstract algebra is realized as an operator algebra on some Hilbert space.

Definition 2 A representation of an involutive unital algebra $\mathfrak{A}$ is a unital *-homomorphism π into the algebra of linear operators on a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$.

Let us recall that an operator A on a Hilbert space $\mathcal{H}$ is defined as a linear map from a subspace $\mathcal{D} \subset \mathcal{H}$ into $\mathcal{H}$. In particular, if $\mathcal{D} = \mathcal{H}$ and A satisfies $||A|| \doteq \sup_{||x||=1}\{||Ax||\} < \infty$, it is called *bounded*. Bounded operators have many nice properties, but in physics many important observables are represented by unbounded ones. The notion of an algebra of bounded operators on a Hilbert space can be abstractly phrased in the definition of a C^*-algebra.

Definition 3 A C^*-algebra is a Banach involutive algebra (Banach algebra with involution satisfying $\|A^*\| = \|A\|$), such that the norm has the C^*-property:

$$\|A^*A\| = \|A\|\|A^*\|, \quad \forall A \in \mathfrak{A}\,.$$

One can prove that every C^*-algebra is isomorphic to a norm closed algebra of bounded operators $\mathcal{B}(\mathcal{H})$ on a (not necessarily separable) Hilbert space $\mathcal{H}$. A representation of a C^*-algebra $\mathfrak{A}$ is a unital *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$.

In the simplest example from quantum mechanics the algebra of observables is the associative involutive complex unital algebra generated by two hermitian[1] elements p and q with the canonical commutation relation

$$[p, q] = -i\hbar 1_{\mathfrak{A}}\,. \tag{1}$$

This algebra can be realized as an operator algebra on some Hilbert space, but the operators corresponding to p and q cannot both be bounded. Therefore it is convenient, to follow the suggestion of Weyl and to replace the unbounded (hence discontinuous) operators p and q by the unitaries[2] (Weyl operators) $W(\alpha, \beta)$, $\alpha, \beta \in \mathbb{R}$. Instead of requiring the canonical commutation relation for p and q one requires the relation (Weyl relation)

$$W(\alpha, \beta)W(\alpha', \beta') = e^{\frac{i\hbar}{2}(\alpha\beta' - \alpha'\beta)} W(\alpha + \alpha', \beta, \beta') \tag{2}$$

The antilinear involution (adjunction)

$$W(\alpha, \beta)^* = W(-\alpha, -\beta)\,. \tag{3}$$

replaces the hermiticity condition on p and q. The Weyl algebra $\mathfrak{A}_W$ is defined as the unique C^*-algebra generated by unitaries $W(\alpha, \beta)$ satisfying the relations (2), with involution defined by (3) and with unit $1_{\mathfrak{A}} = W(0, 0)$.

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[1] An operator A on a Hilbert space $\mathcal{H}$ with a dense domain $D(A) \subset \mathcal{H}$ is called *hermitian* if $D(A) \subset D(A^*)$ and $Ax = A^*x$ for all $x \in D(A)$. It is *selfadjoint* if in addition $D(A^*) \subset D(A)$.

[2] An element A of an involutive Banach algebra with unit is called unitary if $A^*A = 1_{\mathfrak{A}} = AA^*$.

One can show that if the Weyl operators are represented by operators on a Hilbert space such that they depend strongly continuously[3] on the parameters α and β, then p and q can be recovered as selfadjoint generators, i.e.

$$W(\alpha, \beta) = e^{i(\alpha p + \beta q)} ,$$

satisfying the canonical commutation relation (1). As shown by von Neumann, the C^*-algebra $\mathfrak{A}_W$ has up to equivalence only one irreducible representation where the Weyl operators depend strongly continuously on their parameters, namely the Schrödinger representation $(\mathcal{L}^2(\mathbb{R}), \pi)$ with

$$(\pi(W(\alpha, \beta))\Phi)(x) = e^{\frac{i\hbar\alpha\beta}{2}} e^{i\beta x} \Phi(x + \hbar\alpha) , \tag{4}$$

and the reducible representations with the same continuity property are just multiples of the Schrödinger representation. If one does not require continuity there are many more representations, and they have found recently some interest in loop quantum gravity. In quantum field theory the uniqueness results do not apply, and one has to deal with a huge class of inequivalent representations.

For these reasons it is preferable to define the algebra of observables $\mathfrak{A}$ independently of its representation on a specific Hilbert space as a unital C^*-algebra. The observables are the selfadjoint elements, and the possible outcomes of measurements are elements of their spectrum. The spectrum $\mathrm{spec}(A)$ of $A \in \mathfrak{A}$ is the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda 1_{\mathfrak{A}}$ has no inverse in $\mathfrak{A}$. One might suspect that the spectrum could become smaller if the algebra is embedded in a larger one. Fortunately this is not the case; for physics this mathematical result has the satisfactory effect that the set of possible measurement results of an observable is not influenced by the inclusion of additional observables.

Now we know what the possible outcome of an experiment could be, but what concrete value do we get, if we perform a measurement? In QM this is not the right question to ask. Instead, we can only determine the *probability distribution* of getting particular values from a measurement of an observable A. This probability distribution can be obtained, if we know the *state* of our physical system. Conceptually, a state is a prescription for the preparation of a system. This concept entails in particular that experiments can be reproduced and is therefore equivalent to the ensemble interpretation where the statements of the theory apply to the ensemble of equally prepared systems.

A notion of a state can be also defined abstractly, in the following way:

Definition 4 A state on an involutive algebra $\mathfrak{A}$ is a linear functional $\omega : \mathfrak{A} \to \mathbb{C}$, such that:

$$\omega(A^*A) \geq 0 \quad \text{and} \quad \omega(1_{\mathfrak{A}}) = 1 .$$

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[3] A net $\{T_\alpha\}$ of operators on a Hilbert space $\mathcal{H}$ converges strongly to an operator T if and only if $||T_\alpha x - Tx|| \to 0$ for all $x \in \mathcal{H}$.

The first condition can be understood as a positivity condition and the second one is the normalization. The values $\omega(A)$ are interpreted as the expectation values of the observable A in the given state. Given an observable A and a state ω on a C*-algebra $\mathfrak{A}$ we can reconstruct the full probability distribution $\mu_{A,\omega}$ of measured values of A in the state ω from its moments, i.e. the expectation values of powers of A,

$$\int \lambda^n d\mu_{A,\omega}(\lambda) = \omega(A^n) \, .$$

States on C^*-algebras are closely related to representations on Hilbert spaces. This is provided by the famous GNS (Gelfand-Naimark-Segal) theorem:

Theorem 1 *Let ω be a state on the involutive unital algebra $\mathfrak{A}$. Then there exists a representation π of the algebra by linear operators on a dense subspace $\mathcal{D}$ of some Hilbert space $\mathcal{H}$ and a unit vector $\Omega \in \mathcal{D}$, such that*

$$\omega(A) = (\Omega, \pi(A)\Omega) \, ,$$

and $\mathcal{D} = \{\pi(A)\Omega, A \in \mathfrak{A}\}$.

Proof The proof is quite simple. First let us introduce a scalar product on the algebra in terms of the state ω by

$$\langle A, B \rangle \doteq \omega(A^* B) \, .$$

Linearity for the right and antilinearity for the left factor are obvious, hermiticity $\langle A, B \rangle = \overline{\langle B, A \rangle}$ follows from the positivity of ω and the fact that we can write $A^* B$ and $B^* A$ as linear combinations of positive elements:

$$\begin{aligned} 2(A^*B + B^*A) &= (A+B)^*(A+B) - (A-B)^*(A-B) \, , \\ 2(A^*B - B^*A) &= -i(A+iB)^*(A+iB) + i(A-iB)^*(A-iB) \, . \end{aligned}$$

Furthermore, positivity of ω immediately implies that the scalar product is positive semidefinite, i.e. $\langle A, A \rangle \geq 0$ for all $A \in \mathfrak{A}$. We now study the set

$$\mathfrak{N} \doteq \{A \in \mathfrak{A} | \omega(A^* A) = 0\} \, .$$

We show that $\mathfrak{N}$ is a left ideal of $\mathfrak{A}$. Because of the Cauchy-Schwarz inequality $\mathfrak{N}$ is a subspace of $\mathfrak{A}$. Moreover, for $A \in \mathfrak{N}$ and $B \in \mathfrak{A}$ we have, again because of the Cauchy-Schwarz inequality:

$$\omega((BA)^*BA) = \omega(A^*B^*BA) = \left\langle B^*BA, A \right\rangle \leq \sqrt{\langle B^*BA, B^*BA \rangle}\sqrt{\langle A, A \rangle} = 0 \, ,$$

hence $BA \in \mathfrak{N}$. Now we define $\mathcal{D}$ to be the quotient $\mathfrak{A}/\mathfrak{N}$. Per constructionem the scalar product is positive definite on $\mathcal{D}$, thus we can complete it to obtain a Hilbert space $\mathcal{H}$. The representation π is induced by left multiplication of the algebra,

$$\pi(A)(B + \mathfrak{N}) \doteq AB + \mathfrak{N}\,,$$

and we set $\Omega = 1 + \mathfrak{N}$. In case that $\mathfrak{A}$ is a C*-algebra, one can show that the operators $\pi(A)$ are bounded, hence admitting unique continuous extensions to bounded operators on $\mathcal{H}$.

It is also straightforward to see that the construction is unique up to unitary equivalence. Let $(\pi', \mathcal{D}', \mathcal{H}', \Omega')$ be another quadruple satisfying the conditions of the theorem. Then we define an operator $U : \mathcal{D} \to \mathcal{D}'$ by

$$U\pi(A)\Omega \doteq \pi'(A)\Omega'.$$

U is well defined, since $\pi(A)\Omega = 0$ if and only if $\omega(A^*A) = 0$, but then we have also $\pi'(A)\Omega' = 0$. Furthermore U preserves the scalar product and is invertible and has therefore a unique extension to a unitary operator from $\mathcal{H}$ to $\mathcal{H}'$. This shows that π and π' are unitarily equivalent.

The representation π will not be irreducible, in general, i.e. there may exist a non-trivial closed invariant subspace. In this case, the state ω is not pure, which means that it is a convex combination of other states,

$$\omega = \lambda\omega_1 + (1 - \lambda)\omega_2\ ,\ 0 < \lambda < 1\ ,\ \omega_1 \neq \omega_2\ . \tag{5}$$

To illustrate the concept of the GNS representation, let $\pi_{1,2}$ be representations of $\mathfrak{A}$ on Hilbert spaces $\mathcal{H}_{1,2}$, respectively. Choose unit vectors $\Psi_1 \in \mathcal{H}_1$, $\Psi_2 \in \mathcal{H}_2$ and define the states

$$\omega_i(A) = \langle \Psi_i, \pi_i(A)\Psi_i\rangle\ ,\ i = 1, 2\ . \tag{6}$$

Let ω be the convex combination

$$\omega(A) = \frac{1}{2}\omega_1(A) + \frac{1}{2}\omega_2(A)\ . \tag{7}$$

ω is a linear functional satisfying the normalization and positivity conditions and therefore is again a state in the algebraic sense. Now let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ be the direct sum of the two Hilbert spaces and let

$$\pi(A) = \begin{pmatrix} \pi_1(A) & 0 \\ 0 & \pi_2(A) \end{pmatrix} \tag{8}$$

Then the vector

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \tag{9}$$

satisfies the required relation

$$\omega(A) = \langle \Psi, \pi(A)\Psi \rangle \ . \tag{10}$$

For more information on operator algebras see [5, 6, 32].

In classical mechanics one has a similar structure. Here the algebra of observables is commutative and can be identified with the algebra of continuous functions on phase space. In addition, there is a second product, the Poisson bracket. This product is only densely defined. States are probability measures, and pure states correspond to the evaluation of functions at a given point of phase space.

3 Locally Covariant Field Theory

Field theory involves infinitely many degrees of freedom, associated to the points of spacetime. Crucial for the success of field theory is a principle which regulates the way these degrees of freedom influence each other. This is the principle of locality, more precisely expressed by the German word *Nahwirkungsprinzip*. It states that each degree of freedom is influenced only by a relatively small number of other degrees of freedom. This induces a concept of neighborhoods in the set of degrees of freedom.

The original motivation for developing QFT was to combine the QM with special relativity. In this sense we expect to have in our theory some notion of *causality*. Let us briefly describe what it means in mathematical terms. In special relativity space and time are described together with one object, called *Minkowski spacetime* . Since it will be useful later on, we define now a general notion of a *spacetime* in physics.

Definition 5 A spacetime (M, g) is a smooth (4 dimensional) manifold (Hausdorff, paracompact, connected) with a smooth pseudo-Riemannian metric[4] of Lorentz signature (we choose the convention $(+, -, -, -)$).

A spacetime M is said to be orientable if there exists a differential form of maximal degree (a volume form), which does not vanish anywhere. We say that M is time-orientable if there exists a smooth vector field u on M such that for every $p \in M$ it holds $g(u, u)_p > 0$. We will always assume that our spacetimes are orientable and time-orientable. We fix the orientation and choose the time-orientation by selecting a specific vector field u with the above property. Let $\gamma : \mathbb{R} \supset I \to M$ be a smooth curve in M, for I an interval in $\mathbb{R}$. We say that γ is causal (timelike) if it holds $g(\dot{\gamma}, \dot{\gamma}) \geq 0$ (> 0), where $\dot{\gamma}$ is the vector tangent to the curve.

Given the global timelike vector field u on M, one calls a causal curve γ future-directed if $g(u, \dot{\gamma}) > 0$ all along γ, and analogously one calls γ past-directed if $g(u, \dot{\gamma}) < 0$. This induces a notion of time-direction in the spacetime (M, g). For any point $p \in M$, $J^{\pm}(p)$ denotes the set of all points in M which can be connected to x by a future(+)/past(−)-directed causal curve $\gamma : I \to M$ so that $x = \gamma(\inf\ I)$.

[4] a smooth tensor field $g \in \Gamma(T^*M \otimes T^*M)$, s.t. for every $p \in M$, g_p is a symmetric non degenerate bilinear form.

The set $J^+(p)$ is called the causal future and $J^-(p)$ the causal past of p. The boundaries $\partial J^{\pm}(p)$ of these regions are called respectively: the *future/past lightcone*. Two subsets O_1 and O_2 in M are called causally separated if they cannot be connected by a causal curve, i.e. if for all $x \in \overline{O_1}$, $J^{\pm}(x)$ has empty intersection with $\overline{O_2}$. By $O^{\perp}$ we denote the causal complement of O, i.e. the largest open set in M which is causally separated from O.

In the context of general relativity we will also make use of following definitions:

Definition 6 A causal curve is **future inextendible** if there is no $p \in M$ such that:

$$\forall U \subset M \text{open neighborhoods of } p,\ \exists t' \text{ s.t. } \gamma(t) \in U \forall t > t'\,.$$

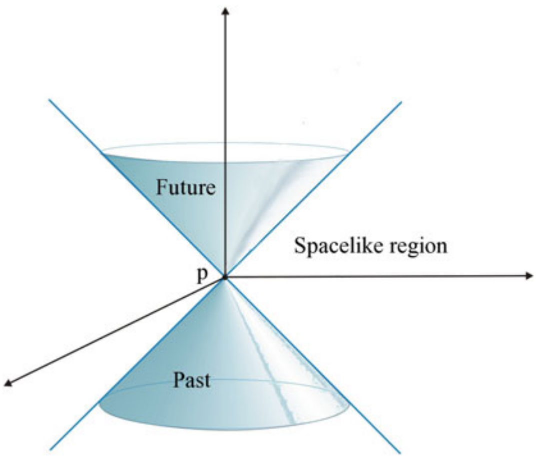
Definition 7 **A Cauchy hypersurface** in M is a smooth subspace of M such that every inextendible causal curve intersects it exactly once.

Definition 8 An oriented and time-oriented spacetime M is called **globally hyperbolic** if there exists a smooth foliation of M by Cauchy hypersurfaces.

For now let us consider a simple case of the *Minkowski spacetime* $\mathbb{M}$ which is just $\mathbb{R}^4$ with the diagonal metric $\eta = \mathrm{diag}(1, -1, -1, -1)$. A *lightcone* with apex p is shown on Fig. 1, together with the future and past of p.

One of the main principles of special relativity tells us that physical systems which are located in causally disjoint regions should in some sense be independent. Here we come to the important problem: *How to implement this principle in quantum theory?* A natural answer to this question is provided by the Haag-Kastler framework [18, 19], which is based on the principle of *locality*. In the previous section we argued that operator algebras are a natural framework for quantum physics. Locality can be realized by identifying the algebras of observables that can be measured in given

Fig. 1 A lightcone in Minkowski spacetime



bounded regions of spacetime. In other words we associate to each bounded $\mathcal{O} \subset M$ a C^*-algebra $\mathfrak{A}(\mathcal{O})$. This association has to be compatible with a physical notion of subsystems. It means that if we have a region $\mathcal{O}$ which lies inside $\mathcal{O}'$ we want the corresponding algebra $\mathfrak{A}(\mathcal{O})$ to be contained inside $\mathfrak{A}(\mathcal{O}')$, i.e. in a bigger region we have more observables. This property can be formulated as the **Isotony** condition for the net $\{\mathfrak{A}(\mathcal{O})\}$ of local algebras associated to bounded regions of the spacetime. In the Haag-Kastler framework one specializes to Minkowski space $\mathbb{M}$ and imposes some further, physically motivated, properties:

- **Locality (Einstein causality)**. Algebras associated to spacelike separated regions commute: $\mathcal{O}_1$ spacelike separated from $\mathcal{O}_2$, then $[A, B] = 0$, $\forall A \in \mathfrak{A}(\mathcal{O}_1)$, $B \in \mathfrak{A}(\mathcal{O}_2)$. This expresses the "independence" of physical systems associated to regions $\mathcal{O}_1$ and $\mathcal{O}_2$.
- **Covariance**. The Minkowski spacetime has a large group of isometrics, namely the Poincaré group. We require that there exists a family of isomorphisms $\alpha_L^{\mathcal{O}} : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}(L\mathcal{O})$ for Poincaré transformations L, such that for $\mathcal{O}_1 \subset \mathcal{O}_2$ the restriction of $\alpha_L^{\mathcal{O}_2}$ to $\mathfrak{A}(\mathcal{O}_1)$ coincides with $\alpha_L^{\mathcal{O}_1}$ and such that: $\alpha_{L'}^{L\mathcal{O}} \circ \alpha_L^{\mathcal{O}} = \alpha_{L'L}^{\mathcal{O}}$,
- **Time slice axiom**: the algebra of a neighborhood of a Cauchy surface of a given region coincides with the algebra of the full region. Physically this correspond to a well-posedness of an initial value problem. We need to determine our observables in some small time interval $(t_0 - \epsilon, t_0 + \epsilon)$ to reconstruct the full algebra.
- **Spectrum condition**. This condition corresponds physically to the positivity of energy. One assumes that there exist a compatible family of faithful representations $\pi_{\mathcal{O}}$ of $\mathfrak{A}(\mathcal{O})$ on a fixed Hilbert space (i.e. the restriction of $\pi_{\mathcal{O}_2}$ to $\mathfrak{A}(\mathcal{O}_1)$ coincides with $\pi_{\mathcal{O}_1}$ for $\mathcal{O}_1 \subset \mathcal{O}_2$) such that translations are unitarily implemented, i.e. there is a unitary representation U of the translation group satisfying

$$U(a)\pi_{\mathcal{O}}(A)U(a)^{-1} = \pi_{\mathcal{O}+a}(\alpha_a(A)),\ A \in \mathfrak{A}(\mathcal{O}),$$

  and such that the joint spectrum of the generators P_μ of translations $e^{iaP} = U(a)$, $aP = a^\mu P_\mu$, is contained in the forward lightcone: $\sigma(P) \subset \overline{V}_+$.

We now want to generalize this framework to theories on generic spacetimes. To start with, we may think of a globally hyperbolic neighborhood U of a spacetime point x in some spacetime M. Moreover, we assume that any causal curve in M with end points in U lies entirely in U. Then we require that the structure of the algebra of observables associated to U should be completely independent of the world outside. We may formalize this idea by requiring that for any embedding $\chi : M \to N$ of a globally hyperbolic manifold M into another one N which preserves the metric, the orientations and the causal structure[5] (these embeddings will be called *admissible*), there exist an injective homomorphism

$$\alpha_\chi : \mathfrak{A}(M) \to \mathfrak{A}(N) \tag{11}$$

[5] The property of *causality preserving* is defined as follows: let $\chi : M \to N$, for any causal curve $\gamma : [a, b] \to N$, if $\gamma(a), \gamma(b) \in \chi(M)$ then for all $t \in]a, b[$ we have: $\gamma(t) \in \chi(M)$.

of the corresponding algebras of observables, moreover if $\chi_1 : M \to N$ and $\chi_2 : N \to L$ are embeddings as above then we require the covariance relation

$$\alpha_{\chi_2 \circ \chi_1} = \alpha_{\chi_2} \circ \alpha_{\chi_1} \,. \tag{12}$$

In this way we described a functor $\mathfrak{A}$ between two categories: the category **Loc** of globally hyperbolic spacetimes with admissible embeddings as arrows and the category **Obs** of algebras (Poisson algebras for classical physics and C^*-algebras for quantum physics) with homomorphisms as arrows.

We may restrict the category of spacetimes to subregions of a given spacetime and the arrows to inclusions. In this way we obtain the Haag-Kastler net of local algebras on a globally hyperbolic spacetime as introduced by Dimock. In case the spacetime has nontrivial isometries, we obtain additional embeddings, and the covariance condition above provides a representation of the group of isometries by automorphisms of the Haag-Kastler net.

The causality requirements of the Haag-Kastler framework, i.e. the commutativity of observables localized in spacelike separated regions, is encoded in the general case in the tensor structure of the functor $\mathfrak{A}$. Namely, the category of globally hyperbolic manifolds has the disjoint union as a tensor product, with the empty set as unit object and where admissible embeddings $\chi : M_1 \otimes M_2 \to N$ have the property that the images $\chi(M_1)$ and $\chi(M_2)$ cannot be connected by a causal curve. On the level of C^*-algebras we may use the minimal tensor product as a tensor structure. See [8] for details.

The solvability of the initial value problem can be formulated as the requirement that the algebra $\mathfrak{A}(N)$ of any neighborhood N of some Cauchy surface Σ already coincides with $\mathfrak{A}(M)$. This is the *time slice axiom* of axiomatic quantum field theory. It can be used to describe the evolution between different Cauchy surfaces. As a first step we associate to each Cauchy surface Σ the inverse limit

$$\mathfrak{A}(\Sigma) = \overleftarrow{\lim_{N \supset \Sigma}} \mathfrak{A}(N) \,. \tag{13}$$

Elements of the inverse limit consist of sequences $A = (A_N)_{L_A \supset N \supset \Sigma}$ with $\alpha_{N \subset K}(A_N) = A_K$, $K \subset L_A$, with the equivalence relation

$$A \sim B \text{ if } A_N = B_N \text{ for all } N \subset L_A \cap L_B \,. \tag{14}$$

The algebra $\mathfrak{A}(\Sigma)$ can be embedded into $\mathfrak{A}(M)$ by

$$\alpha_{\Sigma \subset M}(A) = \alpha_{N \subset M}(A_N) \text{ for some (and hence all) } \Sigma \subset N \subset L_A \,. \tag{15}$$

If we now adopt the time slice axiom we find that each homomorphism $\alpha_{N \subset M}$ is an isomorphism. Hence $\alpha_{\Sigma \subset M}$ is also an isomorphism and we obtain the propagator between two Cauchy surfaces Σ_1 and Σ_2 by

$$\alpha^M_{\Sigma_1 \Sigma_2} = \alpha^{-1}_{\Sigma_1 \subset M} \circ \alpha_{\Sigma_2 \subset M} \tag{16}$$

This construction resembles constructions in topological field theory for the description of cobordisms. But there one associates Hilbert spaces to components of the boundary and maps between these Hilbert spaces to the spacetime itself. This construction relies on the fact that for these theories the corresponding Hilbert spaces are finite dimensional. It was shown [39] that a corresponding construction for the free field in 3 and more dimensions does not work, since the corresponding Boboliubov transformation is not unitarily implementable (Shale's criterion [34] is violated). Instead one may associate to the Cauchy surfaces the corresponding algebras of canonical commutation relations and to the cobordism an isomorphism between these algebras. For the algebra of canonical anticommutation relations for the free Dirac fields the above isomorphism was explicitly constructed [40]. Our general argument shows that the association of a cobordism between two Cauchy surfaces of globally hyperbolic spacetimes to an isomorphism of algebras always exists provided the time slice axiom is satisfied. As recently shown, the latter axiom is actually generally valid in perturbative Algebraic Quantum Field Theory [11].

In the Haag-Kastler framework on Minkowski space an essential ingredient was translation symmetry. This symmetry allowed the comparison of observables in different regions of spacetime and was (besides locality) a crucial input for the analysis of scattering states.

In the general covariant framework sketched above no comparable structure is available. Instead one may use fields which are subject to a suitable covariance condition, termed locally covariant fields. A locally covariant field is a family φ_M of fields on spacetimes M such that for every embedding $\chi : M \to N$ as above

$$\alpha_\chi(\varphi_M(x)) = \varphi_N(\chi(x)) \ . \tag{17}$$

If we consider fields as distributions with values in the algebras of observables, a field φ may be considered as a natural transformation between the functor $\mathfrak{D}$ of test function spaces to the functor $\mathfrak{A}$ of field theory. The functor $\mathfrak{D}$ associates to every spacetime M its space of compactly supported $\mathcal{C}^\infty$-functions,

$$\mathfrak{D}(M) = \mathcal{C}^\infty_{\mathrm{c}}(M, \mathbb{R}) \ , \tag{18}$$

and to every embedding $\chi : M \to N$ of spacetimes the pushforward of test functions $f \in \mathfrak{D}(M)$

$$\mathfrak{D}\chi \equiv \chi_* \ , \ \ \chi_* f(x) = \begin{cases} f(\chi^{-1}(x)) \ , & x \in \chi(M) \\ 0 \ , & \text{else} \end{cases} \tag{19}$$

$\mathfrak{D}$ is a covariant functor. Its target category is the category of locally convex vector spaces **Vec** which contains also the category of topological algebras which is the target category for $\mathfrak{A}$. A natural transformation $\varphi : \mathfrak{D} \to \mathfrak{A}$ between covariant functors with the same source and target categories is a family of morphisms $\varphi_M : \mathfrak{D}(M) \to \mathfrak{A}(M)$, $M \in \mathrm{Obj}(\mathbf{Loc})$ such that

$$\mathfrak{A}\chi \circ \varphi_M = \varphi_N \circ \mathfrak{D}\chi \tag{20}$$

with $\mathfrak{A}\chi = \alpha_\chi$.

4 Classical Field Theory

Before we enter the arena of quantum field theory we show that the concept of local covariance leads to a nice reformulation of classical field theory in which the relation to QFT becomes clearly visible. Let us consider a scalar field theory. On a given spacetime M the possible field configurations are the smooth functions on M. If we embed a spacetime M into another spacetime N, the field configurations on N can be pulled back to M, and we obtain a functor $\mathfrak{E}$ from **Loc** to the category **Vec** of locally convex vector spaces

$$\mathfrak{E}(M) = \mathcal{C}^\infty(M, \mathbb{R}) \ , \ \mathfrak{E}\chi = \chi^* \tag{21}$$

with the pullback $\chi^*\varphi = \varphi \circ \chi$ for $\varphi \in \mathcal{C}^\infty(M, \mathbb{R})$. Note that $\mathfrak{E}$ is contravariant, whereas the functor $\mathfrak{D}$ of test function spaces with compact support is covariant.

The classical observables are real valued functions on $\mathfrak{E}(M)$, i.e. (not necessarily linear) functionals. An important property of a functional is its spacetime support. Is is defined as a generalization of the distributional support, namely as the set of points $x \in M$ such that F depends on the field configuration in any neighbourhood of x.

$$\begin{aligned} \mathrm{supp}\, F \doteq \{x \in M | \forall \text{ neighbourhoods } U \text{ of } x\ \exists \varphi, \psi \in \mathfrak{E}(M), \mathrm{supp}\, \psi \subset U \\ \text{such that } F(\varphi + \psi) \neq F(\varphi)\} \ . \end{aligned} \tag{22}$$

Here we will discuss only compactly supported functionals. Next one has to select a class of functionals which are sufficiently regular such that all mathematical operations one wants to perform are meaningful and which on the other side is large enough to cover the interesting cases.

One class one may consider is the class $\mathfrak{F}_{\mathrm{reg}}(M)$ of regular polynomials

$$F(\varphi) = \sum_{\text{finite}} \int dx_1 \dots dx_n f_n(x_1, \dots, x_n)\varphi(x_1) \dots \varphi(x_n) \tag{23}$$

with test functions $f_n \in \mathfrak{D}(M^n)$. Another class $\mathfrak{F}_{\mathrm{loc}}(M)$ consists of the local functionals

$$F(\varphi) = \int dx \mathcal{L}(x, \varphi(x), \partial\varphi(x), \dots) \tag{24}$$

where $\mathcal{L}$ depends smoothly on x and on finitely many derivatives of φ at x. The local functionals arise as actions and induce the dynamics. The only regular polynomials in this class are the linear functionals

$$F(\varphi) = \int dx f(x)\varphi(x) \ . \tag{25}$$

It turns out to be convenient to characterize the admissible class of functionals in terms of their functional derivatives.

Definition 9 (*After* [30]) Let X and Y be topological vector spaces, $U \subseteq X$ an open set and $f : U \to Y$ a map. The derivative of f at x in the direction of h is defined as

$$df(x)(h) \doteq \lim_{t\to 0} \frac{1}{t} \left(f(x+th) - f(x)\right) \tag{26}$$

whenever the limit exists. The function f is called differentiable at x if $df(x)(h)$ exists for all $h \in X$. It is called continuously differentiable if it is differentiable at all points of U and $df : U \times X \to Y, (x, h) \mapsto df(x)(h)$ is a continuous map. It is called a $\mathcal{C}^1$-map if it is continuous and continuously differentiable. Higher derivatives are defined for $\mathcal{C}^n$-maps by

$$d^n f(x)(h_1, \dots, h_n) \doteq \lim_{t\to 0} \frac{1}{t}\big(d^{n-1} f(x+th_n)(h_1, \dots, h_{n-1}) - d^{n-1} f(x)(h_1, \dots, h_{n-1})\big) \tag{27}$$

In particular it means that if F is a smooth functional on $\mathfrak{E}(M)$, then its n-th derivative at the point $\varphi \in \mathfrak{E}(M)$ is a compactly supported distributional density $F^{(n)}(\varphi) \in \mathcal{E}'(M^n)$. There is a distinguished volume form on M, namely the one provided by the metric: $\sqrt{-\det(g)}d^4x$. We can use it to construct densities from functions and to provide an embedding of $\mathcal{D}(M^n)$ into $\mathcal{E}'(M^n)$. For more details on distributions on manifolds, see Chap. 1 of [2]. Using the distinguished volume form we can identify derivatives $F^{(n)}(\varphi)$ with distributions. We further need some conditions on their wave front sets.

Let us make a brief excursion to the concept of wave front sets and its use for the treatment of distributions. Readers less familiar with these topics can find more details in the appendix 2.7 or refer to [25] or Chap. 4 of [1]. Let $t \in \mathcal{D}'(\mathbb{R}^n)$ and $f \in \mathcal{D}(\mathbb{R}^n)$. The Fourier transform of the product ft is a smooth function. If this function vanishes fast at infinity for all $f \in \mathcal{D}(\mathbb{R}^n)$, t itself is a smooth function. Singularities of t show up in the absence of fast decay in some directions. A point $(x, k) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ is called a regular point of t if there exists a test function f with $f(x) = 1$ such that the Fourier transform of ft decays strongly in an open cone around k. The wave front set of t is now defined as the complement of the set of regular points of t in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$.

On a manifold M the definition of the Fourier transform depends on the choice of a chart. But the property of strong decay in some direction (characterized now by a point (x, k), $k \neq 0$ of the cotangent bundle T^*M) turns out to be independent of the chart. Therefore the wave front set WF of a distribution on a manifold M is a well defined closed conical subset of the cotangent bundle (with the zero section removed).

Let us illustrate the concept of the wave front set in two examples. The first one is the δ-function. We find

$$\int dx f(x)\delta(x)e^{ikx} = f(0) \;, \tag{28}$$

hence $\mathrm{WF}(\delta) = \{(0, k), k \neq 0\}$.

The other one is the function $x \mapsto (x + i\epsilon)^{-1}$ in the limit $\epsilon \downarrow 0$. We have

$$\lim_{\epsilon\downarrow 0} \int dx \frac{f(x)}{x + i\epsilon} e^{ikx} = -i \int_k^\infty dk' \hat{f}(k') \;. \tag{29}$$

Since the Fourier transform $\hat{f}$ of a test function $f \in \mathcal{D}(\mathbb{R})$ is strongly decaying for $k \to \infty$, $\int_k^\infty dk' \hat{f}(k')$ is strongly decaying for $k \to \infty$, but for $k \to -\infty$ we obtain

$$\lim_{k\to-\infty} \int_k^\infty dk' \hat{f}(k') = 2\pi f(0) \;, \tag{30}$$

hence

$$\mathrm{WF}(\lim_{\epsilon\downarrow 0}(x + i\epsilon)^{-1}) = \{(0, k), k < 0\} \;. \tag{31}$$

The wave front sets provide a simple criterion for the pointwise multiplicability of distributions. Namely, let t, s be distributions on an n dimensional manifold M such that the pointwise sum (Whitney sum) of their wave front sets

$$\mathrm{WF}(t) + \mathrm{WF}(s) = \{(x, k + k')|(x, k) \in \mathrm{WF}(t), (x, k') \in \mathrm{WF}(s)\} \tag{32}$$

does not intersect the zero section of T^*M. Then the pointwise product ts can be defined by

$$\langle ts, fg\rangle = \frac{1}{(2\pi)^n} \int dk\, \widehat{tf}(k)\widehat{sg}(-k) \tag{33}$$

for test functions f and g with sufficiently small support and where the Fourier transform refers to an arbitrary chart covering the supports of f and g. The convergence of the integral on the right hand side follows from the fact, that for every $k \neq 0$ either

$\widehat{tf}$ decays fast in a conical neighborhood around k or $\widehat{sg}$ decays fast in a conical neighborhood around $-k$ whereas the other factor is polynomially bounded.

The other crucial property is the characterization of the propagation of singularities. To understand it in more physical terms it is useful to use an analogy with Hamiltonian mechanics. Note that the cotangent bundle T^*M has a natural symplectic structure. The symplectic 2-form is defined as an exterior derivative of the canonical one-form, given in local coordinates as $\theta_{(x,k)} = \sum_{i=1}^n k_i dx^i$ (k_i are coordinates in the fibre). Let P be a partial differential operator with real principal symbol σ_P. Note that σ_P is a function on T^*M and its differential $d\sigma_P$ is a 1-form. On a symplectic manifold 1-forms can be canonically identified with vector fields by means of the symplectic form. Therefore every differentiable function H determines a unique vector field X_H, called the Hamiltonian vector field with the Hamiltonian H. Let X_P be the Hamiltonian vector field corresponding to σ_P. Explicitly it can be written as:

$$X_P = \sum_{j=1}^n \frac{\partial \sigma_P}{\partial k_j}\frac{\partial}{\partial x_j} - \frac{\partial \sigma_P}{\partial x_j}\frac{\partial}{\partial k_j}$$

Let us now consider the integral curves (Hamiltonian flow) of this vector field. A curve $(x_j(t), k_j(t))$ is an integral curve of X_P if it fulfills the system of equations (Hamilton's equations):

$$\frac{dx_j}{dt} = \frac{\partial \sigma_P}{\partial k_j}\,,$$
$$\frac{dk_j}{dt} = -\frac{\partial \sigma_P}{\partial x_j}\,.$$

The set of all such solution curves is called the bicharacteristic flow. Along the Hamiltonian flow it holds $\frac{d\sigma_P}{dt} = X_P(\sigma_P) = 0$ (this is the law of conservation of energy for autonomous systems in classical mechanics), so σ_P is constant under the bicharacteristic flow. If $\sigma_P((x_j(t), k_j(t))) = 0$ we call the corresponding flow null. The set of all such integral curves is called the null bicharacteristics.

Let us now define the characteristics of P as $\mathrm{char}P = \{(x,k) \in T^*M | \sigma(P)(x,k) = 0\}$ of P. Then the theorem on the propagation of singularities states that the wave front set of a solution u of the equation $Pu = f$ with f smooth is a union of orbits of the Hamiltonian flow X_P on the characteristics $\mathrm{char}P$.

In field theory on Lorentzian spacetime we are mainly interested in hyperbolic differential operators. Their characteristics is the light cone, and the principal symbol is the metric on the cotangent bundle. The wave front set of solutions therefore is a union of null geodesics together with their cotangent vectors $k = g(\dot{\gamma}, \cdot)$.

We already have all the kinematical structures we need. Now in order to specify a concrete physical model we need to introduce the dynamics. This can be done by means of a *generalized Lagrangian* . As the name suggests the idea is motivated by Lagrangian mechanics. Indeed, we can think of this formalism as a way to make precise the variational calculus in field theory. Note that since our spacetimes are globally hyperbolic, they are never compact. Moreover we cannot restrict ourselves to compactly supported field configurations, since the nontrivial solutions of globally hyperbolic equations don't belong to this class. Therefore we cannot identify the action with a functional on $\mathfrak{E}(M)$ obtained by integrating the Lagrangian density over the whole manifold. Instead we follow [10] and define a Lagrangian L as a natural transformation between the functor of test function spaces $\mathfrak{D}$ and the functor $\mathfrak{F}_{\text{loc}}$ such that it satisfies $\text{supp}(L_M(f)) \subseteq \text{supp}(f)$ and the additivity rule[6]

$$L_M(f+g+h) = L_M(f+g) - L_M(g) + L_M(g+h)\,,$$

for $f, g, h \in \mathfrak{D}(M)$ and $\text{supp}\, f \cap \text{supp}\, h = \varnothing$. The action $S(L)$ is now defined as an equivalence class of Lagrangians [10], where two Lagrangians L_1, L_2 are called equivalent $L_1 \sim L_2$ if

$$\text{supp}(L_{1,M} - L_{2,M})(f) \subset \text{supp}\, df\,, \tag{34}$$

for all spacetimes M and all $f \in \mathfrak{D}(M)$. This equivalence relation allows us to identify Lagrangians differing by a total divergence. For the free minimally coupled (i.e. $\xi = 0$) scalar field the generalized Lagrangian is given by:

$$L_M(f)(\varphi) = \frac{1}{2}\int\limits_M (\nabla_\mu\varphi\nabla^\mu\varphi - m^2\varphi^2) f\, \text{dvol}_M\,. \tag{35}$$

The equations of motion are to be understood in the sense of [10]. Concretely, the Euler-Lagrange derivative of S is a natural transformation $S' : \mathfrak{E} \to \mathfrak{D}'$ defined as

$$\langle S'_M(\varphi), h\rangle = \left\langle L_M(f)^{(1)}(\varphi), h\right\rangle, \tag{36}$$

with $f \equiv 1$ on $\text{supp} h$. The field equation is now a condition on φ:

$$S'_M(\varphi) = 0\,. \tag{37}$$

Note that the way we obtained the field equation is analogous to variational calculus on finite dimensional spaces. We can push this analogy even further and think of variation of a functional in a direction in configuration space given by an infinite dimensional vector field. This concept is well understood in mathematics and for details one can refer for example to [29, 30]. Here we consider only variations in the directions of compactly supported configurations, so the space of vector fields we are interested in can be identified with $\mathfrak{V}(M) = \{X : \mathfrak{E}(M) \to \mathfrak{D}(M) | X \text{ smooth}\}$. In more precise terms this is the space of vector fields on $\mathfrak{E}(M)$, considered as a

[6] We do not require linearity since in quantum field theory the renormalization flow does not preserve the linear structure; it respects, however, the additivity rule (see [10]).

manifold[7] modeled over $\mathfrak{D}(M)$. The set of functionals

$$\varphi \mapsto \langle S'_M(\varphi), X(\varphi)\rangle \ , \ \ X \in \mathfrak{V}(M) \tag{38}$$

is an ideal $\mathfrak{I}_S(M)$ of $\mathfrak{F}(M)$ with respect to pointwise multiplication,

$$(F \cdot G)(\varphi) = F(\varphi)G(\varphi) \ . \tag{39}$$

The quotient

$$\mathfrak{F}_S(M) = \mathfrak{F}(M)/\mathfrak{I}_S(M) \tag{40}$$

can be interpreted as the space of solutions of the field equation. The latter can be identified with the phase space of the classical field theory.

We now want to equip $\mathfrak{F}_S(M)$ with a Poisson bracket. Here we rely on a method originally introduced by Peierls. Peierls considers the influence of an additional term in the action. Let $F \in \mathfrak{F}_{\mathrm{loc}}(M)$ be a local functional. We are interested in the flow (Φ_λ) on $\mathfrak{E}(M)$ which deforms solutions of the original field equation $S'_M(\varphi) = \omega$ with a given source term ω to those of the perturbed equation $S'_M(\varphi)+\lambda F^{(1)}(\varphi) = \omega$. Let $\Phi_0(\varphi) = \varphi$ and

$$\frac{d}{d\lambda}\left(S'_M(\Phi_\lambda(\varphi)) + F^{(1)}(\Phi_\lambda(\varphi))\right)\Big|_{\lambda=0} = 0 \ . \tag{41}$$

Note that the second variational derivative of the unperturbed action induces an operator $S''_M(\varphi) : \mathfrak{E}(M) \to \mathfrak{D}'(M)$. We define it in the following way:

$$\langle S''_M(\varphi), h_1 \otimes h_2\rangle \doteq \left\langle L^{(2)}_M(f)(\varphi), h_1 \otimes h_2\right\rangle ,$$

where $f \equiv 1$ on the supports of h_1 and h_2. This defines $S''_M(\varphi)$ as an element of $\mathfrak{D}'(M^2)$ and by Schwartz's kernel theorem we can associate to it an operator from $\mathfrak{D}(M)$ to $\mathfrak{D}'(M)$. Actually, since $L_M(f)$ is local, the second derivative has support on the diagonal, so $S''_M(\varphi)$ can be evaluated on smooth functions h_1, h_2, where only one of them is required to be compactly supported, and it induces an operator (the so called linearized Euler-Lagrange operator) $E'[S_M](\varphi) : \mathfrak{E}(M) \to \mathfrak{D}'(M)$.

From (41) it follows that the vector field $\varphi \mapsto X(\varphi) = \frac{d}{d\lambda}\Phi_\lambda(\varphi)|_{\lambda=0}$ satisfies the equation

$$\langle S''_M(\varphi), X(\varphi) \otimes \cdot\rangle + \left\langle F^{(1)}(\varphi), \cdot\right\rangle = 0 \, , \tag{42}$$

which in a different notation can be written as

[7] An infinite dimensional manifold is modeled on a locally convex vector space just as a finite dimensional one is modeled on $\mathbb{R}^n$. For more details see [29, 30].

$$\langle E'[S_M](\varphi), X(\varphi)\rangle + F^{(1)}(\varphi) = 0\,.$$

We now assume that $E'[S_M](\varphi)$ is, for all φ, a normally hyperbolic differential operator $\mathfrak{E}(M) \to \mathfrak{E}(M)$, and let Δ_S^R, Δ_S^A be the retarded and advanced Green's operators, i.e. linear operators $\mathfrak{D}(M) \to \mathfrak{E}(M)$ satisfying:

$$E'[S_M] \circ \Delta_S^{R/A} = \mathrm{id}_{\mathfrak{D}(M)}\,,$$
$$\Delta_S^{R/A} \circ (E'[S_M]\big|_{\mathfrak{D}(M)}) = \mathrm{id}_{\mathfrak{D}(M)}\,.$$

Moreover, with the use of Schwartz's kernel theorem one can identify $\Delta_S^{R/A} : \mathfrak{D}(M) \to \mathfrak{E}(M)$ with elements of $\mathcal{D}'(M^2)$. As such, they are required to satisfy the following support properties:

$$\mathrm{supp}(\Delta^R) \subset \{(x,y) \in M^2 | y \in J^-(x)\}\,, \tag{43}$$
$$\mathrm{supp}(\Delta^A) \subset \{(x,y) \in M^2 | y \in J^+(x)\}\,. \tag{44}$$

Their difference $\Delta_S = \Delta_S^{\mathrm{A}} - \Delta_S^{\mathrm{R}}$ is called the causal propagator of the Klein-Gordon equation. Coming back to equation (42) we have now two distinguished solutions for X,

$$X^{R,A}(\varphi) = \Delta_S^{R,A} F^{(1)}(\varphi)\,. \tag{45}$$

The difference of the associated derivations on $\mathfrak{F}(M)$ defines a product

$$\{F, G\}_S(\varphi) = \langle \Delta_S(\varphi) F^{(1)}(\varphi), G^{(1)}(\varphi)\rangle \tag{46}$$

on $\mathfrak{F}_{\mathrm{loc}}(M)$, the so-called Peierls bracket.

The Peierls bracket satisfies the conditions of a Poisson bracket, in particular the Jacobi identity (for a simple proof see [26]). Moreover, if one of the entries is in the ideal $\mathfrak{I}_S(M)$, also the bracket is in the ideal, hence the Peierls bracket induces a Poisson bracket on the quotient algebra.

In standard cases, the Peierls bracket coincides with the canonical Poisson bracket. Namely let

$$\mathcal{L}(\varphi) = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4\,. \tag{47}$$

Then $S_M'(\varphi) = -\left((\Box + m^2)\varphi + \frac{\lambda}{3!}\varphi^3\right)$ and $S_M''(\varphi)$ is the linear operator

$$-\left(\Box + m^2 + \frac{\lambda}{2}\varphi^2\right) \tag{48}$$

(the last term acts as a multiplication operator).

The Peierls bracket is

$$\{\varphi(x), \varphi(y)\}_S = \Delta_S(\varphi)(x, y) \tag{49}$$

where $x \mapsto \Delta_S(\varphi)(x, y)$ is a solution of the (at φ) linearized equation of motion with the initial conditions

$$\Delta_S(\varphi)(y^0, \mathbf{x}; y) = 0\ , \ \frac{\partial}{\partial x^0}\Delta_S(\varphi)(y^0, \mathbf{x}; y) = \delta(\mathbf{x}, \mathbf{y})\ . \tag{50}$$

This coincides with the Poisson bracket in the canonical formalism. Namely, let φ be a solution of the field equation. Then

$$0 = \{(\Box + m^2)\varphi(x) + \frac{\lambda}{3!}\varphi^3(x), \varphi(y)\} = (\Box + m^2 + \frac{\lambda}{2}\varphi(x)^2))\{\varphi(x), \varphi(y)\} \tag{51}$$

hence the Poisson bracket satisfies the linearized field equation with the same initial conditions as the Peierls bracket.

Let us now discuss the domain of definition of the Peierls bracket. It turns out that it is a larger class of functionals than just $\mathfrak{F}_{\mathrm{loc}}(M)$. To identify this class we use the fact that the WF set of Δ_S is given by

$$\mathrm{WF}(\Delta_S) = \{(x, k; x', -k') \in \dot{T}^*M^2 | (x, k) \sim (x', k')\}\ ,$$

where the equivalence relation $\sim$ means that there exists a null geodesic strip such that both (x, k) and (x', k') belong to it. A null geodesic strip is a curve in T^*M of the form $(\gamma(\lambda), k(\lambda))$, $\lambda \in I \subset \mathbb{R}$, where $\gamma(\lambda)$ is a null geodesic parametrized by λ and $k(\lambda)$ is given by $k(\lambda) = g(\dot{\gamma}(\lambda), \cdot)$. This follows from the theorem on the propagation of singularities together with the initial conditions and the antisymmetry of Δ_S. (See [31] for a detailed argument.)

It is now easy to check, using Hörmander's criterion on the multiplicability of distributions [25] that the Peierls bracket (46) is well defined if F and G are such that the sum of the WF sets of the functional derivatives $F^{(1)}(\varphi), G^{(1)}(\varphi) \in \mathcal{E}'(M)$ and $\Delta \in \mathcal{D}'(M^2)$ don't intersect the 0-section of the cotangent bundle T^*M^2. This is the case if the functionals fulfill the following criterion:

$$\mathrm{WF}(F^{(n)}(\varphi)) \subset \Xi_n, \quad \forall n \in \mathbb{N},\ \forall \varphi \in \mathfrak{E}(M)\ , \tag{52}$$

where Ξ_n is an open cone defined as

$$\Xi_n \doteq T^*M^n \setminus \{(x_1, \ldots, x_n; k_1, \ldots, k_n) | (k_1, \ldots, k_n) \in (\overline{V}_+^n \cup \overline{V}_-^n)_{(x_1,\ldots,x_n)}\}\ , \tag{53}$$

where $(\overline{V}_\pm)_x$ is the closed future/past lightcone understood as a conic subset of T_x^*M. We denote the space of smooth compactly supported functionals, satisfying (52) by $\mathfrak{F}_{\mu c}(M)$ and call them *microcausal functionals*. This includes in particular local functionals. For them the support of the functional derivatives is on the thin diagonal, and the wave front sets satisfy $\sum k_i = 0$.

To see that $\{.,.\}_S$ is indeed well defined on $\mathfrak{F}_{\mu c}(M)$, note that $\mathrm{WF}(\Delta)$ consists of elements (x, x', k, k'), where k, k' are dual to lightlike vectors in $T_x M$, $T_{x'} M$ accordingly. On the other hand, if $(x, k_1) \in \mathrm{WF}(F^{(1)}(\varphi))$, then k_1 is necessarily dual to a vector which is spacelike, so $k_1 + k$ cannot be 0. The same argument is valid for $G^{(1)}(\varphi)$. Moreover it can be shown that $\{F, G\}_S \in \mathfrak{F}_{\mu c}(M)$. The classical field theory is defined as $\mathfrak{A}(M) = (\mathfrak{F}_{\mu c}(M), \{.,.\}_S)$. One can check that $\mathfrak{A}$ is indeed a covariant functor from **Loc** to **Obs**, the category of Poisson algebras.

5 Deformation Quantization of Free Field Theories

Starting from the Poisson algebra $(\mathfrak{F}_{\mu c}(M), \{.,.\}_S)$ one may try to construct an associative algebra $(\mathfrak{F}_{\mu c}(M)[[\hbar]], \star)$ such that for $\hbar \to 0$

$$F \star G \to F \cdot G \tag{54}$$

and

$$[F, G]_\star / i\hbar \to \{F, G\}_S \, . \tag{55}$$

For the Poisson algebra of functions on a finite dimensional Poisson manifold the deformation quantization exists in the sense of formal power series due to a theorem of Kontsevich [28]. In field theory the formulas of Kontsevich lead to ill defined terms, and a general solution of the problem is not known. But in case the action is quadratic in the fields the $\star$-product can be explicitly defined by

$$(F \star G)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \big\langle F^{(n)}(\varphi), (\Delta_S)^{\otimes n} G^{(n)}(\varphi) \big\rangle \, , \tag{56}$$

which can be formally written as $e^{\frac{i\hbar}{2}\left\langle \Delta_S, \frac{\delta^2}{\delta\varphi\delta\varphi'}\right\rangle} F(\varphi) G(\varphi')|_{\varphi'=\varphi}$. This product is well defined (in the sense of formal power series in $\hbar$) for regular functionals $F, G \in \mathfrak{F}_{\mathrm{reg}}(M)$ and satisfies the conditions above. Let for instance

$$F(\varphi) = e^{\int dx \varphi(x) f(x)} \, , \; G(\varphi) = e^{\int dx \varphi(x) g(x)} \, , \tag{57}$$

with test functions $f, g \in \mathfrak{D}(M)$. We have

$$\frac{\delta^n}{\varphi(x_1) \ldots \varphi(x_n)} F(\varphi) = f(x_1) \ldots f(x_n) F(\varphi) \tag{58}$$

and hence

$$(F \star G)(\varphi) \tag{59}$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int dx dy \frac{i\hbar}{2} \Delta_S(x, y) f(x) g(y) \right)^n F(\varphi) G(\varphi) \tag{60}$$

For later purposes we want to extend the product to more singular functionals which includes in particular the local functionals. We decompose

$$\Delta_S = \Delta_S^+ - \Delta_S^- \tag{61}$$

such that the wavefront set of Δ_S is decomposed into two disjoint parts. The wave front set of Δ_S consists of pairs of points x, x' which can be connected by a null geodesic, and of covectors (k, k') where k is the cotangent vector of the null geodesic at x and $-k'$ is the cotangent vector of the same null geodesic at x'. The lightcone with the origin removed consists of two disjoint components, the first one containing the positive frequencies and the other one the negative frequencies. The WF set of the positive frequency part of Δ_S is therefore:

$$\mathrm{WF}(\Delta_+) = \{(x, k; x', -k') \in \dot{T}M^2 | (x, k) \sim (x', k'), k \in (\overline{V}_+)_x\}\,. \tag{62}$$

On Minkowski space one could choose $-i\Delta_S^+$ as the Wightman 2-point-function, i.e. the vacuum expectation value of the product of two fields. This, however, becomes meaningless in a more general context, since a generally covariant concept of a vacuum state does not exist. Nevertheless, such a decomposition always exist, but is not unique and the difference between two different choices of Δ_S^+ is always a smooth symmetric function. Let us write $\Delta_S^+ = \Delta_S + H$. We then consider the linear functional derivative operator

$$\Gamma_H = \langle H, \frac{\delta^2}{\delta\varphi^2} \rangle \tag{63}$$

and define a new $\star$-product by

$$F \star' G = e^{\frac{i\hbar}{2}\Gamma_H} \left((e^{\frac{-i\hbar}{2}\Gamma_H} F) \star (e^{\frac{-i\hbar}{2}\Gamma_H} G) \right) \tag{64}$$

which differs from the original one in the replacement of $\frac{i\hbar}{2}\Delta_S$ by $i\hbar\Delta_S^+$. This $\star$-product can now be defined on a much larger space of functionals, namely the microcausal ones $\mathfrak{F}_{\mu c}(M)$. The transition between these two $\star$-products correspond to normal ordering, is just an algebraic version of Wick's theorem. The map $\alpha_H \doteq e^{\frac{i\hbar}{2}\Gamma_H}$ provides the equivalence between $\star$ and $\star'$ on the space of regular functionals $\mathfrak{F}_{\mathrm{reg}}(M)$. Its image can be then completed to a larger space $\mathfrak{F}_{\mu c}(M)$. We can also build a corresponding (sequential) completion $\alpha_H^{-1}(\mathfrak{F}_{\mu c}(M))$ of the source space. This amounts to extending $\mathfrak{F}_{\mathrm{reg}}(M)$ with all elements of the form $\lim_{n\to\infty}\alpha_H^{-1}(F_n)$, where (F_n) is a convergent sequence in $\mathfrak{F}_{\mu c}(M)$ with respect to the Hörmander topology [10, 25]. We recall now the definition of this topology.

Let us denote the space of compactly supported distributions with WF sets contained in a conical set $C \subset T^*M^n$ by $\mathcal{E}'_C(M^n)$. Now let $C_n \subset \Xi_n$ be a closed cone contained in Ξ_n defined by (53). We introduce (after [1, 10, 25]) the following family of seminorms on $\mathcal{E}'_{C_n}(M^n)$:

$$p_{n,\varphi,\tilde{C},k}(u) = \sup_{k\in\tilde{C}}\{(1+|k|)^k|\widehat{\varphi u}(k)|\},$$

where the index set consists of $(n, \varphi, \tilde{C}, k)$ such that $k \in \mathbb{N}_0$, $\varphi \in \mathcal{D}(M)$ and $\tilde{C}$ is a closed cone in $\mathbb{R}^n$ with $(\mathrm{supp}(\varphi)\times\tilde{C})\cap C_n = \varnothing$. These seminorms, together with the seminorms of the strong topology provide a defining system for a locally convex topology denoted by τ_{C_n}. To control the wave front set properties inside open cones, we take an inductive limit. The resulting topology is denoted by τ_{Ξ_n}. One can show that $\mathcal{D}(M)$ is sequentially dense in $\mathcal{E}'_{\Xi_n}(M)$ in this topology.

For microcausal functionals it holds that $F^{(n)}(\varphi) \in \mathcal{E}'_{\Xi_n}(M)$, so we can equip $\mathfrak{F}_{\mu c}(M)$ with the initial tpopolgy with respect to mappings:

$$\mathcal{C}^\infty(\mathfrak{E}(M),\mathbb{R}) \ni F \mapsto F^{(n)}(\varphi) \in (\mathcal{E}_{\Xi_n}(M), \tau_{\Xi_n}) \quad n \geq 0\,, \tag{65}$$

The locally convex vector space of local functionals $\mathfrak{F}_{\mathrm{loc}}(M)$ is dense in $\mathfrak{F}_{\mu c}(M)$ with respect to τ_Ξ. To see these abstract concepts at work let us consider the example of the Wick square:

Example 1 Consider a sequence $F_n(\varphi) = \int \varphi(x)\varphi(y)g_n(y-x)f(x)$ with a smooth function f and a sequence of smooth functions g_n which converges to the δ distribution in the Hörmander topology. By applying $\alpha_H^{-1} = e^{\frac{-i\hbar}{2}\Gamma_H}$ we obtain a sequence

$$\alpha_H^{-1}F_n = \int (\varphi(x)\varphi(y)g_n(y-x)f(x) - H(x,y)g_n(y-x)f(x))\,,$$

The limit of this sequence can be identified with $\int :\varphi(x)^2: f(x)$, i.e.:

$$\int :\varphi(x)^2: f(x) = \lim_{n\to\infty}\int (\varphi(x)\varphi(y) - H(x,y))g_n(y-x)f(x)$$

We can write it in a short-hand notation as a coinciding point limit:

$$:\varphi(x)^2: = \lim_{x\to y}(\varphi(x)\varphi(y) - H(x,y))\,.$$

We can see that transforming with α_H^{-1} corresponds formally to a subtraction of $H(x, y)$. Now, to recognize the Wick's theorem let us consider a product of two Wick squares $:\varphi(x)^2::\varphi(y)^2:$. With the use of the isomorphism α_H^{-1} this can be written as:

$$\begin{aligned}\int \varphi(x)^2 f_1(x) \star' \int \varphi(y)^2 f_2(y) = &\int \varphi(x)^2\varphi(y)^2 f_1(x) f_2(y)\\ &+ 2i\hbar \int \varphi(x)\varphi(y)\Delta_S^+(x, y) f_1(x) f_2(y)\\ &- \hbar^2 \int (\Delta_S^+(x, y))^2 f_1(x) f_2(y)\,.\end{aligned}$$

Omitting the test functions and using α_H^{-1} we obtain

$$:\varphi(x)^2::\varphi(y)^2:=:\varphi(x)^2\varphi(y)^2:+4:\varphi(x)\varphi(y):\frac{i\hbar}{2}\Delta_S^+(x,y)+2\Big(\frac{i\hbar}{2}\Delta_S^+(x,y)\Big)^2,$$

which is a familiar form of the Wick's theorem applied to $:\varphi(x)^2::\varphi(y)^2:$.

In the next step we want to define the involution on our algebra. Note that the complex conjugation satisfies the relation:

$$\overline{F \star G} = \overline{G} \star \overline{F}\,. \tag{66}$$

Therefore we can use it to define an involution $F^*(\varphi) \doteq \overline{F(\varphi)}$. The resulting structure is an involutive noncommutative algebra $(\mathfrak{F}_{\mu c}(M)[[\hbar]], \star')$, which provides a quantization of $(\mathfrak{F}_{\mu c}(M), \{.,.\}_S)$. To see that this is equivalent to canonical quantization, let us look at the commutator of two smeared fields $\Phi(f)$, $\Phi(g)$, where $\Phi(f)(\varphi) \doteq \int f\varphi\ \mathrm{dvol}_M$. The commutator reads

$$[\Phi(f), \Phi(g)]_{\star'} = i\hbar\langle f, \Delta_S g\rangle\,, \quad f, g \in \mathfrak{D}(\mathbb{M})\,,$$

This indeed reproduces the canonical commutation relations. Here we used the fact that the choice of Δ_S^+ is unique up to a symmetric function, which doesn't contribute to the commutator (which is antisymmetric). In case Δ_S^+ is a distribution of positive type (as in the case of the Wightman 2-point-function) the linear functional on $\mathfrak{F}(M)$

$$\omega(F) = F(0) \tag{67}$$

is a state (the vacuum state in the special case above), and the associated GNS representation is the Fock representation. The kernel of the representation is the ideal generated by the field equation.

6 Interacting Theories and the Time Ordered Product

If we have an action for which S''_M still depends on φ, we choose a particular φ_0 and split

$$S_M(\varphi_0 + \psi) = \frac{1}{2}\langle S''_M(\varphi_0), \psi \otimes \psi\rangle + S_I(\varphi_0, \psi) \,. \tag{68}$$

From now on we drop the subscript M of S_I, since it's clear that we work on a fixed manifold. We now introduce the linear operator

$$\mathcal{T} = e^{i\hbar\langle \Delta_S^D, \frac{\delta^2}{\delta\psi^2}\rangle} \tag{69}$$

which acts on $\mathfrak{F}_{\mathrm{reg}}(M)$ as

$$(\mathcal{T}F)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!}\big\langle (i\Delta_S^D)^{\otimes n}, F^{(2n)}(\varphi)\big\rangle,$$

with the Dirac propagator $\Delta_S^D = \frac{1}{2}(\Delta_S^R + \Delta_S^A)$ at φ_0. Formally, $\mathcal{T}$ may be understood as the operator of convolution with the oscillating Gaussian measure with covariance $i\hbar\Delta_S^D$. By

$$F \cdot_{\mathcal{T}} G = \mathcal{T}\left(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G\right) \tag{70}$$

we define a new product on $\mathfrak{F}_{\mathrm{reg}}(M)$ which is the time ordered product with respect to $\star$ and which is equivalent to the pointwise product of classical field theory. We then define a linear map

$$R_{S_I}F = \left(e_{\mathcal{T}}^{S_I}\right)^{\star -1} \star \left(e_{\mathcal{T}}^{S_I} \cdot_{\mathcal{T}} F\right) \tag{71}$$

where $e_{\mathcal{T}}$ is the exponential function with respect to the time ordered product,

$$e_{\mathcal{T}}^{F} = \mathcal{T}\big(e^{\mathcal{T}^{-1}F}\big) \,. \tag{72}$$

R_{S_I} is invertible with the inverse

$$R_{S_I}^{-1}F = e_{\mathcal{T}}^{-S_I} \cdot_{\mathcal{T}} \left(e_{\mathcal{T}}^{S_I} \star F\right) \tag{73}$$

We now define the $\star$-product for the full action by

$$F \star_S G = R_{S_I}^{-1}\left(R_{S_I}F \star R_{S_I}G\right) \tag{74}$$

7 Renormalization

Unfortunately, the algebraic structures discussed so far are well defined only if S_I is a regular functional. An easy extension is provided by the operation of normal ordering as described in the section of deformation quantization. This operation transforms the time ordering operator $\mathcal{T}$ into another one $\mathcal{T}'$, such that the new time ordered product is now defined with respect to the Feynman propagator Δ_S^F, no longer the Dirac propagator Δ_S^D. Note that the Feynman propagator does depend on the choice of Δ_S^+. Contrary to the $\star'$ product which is everywhere defined due to the wave front set properties of the positive frequency part of Δ_S, the time ordered product is in general undefined since the wave front set of the Feynman propagator contains the wave front set of the δ-function. We want, however, to extend to a larger class which contains in particular all local functionals. As already proposed by Stückelberg [36] and Bogoliubov [3, 4] and carefully worked out by Epstein and Glaser [16], the crucial problem is the definition of time ordered products of local functionals. Let us first consider a special case.

Let $F = \frac{1}{2}\int dx\varphi(x)^2 f(x)$, $G = \frac{1}{2}\int dx\varphi(x)^2$. Then the time ordered product $\cdot_{\mathcal{T}'}$ is formally given by

$$(F \cdot_{\mathcal{T}'} G)(\varphi) = F(\varphi)G(\varphi) + i\hbar \int dxdy\varphi(x)\varphi(y)f(x)g(y)\Delta_S^F(x,y) - \frac{\hbar^2}{2}\int dxdy\Delta_S^F(x,y)^2 f(x)g(y)\,. \tag{75}$$

But the last term contains the pointwise product of a distribution with itself. For $x \neq y$ the covectors (k, k') in the wave front set satisfy the condition that k and $-k'$ are cotangent to an (affinely parametrized) null geodesics connecting x and y. k is future directed if x is in the future of y and past directed otherwise. Hence the sum of two such covectors cannot vanish. Therefore the theorem on the multiplicability of distributions applies and yields a distribution on the complement of the diagonal $\{(x,x)|x \in M\}$. On the diagonal, however, the only restriction is $k = -k'$, hence the sum of the wave front set of Δ_S^F with itself meets the zero section of the cotangent bundle at the diagonal.

In general the time-ordered product $\mathcal{T}_n(F_1,\ldots,F_n) \doteq F_1 \cdot_{\mathcal{T}} \ldots \cdot_{\mathcal{T}} F_n$ of n local functionals is well defined for local entries as long as supports of $F_1,\ldots,F_n$ are pairwise disjoint. The technical problem one now has to solve is the extension of a distribution which is defined outside of a submanifold to an everywhere defined distribution. In the case of QFT on Minkowski space one can exploit translation invariance and reduce the problem in the relative coordinates to the extension problem of a distribution defined outside of the origin in $\mathbb{R}^n$. The crucial concept for this extension problem is Steinmann's scaling degree [37].

Definition 10 Let $U \subset \mathbb{R}^n$ be a scale invariant open subset (i.e. $\lambda U = U$ for $\lambda > 0$), and let $t \in \mathcal{D}'(U)$ be a distribution on U. Let $t_\lambda(x) = t(\lambda x)$ be the scaled distribution. The scaling degree sd of t is

$$\mathrm{sd}\, t = \inf\{\delta \in \mathbb{R} | \lim_{\lambda \to 0} \lambda^{\delta} t_{\lambda} = 0\} . \tag{76}$$

There is one more important concept related to the scaling degree, namely the degree of divergence. It is defined as:

$$\mathrm{div}(t) \doteq \mathrm{sd}(t) - n .$$

Theorem 2 *Let $t \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ with scaling degree $\mathrm{sd}\, t < \infty$. Then there exists an extension of t to an everywhere defined distribution with the same scaling degree. The extension is unique up to the addition of a derivative $P(\partial)\delta$ of the delta function, where P is a polynomial with degree bounded by $\mathrm{div}(t)$ (hence vanishes for $\mathrm{sd}\, t < n$).*

A proof may be found in [13]. In the example above the scaling degree of $\Delta_S^F(x)^2$ is 4 (in 4 dimensions). Hence the extension exists and is unique up to the addition of a multiple of the delta function.

The theorem above replaces the cumbersome estimates on conditional convergence of Feynman integrals on Minkowski momentum space. Often this convergence is not proven at all, instead the convergence of the corresponding integrals on momentum space with euclidean signature is shown. The transition to Minkowski signature is then made after the integration. This amounts not to a computation but merely to a definition of the originally undefined Minkowski space integral.

The generalization of the theorem on the extension of distributions to the situation met on curved spacetimes is due to Brunetti and one of us (K.F.) [7]. It uses techniques of microlocal analysis to reduce the general situation to the case covered by the theorem above.

The construction of time ordered products is then performed in the following way (*causal perturbation theory*). One searches for a family $(\mathcal{T}_n)_{n \in \mathbb{N}_0}$ of n-linear symmetric maps from local functionals to microcausal functionals subject to the following conditions:

T 1. $\mathcal{T}_0 = 1$
T 2. $\mathcal{T}_1 = \mathrm{id}$
T 3. $\mathcal{T}_n(F_1, \ldots, F_n) = \mathcal{T}_k(F_1, \ldots, F_k) \star \mathcal{T}_{n-k}(F_{k+1}, \ldots, F_n)$ if the supports $\mathrm{supp} F_i$, $i = 1, \ldots, k$ of the first k entries do not intersect the past of the supports $\mathrm{supp} F_j$, $j = k+1, \ldots, n$ of the last $n-k$ entries (*causal factorisation property*).

The construction proceeds by induction: when the first n maps $\mathcal{T}_k$, $k = 0, \ldots, n$ have been determined, the map $\mathcal{T}_{n+1}$ is determined up to an $(n+1)$-linear map Z_{n+1} from local functionals to local functionals. This ambiguity corresponds directly to the freedom of adding finite counterterms in every order in perturbation theory.

The general result can be conveniently formulated in terms of the formal S-matrix, defined as the generating function of time ordered products,

$$\mathcal{S}(V) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{T}_n(V, \ldots, V) \,. \tag{77}$$

Then the S-matrix $\hat{\mathcal{S}}$ with respect to an other sequence of time ordered products is related to $\mathcal{S}$ by

$$\hat{\mathcal{S}} = \mathcal{S} \circ Z \tag{78}$$

where Z maps local functionals to local functionals, is analytic with vanishing zero order term and with the first order term being the identity. The maps Z form the renormalization group in the sense of Petermann and Stückelberg. They are formal diffeomorphisms on the space of local functionals and describe the allowed finite renormalization.

In order to illustrate the methods described above we work out the combinatorics in terms of Feynman diagrams (graphs). Let D be the second order functional differential operator $D = i\hbar\langle \Delta_S^F, \frac{\delta^2}{\delta\varphi^2}\rangle$. The time ordered product of n factors is formally given by

$$F_1 \cdot_{\mathcal{T}} \ldots F_n \equiv \mathcal{T}_n(F_1, \ldots F_n) = e^{\frac{1}{2}D}(e^{-\frac{1}{2}D} F_1 \cdot \ldots e^{-\frac{1}{2}D} F_n)$$

Using Leibniz' rule and the fact that D is of second order we find

$$(F_1 \cdot_{\mathcal{T}} \ldots F_n)(\varphi) = e^{\sum_{i<j} D_{ij}} F_1(\varphi_1) \cdots F_n(\varphi_n)|_{\varphi_1 = \ldots \varphi_n = \varphi} \tag{79}$$

with $D_{ij} = i\hbar\langle \Delta_S^F, \frac{\delta^2}{\delta\varphi_i \delta\varphi_j}\rangle$. The expansion of the exponential function of the differential operator yields

$$e^{\sum_{i<j} D_{ij}} = \prod_{i<j} \sum_{l_{ij}=0}^{\infty} \frac{D_{ij}^{l_{ij}}}{l_{ij}!} \tag{80}$$

The right hand side may now be written as a sum over all graphs Γ with vertices $V(\Gamma) = \{1, \ldots, n\}$ and l_{ij} lines $e \in E(\Gamma)$ connecting the vertices i and j. We set $l_{ij} = l_{ji}$ for $i > j$ and $l_{ii} = 0$ (no tadpoles). If e connects i and j we set $\partial e := \{i, j\}$. Then we obtain

$$\mathcal{T}_n = \sum_{\Gamma \in G_n} \mathcal{T}_\Gamma \tag{81}$$

with G_n the set of all graphs with vertices $\{1, \ldots n\}$ and $\mathcal{T}_\Gamma = \frac{1}{\mathrm{Sym}}(\Gamma)\langle \widetilde{S}_\Gamma, \delta_\Gamma\rangle$ where

$$\widetilde{S}_\Gamma = \prod_{e \in E(\Gamma)} \Delta_S^F(x_{e,i}, i \in \partial e)$$

$$\delta_\Gamma = \frac{\delta^{|E(\Gamma)|}}{\prod_{i\in V(\Gamma)} \prod_{e:i\in\partial e} \delta\varphi_i(x_{e,i})}|_{\varphi_1=\dots=\varphi_n}$$

and the symmetry factor is $\mathrm{Sym}(\Gamma) = \prod_{i<j} l_{ij}!$. Note that $\widetilde{S}_\Gamma$ is a well-defined distribution in $\mathcal{D}'((\mathbb{M}^2\backslash\mathrm{Diag})^{|E(\Gamma)|})$ (Diag denotes the thin diagonal) that can be uniquely extended to $\mathcal{D}'(\mathbb{M}^{2|E(\Gamma)|})$, since the Feynman fundamental solution has a unique extension with the same scaling degree. More explicitly we can write (81) as:

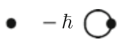
$$\mathcal{T}_n(F_1,\dots,F_n) = \sum_{\Gamma\in G_n} \frac{1}{\mathrm{Sym}}(\Gamma)\langle \widetilde{S}_\Gamma, \delta_\Gamma(F_1,\dots,F_n)\rangle \tag{82}$$

Graphically we represent F with a vertex • and D_{ij} with a dumbbell ∘—∘, so each empty circle corresponds to a functional derivative. Applying the derivative on a functional can be pictorially represented as filling the circle with the vertex. Note that the expansion in graphs is possible due to the fact that the action used as a starting point is quadratic, so D is a second order differential operator. If it were of order $k > 2$, instead of lines we would have had to use $k-1$ simplices to represent it. Let us illustrate the concepts which we introduced here on a simple example.

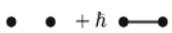
Example 2 (removing tadpoles) Let us look at the definition of the time ordered product of F and G in low orders in $\hbar$. We can write $D(F\cdot G)$ diagramatically as:

$$\frac{1}{i\hbar}D(F\cdot G) = \left\langle \Delta_S^F, F^{(2)}\right\rangle G + F\left\langle \Delta_S^F, G^{(2)}\right\rangle + 2\left\langle \Delta_S^F, F^{(1)}\otimes G^{(1)}\right\rangle \tag{83}$$

Here we see that the *tadpoles* are present. The lowest order contributions to $e^{-\frac{1}{2}D}F$ can be written as:

$$e^{-\frac{1}{2}D}F = F - \tfrac{1}{2}DF + \mathcal{O}(\hbar^2) =$$  $+\mathcal{O}(\hbar^2)$.

Now we write the expression for $F \cdot_{\mathcal{T}} G$ up to the first order in $\hbar$:

$$\left(1+\tfrac{1}{2}D\right)\left[\left(1-\tfrac{1}{2}D\right)F\cdot\left(1-\tfrac{1}{2}D\right)G\right] =$$  $+\mathcal{O}(\hbar^2)$.

All the loop terms cancel out. We can see that applying $e^{-\frac{1}{2}D}$ on G and F reflects what is called in physics "removing the tadpoles". In formula (80) it is reflected by the fact that we set $l_{ii} = 0$.

As long as the formula (79) is applied to regular functionals there is no problem, since their functional derivatives are by definition test functions. But the relevant functionals are the interaction Lagrangians which are local functionals and therefore have derivatives with support on the thin diagonal, hence all but the first derivative are singular. As a typical example consider

$$F(\varphi) = \int dz f(z) \frac{\varphi(z)^k}{k!} \, .$$

Its derivatives are

$$F^{(l)}[\varphi](x_1, \ldots, x_l) = \int dz f(z) \frac{\varphi(z)^{k-l}}{(k-l)!} \prod_i \delta(z - x_i) \, . \tag{84}$$

In general, the functional derivatives of a local functional have the form

$$F^{(l)}[\varphi](x_1, \ldots, x_l) = \int dz \sum_j f_j[\varphi](z) p_j(\partial_{x_1}, \ldots, \partial_{x_l}) \prod_{i=1}^{l} \delta(z - x_i)$$

with polynomials p_j and φ-dependent test functions $f_j[\varphi]$. The integral representation above is not unique since one can add total derivatives. This amounts to the relation

$$\int dz q(\partial_z) f(z) p(\partial_{x_1}, \ldots, \partial_{x_l}) \prod_i \delta(z - x_i) = \int dz f(z) q(\partial_{x_1} + \cdots \partial_{x_l}) \\ p(\partial_{x_1}, \ldots, \partial_{x_l}) \prod_i \delta(z - x_i) \, . \tag{85}$$

We insert the integral representation (84) into the formula (82) for the time ordered product and in each term we obtain:

$$\langle \widetilde{S}_\Gamma, \delta_\Gamma(F_1, \ldots, F_n) \rangle = \int d\mathbf{x} d\mathbf{z} \prod_{v \in V(\Gamma)} \Big(\sum_{j_v} f^v_{j_v}[\varphi](z_v) \\ p_{j_v}(\partial_{x_{e,v}} | v \in \partial e) \prod_{e: v \in \partial e}^{\alpha_v} \delta^4(z_v - x_{e,v}) \Big) \widetilde{S}_\Gamma \, ,$$

where α_v is the number of lines adjacent at vertex v and we use the notation $\mathbf{x} = (x_{e,v} | e \in E(\Gamma), v \in \partial e)$, $\mathbf{z} = (z_v | v \in V(\Gamma))$. We can move the partial derivatives $\partial_{x_{e,v}}$ by formal partial integration to the distribution $\widetilde{S}_\Gamma$. Next we integrate over the delta distributions, which amounts to the pullback of a derivative of $\widetilde{S}_\Gamma$ with respect to the map $\rho_\Gamma : \mathbb{M}^{|V(\Gamma)|} \to \mathbb{M}^{2|E(\Gamma)|}$ given by the prescription

$$(\rho_\Gamma(z))_{e,v} = z_v \quad \text{if } v \in \partial e \, .$$

Let p be a polynomial in the derivatives with respect to the partial derivatives $\partial_{x_{e,v}}$, $v \in \partial e$. The pullback ρ_Γ^* of $p\widetilde{S}_\Gamma$ is well defined on $\mathbb{M}^{|V(\Gamma)|}\backslash$DIAG, where DIAG is the large diagonal:

$$\text{DIAG} = \left\{ z \in \mathbb{M}^{|V(\Gamma)|} \,|\, \exists v, w \in V(\Gamma), v \neq w : z_v = z_w \right\} .$$

The problem of renormalization now amounts to finding the extensions of $\rho_\Gamma^* p\widetilde{S}_\Gamma$ to everywhere defined distributions $S_{\Gamma,p} \in \mathcal{D}'(\mathbb{M}^{|V(\Gamma)|})$ which depend linearly on p. These extensions must satisfy the relation

$$\partial_{z_v} S_{\Gamma,p} = S_{\Gamma,(\sum_e \partial_{x_{e,v}})p} \tag{86}$$

We present now the inductive procedure of Epstein and Glaser that allows to define the desired extension of $\rho_\Gamma^* p\widetilde{S}_\Gamma$. For the simplicity of notation we first consider the case where no derivative couplings are present.

Let us define an *Epstein-Glaser subgraph (EG subgraph)* $\gamma \subseteq \Gamma$ to be a subset of the set of vertices $V(\gamma) \subseteq V(\Gamma)$ together with all lines in Γ connecting them,

$$E(\gamma) = \{e \in E(\Gamma) : \partial e \subset V(\gamma)\} .$$

The first step of the Epstein-Glaser induction is to choose extensions for all EG subgraphs with two vertices, $|V(\gamma)| = 2$. In this case we have translation invariant distributions in $\mathcal{D}'(\mathbb{M}^2\backslash\text{Diag})$, which correspond in relative coordinates to generic distributions $\widetilde{t}_\gamma$ in $\mathcal{D}'(\mathbb{M}\backslash\{0\})$. The scaling degree of these distributions is given by $|E(\gamma)|\,(d-2)$, and we can choose a (possibly unique) extension according to Theorem 2. By translation invariance this gives extensions $t_\gamma \in \mathcal{D}'(\mathbb{M}^2)$.

Now we come to the induction step. For a generic EG subgraph $\gamma \subseteq \Gamma$ with n vertices we assume that the extensions of distributions corresponding to all EG subgraphs of γ with less than n vertices have already been chosen. The causality condition **T3.** then gives a translation invariant distribution in $\mathcal{D}'(\mathbb{M}^{|V(\gamma)|}\backslash\text{Diag})$ which corresponds to a generic distribution $\widetilde{t}_\gamma \in \mathcal{D}'(\mathbb{M}^{|V(\gamma)|-1}\backslash\{0\})$. The scaling degree and hence the degree of divergence of this distribution is completely fixed by the structure of the graph:

$$\text{div}(\gamma) = |E(\gamma)|\,(d-2) - (|V(\gamma)|-1)\,d\,, \qquad d = \dim(\mathbb{M})\,. \tag{87}$$

We call γ superficially convergent if $\text{div}(\gamma) < 0$, logarithmically divergent if $\text{div}(\gamma) = 0$ and divergent of degree $\text{div}(\gamma)$ otherwise. Again by Theorem 2 there is a choice to be made in the extension of $\widetilde{t}_\gamma$ in the case $\text{div}(\gamma) \geq 0$.

Let us now come back to the case where derivative couplings are present. The scaling degree of $p\widetilde{S}_\Gamma$ fulfills:

$$\text{sd}(p\widetilde{S}_\Gamma) \leq \text{sd}(\widetilde{S_\Gamma}) + |p|\,,$$

where $|p|$ is the degree of the polynomial p. We can see that p encodes the derivative couplings appearing in the graph γ. In the framework of Connes-Kreimer Hopf algebras it is called the *external structure of the graph*. The presence of derivative couplings introduces an additional freedom in the choice of the extension in each step of the Epstein-Glaser induction and one has to use it to fulfill (86). This relation follows basically from the Action Ward Identity, as discussed in [14, 15]. It can be also seen as a consistency condition implementing the Leibniz rule, see [24].

Let us now remark on the relation of the Epstein-Glaser induction to a more conventional approach to renormalization. Firstly we show, how the EG renormalization relates to the regularization procedure. We are given an EG subgraph γ with n vertices and we assume that all the subgraphs with $n-1$ vertices are already renormalized. Let

$$\mathcal{D}_\lambda(\mathbb{M}^{n-1}) := \{f \in \mathcal{D}(\mathbb{M}^{n-1}) \,|\, (\partial^\alpha f)(0) = 0 \;\; \forall|\alpha| \le \lambda\} \tag{88}$$

be the space of functions with derivatives vanishing up to order λ and let $\mathcal{D}'_\lambda(\mathbb{M}^{n-1})$ be the corresponding space of distributions. Theorem 2 tells us that the distribution $\widetilde{t_\gamma} \in \mathcal{D}'(\mathbb{M}^{n-1})$ associated with the EG subgraph γ has a unique extension to an element of $\mathcal{D}'_{\mathrm{div}(\gamma)}(\mathbb{M}^{n-1})$. An extension to a distribution on the full space $\mathcal{D}(\mathbb{M}^{n-1})$ can be therefore defined by a choice of the projection:

$$W : \mathcal{D}(\mathbb{M}^{n-1}) \to \mathcal{D}_{\mathrm{div}(\gamma)}(\mathbb{M}^{n-1})\,.$$

There is a result proven in [13], which characterizes all such projections:

Proposition 1 *There is a one-to-one correspondence between families of functions*

$$\left\{w_\alpha \in \mathcal{D}\,|\;\; \forall\,|\beta| \le \lambda : \partial^\beta w_\alpha(0) = \delta^\beta_\alpha,\; |\alpha| \le \lambda\right\} \tag{89}$$

and projections $W : \mathcal{D} \to \mathcal{D}_\lambda$. The set (89) defines a projection W by

$$Wf := f - \sum_{|\alpha|\le\lambda} f^{(\alpha)}(0)\, w_\alpha\,. \tag{90}$$

Conversely a set of functions of the form (89) is given by any basis of $\mathrm{ran}(1-W)$ *dual to the basis* $\left\{\delta^{(\alpha)} : |\alpha| \le \lambda\right\}$ *of* $\mathcal{D}^\perp_\lambda \subset \mathcal{D}'$.

Let us now define, following [27], what we mean by a regularization of a distribution.

Definition 11 (*Regularization*) Let $\tilde{t} \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ be a distribution with degree of divergence λ, and let $\bar{t} \in \mathcal{D}'_\lambda(\mathbb{R}^n)$ be the unique extension of $\tilde{t}$ with the same degree of divergence. A family of distributions $\{t^\zeta\}_{\zeta\in\Omega\setminus\{0\}}$, $t^\zeta \in \mathcal{D}'(\mathbb{R}^n)$, with $\Omega \subset \mathbb{C}$ a neighborhood of the origin, is called a regularization of $\tilde{t}$, if

$$\forall g \in \mathcal{D}_\lambda(\mathbb{R}^n) : \quad \lim_{\zeta\to 0}\langle t^\zeta, g\rangle = \langle \bar{t}, g\rangle\,. \tag{91}$$

The regularization $\{t^\zeta\}$ is called analytic, if for all functions $f \in \mathcal{D}(\mathbb{R}^n)$ the map

$$\Omega \setminus \{0\} \ni \zeta \mapsto \langle t^\zeta, f \rangle \tag{92}$$

is analytic with a pole of finite order at the origin. The regularization $\{t^\zeta\}$ is called finite, if the limit $\lim_{\zeta\to 0}\langle t^\zeta, f\rangle \in \mathbb{C}$ exists $\forall f \in \mathcal{D}(\mathbb{R}^n)$; in this case $\lim_{\zeta\to 0} t^\zeta \in \mathcal{D}'(\mathbb{R}^n)$ is called an extension or renormalization of $\tilde{t}$.

For a finite regularization the limit $\lim_{\zeta\to 0} t^\zeta$ is indeed a solution t of the extension problem. Given a regularization $\{t^\zeta\}$ of t, it follows from (91) that for any projection $W : \mathcal{D} \to \mathcal{D}_\lambda$

$$\langle \bar{t}, Wf \rangle = \lim_{\zeta\to 0} \langle t^\zeta, Wf \rangle \quad \forall f \in \mathcal{D}(\mathbb{R}^n)\,. \tag{93}$$

Any extension $t \in \mathcal{D}'(\mathbb{R}^n)$ of $\tilde{t}$ with the same scaling degree is of the form $\langle t, f\rangle = \langle \bar{t}, Wf\rangle$ with some W-projection of the form (90). Since $t^\zeta \in \mathcal{D}'(\mathbb{R}^n)$ we can write (93) in the form

$$\langle \bar{t}, Wf \rangle = \lim_{\zeta\to 0} \left[\langle t^\zeta, f\rangle - \sum_{|\alpha|\le \mathrm{sd}(t)-n} \langle t^\zeta, w_\alpha\rangle\; f^{(\alpha)}(0) \right]. \tag{94}$$

In general the limit on the right hand side cannot be split, since the limits of the individual terms might not exist. However, if the regularization $\{t^\zeta, \zeta \in \Omega \setminus \{0\}\}$ is analytic, each term can be expanded in a Laurent series around $\zeta = 0$, and since the overall limit is finite, the principal parts (pp) of these Laurent series must coincide. It follows that the principal part of any analytic regularization $\{t^\zeta\}$ of a distribution $t \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ is a local distribution of order $\mathrm{sd}(t)-n$. We can now give a definition of the minimal subtraction in the EG framework.

Corollary 1 *(*Minimal Subtraction*) The regular part (*$\mathrm{rp} = 1 - \mathrm{pp}$*) of any analytic regularization* $\{t^\zeta\}$ *of a distribution* $\tilde{t} \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ *defines by*

$$\langle t^{\mathrm{MS}}, f\rangle := \lim_{\zeta\to 0} \mathrm{rp}(\langle t^\zeta, f\rangle) \tag{95}$$

an extension of $\tilde{t}$ *with the same scaling degree,* $\mathrm{sd}(t^{\mathrm{MS}}) = \mathrm{sd}(\tilde{t})$*. The extension* t^{MS} *defined by (95) is called "minimal subtraction".*

To finish this discussion we want to remark on the difference between the Epstein-Glaser procedure and the BPHZ scheme. It is best seen on the example of the rising sun diagram of the φ^4 theory. In the framework of BPHZ, it contains three logarithmically divergent subdiagrams, which have to be renormalized first. In the perspective of EG, however, it is a diagram with two vertices and, hence, contains no divergent subdiagram at all. This way one saves some work computing contributions, which, as shown by Zimmermann [42] cancel out in the end.

We have just seen how to define the n-fold time ordered products (i.e. multilinear maps $\mathcal{T}_n$) by the procedure of Epstein and Glaser. An interesting question is whether the renormalized time ordered product defined by such a sequence of multilinear maps can be understood as an iterated binary product on a suitable domain. Recently we proved in [17] that this is indeed the case. The crucial observation is that multiplication of local functionals is injective. More precisely, let $\mathfrak{F}_0(M)$ be the set of local functionals vanishing at some distinguished field configuration (say $\varphi = 0$). Iterated multiplication m is then a linear map from the symmetric Fock space over $\mathfrak{F}_0(M)$ onto the algebra of functionals which is generated by $\mathfrak{F}_0(M)$. Then there holds the following assertion:

Proposition 2 *The multiplication $m : S^\bullet \mathfrak{F}_0(M) \to \mathfrak{F}(M)$ is bijective (where S^k denotes the symmetrised tensor product of vector spaces).*

Let $\beta = m^{-1}$. We now define the renormalized time ordering operator on the space of multilocal functionals $\mathfrak{F}(M)$ by

$$\mathcal{T}_\mathrm{r} := (\bigoplus_n \mathcal{T}_n) \circ \beta \tag{96}$$

This operator is a formal power series in $\hbar$ starting with the identity, hence it is injective. The renormalized time ordered product is now defined on the image of $\mathcal{T}_\mathrm{r}$ by

$$A \cdot_{\mathcal{T}_\mathrm{r}} B \doteq \mathcal{T}_\mathrm{r}(\mathcal{T}_\mathrm{r}^{-1} A \cdot \mathcal{T}_\mathrm{r}^{-1} B)\,, \tag{97}$$

This product is equivalent to the pointwise product and is in particular associative and commutative. Moreover, the n-fold time ordered product of local functionals coincides with the n-linear map $\mathcal{T}_n$ of causal perturbation theory.

Appendix—Distributions and Wavefront Sets

We recall same basic notions from the theory of distributions on $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be an open subset and $\mathcal{E}(\Omega) \doteq \mathcal{C}^\infty(\Omega, \mathbb{R})$ the space of smooth functions on it. We equip this space with a Fréchet topology generated by the family of seminorms:

$$p_{K,m}(\varphi) = \sup_{\substack{x \in K \\ |\alpha| \le m}} |\partial^\alpha \varphi(x)|\,, \tag{98}$$

where $\alpha \in \mathbb{N}^N$ is a multiindex and $K \subset \Omega$ is a compact set. This is just the topology of uniform convergence on compact sets, of all the derivatives.

The space of smooth compactly supported functions $\mathcal{D}(\Omega) \doteq \mathcal{C}_c^\infty(\Omega, \mathbb{R})$ can be equipped with a locally convex topology in a similar way. The fundamental system of seminorms is given by [35]:

$$p_{\{m\},\{\epsilon\},a}(\varphi) = \sup_{\nu} \Big(\sup_{\substack{|x|\geq\nu,\\ |p|\leq m_\nu}} \big|D^p\varphi^a(x)\big|/\epsilon_\nu \Big) , \tag{99}$$

where $\{m\}$ is an increasing sequence of positive numbers going to $+\infty$ and $\{\epsilon\}$ is a decreasing one tending to 0.

The space of ***distributions*** is defined to be the dual $\mathcal{D}'(\Omega)$ of $\mathcal{D}(\Omega)$ with respect to the topology given by (99). Equivalently, given a linear map L on $\mathcal{D}(\Omega)$ we can decide if it is a distribution by checking one of the equivalent conditions given in the theorem below [25, 33, 41].

Theorem 3 *A linear map u on $\mathcal{E}(\Omega)$ is a distribution if it satisfies the following equivalent conditions:*

1. *To every compact subset K of Ω there exists an integer m and a constant $C > 0$ such that for all $\varphi \in \mathcal{D}$ with support contained in K it holds:*

   $$|u(\varphi)| \leq C \max_{p\leq k} \sup_{x\in\Omega} |\partial^p\varphi(x)| .$$

   We call $||u||_{\mathcal{C}^k(\Omega)} \doteq \max_{p\leq k}\sup_{x\in\Omega} |\partial^p\varphi(x)|$ the $\mathcal{C}^k$-norm and if the same integer k can be used in all K for a given distribution u, then we say that u is of order k.
2. *If a sequence of test functions $\{\varphi_k\}$, as well as all their derivatives converge uniformly to 0 and if all the test functions φ_k have their supports contained in a compact subset $K \subset \Omega$ independent of the index k, then $u(\varphi_k) \to 0$.*

An important property of a distribution is its support. If $U' \subset U$ is an open subset then $\mathcal{D}(U')$ is a closed subspace of $\mathcal{D}(U)$ and there is a natural restriction map $\mathcal{D}'(U) \to \mathcal{D}'(U')$. We denote the restriction of a distribution u to an open subset U' by $u|_{U'}$.

Definition 12 The support supp$u$ of a distribution $u \in \mathcal{D}'(\Omega)$ is the smallest closed set $\mathcal{O}$ such that $u|_{\Omega\setminus\mathcal{O}} = 0$. In other words:

$$\mathrm{supp}u \doteq \{x \in \Omega | \,\forall U \text{ openneigh.of } x,\ U \subset \Omega\, \exists\varphi \in \mathcal{D}(\Omega),\, \mathrm{supp}\varphi \subset U,\ \text{s.t.} < u, \varphi > \neq 0\} .$$

Distributions with compact support can be characterized by means of a following theorem:

Theorem 4 *The set of distributions in Ω with compact support is identical with the dual $\mathcal{E}'(\Omega)$ of $\mathcal{E}(\Omega)$ with respect to the topology given by* (98).

Now we discuss the singularity structure of distributions. This is mainly based on [25] and Chap. 4 of [1].

Definition 13 The singular support sing supp u of $u \in \mathcal{D}'(\Omega)$ is the smallest closed subset $\mathcal{O}$ such that $u|_{\Omega\setminus\mathcal{O}} \in \mathcal{E}(\Omega\setminus\mathcal{O})$.

We recall an important theorem giving the criterion for a compactly distribution to have an empty singular support:

Theorem 5 *A distribution $u \in \mathcal{E}'(\Omega)$ is smooth if and only if for every N there is a constant C_N such that:*

$$|\hat{u}(k)| \leq C_N(1+|k|)^{-N},$$

where $\hat{u}$ denotes the Fourier transform of u.

We can see that a distribution is smooth if its Fourier transform decays fast at infinity. If a distribution has a nonempty singular support we can give a further characterization of its singularity structure by specifying the direction in which it is singular. This is exactly the purpose of the definition of a wave front set.

Definition 14 For a distribution $u \in \mathcal{D}'(\Omega)$ the wavefront set $\mathrm{WF}(u)$ is the complement in $\Omega \times \mathbb{R}^n \setminus \{0\}$ of the set of points $(x,k) \in \Omega \times \mathbb{R}^n \setminus \{0\}$ such that there exist

- a function $f \in \mathcal{D}(\Omega)$ with $f(x) = 1$,
- an open conic neighborhood C of k, with

$$\sup_{k\in C}(1+|k|)^N|\widehat{f\cdot u}(k)| < \infty \qquad \forall N \in \mathbb{N}_0\,.$$

On a manifold M the definition of the Fourier transform depends on the choice of a chart, but the property of strong decay in some direction (characterized now by a point (x,k), $k \neq 0$ of the cotangent bundle T^*M) turns out to be independent of this choice. Therefore the wave front set (WF) of a distribution on a manifold M is a well defined closed conical subset of the cotangent bundle (with the zero section removed).

The wave front sets provide a simple criterion for the existence of point-wise products of distributions. Before we give it, we prove a more general result concerning the pullback. Here we follow closely [1, 25]. Let $F : X \to Y$ be a smooth map between $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$. We define the normal set N_F of the map F as:

$$N_F \doteq \{(F(x),\eta) \in Y \times \mathbb{R}^n | (dF_x)^T(\eta) = 0\}\,,$$

where $(dF_x)^T$ is the transposition of the differential of F at x.

Theorem 6 *Let Γ be a closed cone in $Y \times (\mathbb{R}^n\{0\})$ and $F : X \to Y$ as above, such that $N_F \cap \Gamma = \varnothing$. Then the pullback of functions $F^* : \mathcal{E}(X) \to \mathcal{E}(Y)$ has a unique, sequentially continuous extension to a sequentially continuous map $\mathcal{D}'_\Gamma(Y) \to \mathcal{D}'(X)$, where $\mathcal{D}'_\Gamma(Y)$ denotes the space of distributions with WF sets contained in Γ.*

Proof Here we give only an idea of the proof. Details can be found in [1, 25]. Firstly, one has to show that the problem can be reduced to a local construction. Let $x \in X$. We assumed that $N_F \cap \Gamma = \varnothing$, so we can choose a compact neighborhood K of $F(x)$ and an open neighborhood $\mathcal{O}$ of x such that $\overline{F(\mathcal{O})} \subset \mathrm{int}(K)$ and the following condition holds:

$$\exists\epsilon > 0 \text{ s.t. } V \doteq \overline{\bigcup_{x\in\mathcal{O}}\{k|(dF_x)^T k\}} \text{ satisfies } (K \times V) \cap \Gamma = \varnothing\,.$$

Such neighborhoods define a cover of X and we choose its locally finite refinement which we denote by $\{\mathcal{O}_\alpha\}_{\alpha\in A}$, where A is some index set. To this cover we have the associated family of compact sets $K_\lambda \subset Y$ and we choose a partition of unity $\sum_{\alpha\in A} g_\alpha = 1$, $\text{supp} g_\alpha \subset \mathcal{O}_\alpha$ and a family $\{f_\alpha\}_{\alpha\in A}$ of functions on Y with supp $f_\alpha = K_\alpha$ and $f_\alpha \equiv 1$ on $F(\text{supp} g_\alpha)$. Then:

$$F^*(\varphi) = \sum_{\alpha\in A} g_\alpha F^*(f_\alpha \varphi)\,.$$

This way the problem reduces to finding an extension of $F^*_\alpha \doteq (F\big|_{\mathcal{O}_\alpha})^*$: $\mathcal{C}^\infty_c(K_\alpha, \mathbb{R}) \to \mathcal{C}^\infty(\mathcal{O}_\alpha, \mathbb{R})$ to a map on $\mathcal{D}'_\Gamma(K_\alpha)$. Note that for $\varphi \in \mathcal{C}^\infty_c(K_\alpha)$, $\text{supp}\chi \subset \mathcal{O}_\alpha$, we can write the pullback as:

$$\langle F^*_\alpha(\varphi), \chi\rangle = \int \varphi(F_\alpha(x))\chi(x)dx = \int \hat{\varphi}(\eta)e^{i\langle F_\alpha(x),\eta\rangle}\chi(x)dxd\eta = \int \hat{\varphi}(\eta)T_\chi(\eta)d\eta\,,$$

where we denoted $T_\chi(\eta) \doteq \int e^{i\langle F_\alpha(x),\eta\rangle}\chi(x)dx$. We can use this expression to define the pullback for $u \in \mathcal{D}'_\Gamma(K_\alpha)$, by setting:

$$\langle F^*_\alpha(u), \chi\rangle \doteq \int \hat{u}(\eta)T_\chi(\eta)d\eta\,.$$

To show that this integral converges, we can divide it into two parts: integration over V_α and over $\mathbb{R}^n \setminus V_\alpha$, i.e.:

$$\langle F^*_\alpha(u), \chi\rangle = \int\limits_{V_\alpha} \hat{u}(\eta)T_\chi(\eta)d\eta + \int\limits_{\mathbb{R}^n\setminus V_\alpha} \hat{u}(\eta)T_\chi(\eta)d\eta\,.$$

The first integral converges since $K_\alpha \times V_\alpha \cap \Gamma = \varnothing$ and therefore $\hat{u}(\eta)$ decays rapidly on V_α, whereas $|T_\chi(\eta)| \le \int |\chi(x)|dx$. The second integral also converges. To prove it, first we note that $\hat{u}(\eta)$ is polynomially bounded i.e. $\hat{\varphi}(\eta) \le C(1+|\eta|)^N$ for some N and appropriately chosen constant C. Secondly, we have a following estimate on $T_\chi(\eta)$: for ever $k \in \mathbb{N}$ and a closed conic subset $V \subset \mathbb{R}^n$ such that $(dF_x)^T\eta \neq 0$ for $\eta \in V$, there exists a constant $C_{k,V}$ for which it holds[8]

$$|T_\chi(\eta)| \le C_{k,V}(1+|\eta|)^{-k}\,,$$

[8] For the proof of this estimate see [1, 25]

Since for $\eta \in V_\alpha$ it holds $(dF_x)^T\eta > \epsilon > 0$, we can use this estimate to prove the convergence of the second integral.

We already proved that $F^* : \mathcal{D}'_\Gamma(Y) \to \mathcal{D}'(X)$ exists. Now it remains to show its sequential continuity. This can be easily done, with the use of estimates provided above and the uniform boundedness principle.

Using this theorem we can define the pointwise product of two distributions t, s on an n-dimensional manifold M as a pullback by the diagonal map $D : M \to M \times M$ if the pointwise sum of their wave front sets

$$\mathrm{WF}(t) + \mathrm{WF}(s) = \{(x, k + k')|(x, k) \in \mathrm{WF}(t), (x, k') \in \mathrm{WF}(s)\},$$

does not intersect the zero section of $\dot{T}^*M$. This is the theorem 8.2.10 of [25]. To see that this is the right criterion, note that the set of normals of the diagonal map $D : x \mapsto (x, x)$ is given by $N_D = \{(x, x, k, -k)|x \in M, k \in T^*M\}$. The product ts is defined by: $ts = D^*(t \otimes s)$ and if one of t, s is compactly supported, then so is ts and we define the contraction by $\langle t, s\rangle \doteq \widehat{ts}(0)$.

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Lectures on Mathematical Aspects of (twisted) Supersymmetric Gauge Theories

Kevin Costello and Claudia Scheimbauer

Abstract Supersymmetric gauge theories have played a central role in applications of quantum field theory to mathematics. Topologically twisted supersymmetric gauge theories often admit a rigorous mathematical description: for example, the Donaldson invariants of a 4-manifold can be interpreted as the correlation functions of a topologically twisted $\mathcal{N} = 2$ gauge theory. The aim of these lectures is to describe a mathematical formulation of partially-twisted supersymmetric gauge theories (in perturbation theory). These partially twisted theories are intermediate in complexity between the physical theory and the topologically twisted theories. Moreover, we will sketch how the operators of such a theory form a two complex dimensional analog of a vertex algebra. Finally, we will consider a deformation of the $\mathcal{N} = 1$ theory and discuss its relation to the Yangian, as explained in [8, 9].

These are lecture notes of a minicourse given by the first author at the Winter school in Mathematical Physics 2012 in Les Houches on minimal (or holomorphic) twists of supersymmetric gauge theories.

Supersymmetric gauge theories in general are very difficult to study, whereas topologically twisted supersymmetric gauge theories have been well-studied. Our object of interest lies somewhere in between:

$$
\begin{array}{c}
\text{Supersymmetric gauge theories} \\
\cup\!| \\
\boxed{\textbf{Minimal (or holomorphic) twists of supersymmetric gauge theories}} \\
\cup\!| \\
\text{Topologically twisted supersymmetric gauge theories (e.g. Donaldson theory)}
\end{array}
$$

K. Costello (✉)
Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208-2730, USA
e-mail: costello@math.northwestern.edu

C. Scheimbauer
Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland
e-mail: scheimbauer@math.ethz.ch

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In the first section, we recall basics of supersymmetry. We define and describe holomorphic twists of $\mathcal{N} = 1, 2, 4$ supersymmetric theories. In the second section, we discuss the structure of the observables of field theories. In the last section, we examine the structure of the observables of twisted SUSY gauge theories more closely and explain a relation to vertex algebras. Moreover, we consider a deformation of the $\mathcal{N} = 1$ theory and discuss its relation to the Yangian and (conjecturally) to the quantum loop algebra. In a short appendix, we briefly summarize the framework set up in [7] relating perturbative field theories, moduli problems, and elliptic L_∞-algebras.

1 Basics of Supersymmetry

In these lectures, we consider gauge theories on $\mathbb{R}^4$. Everything in this first section is essentially standard, references for this material are [7, 11, 13].

1.1 Super-Translation Lie Algebra and Supersymmetric Field Theories

Recall that there is an isomorphism of groups

$$\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2).$$

Let S^+ and S^- be the fundamental representations of the two $\mathrm{SU}(2)$'s. More precisely, referring to the two copies of $\mathrm{SU}(2)$ in $\mathrm{Spin}(4)$ as $\mathrm{SU}(2)_\pm$, let S^+ be the 2-dimensional complex fundamental representation of $\mathrm{SU}(2)_+$ endowed with trivial $\mathrm{SU}(2)_-$ action. Thus, S^+ is a 2-dimensional complex representation of $\mathrm{Spin}(4)$, and similarly, so is S^-.

Let $V_\mathbb{R} = \mathbb{R}^4$ and $V_\mathbb{C} = V_\mathbb{R} \otimes \mathbb{C}$. Then $V_\mathbb{R} = \mathbb{R}^4$ is the defining 4-dimensional real representation of $\mathrm{SO}(4)$ and

$$V_\mathbb{C} \cong S^+ \otimes S^-$$

as complex $\mathrm{Spin}(4)$ representations.

Definition 1 The *super-translation Lie algebra* $T^{\mathcal{N}=1}$ is the complex $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra[1]

$$T^{\mathcal{N}=1} = V_\mathbb{C} \oplus \Pi(S^+ \oplus S^-),$$

where the Lie bracket is defined by $[Q^+, Q^-] = Q^+ \otimes Q^- \in V_\mathbb{C}$ for $Q^+ \in S^+$, $Q^- \in S^-$, and is zero otherwise.

[1] Here ΠC means that the vector space C has odd degree. So $T^{\mathcal{N}=1}$ consists of $V_\mathbb{C}$ in degree 0 and $S^+ \oplus S^-$ in degree 1.

To encode more supersymmetry, we extend this definition to the following.

Definition 2 Let W be a complex vector space. Define the *super-translation Lie algebra* T^W to be the complex $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra

$$T^W = V_{\mathbb{C}} \oplus \Pi(S^+ \otimes W \oplus S^- \otimes W^*),$$

where the Lie bracket is defined by $[Q^+ \otimes w, Q^- \otimes w^*] = (Q^+ \otimes Q^-)\langle w, w^* \rangle \in V_{\mathbb{C}}$ for $Q^+ \otimes w \in S^+ \otimes W$, and $Q^- \otimes w^* \in S^+ \otimes W^*$. The *number of supersymmetries* is the dimension of W. For $W = \mathbb{C}^k$ we use the notation

$$T^{\mathcal{N}=k} = T^{\mathbb{C}^k}.$$

Note that $\mathrm{Spin}(4)$ acts on T^W.

Definition 3 A *supersymmetric (SUSY) field theory*[2] *on* $\mathbb{R}^4$ is a field theory on $\mathbb{R}^4 = V_{\mathbb{R}}$, equivariant under the action of $\mathrm{Spin}(4) \ltimes V_{\mathbb{R}}$ on $\mathbb{R}^4$, and where the action of the Lie algebra $V_{\mathbb{R}}$ of translations is extended to an action of the Lie algebra of super-translations, in a way compatible with the $\mathrm{Spin}(4)$-action.

Observe that $\mathrm{GL}(W)$ acts on T^W naturally. If $G_R \subseteq \mathrm{GL}(W)$, one can ask that a SUSY field theory has a compatible action of G_R and T^W. In physics parlance, G_R is the *R-symmetry group* of the theory.

1.2 Twisting

The general yoga of deformation theory [18, 22, 23] tells us that symmetries[3] of any mathematical object of cohomological degree 1 correspond to first order deformations. More generally, symmetries of degree k give first-order deformations over the base ring $\mathbb{C}[\varepsilon]/\varepsilon^2$, where ε is of degree $1-k$. The idea is the following. Suppose we are dealing with a differential-graded mathematical object, such as a differential graded algebra A with differential d. A symmetry of A of degree k is a derivation X of A of degree k. The corresponding deformation is given by changing the differential to $\mathrm{d} + \varepsilon X$, where as above ε has degree $1-k$ and we work modulo ε^2.

Suppose that we have a supersymmetric field theory, acted on by the supersymmetry Lie algebra T^W. Let us pick an odd element $Q \in T^W$. In the supersymmetric world, things are bi-graded, by $\mathbb{Z}$ and $\mathbb{Z}/2$. We have both a cohomological degree and a "super" degree. The symmetry Q of our theory is of bi-degree $(0, 1)$; i.e. it is of cohomological degree 0 and super degree 1. Thus, Q will define a deformation of this theory over the base ring $\mathbb{C}[t]/t^2$, where the parameter t is of bi-degree $(1, 1)$ (and thus even).

[2] For simplicity, we omit formal definitions here. See the Appendix or [7] for more details.

[3] In order for this relationship to be a bijection, the word "symmetry" needs to be understood homotopically: e.g. by considering symmetries of a free resolution of an algebraic object.

Concretely, this deformation of our theory is obtained by adding tQ to the BRST differential of the theory. For example, if the theory is described by a factorization algebra (as we will discuss later), we are adding tQ to the differential of the factorization algebra.

In general, first order deformations (corresponding to symmetries of degree 1) extend to all-order deformations if they satisfy the Maurer-Cartan equation. In the example of a differential graded algebra described above, a derivation X of A of degree 1 satisfies the Maurer-Cartan equation if

$$dX + \frac{1}{2}[X, X] = 0.$$

This equation implies that the differential $d + \varepsilon X$ has square zero, where we are working over the base ring $\mathbb{C}[[\varepsilon]]$.

The Lie algebra T^W has zero differential, so that the Maurer-Cartan equation for an odd element $Q \in T^W$ is the equation $[Q, Q] = 0$. Therefore, if Q satisfies this equation, then it gives rise to a deformation of our theory over the base ring $\mathbb{C}[[t]]$, where again t is of bi-degree $(1, 1)$. The twisted theory will be constructed from this deformation.

However, now we see that there is a problem: we would like our twisted theory to be a single $\mathbb{Z} \times \mathbb{Z}/2$-graded theory, not a family of theories over $\mathbb{C}[[t]]$ where t has bi-degree $(1, 1)$. (The fact that t has this bi-degree means that, even if we could set $t = 1$, the resulting theory would not be $\mathbb{Z} \times \mathbb{Z}/2$-graded.)

To resolve this difficulty, we use a $\mathbb{C}^\times$ action to change the grading.

Definition 4 *Twisting data* for a supersymmetric field theory consists of an odd element $Q \in T^W$ and a group homomorphism $\rho : \mathbb{C}^\times \to G_R$ such that

$$\rho(\lambda)(Q) = \lambda Q \quad \forall \lambda \in \mathbb{C}^\times$$

and such that $[Q, Q] = 0$.

Suppose we have such twisting data, and that we have a theory acted on by T^W with R-symmetry group G_R. Then we can, as above, form a family of theories over $\mathbb{C}[[t]]$ by adding tQ to the BRST differential. We can now, however, use the action of $\mathbb{C}^\times$ on everything to change the grading. Indeed, this $\mathbb{C}^\times$ action lifts the bi-grading by $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ to a tri-grading by $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where the first $\mathbb{Z}$ is the weight under the $\mathbb{C}^\times$ action. Since Q has weight 1 under this $\mathbb{C}^\times$ action, t has weight -1 and so tri-degree $(-1,1,1)$.

From this tri-grading we construct a new $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ grading, by declaring that an element with tri-degree (a, b, c) has new bi-degree $(b + a, c + a)$. This change of grading respects signs.

After this change of grading, we see that we have a family of theories over $\mathbb{C}[[t]]$ where t is now of bi-degree $(0, 0)$ i.e. it is of cohomological degree 0 and super degree 0. We still have the $\mathbb{C}^\times$ action, and this acts on t by sending $t \to \lambda^{-1} t$. Therefore this family of theories is independent of t, and we can set $t = 1$.

Thus, our twisting data defines a *twisted field theory* with BRST operator $d+Q$ if d is the original BRST operator. For details on the construction, see in [7, Sect. 13].

Remark 1 Our twisted field theory is a $\mathbb{C}^{\times}$-equivariant family of theories over $\mathbb{C}$ with BRST operator $d+tQ$. By the Rees construction, this is the same as the data of a filtration on the twisted field theory, whose associated graded is the untwisted theory with a shift of grading. It follows that there is a spectral sequence from the cohomology of the observables of the untwisted theory to that of the twisted theory.

One might think that the cohomology of observables of the twisted theory (in the sense above) is a subset of the cohomology of observables of the untwisted theory, because one is looking at the Q-closed modulo Q-exact observables of the original theory. This is not really true, however, because this fails to take account of the differential (the BRST operator) on the observables of the untwisted theory. The best that one can say in general is that there is a spectral sequence relating twisted and untwisted observables.

There are examples (obtained by applying further twists to theories which are already partially twisted) where this spectral sequence degenerates, so that the cohomology of twisted observables has a filtration whose associated graded is the cohomology of untwisted observables. In such cases, twisted and untwisted observables are the "same size", and twisted observables are definitely not a subset of untwisted observables (at the level of cohomology).

1.3 Minimally Twisted $\mathcal{N}=1,2,4$ SUSY Theories Are Holomorphic

We begin with the case $\mathcal{N}=1$. Choosing an element $Q \in S^{+}$ is the same as choosing a complex structure on the linear space $\mathbb{R}^4$, with the property that the standard Riemannian metric on $\mathbb{R}^4$ is Kähler for this complex structure and that the induced orientation on $\mathbb{R}^4$ is the standard one. (Elements in S^{-} give rise to such complex structures which induce the opposite orientation on $\mathbb{R}^4$).

One can see this as follows. Given $Q \in S^{+}$, the stabilizer $\mathrm{Stab}(Q) \subseteq \mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ is $\mathrm{SU}(2)_{-}$, so Q provides a reduction of the structure group to $\mathrm{SU}(2)$. Concretely, $Q \otimes S^{-} \subseteq V_{\mathbb{C}} = \mathbb{C}^4$ is the $(0,1)$ part, i.e. the $-i$ eigenspace of the complex structure, and its complex conjugate is the $(1,0)$ part.

The complexified R-symmetry group for $\mathcal{N}=1$ supersymmetry is $\mathbb{C}^{\times}$, which acts on supercharges in S^{+} with weight 1 and in S^{-} with weight -1. As we explained earlier, we will use this R-symmetry action to change gradings, so that supercharges in S^{+} have cohomological degree 1 and those in S^{-} have cohomological degree -1.

After we change the grading in this way, the $\mathbb{Z}$-graded version of the supertranslation Lie algebra

$$T^{\mathcal{N}=1} = S^{-}[1] \oplus V_{\mathbb{C}} \oplus S^{+}[-1]$$

acts on the untwisted theory. If we twist by an element $Q \in S^+$, then the dg Lie algebra $(T^{\mathcal{N}=1}, [Q, -])$ acts on the Q-twisted theory.

Let $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}$ denote a basis for $V_{\mathbb{C}}$ where we are using the complex structure on $V_{\mathbb{R}}$ induced by Q. Then the map

$$[Q, -] : S^- \to V_{\mathbb{C}}$$

has image $[Q, S^-] = Q \otimes S^- = V_{\mathbb{C}}^{(0,1)}$, which is the subspace generated by the $\frac{\partial}{\partial \bar{z}_i}$'s. Thus, translations in the $\frac{\partial}{\partial \bar{z}_i}$ directions are homotopically trivial in the twisted theory.

This means that the twisted theory is *holomorphic*. Let us briefly explain this idea. Recall that the energy-momentum tensor of a field theory arises from the action of the translation group $V_{\mathbb{R}}$ on the field theory. One (quite weak) way to say that a field theory is topological is that the energy-momentum tensor is trivial. This implies, for instance, that correlation functions are independent of position. Our definition of holomorphic is that the action of $V_{\mathbb{C}}^{(0,1)}$ is (homotopically) trivial. This will mean that correlation functions are holomorphic functions of position.

In fact, for $\mathcal{N} = 1, 2, 4$, any twist by a Q of the form $Q^+ \otimes w \in S^+ \otimes W$ (a decomposable tensor) produces a holomorphic field theory. Twists by such elements are called *minimal twists*.

Examples of such a minimally twisted supersymmetric gauge theory can be obtained by twisting the anti-self-dual $\mathcal{N} = 1, 2, 4$ supersymmetric gauge theories[4] on $\mathbb{R}^4$. In fact, these twisted field theories arise as cotangent theories, which means that the space of solutions to the equations of motion is described as a -1-shifted cotangent bundle:

$\mathcal{N} = 1$ $\quad T^*[-1]$(holomorphic G-bundles)
$\mathcal{N} = 2$ $\quad T^*[-1]$(holomorphic G-bundles + $\psi \in H^0(\mathfrak{g}_P)$)
$\mathcal{N} = 4$ $\quad T^*[-1]$(holomorphic G-bundles + $\psi_1, \psi_2 \in H^0(\mathfrak{g}_P)$ s.t. $[\psi_1, \psi_2] = 0$)

If we work perturbatively (as we do for most of this note), we consider solutions to the equations of motion which lie in a formal neighbourhood of a given solution. It is possible to glue together the perturbative descriptions over the moduli space of classical solutions, but we do not consider this point in this note. Here G is a semi-simple algebraic group. We denote by $\mathfrak{g}_P = P \times_G \mathfrak{g}$ the adjoint bundle of Lie algebras associated to P.

They admit an explicit description, as derived in [7]. The fields of these theories can be described in the BV formalism as follows:

$\mathcal{N} = 1$ The fields are

$$\Omega^{0,*}(\mathbb{C}^2, \mathfrak{g})[1] \oplus \Omega^{2,*}(\mathbb{C}^2, \mathfrak{g}^\vee),$$

[4] We will refer to these as "the $\mathcal{N} = 1, 2, 4$ twisted SUSY gauge theory" in the rest of these notes.

and the action on the space of fields is given by

$$\int_{\mathbb{C}^2} Tr(\beta \wedge (\bar{\partial}\alpha + \frac{1}{2}[\alpha, \alpha])),$$

where $\alpha \in \Omega^{0,*}(\mathbb{C}^2, \mathfrak{g})[1]$ and $\beta \in \Omega^{2,*}(\mathbb{C}^2, \mathfrak{g}^\vee)$. This theory is a *holomorphic BF theory*, see [8]. It is equivalent to holomorphic Chern-Simons theory on the supermanifold $\mathbb{C}^{2|1}$.

$\mathcal{N} = 2$ We get something similar, replacing $\mathfrak{g}$ by $\mathfrak{g}[\varepsilon]$, where ε is a square-zero parameter of degree 1. Thus, the field α is an element $\Omega^{0,*}(\mathbb{C}^2, \mathfrak{g}[\varepsilon])[1]$ and the field β is an element of $\Omega^{2,*}(\mathbb{C}^2, (\mathfrak{g}[\varepsilon])^\vee)$.

$\mathcal{N} = 4$ Again, we get something similar, replacing $\mathfrak{g}$ by $\mathfrak{g}[\varepsilon_1, \varepsilon_2]$, where $\varepsilon_1, \varepsilon_2$ are square-zero parameters of degrees 1 and -1 respectively. This is equivalent to holomorphic Chern-Simons theory on $\mathbb{C}^{2|3}$.

This result from [7] allows an explicit calculation (at the classical level) of the spaces of observables of these supersymmetric gauge theories. (We will discuss the structure on observables using the language of factorization algebras shortly). For instance, for the $\mathcal{N} = 4$ theory, the space of observables supported at the origin in $\mathbb{C}^2$ is

$$C^*(\mathfrak{g}[[z_1, z_2, \varepsilon_1, \varepsilon_2, \varepsilon_3]])$$

where the ε_i are three odd parameters. This result was also derived in [10], using different methods.

2 Factorization Algebras in Perturbative Quantum Field Theory

In the book [6], a definition of a quantum field theory based on Wilsonian effective action and the BV formalism is given. The main result is that we can construct, using renormalization, such perturbative quantum field theories starting from a classical field theory and working term by term in $\hbar$, using obstruction theory. Let $\mathcal{E}$ be the space of fields of a classical field theory and let $\mathcal{O}(\mathcal{E})$ be the functionals on $\mathcal{E}$. If we have a quantization modulo $\hbar^n$, there may be an obstruction $O_n \in H^1(\mathcal{O}_{loc}(\mathcal{E}))$ to quantize to the next order. Here $\mathcal{O}_{loc}(\mathcal{E})$ denotes the subcomplex of $\mathcal{O}(\mathcal{E})$ consisting of local functionals, i.e. functionals which can be written as sums of integrals over differential operators. If O_n vanishes, we can quantize to the next order, and the possible lifts are a torsor for $H^0(\mathcal{O}_{loc}(\mathcal{E}))$.

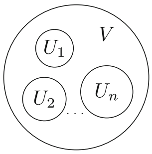
2.1 Factorization Algebras

In [4], Costello and Gwilliam analyze the structure of observables of a quantum field theory in the language of factorization algebras. The notion of factorization algebra was introduced in the algebro-geometric context by Beilinson and Drinfeld in [2].

The approach used by Costello-Gwilliam is very similar to how observables and the operator product are encoded in Segal's axioms for quantum field theory [26–28].

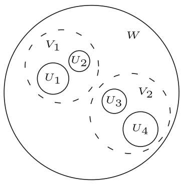
Definition 5 Let M be a topological space and let $\mathcal{C}$ be a symmetric monoidal category (in examples from field theory $\mathcal{C}$ will be cochain complexes or some variant). A *prefactorization algebra* $\mathcal{F}$ *on* M *(with values in* $\mathcal{C}$*)* consists of the following data.

1. For every open subset $U \subseteq M$, an object $\mathcal{F}(U) \in \mathrm{Ob}(\mathcal{C})$.
2. If $U_1, \ldots, U_n$ are pairwise disjoint open subsets of an open set V, we have a morphism



$$\rightsquigarrow \qquad \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(V),$$

such that if $U_1 \amalg \cdots \amalg U_{n_i} \subseteq V_i$ and $V_1 \amalg \cdots \amalg V_k \subseteq W$, the following diagram commutes.



(for $k = n_1 = n_2 = 2$)

$$\rightsquigarrow \qquad \begin{array}{ccc} \bigotimes_{i=1}^{k} \bigotimes_{j=1}^{n_i} \mathcal{F}(U_j) & \longrightarrow & \bigotimes_{i=1}^{k} \mathcal{F}(V_i) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}(W) & \end{array}$$

A *factorization algebra on* M is a prefactorization algebra on M which additionally satisfies a gluing condition saying that given an open cover $\{U_i\}$ of V satisfying certain conditions, $\mathcal{F}(V)$ can be recovered from the $\mathcal{F}(U_i)$'s.This glueing condition is analogous to the one for (homotopy) (co-)sheaves.
For the exact gluing condition and more details on the theory of factorization algebras we refer to [4] and to Grégory Ginot's contribution [15].

Although the definition makes sense for an arbitrary topological space, we will only consider factorization algebras on manifolds.

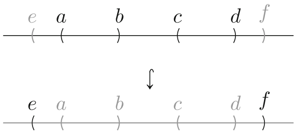
2.2 Associative Algebras are Factorization Algebras

Actually, associative algebras are a special case of factorization algebras with values in chain complexes. Suppose that we have a factorization algebra $\mathcal{F}$ on $\mathbb{R}$ with the property that for any interval $(a, b) \subseteq \mathbb{R}$ the map

$$\mathcal{F}((a, b)) \xrightarrow{\simeq} \mathcal{F}(\mathbb{R})$$

is a quasi-isomorphism, i.e. an isomorphism on cohomology.[5] Then $\mathcal{F}$ defines an associative algebra (up to homotopy). Let $A = \mathcal{F}(\mathbb{R}) \simeq \mathcal{F}((a, b))$ for any $(a, b) \subseteq \mathbb{R}$. If $(a, b) \amalg (c, d) \subseteq (e, f)$ with $e < a < b < c < d < f$, the factorization algebra structure gives us a map

$$\mathcal{F}((a,b)) \xrightarrow{\simeq} \mathcal{F}(\mathbb{R})$$



$$\rightsquigarrow \quad \begin{array}{ccccc} \mathcal{F}((a,b)) & \otimes \ \mathcal{F}((c,d)) & \longrightarrow & \mathcal{F}((e,f)) \\ \wr\mid & \wr\mid & & \wr\mid \\ A \ \otimes & A & \xrightarrow{m} & A \end{array}$$

Conversely, any associative algebra defines a locally constant factorization algebra on $\mathbb{R}$.

Remark 2 Such factorization algebras really appear in quantum mechanics. Quantum mechanics is the field theory with fields $\phi \in C_c^\infty(\mathbb{R})$ and action functional $S(\phi) = \int \phi \Delta \phi$, $\phi \in C_c^\infty(\mathbb{R})$. Then the equations of motion say that ϕ is harmonic, i.e. $\Delta\phi = 0$. Harmonic functions on (a, b) extend uniquely to harmonic functions on $\mathbb{R}$: this implies that the factorization algebra constructed from this example has the property that the map $\mathcal{F}((a, b)) \to \mathcal{F}(\mathbb{R})$ is an isomorphism. This example will be explained in more detail in 2.4.

2.3 The Factorization Algebra of Observables

It is shown in [4] that observables of a (perturbative) quantum field theory in Euclidean signature turn out to have the structure of a factorization algebra with values in the category of cochain complexes of $\mathbb{C}[[\hbar]]$-modules, flat over $\mathbb{C}[[\hbar]]$. These cochain complexes are built from spaces of smooth functions and distributions on the spacetime manifold. Technically, these cochain complexes are endowed with a "diffeological" structure, which is something a little weaker than a topology; this reflects their analytical origin.

Observables of a classical field theory also form a factorization algebra. Starting from the quantum observables, the classical observables are

$$Obs^{cl}(V) := Obs^q(V)/\hbar = \left\{ \begin{array}{c} \text{functions on the "derived" moduli space of} \\ \text{solutions to the Euler-Lagrange equations on } V \end{array} \right\}.$$

Taking the derived space of solutions to the Euler-Lagrange equations amounts to one version of the BV classical formalism. The antifields of the BV formalism correspond to taking the Koszul complex associated to the equations of motion and the ghosts

[5] Such a factorization algebra is called *locally constant*.

correspond to taking the quotient by the gauge group in a homological way. For more details, see [6].

The factorization algebra of quantum observables deforms that of classical observables, in that quantum observables are a factorization algebra over $\mathbb{C}[[\hbar]]$ and restrict to classical observables modulo $\hbar$. To first order, this deformation is closely related to the BV antibracket on the classical observables.

We should emphasize that quantum observables, for a general quantum field theory in Euclidean signature, *do not* form an associative algebra.[6] Associative algebras arise when one studies factorization algebras on the real line (associated to 1-dimensional quantum field theories). The associative product is the operator product of observables in the time direction. For a factorization algebra on a higher-dimensional manifold, there is no specified "time" direction which allows one to define the associative product, rather there is a kind of "product" for every direction in space-time.

Furthermore, quantum field theories in dimension larger than one rarely satisfy the locally-constant condition which was satisfied by the observables of quantum mechanics. The exception to this rule is the observables of a topological field theory. In this case, however, we find that observables from an E_n-algebra (a structure studied by topologists to encode the product in an n-fold loop space) rather than simply an associative algebra. This is a result of Lurie [24], who shows that there is an equivalence between locally-constant factorization algebras on $\mathbb{R}^n$ and E_n-algebras.

Another point to emphasize is that factorization algebras are only the right language to capture the structure of observables of a QFT in Euclidean signature. In Lorentzian signature, the operator product (at least for massless theories) has singularities on the light-cone and not just on the diagonal, so that we would only expect to be able to define the factorization product for pairs of open subsets which are not just disjoint, but which can not be connected by a path in the light cone.

2.3.1 More structures on the factorization algebra of observables

Translation invariance: If we additionally have translation invariance on a locally constant factorization algebra on $\mathbb{R}$, we get an associative algebra endowed with an infinitesimal automorphism, i.e. a derivation. This derivation encodes the Hamiltonian of the field theory.

Poisson bracket: The classical observables $Obs^{cl}(U)$ form a commutative dg algebra. Moreover, we have a Poisson bracket of cohomological degree one,[7] the "antibracket" $\{\,,\}$ on $Obs^{cl}(U)$.

[6] Note that we work in Euclidean signature. Some axiom systems in Lorentzian signature have an asssociative structure on observables: see Klaus Fredenhagen's lectures in the same volume.

[7] This means that Obs^{cl} has the structure of a P_0 *factorization algebra*, where P_0 is the operad describing commutative dg algebras with a Poisson bracket of degree 1.

(Weak) Quantization condition: In deformation quantization, the non-commutative algebra structure to first order must be related to the Poisson bracket. We have a similar condition[8] relating the factorization algebras of quantum and classical observables.

The differential d on $Obs^q(U)$ should satisfy

1. Modulo $\hbar$, d coincides with the differential d_0 on $Obs^{cl}(U)$.
2. Let

$$d_1 : H^i(Obs^{cl}(U)) \to H^{i+1}(Obs^{cl}(U))$$

be the boundary map coming from the exact sequence of complexes

$$\hbar Obs^{cl}(U) \longrightarrow Obs^q(U) \mod \hbar^2 \longrightarrow Obs^{cl}(U).$$

d_1 lifts to a cochain map of degree 1 $Obs^{cl}(U) \to Obs^{cl}(U)$, which we continue to call d_1. Then, if we define a bilinear map on $Obs^{cl}(U)$ by

$$\{a, b\}^{d_1} = d_1(ab) \mp a d_1 b - (d_1 a)b$$

we ask that there is a homotopy between $\{a, b\}^{d_1}$ and the original bracket $\{a, b\}$. (In particular, these two brackets must coincide at the level of cohomology).

2.4 Example: The Free Scalar Field

Let M be a compact Riemannian manifold. We will consider the field theory where the fields are $\phi \in C^\infty(M)$, and the action functional is

$$S(\phi) = \int_M \phi \Delta \phi,$$

where Δ is the Laplacian on M.

2.4.1 Classical Observables

If $U \subseteq M$ is an open subset, then the space of solutions of the equations of motion on U is the space of harmonic functions on U,

$$\{\phi \in C^\infty(U) | \Delta\phi = 0\}.$$

[8] We present here a weak version of the condition. A stronger version, discussed in [4], is that Obs^q is a BD *factorization algebra*, where BD is the Beilinson-Drinfeld operad. The BD operad is an operad over $\mathbb{C}[[\hbar]]$ deforming the P_0 operad: $BD \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \simeq P_0$.

As discussed above, we consider the *derived* space of solutions of the equations of motion. This is a linear dg manifold, i.e. a cochain complex. For more details about the derived philosophy, the reader should consult [4]. In this simple situation, the derived space of solutions to the free field equations, on an open subset $U \subseteq M$, is the two-term complex

$$\mathcal{E}(U) = \left(C^\infty(U) \xrightarrow{\Delta} C^\infty(U)[-1] \right)$$

The classical observables of a field theory on an open subset $U \subseteq M$ should be functions on the derived space of solutions to the equations of motion on U, and thus the symmetric algebra of the dual.[9] The dual to the two-term complex $\mathcal{E}(U)$ above is the complex

$$\mathcal{E}_c^\vee(U) = \left(\mathcal{D}_c(U)[1] \xrightarrow{\Delta} \mathcal{D}_c(U) \right),$$

where $\mathcal{D}_c(U)$ indicates the space of compactly supported distributions on U.

We would like to define $Obs^{cl} = \mathcal{O}(\mathcal{E}) = \mathrm{Sym}(\mathcal{E}^\vee)$, but in order to define a Poisson structure on Obs^{cl}, instead we need to use a version of elliptic regularity, which we call the Atiyah-Bott lemma [1]. Let

$$\mathcal{E}_c^!(U) = (C_c^\infty(U)[1] \longrightarrow C_c^\infty(U)).$$

The Atiyah-Bott lemma states that the map of cochain complexes

$$\mathcal{E}_c^!(U) \longrightarrow \mathcal{E}_c^\vee(U),$$

given by viewing a compactly supported function as a distribution, is a continuous homotopy equivalence.

Thus, we define our classical observables to be

$$Obs^{cl}(U) = \mathrm{Sym}\left(\mathcal{E}_c^!(U) \right) = \oplus_n \mathrm{Sym}^n \mathcal{E}_c^!(U).$$

By $\mathrm{Sym}^n \mathcal{E}_c^!(U)$ we mean the S_n-invariants in the complex of compactly supported sections of the bundle $(E^!)^{\boxtimes n}$ on U^n. Equivalently, we can view $\mathrm{Sym}^n \mathcal{E}_c^!(U)$ as the symmetric product of the topological vector space $\mathcal{E}_c^!(U)$ using the completed inductive (or bornological) tensor product.[10]

[9] For free theories, it is enough to consider polynomial functions.

[10] These tensor products both have the property that $C_c^\infty(M)\widehat{\otimes}C_c^\infty(N) = C_c^\infty(M \times N)$, and similarly for compactly supported smooth sections of a vector bundle on M. The more familiar projective tensor product does not (at least not obviously) have this property. See [16] for a discussion of the inductive tensor product and [20] for the bornological tensor product. The reader with no taste for functional analysis should just take the fact that $\mathcal{E}_c(U)^{\otimes n} = \Gamma_c(U, E^{\boxtimes n})$ as a definition of $\mathcal{E}_c(U)^{\otimes n}$.

It is clear that classical observables form a prefactorization algebra. Indeed, $Obs^{cl}(U)$ is a differential graded commutative algebra. If $U \subseteq V$, there is a natural algebra homomorphism

$$i_V^U : Obs^{cl}(U) \to Obs^{cl}(V),$$

which on generators is just the natural map $C_c^\infty(U) \to C_c^\infty(V)$ given by extending a continuous compactly supported function on U by zero on $V \backslash U$.

If $U_1, \ldots, U_n \subseteq V$ are disjoint open subsets, the prefactorization structure map is the continuous multilinear map

$$\begin{aligned} Obs^{cl}(U_1) \times \cdots \times Obs^{cl}(U_n) &\to Obs^{cl}(V) \\ \alpha_1 \times \cdots \times \alpha_n &\mapsto \sum_{i=1}^n i_V^{U_i} \alpha_i \in Obs^{cl}(V). \end{aligned}$$

In dimension one, this is particularly simple.

Lemma 1 [4] *If $U = (a, b) \subset \mathbb{R}$ is an interval in $\mathbb{R}$, then*

1. *For any $x \in (a, b)$, the complex*

$$\mathcal{E}_c^!((a,b)) = \left(C_c^\infty((a,b))[1] \xrightarrow{\Delta} C_c^\infty((a,b)) \right)$$

is quasi-isomorphic to $\mathbb{R}^2$ situated in degree 0, i.e. the cohomology is

$$H^* \left(\mathcal{E}_c^!((a,b)) \right) = \mathbb{R}^2.$$

2. *The algebra of classical observables for the free field has cohomology*

$$H^* \left(Obs^{cl}((a,b)) \right) = \mathbb{R}[p, q],$$

the free algebra on two variables.

Proof We first show that for any $x_0 \in (a, b)$, the map

$$\begin{aligned} \mathcal{E}((a,b)) = \left(C^\infty((a,b)) \xrightarrow{\Delta} C^\infty((a,b))[-1] \right) &\xrightarrow{\pi} \mathbb{R}^2 \\ (\phi, \psi) &\longmapsto (\phi(x_0), \phi'(x_0)) \end{aligned}$$

is a quasi-isomorphism. To show this, consider the inclusion

$$\begin{aligned} \mathbb{R}^2 &\overset{i}{\hookrightarrow} \mathcal{E}((a,b)) \\ (a,b) &\longmapsto (a + b(x - x_0), 0). \end{aligned}$$

Then $\pi \circ i = \mathrm{id}|_{\mathbb{R}^2}$ and $i \circ \pi : (\phi, \psi) \mapsto (\phi(x_0) + \phi'(x_0)(x - x_0), 0) \in \mathcal{E}((a, b))$. Thus, a homotopy between the identity and $i \circ \pi$ is given by

$$(\phi, \psi) \longmapsto S(\phi, \psi)(x) = \left(\int_{y=a}^{x} \int_{u=a}^{y} \psi(u)\, \mathrm{d}u \mathrm{d}y, 0 \right),$$

as $\mathrm{id} - i \circ \pi = [\Delta, S]$.

This implies that the dual $\mathcal{E}^\vee((a, b))$ also is quasi-isomorphic to $\mathbb{R}^2$ and, by elliptic regularity, $\mathcal{E}_c^!((a, b)) \simeq \mathcal{E}^\vee((a, b)) \simeq \mathbb{R}^2$.

The second part follows directly from the first and the exactness of $\mathrm{Sym}_{\mathbb{R}}$.

Remark 3 The quasi-isomorphism π from the proof induces the desired quasi-isomorphism $\pi^\vee : (\mathbb{R}^2)^\vee \to \mathcal{E}^\vee$. So,

$$\begin{aligned} \pi^\vee(1, 0)(\phi, \psi) &= \phi(x_0) = \delta_{x_0}(\phi) = q(\phi, \psi) \\ \pi^\vee(0, 1)(\phi, \psi) &= \phi'(x_0) = \delta'_{x_0}(\phi) = p(\phi, \psi), \end{aligned}$$

the position and the momentum observables, respectively. Thus, the cohomology of $\mathcal{E}^\vee((a, b))$ is generated by q and p.

Recall that the classical observables are endowed with a Poisson bracket of degree 1. For $\alpha \in C_c^\infty(U)$ and $\beta \in C_c^\infty(U)[1]$, we have

$$\{\alpha, \beta\} = \int_U \alpha\beta\, dVol.$$

This extends uniquely to a continuous Poisson bracket on $Obs^{cl}(U)$.

2.4.2 Quantizing free field theories

Our philosophy is that we should take a P_0 factorization algebra Obs^{cl} (e.g. the observables of a classical field theory) and deform it into a BD factorization algebra Obs^q. This is a strong version of the quantization condition. For a general (interacting) field theory, with the current state of technology we can only construct a weak quantization as defined in Sect. 2.3. However, in the case of a free field theory, we can show that the quantization satisfies this strong quantization condition.

Now we will construct such a quantization of the classical observables of our free field theory, i.e. a factorization algebra Obs^q with the property that

$$Obs^q(U) = Obs^{cl}(U)[\hbar]$$

as $\mathbb{C}[\hbar]$-modules and with a differential d such that

1. Modulo $\hbar$, d coincides with the differential on $Obs^{cl}(U)$,

2. d satisfies

$$\mathrm{d}(ab) = (\mathrm{d}a)b + (-1)^{|a|}a(\mathrm{d}b) + \hbar\{a, b\},$$

where the multiplication arises from that on $Obs^{cl}(U)$.

The construction (see also [17]) starts with a certain graded Heisenberg Lie algebra. Let

$$\mathcal{H}(U) = \left(C_c^{\infty}(U) \xrightarrow{\Delta} C_c^{\infty}(U)[-1]\right) \oplus \mathbb{R}\hbar[-1],$$

where $\mathbb{R}$ is situated in degree 1. Let us give $\mathcal{H}(U)$ a Lie bracket by saying that, if $\alpha \in C_c^{\infty}(U)$ and $\beta \in C_c^{\infty}(U)[-1]$, then

$$[\alpha, \beta] = \hbar \int_U \alpha\beta.$$

Let

$$Obs^q(U) = C_{-*}(\mathcal{H}(U))$$

be the Chevalley-Eilenberg Lie algebra chain complex of $\mathcal{H}(U)$ with the grading reversed. The tensor product that is used to define the Chevalley chain complex is, as before, the completed inductive (or bornological) tensor product of topological vector spaces.

Thus,

$$\begin{aligned} Obs^q(U) &= (\mathrm{Sym}^*(\mathcal{H}(U)[1]), \mathrm{d}) \\ &= (Obs^{cl}(U)[\hbar], \mathrm{d}) \\ &= \left(\oplus_n \Gamma_c(U^n, (E^!)^{\boxtimes n})^{S_n}\right)[\hbar] \end{aligned}$$

where, in the last line, $E^!$ is the direct sum of the trivial vector bundles in degrees 0 and -1. The differential d is defined by first extending the Lie bracket by

$$[\alpha, \beta \wedge \gamma] = [\alpha, \beta] \wedge \gamma + (-1)^{|\beta|(|\alpha|+1)}\beta \wedge [\alpha, \gamma],$$

and then defining

$$\mathrm{d}(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta + \hbar[\alpha, \beta].$$

Thus, by definition, $Obs^q(U)$ is a BD-algebra and Obs^q has the structure of a factorization algebra in BD-algebras by extending the natural map $C^{\infty}(U) \to C^{\infty}(V)$ by the identity to the central extension.

Finally, one can prove that our construction of the factorization algebra for a free field theory, when restricted to dimension one, reconstructs the Weyl algebra associated to quantum mechanics.

Proposition 1 *Let Obs^q denote the factorization algebra on $\mathbb{R}$ constructed from the free field theory, as above. Then,*

1. *The cohomology $H^*(Obs^q)$ is locally constant.*
2. *The corresponding associative algebra is the Weyl algebra, generated by $p, q, \hbar$ with the relation $[p, q] = \hbar$. Classically, p is the observable which sends a field $\phi \in C^\infty(\mathbb{R})$ to $\phi'(0)$, and q sends ϕ to $\phi(0)$.*
3. *The fact that this factorization algebra is translation invariant means that the corresponding associative algebra is equipped with a derivation which we call H. This derivation is given by the Lie bracket with the Hamiltonian,*

$$H = \tfrac{1}{2}\hbar^{-1}[p^2, \ \ ].$$

A proof can be found in the section on quantum mechanics and the Weyl algebra in [4].

3 Factorization Algebras Associated to SUSY Gauge Theories

In this section, we will mostly consider the $\mathcal{N} = 1$ twisted SUSY gauge theory.

3.1 Replacing $\mathbb{C}^2$ by a Complex Surface

In Sect. 1, we discussed twistings of SUSY gauge theories. They arose as parts of a theory invariant under some $Q \in S^+$ and gave a holomorphic field theory on $\mathbb{C}^2$.

Recall that for $\mathcal{N} = 1$, the fields $\mathcal{E}$ of the twisted theory on $\mathbb{C}^2$ are built from a Lie algebra $\mathfrak{g}$ with associated elliptic Lie algebra $\mathcal{L}_{\mathcal{N}=1} = \mathcal{E}[-1]$,

$$\mathcal{L}_{\mathcal{N}=1} = \Omega^{0,*}(\mathbb{C}^2, \mathfrak{g}) \oplus \Omega^{2,*}(\mathbb{C}^2, \mathfrak{g}^\vee)[-1] \cong \Omega^{0,*}(\mathbb{C}^2, \mathfrak{g} \oplus \mathfrak{g}^\vee[-1]), \quad (1)$$

with differential $\bar{\partial}$, and Lie bracket determined by

$$[\beta, \beta'] = 0, \quad [\alpha, \beta] = \mathrm{ad}^*_\alpha(\beta) \in \Omega^{2,*}(\mathbb{C}^2, \mathfrak{g}^\vee)[-1], \quad [\alpha, \alpha'] \in \Omega^{0,*}(\mathbb{C}^2, \mathfrak{g}),$$

for $\alpha, \alpha' \in \Omega^{0,*}(\mathbb{C}^2, \mathfrak{g})$, $\beta, \beta' \in \Omega^{2,*}(\mathbb{C}^2, \mathfrak{g}^\vee)[-1]$. The invariant pairing on $\mathcal{L}_{\mathcal{N}=1}$ is given by

$$\langle \phi \otimes X, \psi \otimes Y \rangle = \int \phi \wedge \psi \, \langle X, Y \rangle_{\mathfrak{g}},$$

where $X \in \mathfrak{g}$, $Y \in \mathfrak{g}^\vee$, and $\phi, \psi \in \Omega^{0,*}(\mathbb{C}^2)$.[11]

Our Chern-Simons action functional is

$$S(\chi) = \frac{1}{2}\langle \chi, \bar{\partial}\chi \rangle + \frac{1}{6}\langle \chi, [\chi, \chi] \rangle$$

for a general field χ.

We saw at the end of Sect. 1 that this twisted $\mathcal{N} = 1$ theory is the space of solutions to the equations of motion for the cotangent theory to the pointed moduli problem of holomorphic principal G-bundles on $\mathbb{C}^2$. This theory makes sense on a general complex surface X, and is the *cotangent theory to the pointed moduli problem of holomorphic principal G -bundles on X*,

$$T^*[-1]\,\mathrm{Bun}_G(X).$$

Similarly to before, we find that the moduli space of solutions to the equations of motion is

$$\left\{ (P, \phi) | P \text{ a principal } G\text{-bundle on } X, \phi \in H^0_{\bar{\partial}}(X, K_X \otimes \mathfrak{g}^\vee_{\mathrm{ad}}) \right\},$$

where K_X is the canonical bundle on X and $\mathfrak{g} = \mathrm{Lie}(G)$. This problem corresponds to the elliptic Lie algebra

$$\mathcal{L}_{\mathcal{N}=1}(X) = \Omega^{0,*}(X, \mathfrak{g}) \oplus \Omega^{0,*}(X, \mathfrak{g}^\vee \otimes K_X)[-1].$$

This is because the equations of motion for $\mathcal{L}_{\mathcal{N}=1}(X)$ give

$$\bar{\partial}\alpha + \frac{1}{2}[\alpha, \alpha]$$

for $\alpha \in \Omega^{0,1}(X, \mathfrak{g})$, which is the Maurer-Cartan equation, and for $\beta \in \Omega^{2,0}(X, \mathfrak{g}^\vee)$,

$$\bar{\partial}_\alpha \beta := \bar{\partial}\beta + [\alpha, \beta] = 0.$$

[11] This is actually only well-defined for compactly supported sections, but this technical difficulty can be overcome by passing to a quasi-isomorphic chain complex similar to what we did in 2.4. See [4] for details.

Thus, the principal bundle corresponds to a Maurer-Cartan element in $\Omega^{0,*}(X, \mathfrak{g})$ and ϕ to the element β.

Recall that the classical observables are functions on the (derived) space of solutions to the equations of motion. If our theory is given by the elliptic Lie algebra $\mathcal{L}$, the solutions to the equations of motion are given by Maurer-Cartan elements of $\mathcal{L}$, i.e. $\chi \in \mathcal{L}$ such that $\bar{\partial}\chi + \frac{1}{2}[\chi, \chi] = 0$. If $U \subseteq \mathbb{C}^2$ is open, then the classical observables on U are Lie algebra cochains of $\mathcal{L}(U)$,

$$Obs^{cl}(U) = C^*(\mathcal{L}(U)) = \widehat{\mathrm{Sym}}^*(\mathcal{L}(U)^\vee[-1]).$$

In our case, $\mathcal{L} = \mathcal{L}_{\mathcal{N}=1}(X)$ is the semi-direct product, i.e. the split-zero extension $\mathfrak{h} \ltimes M$ of $\mathfrak{h} = \Omega^{0,*}(\mathbb{C}^2, \mathfrak{g})$ with $M = \Omega^{2,*}(\mathbb{C}^2, \mathfrak{g})$, so if U is a ball, we essentially get

$$Obs^{cl}(U) = C^*(\mathfrak{h} \ltimes M) = C^*\left(\mathrm{Hol}(U) \otimes \mathfrak{g}, \widehat{\mathrm{Sym}}^*\left((\mathrm{Hol}(U)\,\mathrm{d}z_1\mathrm{d}z_2 \otimes \mathfrak{g})^\vee\right)\right),$$

a fancy (derived) version of functions on $\{\phi \in \mathrm{Hol}(U)\,\mathrm{d}z_1\mathrm{d}z_2 \otimes \mathfrak{g}\}/\mathrm{Gauge}$.

3.2 *Quantization*

Recall that by quantization, we mean that we deform the commutative factorization algebra of classical observables to a quantum one. Essentially the differential is deformed by using the BV Laplacian and by replacing the classical action of our field theory by a quantum one which satisfies a renormalized BV quantum master equation. In our case, we find that there is a quantization, and it even is unique:

Theorem 1 *The $\mathcal{N} = 1$ minimally twisted supersymmetric gauge theory on a complex surface X with trivial canonical bundle, perturbing around any holomorphic G-bundle for a simple algebraic group G, admits a unique quantization compatible with certain natural symmetries.*

The proof of this is given in [8]. It relies on the renormalization theory from [6] which reduces it to a cohomological calculation. More precisely, [6] tells us that, order by order in $\hbar$, the obstruction to quantizing to the next order lies in $H^1(\mathcal{O}_{loc})$, and the ambiguity in quantizing is given by $H^0(\mathcal{O}_{loc})$. Similar techniques can be used to show that the twisted $\mathcal{N} = 2$ and 4 theories can be quantized on any complex surface X.

Remark 4 This theorem is a special case of a very general result. Recall that the fields of the twisted $\mathcal{N} = 1$ theory are $T^*[-1]\,\mathrm{Bun}_G(X)$. Since X has trivial canonical bundle, $\mathrm{Bun}_G(X)$ already has a symplectic form. The general result is that the cotangent theory to an elliptic moduli problem which is already symplectic has a natural quantization (the unique quantization compatible with certain symmetries).

3.3 The Relation to Vertex Algebras

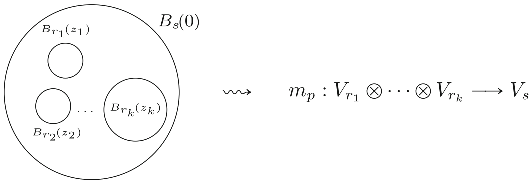
As we saw in the previous section, quantization gives a factorization algebra of quantum observables, which looks like

$$Obs^q : U \longmapsto \left(C^*(\mathcal{L}(U))[\![\hbar]\!], \mathrm{d}\right),$$

where modulo $\hbar$, d coincides with the differential d_{CE} on $C^*(\mathcal{L}(U))$, cf. Sect. 2.4.2. In particular, the factorization algebra structure of our minimally twisted $\mathcal{N} = 1, 2, 4$ theory on $\mathbb{C}^2$ associates a product to each configuration of k balls inside a larger one in $\mathbb{C}^2$. Now consider the situation where the big ball is centered at the origin. By translation invariance of our theory, up to isomorphism $Obs^q(B_r(z))$ is independent of z, and we call this V_r. Thus, for every $p = (z_1, \ldots, z_k)$, a point in the parameter space

$$P(r_1, \ldots, r_k|s) = \{k \text{ disjoint balls of radii } r_1, \ldots, r_k \text{ inside } V_s\},$$

we get a map $m_p : V_{r_1} \otimes \cdots \otimes V_{r_k} \longrightarrow V_s$.



This map depends smoothly on the parameter, i.e.

$$V_{r_1} \otimes \cdots \otimes V_{r_k} \longrightarrow V_s \otimes C^\infty(P(r_1, \ldots, r_k|s)).$$

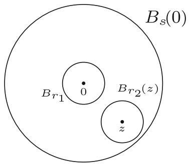
If the field theory is holomorphic (as in our case), this map lifts to a cochain map of degree 0 compatible with composition

$$V_{r_1} \otimes \cdots \otimes V_{r_k} \longrightarrow V_s \otimes \Omega^{0,*}(P(r_1, \ldots, r_k|s)),$$

which leads to a map in cohomology,

$$H^*(V_{r_1}) \otimes \cdots \otimes H^*(V_{r_k}) \longrightarrow H^*(V_s) \otimes H^*_{\bar\partial}\left(P(r_1, \ldots, r_k|s)\right). \tag{2}$$

Remark 5 In complex dimension 1, the analogous structure is that of a vertex algebra. Consider the multiplication arising from the factorization algebra structure for 2 balls inside a larger one. Assume that one is centered at the origin, $B_{r_1}(z_1) = B_{r_1}(0) =: B_{r_1}$. Let $P'(r_1, r_2 \mid s)$ denote the subspace of $P(r_1, r_2 \mid s)$ where the first ball is centered at the origin.



$$\rightsquigarrow \qquad V_{r_1} \otimes V_{r_2} \longrightarrow V_s \otimes \Omega^{0,*}(P(r_1, r_2|s)) \tag{3}$$

Assuming B_{r_1} to be centered at the origin, the center of the second disk $z = z_2$ of radius r_2 can be anywhere in the annulus of radii r_1+r_2 and $s-r_2$. Thus, our parameter space $P'(r_1, r_2 \mid s)$ consists of this annulus. Let us look at the maps analogous to (2). The 0th cohomology of $P'(r_1, r_2|s)$ consists of holomorphic functions on the annulus which are just power series in z and z^{-1} which converge on the annulus. So we get a map of degree zero

$$H^*(V_{r_1}) \otimes H^*(V_{r_2}) \longrightarrow H^*(V_s) \otimes \mathbb{C}\{z, z^{-1}\},$$

where $\mathbb{C}\{z, z^{-1}\}$ refers to those Laurent series which converge on the appropriate annulus. Moreover, in one dimension, all higher cohomologies vanish, so this is all we get. See [17] for the worked out example of the $\beta\gamma$ system, and [4] for the general theorem that holomorphically translation-invariant factorization algebras in one complex dimension have the structure of a vertex algebra on their cohomology.

Remark 6 For holomorphic theories in complex dimension > 1, it turns out to be better to use polydiscs instead of discs. Thus, $B_r \subset \mathbb{C}^2$ should be understood as the product $D_r \times D_r$ of two discs of radius r in $\mathbb{C}$.

In our case of a twisted SUSY gauge theory on $\mathbb{C}^2$, we get a two dimensional analog of vertex algebras, i.e. for $\mathcal{N} = 1, 2, 4$ and for any semi-simple Lie algebra we get a two dimensional vertex algebra. The problem is to compute this object. Recall from (2) that on cohomology, we get a map of degree zero. As in dimension 1, we restrict to the space $P'(r_1, r_2 \mid s)$ where the first polydisc is centered at the origin. We can identify $P'(r_1, r_2 \mid s)$ with the two complex dimensional analog of an annulus: the complement of one polydisc in another.

Again, we can compute the Dolbeault cohomology of this space. By Hartogs' theorem, every holomorphic function on P' extends to zero, so as the zeroth cohomology we just get holomorphic functions on the polydisc and therefore a map of degree zero

$$H^*(V_{r_1}) \otimes H^*(V_{r_2}) \longrightarrow H^*(V_s) \otimes \mathbb{C}\{z_1, z_2\}, \tag{4}$$

where we use the notation $\mathbb{C}\{z_1, z_2\}$ to refer to series which converge on the appropriate polydisc.

This map extends to $z_1 = z_2 = 0$; this allows us to construct a commutative algebra from the spaces $H^*(V_r)$. Let us assume (as happens in practice) that the map $H^*(V_r) \to H^*(V_s)$ (with $r < s$) associated to the inclusion of one disc centered at the origin into another is injective. Then, let

$$\mathcal{V} = \bigcup H^*(V_r)$$

and let $F^r\mathcal{V} = H^*(V_r)$. Then, the map (4) gives $\mathcal{V}$ the structure of a commutative algebra with an increasing filtration by $\mathbb{R}_{>0}$.

The vector fields $\frac{\partial}{\partial z_i}$ act on $H^*(V_r)$; they extend to commuting derivations of the commutative algebra $\mathcal{V}$. The map in equation (4) (or rather its completion where we use $\mathbb{C}[\![z_1, z_2]\!]$) is completely encoded by the filtered commutative algebra $\mathcal{V}$ with its commuting derivations.

However, there is more structure. We have the following identification:

$$H^1_{\bar{\partial}}(P') = z_1^{-1}z_2^{-1}\mathbb{C}\{z_1^{-1}, z_2^{-1}\}.$$

Thus, the first Dolbeault cohomology of the two complex dimensional annulus consists of series in z_i^{-1} with certain convergence properties. So we find that there is a map

$$\mu : H^*(V_{r_1}) \otimes H^*(V_{r_2}) \longrightarrow H^*(V_s) \otimes z_1^{-1}z_2^{-1}\mathbb{C}\{z_1^{-1}, z_2^{-1}\}. \tag{5}$$

Thus, at the level of cohomology, (3), resp. (4) and (5), form an analog of the operator product expansion of a vertex algebra.

One can check that the structure given by (5) is a kind of Poisson bracket with respect to the commutative product obtained from (4). To define this we need some notation. If $\alpha, \beta \in H^*(V_r)$, then $\mu(\alpha, \beta)$ is a class in $H^1_{\bar{\partial}}(P') \otimes \mathcal{V}$. Recall that $P' \subset \mathbb{C}^2$ is an open subset which is the complement of one polydisc in another. Thus, P' retracts onto a 3-sphere $S^3 \subset P'$. Then, for every $f \in \mathbb{C}[z_1, z_2]$, one can define a bracket by

$$\{\alpha, \beta\}_f = \int_{S^3} \mu(\alpha, \beta) f \mathrm{d}z_1 \mathrm{d}z_2 \in \mathcal{V}.$$

This makes sense, as $\mu(\alpha, \beta)\mathrm{d}z_1\mathrm{d}z_2 f$ is a closed 3-form on P' with coefficients in $\mathcal{V}$. The integral only depends on the homology class of the sphere $S^3 \subset P'$, which we choose to be the fundamental class.

This bracket gives a map

$$\{-, -\}_f : H^*(V_{r_1}) \otimes H^*(V_{r_2}) \to \mathcal{V}$$

and extends to a map

$$\{-, -\}_f : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}.$$

One can check that $\{-, -\}_f$ is a derivation in the first factor for the commutative product on $\mathcal{V}$, and satisfies an identity similar to the Jacobi identity. Let us explain the Jacobi identity we find. If $g \in \mathbb{C}[z_1, z_2]$ let us use the notation

$$g(z_1 + w_1, z_2 + w_2) = \sum g'(z_1, z_2) g''(w_1, w_2).$$

That is, $\mathbb{C}[z_1, z_2]$ is a Hopf algebra, with coproduct coming from addition on the plane $\mathbb{C}^2$, and we are using the Sweedler notation to write the coproduct $\delta(g)$ of an element g as $\delta(g) = \sum g' \otimes g''$.

Then, the analog of the Jacobi identity in our situation is the following:

$$\{\{\alpha, \beta\}_f, \gamma\}_g = \{\{\alpha, \gamma\}_g, \beta\}_f + \sum \{\alpha, \{\beta, \gamma\}_{g'}\}_{fg''}.$$

Note that in the case $f = g = 1$, this is the usual Jacobi identity.

All these relations follow from the axioms of a holomorphically translation-invariant factorization algebra using Stokes' theorem.

We have not presented all details of the structure of a higher-dimensional cohomological vertex algebra (i.e. the structure present on the cohomology of a holomorphically-translation invariant factorization algebra). Hopefully this will be developed in full elsewhere. The interested reader might consider working out and writing down the entire structure, including all relations satisfied by the Poisson brackets described above.

Remark 7 There is a similar story for topological field theories on $\mathbb{R}^k$. There, one finds that the 2-point operator product is, at the level of cohomology, a map

$$H^*(V) \otimes H^*(V) \to H^*(V) \otimes H^*_{dR}(\mathbb{R}^k \setminus \{0\}).$$

Here V is the complex $Obs^q(D)$ for any disc D in $\mathbb{R}^k$. On the right hand side of this expression we find the de Rham cohomology of a thickened sphere in $\mathbb{R}^k$, whereas in the holomorphic case we found the Dolbeault cohomology of a simliar region.

At the cochain level, this operator product gives the complex V the structure of an E_k-algebra. If $k > 1$, then the class in $H^0(\mathbb{R}^k \setminus \{0\})$ gives $H^*(V)$ the structure of a commutative algebra, and the class in $H^{k-1}(\mathbb{R}^k \setminus \{0\})$ gives $H^*(V)$ a Poisson bracket of cohomological degree $1 - k$.

The holomorphic situation is very analogous: in dimension $k > 1$, we have a commutative algebra with an infinite family of compatible Poisson brackets.

Examples of 2-dimensional topological field theories are given by topological twists of 2-dimensional supersymmetric gauge theories. For example, the B-twist of the 2-dimensional $\mathcal{N} = (2, 2)$ gauge theory gives the theory described by the elliptic Lie algebra

$$(\Omega^*(\mathbb{R}^2, \mathfrak{g}[\varepsilon]), \mathrm{d}_{dR}, [\, ,]),$$

where ε is a square-zero parameter of cohomological degree 1. This is entirely analogous to the fact that the 2-complex dimensional theory arises as a twist of the $\mathcal{N} = 1$ gauge theory; the only difference is that the Dolbeault complex on $\mathbb{C}^2$ has been replaced by the de Rham complex on $\mathbb{R}^2$.

Table 1 Examples of the structure of observables

Space	Q	Structures on the cohomology of observables	Examples/References
$\mathbb{R}$	d_{dR}	Associative product	Topological quantum mechanics [14]
$\mathbb{R}^2$	d_{dR}	Commutative product and Degree-1 bracket	Poisson-Sigma model [3], topologically twisted $\mathcal{N} = (2, 2)$-gauge theory, the B-model
$\mathbb{C}$	$\bar{\partial}$	Holomorphic analog of a vertex algebra	Minimal twists of 2d SUSY field theories, for example [5], [7], $\beta\gamma$ system, [17]
$\mathbb{C}^2$	$\bar{\partial}$	Commutative product, 2 commuting derivations, and family of Poisson brackets of degree 1 parametrized by $f \in \mathbb{C}[z_1, z_2]$.	$\mathcal{N} = 1, 2, 4$ minimally twisted SUSY gauge theories [7, 8]

Further examples arise from topological σ-models, such as topological quantum mechanics [14], the Poisson σ-model [3, 21], and the B-model. (At the perturbative level, the factorization algebra associated to the A-model is uninteresting). Summarizing, we get Table 1.

One can explicitly compute the map μ in (5) for the simplest case.

Example 1 Consider the abelian $\mathcal{N} = 1$ gauge theory, i.e. $\mathfrak{g} = \mathbb{C}$. Then

$$H^*(V_r) = \mathscr{O}(\mathrm{Hol}(B_r) \oplus \mathrm{Hol}(B_r)[-1]),$$

where $\mathscr{O}$ indicates the algebra of formal power series. We will use ϕ, ψ to denote elements of the two copies of $\mathrm{Hol}(B_r)$: ϕ is of degree 0 and ψ is of degree 1. Let α, β be the observables defined by

$$\alpha(\phi, \psi) = \phi(0)$$
$$\beta(\phi, \psi) = \psi(0).$$

Then, one finds that the commutative product does not change, but that the map μ in (5) is given by the formula

$$\mu(\phi, \psi) = z_1^{-1} z_2^{-1} \hbar c$$

for a certain constant c.

3.4 A Deformation of the Theory

Finally, the $\mathcal{N} = 1$ theory has a deformation which is holomorphic in two real dimensions, and topological in the two other real dimensions. This gives a new relation between the $\mathcal{N} = 1$ gauge theory and the Yangian (see [8] for a detailed discussion of this story).

We deform the action functional on the space of fields $\Omega^{0,*}(\mathbb{C}^2, \mathfrak{g}[1]) \oplus \Omega^{2,*}(\mathbb{C}^2, \mathfrak{g}^\vee)$ to

$$S^{new}(\alpha, \beta) = S^{old}(\alpha, \beta) + \int \alpha \, \mathrm{d}z_1 \wedge \partial\alpha,$$

for $\alpha \in \Omega^{0,*}(\mathbb{C}^2, \mathfrak{g}[1])$, $\beta \in \Omega^{2,*}(\mathbb{C}^2, \mathfrak{g}^\vee)$. Note that this is not invariant under $\mathrm{SL}(2, \mathbb{C})$ anymore. The moduli space of solutions to the equations of motion of this deformed theory turns out to be holomorphic G-bundles on $\mathbb{C}^2$ along with a compatible flat holomorphic connection in the z_2 direction.

More generally, such a deformation can be defined on any complex surface with a closed 1-form; in the previous case, we took the complex surface $\mathbb{C}^2$ with the closed 1-form $\mathrm{d}z_1$. As another example, consider the complex surface $\mathbb{C}^* \times \mathbb{C}$ with the closed 1-form $\frac{\mathrm{d}z_1}{z_1}$. The deformed theory on this complex surface can be projected down to $\mathbb{R}_{>0} \times \mathbb{C}$, where we find a theory we could call "Chern-Simons theory for the loop group".

The quantum observables of the deformed theory on the surface $\mathbb{C}^2$ on a formal disc are

$$C^*(\mathfrak{g}[\![z_1]\!])[\![\hbar]\!] = Sym^*(\mathfrak{g}[\![z_1]\!]^\vee)[\![\hbar]\!].$$

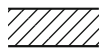
We would like to relate this to the Yangian of the Lie algebra $\mathfrak{g}$, which is a quantization of the Hopf algebra $U(\mathfrak{g}[\![z]\!])$. Here, we have cochains of $\mathfrak{g}[\![z]\!]$ instead of the universal enveloping algebra. These are related by Koszul duality[12]: For a Lie algebra $\mathfrak{h}$,

$$\mathbb{C} \otimes^{\mathbb{L}}_{C^*(\mathfrak{h})} \mathbb{C} = (U(\mathfrak{h}))^\vee.$$

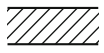
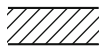
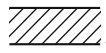
The deformed field theory is topological in the second complex direction, so, by fixing a formal disk in the other coordinate, it gives a locally constant factorization algebra on $\mathbb{R}^2$. Locally constant factorization algebras on $\mathbb{R}^2$ are (by a theorem of Lurie [24]) the same as E_2-algebras. A theorem of Dunn (proved in [24] in the context we need) says that E_2-algebras are the same as E_1-algebras in E_1-algebras. The way to view an E_2-algebra as an E_1-algebra in E_1-algebras is by considering sections of the associated locally constant factorization algebra on an open square (or a strip). Then, the factorization algebra structure gives us two products, namely by including two squares into a third (horizontally) next to each other, or by including them (vertically) above each other. Now we can apply a version of Koszul duality to turn the second E_1-algebra structure of the E_2-algebra into that of a co-E_1-algebra, so we get an E_1-algebra in co-E_1-algebras, i.e. a bialgebra, which, in fact, is a Hopf

[12] There are some delicate issues with this Koszul duality statement: to make it work, we need to treat $C^*(\mathfrak{g})$ as a filtered commutative dga. Details are given in [8].

algebra. This gives us a Hopf algebra deforming $U(\mathfrak{g}[[z]])$: in [8] it is shown that this Hopf algebra is the Yangian.

At least heuristically, the (partial) Koszul duality geometrically amounts to taking only the fields which vanish on the top and bottom of the strip. Thus, we consider observables on $D_{strip} = D_z \times$  where D_z is a (formal) disk with coordinate z. Note that the observables on D_{strip} are $Sym^*(\mathfrak{g}[[z]]^\vee)$. Inclusion of strips vertically and horizontally give us two operations on the (partial) Koszul dual.

1. horizontally: The horizontal inclusion of strips  gives an inclusion map

   $D_z \times$  $\amalg\ D_z \times$  $\longrightarrow D_z \times$ .

   The factorization algebra structure thus gives an associative product on observables. Essentially this product is given by restricting fields from the large strip to the small strips inside, which gives a coalgebra structure on the fields. Then taking the dual when passing to the observables we get the algebra structure.
2. vertically: Now consider the vertical inclusion of strips

   In this case, the condition that the fields vanish on the top and bottom of the strip give us the (additional) coalgebra structure. We cannot restrict fields to the smaller strips because of the boundary condition, but instead, we can extend fields by 0 in between the strips and thus get an algebra structure on fields. Here one has to be careful about taking the dual to get the coalgebra structure on the observables, but this can be done in this case.

Moreover, these operations are compatible and give $Obs^q(D_{strip}) = \mathrm{Sym}(\mathfrak{g}[[z]]^\vee)[[\hbar]]$ the structure of a Hopf algebra.

Theorem 2 *The Hopf algebra $Obs^q(D_{strip})$ obtained in this way is dual to the Yangian Hopf algebra.*

So far we have seen how the operator product in the topological direction is encoded by the Yangian Hopf algebra. In [8] it is shown that the operator product in the holomorphic direction gives rise to a monoidal OPE functor

$$F_{OPE} : \mathrm{Fin}(Y(\mathfrak{g})) \times \mathrm{Fin}(Y(\mathfrak{g})) \to \mathrm{Fin}(Y(\mathfrak{g}))((\lambda)) \qquad (\dagger)$$

which is encoded by the R-matrix of the Yangian. Here, $\mathrm{Fin}(Y(\mathfrak{g}))$ refers to the monoidal category of finite-rank modules over the Yangian (we use the version of the Yangian which quantizes $\mathfrak{g}[[z]]$). This OPE functor should be thought of as a relative of the OPE in the theory of vertex algebras, except that it takes place in the world of monoidal categories rather than that of vector spaces.

As explained in [8, 9] these two results allows one to calculate exactly expectation values of certain Wilson operators in the twisted, deformed $\mathcal{N} = 1$ supersymmetric gauge theory. The answer is expressed in terms of the integrable lattice model constructed from the R-matrix of the Yangian.

3.5 Other Riemann Surfaces

The field theory we are considering can be put on $\Sigma \times \mathbb{R}^2$ where Σ is any Riemann surface equipped with a nowhere-vanishing holomorphic 1-form. This construction will associate an E_2-algebra to any such Riemann surface. In this subsection we will briefly discuss some conjectures about these E_2-algebras and related objects.

If we take our Riemann surface to be $\mathbb{C}^\times$, equipped with the holomorphic volume form $\mathrm{d}z/z$, we find an E_2-algebra to which we can apply the Koszul duality considerations above to produce a Hopf algebra.

Conjecture 1 *The Hopf algebra Koszul dual to the E_2-algebra $Obs^q(\mathbb{C}^\times \times \mathbb{R}^2)$ is dual to the quantum loop algebra $U_\hbar(\mathfrak{g}\{z, z^{-1}\})$.*

Note that there are some subtle issues which we have not addressed which have to do with which completion of $\mathfrak{g}[z, z^{-1}]$ one should use to get precisely the Koszul dual of the E_2-algebra associated to the cylinder $\mathbb{C}^\times$.

Next, let us discuss the case of an elliptic curve E equipped with a holomorphic volume form. Modules for the E_2-algebra associated to an elliptic curve form a monoidal category which deforms the category of sheaves on the formal neighbourhood of the trivial bundle in the stack $\mathrm{Bun}_G(E)$ of G-bundles on E. It is natural to conjecture that this monoidal category should globalize to a monoidal deformation of the category of quasi-coherent sheaves on $\mathrm{Bun}_G(E)$. (Quantizations of categories of sheaves like this are considered in [25], where it is shown that the stack $\mathrm{Bun}_G(E)$ has a 1-shifted symplectic form).

Let us denote this putative quantization by $\mathrm{QC}^q(\mathrm{Bun}_G(E))$. We conjecture that this monoidal category forms part of a kind of categorified two-dimensional field theory, so that there are analogs of familiar objects such as correlation functions.

More precisely, we conjecture the following.

1. For every collection of distinct points $p_1, \ldots, p_n \in E$, there is a monoidal "correlation functor"

$$\mathrm{Fin}(Y(\mathfrak{g})) \times \cdots \times \mathrm{Fin}(Y(\mathfrak{g})) \to \mathrm{QC}^q(\mathrm{Bun}_G(E)).$$

   Here, $\mathrm{Fin}(Y(\mathfrak{g}))$ refers to the monoidal category of finite-rank modules over the Yangian.
   If $M_1, \ldots, M_n$ are modules for the Yangian, we denote by

$$\langle M_1(p_1), \ldots, M_n(p_n) \rangle \in \mathrm{QC}^q(\mathrm{Bun}_G(E))$$

the image of $M_1 \times \cdots \times M_n$ under the correlation functor.

2. These correlation functors should quantize the pull-back map on sheaves associated to the map of stacks

$$\mathrm{Bun}_G(E) \to (BG[\![z]\!])^n$$

obtained by restricting a G-bundle on E to the formal neighbourhood of the points p_i, where each such formal neighbourhood is equipped with its canonical coordinate arising from the 1-form on E.

3. All this data should vary algebraically with the positions of the points p_i as well as over the moduli of elliptic curves equipped with a non-zero 1-form.
4. The correlation functors should have a compatibility with the OPE-functor (†) in the same way that ordinary correlation functions of a conformal field theory are compatible with the OPE. For example, if p, $p + \lambda$ are two points in E where λ is a formal parameter, and M, N are two modules for the Yangian, we expect that there is a monoidal natural isomorphism

$$\langle M(p), N(p + \lambda)\rangle \cong \langle F_{OPE}(M, N)(p)\rangle \in \mathrm{QC}^q(\mathrm{Bun}_G(E))(\!(\lambda)\!).$$

(In the last line, by $\mathrm{QC}^q(\mathrm{Bun}_G(E))(\!(\lambda)\!)$ we mean an appropriate category of $\mathbb{C}(\!(\lambda)\!)$-modules in $\mathrm{QC}^q(\mathrm{Bun}_G(E))$).

Note that if we replace $\mathrm{Bun}_G(E)$ by its formal completion near the trivial bundle, all of this follows from the results of [8]. Globalizing is the challenge.

It is natural to speculate that there is some relationship between the desired quantization of $\mathrm{Bun}_G(E)$ and elliptic quantum groups, but this is currently unclear.

In a similar way, for a general surface Σ equipped with a nowhere-vanishing holomorphic 1-form, one can also speculate the E_2-algebra of observables of our theory on Σ times a disc is related to the quasi-Hopf algebras constructed by Enriquez and Rubtzov [12].

Kapustin [19] has shown that any four-dimensional $\mathcal{N} = 2$ theory admits such a twist. We hope that there is a similarly rich, and largely unexplored, mathematical story describing such theories.

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Appendix

A. Moduli Problems and Field Theories

Throughout this text, field theories are described in terms of elliptic moduli problems which in turn are encoded as elliptic L_∞-algebras. These terms and their relations are constantly used. However, we only defined SUSY field theories in an informal way, so we will give some ideas and definitions here. For the full definitions and detailed explanations, see [7].

Let M be a manifold. The ideal definition of a classical field theory would be to say that a classical field theory on M is a sheaf of derived stacks (of critical loci, the derived spaces of solutions to the equations of motion) on M equipped with a Poisson bracket of degree one (coming from the BV formalism). To simplify things, we make two observations.

1. If X is a derived stack and $x \in X$, then $T_x X[-1]$ has an L_∞ structure, and this completely describes the formal neighborhood of x [18, 22, 23, 29]. Thus, near a given section, a sheaf of derived stacks can be described by a sheaf of L_∞-algebras.
2. If X is a derived stack which is n-symplectic in the sense of [25] , then $T_x X$ has an anti-symmetric pairing (of degree n), so $T_x X[-1]$ has a symmetric pairing (of degree $n-2$). One can show that the L_∞-structure on $T_x X[-1]$ can be chosen so that the pairing is invariant. More precisely, one can prove a formal Darboux theorem showing that formal symplectic derived stacks are the same as L_∞-algebras with an invariant pairing.

From these observations it makes sense to define a perturbative classical field theory (perturbing around a given solution to the equations of motion) to be a sheaf of L_∞-algebras with some sort of an invariant pairing, which we will define below. Moreover, we are interested in the situation where the equations of motion (or equivalently our moduli problem) are described by a system of elliptic partial differential equations, which lead to the following notion.

Definition 6 An *elliptic L_∞-algebra* $\mathcal{L}$ on M consists of

- a graded vector bundle L on M, whose space of sections in $\mathcal{L}$,
- a differential operator $\mathrm{d} : \mathcal{L} \to \mathcal{L}$ of cohomological degree 1 and square 0, which makes $\mathcal{L}$ into an elliptic complex,
- a collection of polydifferential operators $l_n : \mathcal{L}^{\otimes n} \to \mathcal{L}$ which are alternating, of cohomological degree $2-n$, and which give $\mathcal{L}$ the structure of an L_∞-algebra.

An *invariant pairing of degree k* on an elliptic L_∞-algebra $\mathcal{L}$ is an isomorphism of $\mathcal{L}$-modules

$$\mathcal{L} \cong \mathcal{L}^![-k],$$

which is symmetric, where $\mathcal{L}^!(U) = \Gamma(U, L^\vee \otimes Dens_M)$.

Remark 8 Note that the sheaf $\mathcal{L}^!$ is homotopy equivalent to the continuous Verdier dual, which assigns to U the linear dual of $\mathcal{L}_c(U)$.

Such an invariant pairing yields an invariant pairing on the space $\mathcal{L}_c(U)$ for every open U in M. The fact that the pairing on $\mathcal{L}_c(U)$ is invariant follows from the fact that the map $\mathcal{L} \to \mathcal{L}^![-k]$ is an isomorphism of $\mathcal{L}$-modules.

From deformation theory, we know that there is an equivalence of $(\infty, 1)$-categories between the category of differential graded Lie algebras and the category of formal pointed derived moduli problems (see [18, 22, 23]). Here pointed means that we are deforming a given solution to the equations of motion. Thus, the following definitions make sense.

Definition 7 A *formal pointed elliptic moduli problem with a symplectic form of cohomological degree k on M* is an elliptic L_∞-algebra on M with an invariant pairing of cohomological degree $k-2$.

Definition 8 A *perturbative classical field theory* on M is a formal pointed elliptic moduli problem on M with a symplectic form of cohomological degree -1. The *space of fields* $\mathcal{E}$ of a classical field theory arises as a shift of the L_∞-algebra encoding the theory, $\mathcal{E} = \mathcal{L}[1]$.

The field theories we consider in this text all arise as cotangent theories.

Definition 9 Let $\mathcal{L}$ be an elliptic L_∞-algebra on M corresponding to a sheaf of formal moduli problems $\mathcal{M}_\mathcal{L}$ on M. Then the *cotangent field theory* associated to $\mathcal{L}$ is the classical field theory $\mathcal{L} \oplus \mathcal{L}^![-3]$ (with its obvious pairing). Its moduli problem is denoted by $T^*[-1]\mathcal{M}_\mathcal{L}$.

B. Supersymmetry

In supersymmetry, we have two gradings: one by $\mathbb{Z}/2\mathbb{Z}$ (= *fermionic grading*), and one by $\mathbb{Z}$ (= *cohomological grading, "ghost number"*). So one extends the definitions from Appendix A to this bi-graded (= super) setting.

In this super-setting, we want all algebraic structures to preserve the fermion degree and have the same cohomological degree as in the ordinary setting. Thus, the differential of a *super cochain complex* is of degree (0, 1) and the structure maps of a *super L_∞-algebra L*, $l_n : L^{\otimes n} \to L$, are of bi-degree $(0, 2-n)$, satisfying the same relations as in the ordinary case. The other notions from Appendix A carry over similarly.

Definition 10 A *perturbative classical field theory with fermions* on M is a super elliptic L_∞-algebra $\mathcal{L}$ on M with an invariant pairing of bi-degree $(0, -3)$, i.e. of cohomological degree -3 and fermionic degree 0.

Definition 11 A *formal pointed super elliptic moduli problem with a symplectic form of cohomological degree k on M* is a super elliptic L_∞-algebra on M with an invariant pairing of bi-degree $(0, k-2)$.

Now we can encode supersymmetry.

Definition 12 A *field theory on* $\mathbb{R}^4$ *with* $\mathcal{N} = k$ *supersymmetries* is a $\mathrm{Spin}(4) \ltimes \mathbb{R}^4$-invariant super elliptic moduli problem $\mathcal{M}$ defined over $\mathbb{C}$ with a symplectic form of cohomological degree -1; together with an extension of the action of the complexified Euclidean Lie algebra $\mathfrak{so}(4, \mathbb{C}) \ltimes V_{\mathbb{C}}$ to an action of the complexified super-Euclidean Lie algebra $\mathfrak{so}(4, \mathbb{C}) \ltimes T^{\mathcal{N}=k}$.

Given any complex Lie subgroup $G \subseteq \mathrm{GL}(k, \mathbb{C})$, we say that such a supersymmetric field theory has *R-symmetry group G* if the group G acts on the theory in a way covering the trivial action on space-time $\mathbb{R}^4$, and compatible with the action of G on $T^{\mathcal{N}=k}$.

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Snapshots of Conformal Field Theory

Katrin Wendland

Abstract In snapshots, this exposition introduces conformal field theory, with a focus on those perspectives that are relevant for interpreting superconformal field theory by Calabi-Yau geometry. It includes a detailed discussion of the elliptic genus as an invariant which certain superconformal field theories share with the Calabi-Yau manifolds. K3 theories are (re)viewed as prime examples of superconformal field theories where geometric interpretations are known. A final snapshot addresses the K3-related Mathieu Moonshine phenomena, where a lead role is predicted for the chiral de Rham complex.

1 Introduction

Conformal quantum field theory (CFT) became popular in physics thanks to the work by Belavin, Polyakov and Zamolodchikov. In their seminal paper [7], on the one hand, they lay the mathematical foundations of axiomatic CFT, and on the other hand, they show the physical significance of CFT for surface phenomena in statistical physics by describing certain phase transitions of second order through CFT.

Another common source of conformal field theories is string theory, which is many theoreticians' favorite candidate for the unification of all interactions, including gravity. Here, particles are described by strings that move in some potentially complicated background geometry. The string dynamics are governed by a so-called non-linear sigma model, such that conformal invariance yields the string equations of motion. The quantum field theory living on the worldsheet of the string then is a CFT. This implies deep relations between CFT and geometry, which have already led to a number of intriguing insights in geometry, demanding for a more resilient bridge between mathematics and physics.

For example, in the early 90s mirror symmetry provided a first success story for the interaction between mathematics and physics in the context of CFT

K. Wendland (✉)
Mathematics Institute, Freiburg University,
Eckerstr. 1, 79104 Freiburg im Breisgau, Germany
e-mail: katrin.wendland@math.uni-freiburg.de

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[14, 15, 57, 68]. However, a rigorous approach to those types of CFTs that are relevant for such deep insights in algebraic geometry was not available, at the time. As a result, the interaction between mathematics and physics in many cases amounted to a rather imbalanced division of work, where theoretical physicists provided the most amazing predictions and left them to the mathematicians for a proof, who in turn successfully detached their theories from their origins in physics.

With the advent of Monstrous Moonshine [9, 20, 42, 49, 88, 89], and with Borcherds' Fields Medal in 1998 "for his contributions to algebra, the theory of automorphic forms, and mathematical physics, including the introduction of vertex algebras and Borcherds' Lie algebras, the proof of the Conway-Norton moonshine conjecture and the discovery of a new class of automorphic infinite products" [71], the subject of conformal field theory, per se, began to become more popular in mathematics. Indeed, the comparatively new notion of vertex algebras provided a rigorous mathematical foundation to the most basic ingredients of conformal quantum field theory and thereby offered a viable approach to CFT for mathematicians. Nevertheless, the quest to fill the gap between abstract mathematical approaches to CFT and those types of models that are of interest in physics, and that are relevant for deeper insights in algebraic and enumerative geometry, has not yet been completed. The present work attempts to make a contribution to this quest.

Since this exposition can certainly only provide some snapshots of CFT, it has to follow a subjective selection and presentation of material. The guiding principle is the conviction that on the one hand, the foundation of the discussion has to be a mathematically rigorous definition of CFT, which is independent of string theory, while on the other hand, those predictions from CFT which affect the geometry of Calabi-Yau manifolds are among the most intriguing ones. To state and understand the latter, one needs to work with a mathematical formulation of CFT which allows to make contact with the non-linear sigma models in physics, thus sadly excluding a number of popular approaches to CFT. Moreover, the discussion is restricted to so-called two-dimensional Euclidean unitary CFTs.

In more detail, this work is structured as follows:

Section 2 provides a definition of some of the ingredients of CFT. The conformal vertex algebras serve as our point of entry in Sect. 2.1, since this part of CFT is probably the most natural for mathematicians. We proceed in Sect. 2.2 by listing the crucial ingredients that underlie a definition of superconformal field theory, along with additional required properties. The presentation makes no claim for completeness, but according to our declared conviction, we focus on those aspects that are relevant for the discussion of geometric interpretations as introduced later. This in particular restricts our attention to the so-called $N = (2, 2)$ superconformal field theories with space-time supersymmetry. A useful class of examples, which is well understood, is given by the toroidal $N = (2, 2)$ superconformal field theories presented in Sect. 2.3. We summarize the definition and properties of the chiral de Rham complex in Sect. 2.5, as an example of a sheaf of conformal vertex algebras on an arbitrary smooth algebraic variety, which thus provides a link between standard ingredients of CFT and geometric quantities. Since this link is not entirely understood to the very day, for clarification, our discussion rests on the special role of the elliptic genus as an invariant that certain superconformal field theories share with the Calabi-Yau manifolds, as is discussed in some detail in Sect. 2.4.

The elliptic genus is also crucial for our definition of K3 theories in Sect. 3. This class of CFTs deserves some attention, as it provides the only examples of non-linear sigma models on Calabi-Yau manifolds other than tori, where at least there are precise predictions on the global form of the moduli space, implying some very explicit relations between quantities in geometry and CFT. We motivate the definition of K3 theories in detail, and we summarize some of the known properties of these theories. In particular, Proposition 2 recalls the dichotomy of $N = (2, 2)$ superconformal field theories at central charges $c = 6$, $\overline{c} = 6$ with space-time supersymmetry and integral $U(1)$-charges. Indeed, these theories fall into two classes, namely the toroidal and the K3 theories. Thus Proposition 2 is the conformal field theoretic counter part of the classification of Calabi-Yau 2-manifolds into complex two-tori, on the one hand, and K3 surfaces, on the other. Our proof [92, Sect. 7.1], which is little known, is summarized in the Appendix.

The final Sect. 4 is devoted to recent developments in the study of K3 theories, related to the mysterious phenomena known as Mathieu Moonshine. We recall the route to discovery of these phenomena, which also proceeds via the elliptic genus. We offer some ideas towards a geometric interpretation, arguing that one should expect the chiral de Rham complex to be crucial in unraveling the Mathieu Moonshine mysteries. The section closes with an open conjecture, which is related to Mathieu Moonshine, which however is formulated neither alluding to moonshine nor to CFT, and which therefore is hoped to be of independent interest.

2 Ingredients of Conformal Field Theory

The present section collects ingredients of conformal field theory (CFT), more precisely of *two-dimensional Euclidean unitary conformal field theory*. These adjectives translate into the properties of the underlying quantum field theory as follows: First, all fields are parametrized on a *two-dimensional worldsheet*, which comes equipped with a *Euclidean metric*. Second, the fields transform covariantly under *conformal maps* between such worldsheets. Furthermore, the space of states in such a CFT is equipped with a positive definite metric, with respect to which the infinitesimal conformal transformations act *unitarily*.

We begin by describing the simplest fields in our CFTs in terms of the so-called *vertex algebras* in Sect. 2.1. Next, Sect. 2.2 summarizes a definition of conformal field theory, with the toroidal conformal field theories presented as a class of examples in Sect. 2.3. In Sects. 2.4 and 2.5 the related notions of the elliptic genus and the chiral de Rham complex are discussed in the context of superconformal field theories. As such, the present section collects ingredients of CFT, with a focus on those ingredients that are under investigation to the very day.

2.1 Conformal and Superconformal Vertex Algebras

We begin by recalling the notion of *fields*, following [61]. The theory is built on the earlier results [9, 41, 67], see also [38] for a very readable exposition. This definition is most convenient, because it naturally implements the representation theory inherent to CFTs. As we shall see at the end of this section, for the chiral states of CFTs it also allows a straightforward definition of the n-point functions.

Definition 1 Consider a $\mathbb{C}$-vector space $\mathbb{H}$.

- $\mathbb{H}[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]]$ denotes the vector space of formal power series

$$v(z_1, \ldots, z_n) = \sum_{i_1,\ldots,i_n \in \mathbb{Z}} \widehat{v}_{i_1,\ldots,i_n} z_1^{i_1} \cdots z_n^{i_n}, \quad \widehat{v}_{i_1,\ldots,i_n} \in \mathbb{H}.$$

- For $A \in \mathrm{End}_{\mathbb{C}}(\mathbb{H})[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]]$, and for $\alpha \in \mathbb{H}^* := \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{C})$ and $v \in \mathbb{H}$, we set

$$\langle \alpha, A(z_1, \ldots, z_n) v \rangle := \sum_{i_1,\ldots,i_n \in \mathbb{Z}} \langle \alpha, \widehat{A}_{i_1,\ldots,i_n} v \rangle z_1^{i_1} \cdots z_n^{i_n} \quad \in \mathbb{C}[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]],$$

  where on the right hand side, $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathbb{H}^*$ and $\mathbb{H}$.
- If $A(z) \in \mathrm{End}_{\mathbb{C}}(\mathbb{H})[[z^{\pm 1}]]$ with $A(z) = \sum_n \widehat{A}_n z^n$, then ∂A denotes the formal derivative of A,

$$\partial A(z) = \sum_{n \in \mathbb{Z}} n \widehat{A}_n z^{n-1} \quad \in \mathrm{End}_{\mathbb{C}}(\mathbb{H})[[z^{\pm 1}]].$$

- A formal power series $A(z) \in \mathrm{End}_{\mathbb{C}}(\mathbb{H})[[z^{\pm 1}]]$ is called a FIELD on $\mathbb{H}$ if $A(z) = \sum_n \widehat{A}_n z^n$ obeys

$$\forall v \in \mathbb{H} : \quad \exists N \in \mathbb{Z} \text{ such that } \widehat{A}_n v = 0 \;\; \forall n < N.$$

  The endomorphisms $\widehat{A}_n$ are called the MODES of the field A.

In other words, if A is a field on $\mathbb{H}$, then for every $v \in \mathbb{H}$ the expression $A(z)v = \sum_n (\widehat{A}_n v) z^n$ is a formal Laurent series with coefficients in $\mathbb{H}$ and with only finitely many non-zero contributions $(\widehat{A}_n v) z^n$ with $n < 0$. In the context of CFTs one can introduce a completion $\overline{\mathbb{H}}$ of $\mathbb{H}$ with respect to an appropriate topology and then for every $z \in \mathbb{C}^*$ view $A(z)$ as a linear operator from $\mathbb{H}$ to $\overline{\mathbb{H}}$, see for example [38, Sect. 1.2.1]. Accordingly, we call a field $A(z) = \sum_n \widehat{A}_n z^n$ *constant* if $\widehat{A}_n = 0$ for all $n \neq 0$. Similarly, if for $v \in \mathbb{H}$ we have $\widehat{A}_n v = 0$ for all $n < 0$, then we say that $A(z)v$ *is well-defined in* $z = 0$, and $A(z)v_{|z=0} = \widehat{A}_0 v$. Note that a field A according to Definition 1 can be viewed as an operator valued distribution, as usual in quantum field theory. Indeed, by means of the residue, $A(z)$ yields a linear map from complex polynomials into $\mathbb{H}$. By definition, the space $\mathbb{H}$ carries a representation of the Lie algebra generated by the modes of every field on $\mathbb{H}$, with the Lie bracket that is inherited from $\mathrm{End}_{\mathbb{C}}(\mathbb{H})$, namely the commutator.

Let us now consider two fields A, B on $\mathbb{H}$. While the expressions $A(z)B(w)$ and $B(w)A(z)$ make sense as formal power series in $\mathrm{End}_{\mathbb{C}}(\mathbb{H})[[z^{\pm 1}, w^{\pm 1}]]$, a priori it is impossible to interpret them as fields. In general, we expect singular behavior for the coefficients when we insert $w = z$, and in fact the form of this singularity captures the most important aspects of CFT. Here, the notions of *locality* and *normal ordered products* come to aid:

Definition 2

1. Let ∂_w denote the formal derivative with respect to w in $\mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$. On $\mathbb{C}[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$, we define the $\mathbb{C}[[z, w]][z^{-1}, w^{-1}]$-linear operators $\iota_{z>w}$ and $\iota_{w>z}$ into $\mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$ with

$$\text{for } k \in \mathbb{N}: \quad \iota_{z>w}\left(k!(z-w)^{-k-1}\right) = \partial_w^k \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n,$$

$$\iota_{w>z}\left(k!(z-w)^{-k-1}\right) = -\partial_w^k \frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n.$$

2. Fields A, B on $\mathbb{H}$ are called LOCAL WITH RESPECT TO EACH OTHER if there exist a so-called NORMAL ORDERED PRODUCT $:A(z)B(w): \in \mathrm{End}_{\mathbb{C}}(\mathbb{H})[[z^{\pm 1}, w^{\pm 1}]]$ and fields $X_0, \ldots, X_{N-1}$ and $:AB:$ on $\mathbb{H}$, such that for every $\alpha \in \mathbb{H}^*$ and $v \in \mathbb{H}$,

   - we have $\langle \alpha, :A(z)B(w): v\rangle \in \mathbb{C}[[z, w]][z^{-1}, w^{-1}]$,
   - in $\mathrm{End}_{\mathbb{C}}(\mathbb{H})[[z^{\pm 1}]]$, we have $:AB:(z) = :A(z)B(w):_{|w=z}$,
   - in $\mathbb{C}[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$, we have

$$\iota_{z>w}^{-1}\left(\langle \alpha, A(z)B(w)v\rangle\right) = \sum_{j=0}^{N-1} \frac{\langle \alpha, X_j(w)v\rangle}{(z-w)^{j+1}} + \langle \alpha, :A(z)B(w): v\rangle$$
$$= \iota_{w>z}^{-1}\left(\langle \alpha, B(w)A(z)v\rangle\right).$$

   As a shorthand notation one writes the so-called OPERATOR PRODUCT EXPANSION (OPE)

$$A(z)B(w) \sim \sum_{j=0}^{N-1} \frac{X_j(w)}{(z-w)^{j+1}}.$$

For the special fields that feature in CFTs, the formal power series in the above definition yield convergent functions in complex variables z and w on appropriate domains in $\mathbb{C}$. Then the operators $\iota_{z>w}$ and $\iota_{w>z}$ implement the Taylor expansions about $z = w$ in the domains $|z| > |w|$ and $|w| > |z|$, respectively. We therefore refer to these operators as *(formal) Taylor expansions*. The OPE thus captures the singular behavior of the expressions $\langle \alpha, A(z)B(w)v\rangle$ when $z \sim w$, where locality of the fields A and B with respect to each other restricts the possible singularities to poles at $z = w$. For $[A(z), B(w)] := A(z)B(w) - B(w)A(z) \in \mathrm{End}_{\mathbb{C}}(\mathbb{H})[[z^{\pm 1}, w^{\pm 1}]]$, the observation that, in general, $\langle \alpha, [A(z), B(w)]v\rangle$ does not vanish, accounts for the fact that $(z - w)^{-1}$ and its derivatives have different Taylor expansions in the domains $|z| > |w|$ and $|w| > |z|$, respectively. Hence the modes of the fields X_j in the OPE encode the commutators $[\widehat{A}_n, \widehat{B}_m]$ of the modes of A and B.

The Definition 2 of the normal ordered product $:A(z)B(w):$ of two fields $A,\ B$ on $\mathbb{H}$ yields $:A(z)B(w):= A_+(z)B(w) + B(w)A_-(z)$ if $A(z) = A_+(z) + A_-(z)$, where $A_+(z) := \sum_{n\geq 0} \widehat{A}_n z^n$ and $A_-(z) := \sum_{n<0} \widehat{A}_n z^n$. Hence our definition of normal ordered product amounts to a choice in decomposing $A(z) = A_+(z) + A_-(z)$ as stated, which accrues from the choice of decomposing the formal power series

$$\sum_{m=-\infty}^{\infty} z^m w^{-m-1} = \iota_{z>w}\left((z-w)^{-1}\right) - \iota_{w>z}\left((z-w)^{-1}\right) \quad \in \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]].$$

In the context of superconformal field theories, these notions are generalized to include *odd* fields; if both A and B are odd, then locality amounts to

$$\begin{aligned} \iota_{z>w}^{-1}\left(\langle \alpha, A(z)B(w)v\rangle\right) &= \sum_{j=0}^{N-1} \frac{\langle \alpha, X_j(w)v\rangle}{(z-w)^{j+1}} + \langle \alpha, :A(z)B(w):\, v\rangle \\ &= -\iota_{w>z}^{-1}\left(\langle \alpha, B(w)A(z)v\rangle\right), \end{aligned}$$

abbreviated by the same OPE as in Definition 2, and the bracket $[\cdot, \cdot]$ in the above argument is replaced by a superbracket with $[A(z), B(w)] = A(z)B(w) + B(w)A(z)$ for odd fields $A,\ B$. The space $\mathbb{H}$, accordingly, furnishes a representation of the super-Lie algebra generated by the modes of the fields on $\mathbb{H}$.
The following list of examples provides some basic fields in the simplest CFTs:

Example 1 (*U(1)-current*)
We consider the complex Lie algebra $\mathcal{A}$ with $\mathbb{C}$-vector space basis $\{C;\ a_n,\ n \in \mathbb{Z}\}$, where C is a central element and the Lie bracket obeys

$$\forall m,\ n \in \mathbb{Z}: \quad [a_n, a_m] = m\delta_{n+m,0} \cdot \frac{C}{3}.$$

Choose some $c \in \mathbb{R}$ and let $\mathbb{H}$ denote the $\mathcal{A}$-module which under the $\mathcal{A}$-action is generated by a single non-zero vector Ω, with submodule of relations generated by

$$a_n\Omega = 0\ \forall n \leq 0, \quad C\Omega = c\Omega.$$

The space $\mathbb{H}$ can be viewed as polynomial ring in the a_n with $n > 0$. One then checks that the so-called $U(1)$-*current*

$$J(z) := \sum_{n=-\infty}^{\infty} a_n z^{n-1}$$

is a well-defined field on $\mathbb{H}$ which obeys the OPE

$$J(z)J(w) \sim \frac{c/3}{(z-w)^2}.$$

In particular, J is local with respect to itself. Here and in the following, a constant field which acts by multiplication by $\Lambda \in \mathbb{C}$ on $\mathbb{H}$ is simply denoted by Λ.

Example 2 (*Virasoro field*)
For the $U(1)$-current J on the vector space $\mathbb{H}$ introduced in the previous example, assume $c \neq 0$ and let $T(z) := \frac{3}{2c} :JJ:(z)$. One checks that with $c^\bullet = 1$ this field on $\mathbb{H}$ obeys the OPE

$$T(z)T(w) \sim \frac{c^\bullet/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{1}$$

which for the modes of $T(z) = \sum_n L_n z^{n-2}$ translates into

$$\forall n,\, m \in \mathbb{Z}: \quad [L_n, L_m] = (m-n)L_{m+n} + \delta_{n+m,0}\, \tfrac{c^\bullet}{12}\, m(m^2-1). \tag{2}$$

The above Eq. (2) defines the *Virasoro algebra* at central charge $c^\bullet$, whose underlying vector space has $\mathbb{C}$-vector space basis $\{c^\bullet;\ L_n,\ n \in \mathbb{Z}\}$. This Lie algebra is the central extension by $\mathrm{span}_{\mathbb{C}}\{c^\bullet\}$ of the Lie algebra of infinitesimal conformal transformations of the punctured Euclidean plane $\mathbb{C}^*$.

Example 3 (*$bc-\beta\gamma$-system*)
Let $D \in \mathbb{N}$, and consider the super-Lie algebra $\mathcal{A}_D$ with $\mathbb{C}$-vector space basis $\{C;\ a_n^i,\ b_n^i,\ \varphi_n^i,\ \psi_n^i,\ n \in \mathbb{Z},\ i \in \{1,\dots,D\}\big\}$, where the $a_n^i,\ b_n^i$ and the central element C are even, while the $\varphi_n^i,\ \psi_n^i$ are odd, and the only non-vanishing basic super-Lie brackets are

$$\begin{aligned}\forall m,\, n \in \mathbb{Z},\ i,\ j \in \{1,\dots,D\}: \quad [a_n^i, b_m^j] &= \delta^{i,j}\delta_{n+m,0}\cdot C,\\ \{\psi_n^i, \varphi_m^j\} &= \delta^{i,j}\delta_{n+m,0}\cdot C.\end{aligned} \tag{3}$$

Here, $\{\cdot,\cdot\}$ denotes the super-Lie bracket between odd elements of $\mathcal{A}_D$, as is customary in the physics literature. Let $\mathbb{H}$ denote the $\mathcal{A}_D$-module which under the $\mathcal{A}_D$-action is generated by a single non-zero vector Ω, with submodule of relations generated by

$$\begin{aligned}\forall n \leq 0,\ m < 0,\ i,\ j \in \{1,\dots,D\}: \quad a_n^i\Omega &= 0, \quad b_m^j\Omega = 0,\\ \psi_n^i\Omega &= 0, \quad \varphi_m^j\Omega = 0; \qquad C\Omega = \Omega.\end{aligned}$$

Generalizing Examples 1 and 2 above, one checks that

$$\begin{aligned} a^i(z) &:= \sum_{n=-\infty}^{\infty} a^i_n z^{n-1}, \quad b^i(z) := \sum_{m=-\infty}^{\infty} b^i_m z^m, \\ \psi^i(z) &:= \sum_{n=-\infty}^{\infty} \psi^i_n z^{n-1}, \quad \varphi^i(z) := \sum_{m=-\infty}^{\infty} \varphi^i_m z^m, \quad i \in \{1, \dots, D\} \end{aligned}$$

defines pairwise local fields $a^i, b^i, \psi^i, \varphi^i$ on $\mathbb{H}$. Moreover, one finds the OPEs

$$a^i(z)b^j(w) \sim \frac{\delta^{i,j}}{z-w}, \quad \varphi^i(z)\psi^j(w) \sim \frac{\delta^{i,j}}{z-w} \quad \forall\, i,\ j \in \{1, \dots, D\},$$

while all other basic OPEs vanish, and the field

$$T^{\text{top}}(z) := \sum_{j=1}^{D} \Big(:\partial b^j a^j:(z) + :\partial\varphi^j \psi^j:(z) \Big) \tag{4}$$

is a Virasoro field obeying (1) at central charge $c^\bullet = 0$.

Example 4 (*Topological $N = 2$ superconformal algebra*)
With $\mathcal{A}_D$, $\mathbb{H}$, and the fields of the $bc - \beta\gamma$-system defined in the above Example 3, let

$$J(z) := \sum_{j=1}^{D} :\varphi^j \psi^j:(z), \quad Q(z) := \sum_{j=1}^{D} :a^j \varphi^j:(z), \quad G(z) := \sum_{j=1}^{D} :\psi^j \partial b^j:(z). \tag{5}$$

These fields obey the so-called *topological $N = 2$ superconformal algebra* at central charge $c = 3D$:

$$T^{\text{top}}(z)T^{\text{top}}(w) \sim \frac{2T^{\text{top}}(w)}{(z-w)^2} + \frac{\partial T^{\text{top}}(w)}{z-w}, \tag{6}$$

$$\begin{aligned} T^{\text{top}}(z)J(w) &\sim -\frac{c/3}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}, \quad J(z)J(w) \sim \frac{c/3}{(z-w)^2}, \\ T^{\text{top}}(z)Q(w) &\sim \frac{Q(w)}{(z-w)^2} + \frac{\partial Q(w)}{z-w}, \quad Q(z)Q(w) \sim 0, \quad J(z)Q(w) \sim \frac{Q(w)}{z-w}, \\ T^{\text{top}}(z)G(w) &\sim \frac{2G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w}, \quad G(z)G(w) \sim 0, \quad J(z)G(w) \sim -\frac{G(w)}{z-w}, \\ Q(z)G(w) &\sim \frac{c/3}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T^{\text{top}}(w)}{z-w}. \end{aligned} \tag{7}$$

Example 5 ($N = 2$ *superconformal algebra*)
Consider a $\mathbb{C}$-vector space $\mathbb{H}$ and pairwise local fields $T^{\text{top}}(z),\ J(z),\ Q(z),\ G(z)$ on $\mathbb{H}$ which obey the topological $N = 2$ superconformal algebra (6)–(7) at central charge c. Now let

$$T(z) := T^{\text{top}}(z) - \frac{1}{2}\partial J(z), \quad G^+(z) := Q(z), \quad G^-(z) := G(z). \tag{8}$$

Then $T(z)$ is another Virasoro field as in (1), but now with central charge $c^\bullet = c$, and the fields $T(z),\ J(z),\ G^+(z),\ G^-(z)$ on $\mathbb{H}$ obey the so-called $N = 2$ *superconformal algebra* at central charge c,

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{9}$$

$$\begin{aligned}
T(z)J(w) &\sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}, \quad J(z)J(w) \sim \frac{c/3}{(z-w)^2},\\
T(z)G^\pm(w) &\sim \frac{3/2G^\pm(w)}{(z-w)^2} + \frac{\partial G^\pm(w)}{z-w}, \quad J(z)G^\pm(w) \sim \pm\frac{G^\pm(w)}{z-w},\\
G^\pm(z)G^\mp(w) &\sim \frac{c/3}{(z-w)^3} \pm \frac{J(w)}{(z-w)^2} + \frac{T(w) \pm \frac{1}{2}\partial J(w)}{z-w}, \; G^\pm(z)G^\pm(w) \sim 0.
\end{aligned} \tag{10}$$

Equation (8) is referred to by the statement that the fields $T^{\text{top}}(z),\ J(z),\ Q(z),\ G(z)$ are obtained from the fields $T(z),\ J(z),\ G^+(z),\ G^-(z)$ by a *topological A-twist*. Analogously, fields $T^{\text{top}}(z),\ -J(z),\ Q(z),\ G(z)$ which obey a topological $N = 2$ superconformal algebra at central charge c are obtained from fields $T(z),\ J(z),\ G^+(z),\ G^-(z)$ which obey an $N = 2$ superconformal algebra at central charge c by a *topological B-twist* iff $T^{\text{top}}(z) = T(z) - \frac{1}{2}\partial J(z),\ Q(z) = G^-(z),\ G = G^+(z)$, see [35, 98]. On the level of the $N = 2$ superconformal algebras, the transition between topological A-twist and topological B-twist is induced by $(T,\ J,\ G^+,\ G^-) \mapsto (T,\ -J,\ G^-,\ G^+)$, an automorphism of the superconformal algebra. This automorphism is at the heart of *mirror symmetry* [68].

We are now ready to define one of the fundamental ingredients of CFT, namely the notion of *conformal vertex algebra*. The definition is taken from [38] and follows [9, 39, 61]:

Definition 3 A CONFORMAL VERTEX ALGEBRA AT CENTRAL CHARGE $c \in \mathbb{C}$ is given by the following data:

- A $\mathbb{Z}$-graded $\mathbb{C}$-vector space $W = \oplus_{m\in\mathbb{Z}} W_m$ called the SPACE OF STATES.
- A special vector $\Omega \in W_0$ called the VACUUM.
- A linear operator $L\colon W \to W$ called the TRANSLATION OPERATOR.
- A special vector $T \in W_2$ called the CONFORMAL VECTOR.

- A linear map
$$Y(\cdot, z)\colon W \longrightarrow \mathrm{End}(W)[[z^{\pm 1}]],$$
called the STATE- FIELD CORRESPONDENCE, which assigns to every $A \in W$ a field $A(z) := Y(A, z)$ on W.

These data obey the following axioms:

- The VACUUM AXIOM: We have $\Omega(z) = 1$, and for every $A \in W$ and $A(z) = \sum_n \widehat{A}_n z^n$, we obtain $A(z)\Omega \in W[[z]]$, such that $A(z)\Omega$ is well-defined in $z = 0$ and
$$A(z)\Omega_{|z=0} = \widehat{A}_0\Omega = A \in W.$$
One says: The field $A(z)$ CREATES the state A from the vacuum.
- The TRANSLATION AXIOM:
$$L\Omega = 0 \quad \text{and} \quad \forall A \in W\colon \quad [L, A(z)] = \partial A(z).$$
- The LOCALITY AXIOM:
All fields $A(z)$ with $A \in W$ are local with respect to each other.

The (ungraded) vector space W with Ω, L, and the map Y is called a VERTEX ALGEBRA. In a CONFORMAL vertex algebra, in addition

- The field $T(z) = \sum_{n=-\infty}^{\infty} L_n z^{n-2}$ associated to the conformal vector T by the state-field correspondence is a Virasoro field obeying the OPE (1) with central charge $c^\bullet = c$.
- The translation operator L is given by $L = L_1$ and has degree 1.
- For all $m \in \mathbb{Z}$, $L_{0|W_m} = m$, and for $A \in W_m$, the field $A(z)$ has weight m, i.e. $A(z) = \sum_{n=-\infty}^{\infty} A_n z^{n-m}$ with $A_n \in \mathrm{End}(W)$ of degree n.

In the context of superconformal field theories, the notion of conformal vertex algebras of Definition 3 is generalized to superconformal vertex algebras. For an $N = 2$ superconformal vertex algebra, the vector space W in the above Definition is graded by $\frac{1}{2}\mathbb{Z}$ instead of $\mathbb{Z}$, one needs to allow odd fields $A(z) = Y(A, z)$, which can have mode expansions in $z^{1/2}\cdot\mathrm{End}(W)[[z^{\pm 1}]]$, and one needs to generalize the notion of locality to such fields, as explained in the discussion of Definition 2. Finally, one needs to assume that there exist special states $J \in W_1$ and $G^\pm \in W_{3/2}$ such that the associated fields $J(z)$, $G^\pm(z)$ obey the $N = 2$ superconformal algebra (9)–(10).

An important ingredient of CFT are the so-called n-point functions, which associate a function in n complex variables to every n-tuple of states in the CFT. These n-point functions are naturally related to the notion of vertex algebras, as we shall illustrate now. Assume that W is the $\mathbb{Z}$-graded vector space which underlies a conformal vertex algebra, with notations as in Definition 3. Furthermore, assume that W comes equipped with a positive definite scalar product $\langle\cdot,\cdot\rangle$, such that $W = \oplus_{m\in\mathbb{Z}} W_m$ is an orthogonal direct sum. Let $A(z)$, $B(w)$ denote the fields associated to $A, B \in W$ by the state-field correspondence, which are local with respect to each other by the locality axiom. Hence by the very Definition 2 of locality, the formal power series $\langle \Omega, A(z)B(w)\Omega \rangle$ and $\langle \Omega, B(w)A(z)\Omega \rangle$ are obtained from the same series in $\mathbb{C}[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$ by means of the (formal) Taylor expansions $\iota_{z>w}$ and $\iota_{w>z}$, respectively. This series is denoted by $\langle A(z)B(w) \rangle \in \mathbb{C}[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$, such that

$$\iota_{z>w}\left(\langle A(z)B(w)\rangle\right) = \langle \Omega, A(z)B(w)\Omega \rangle, \ \iota_{w>z}\left(\langle A(z)B(w)\rangle\right) = \langle \Omega, B(w)A(z)\Omega \rangle.$$

Then $\langle A(z)B(w) \rangle$ is an example of a 2-point function, and for $A_1, \ldots, A_n \in W$ one analogously defines the n-point functions $\langle A_1(z_1) \cdots A_n(z_n) \rangle$ by successive OPE. The additional properties of CFTs ensure that these n-point functions define meromorphic functions in complex variables $z_1, \ldots, z_n \in \mathbb{C}$, whose possible poles are restricted to the partial diagonals $z_i = z_j$, $i \neq j$.

2.2 *Defining Conformal Field Theories*

This section summarizes an axiomatic approach to conformal field theory. Instead of a full account, the focus lies on those ingredients of CFTs that are relevant for the remaining sections of this exposition. More details can be found e.g. in [91, 94]. We list the ingredients and defining properties of a two-dimensional Euclidean unitary conformal field theory at central charges c, $\overline{c}$:

Ingredient I. [The *space of states* $\mathbb{H}$]
The space $\mathbb{H}$ is a $\mathbb{C}$-vector space with positive definite scalar product $\langle\cdot,\cdot\rangle$ and with a compatible real structure $v \mapsto v^*$. Furthermore, there are two Virasoro fields $T(z)$, $\overline{T}(\overline{z})$ of central charges c, $\overline{c}$ on $\mathbb{H}$, see Eq. (1), where the OPE between T and $\overline{T}$ is trivial:

$$T(z)\overline{T}(\overline{z}) \sim 0.$$

The space of states of a CFT must have a number of additional properties:

Property A. The space of states $\mathbb{H}$ furnishes a *unitary* representation of the two commuting copies of a Virasoro algebra generated by the modes L_n, $\overline{L}_n$, $n \in \mathbb{Z}$, of the Virasoro fields $T(z)$ and $\overline{T}(\overline{z})$, which is *compatible with the real structure* of $\mathbb{H}$. The central elements c, $\overline{c}$ act by multiplication with fixed, real constants, also denoted c, $\overline{c} \in \mathbb{R}$. The operators L_0 and $\overline{L}_0$ are self-adjoint and positive semidefinite, and $\mathbb{H}$ decomposes into a direct sum of their simultaneous eigenspaces indexed by $R \subset \mathbb{R}^2$,

$$\mathbb{H} = \bigoplus_{(h,\overline{h})\in R} \mathbb{H}_{h,\overline{h}}, \quad \mathbb{H}_{h,\overline{h}} := \ker(L_0 - h \cdot \mathrm{id}) \cap \ker\left(\overline{L}_0 - \overline{h} \cdot \mathrm{id}\right).$$

By this we mean that R does not have accumulation points, and that every vector in $\mathbb{H}$ is a sum of contributions from finitely many different eigenspaces $\mathbb{H}_{h,\overline{h}}$. Moreover, every $\mathbb{H}_{h,\overline{h}}$ is finite dimensional.

Property A ensures that the space of states $\mathbb{H}$ of every conformal field theory furnishes a very well-behaved representation of two commuting copies of a Virasoro algebra. In addition, we need to assume that the *character* of this representation has favorable properties:

Property B. For $\tau \in \mathbb{C}$, $\Im(\tau) > 0$, let $q := \exp(2\pi i\tau)$; the *partition function*

$$Z(\tau) := \sum_{(h,\overline{h})\in R} \left(\dim_{\mathbb{C}}\mathbb{H}_{h,\overline{h}}\right) q^{h-c/24}\overline{q}^{\overline{h}-\overline{c}/24} = \mathrm{Tr}_{\mathbb{H}}\left(q^{L_0-c/24}\overline{q}^{\overline{L}_0-\overline{c}/24}\right)$$

is well defined for all values of τ in the complex upper halfplane, and it is invariant under modular transformations

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

Since by Property B the partition function is modular invariant, it in particular is invariant under the translation $\tau \mapsto \tau + 1$ of the modular parameter. This implies that for every pair $(h, \overline{h}) \in R$ of eigenvalues of L_0 and $\overline{L}_0$, we have $h - \overline{h} \in \mathbb{Z}$. Hence the subspaces $W := \ker\left(\overline{L}_0\right)$ and $\overline{W} := \ker(L_0)$ are $\mathbb{Z}$-graded by L_0 and $\overline{L}_0$, respectively. To obtain a CFT, these subspaces are required to carry additional structure, which we are already familiar with:

Property C. The subspaces $W := \ker\left(\overline{L}_0\right)$ and $\overline{W} := \ker(L_0)$ of $\mathbb{H}$ carry the structure of conformal vertex algebras, see Definition 3, with $T(z)$ and $\overline{T}(\overline{z})$ the fields associated to the respective conformal vectors by the state-field correspondence. Moreover, the vacuum vector Ω of the conformal vertex algebra W agrees with the vacuum vector of $\overline{W}$, and Ω is a real unit vector yielding a basis of $W \cap \overline{W} = \mathbb{H}_{0,0}$.
The vertex algebras with underlying vector spaces W and $\overline{W}$ are called the *chiral algebras* of the CFT, and to simplify the terminology, we also refer to W and $\overline{W}$ as the chiral algebras.

As was discussed at the end of Sect. 2.1, in this setting there is a natural definition of n-point functions for the fields in the chiral algebras associated to W and $\overline{W}$. This definition, however, is not sufficient to capture the general n-point functions of conformal field theory. The notion is generalized along the following lines:

Ingredient II. [The system $\langle\cdots\rangle$ of n-point functions]
The space of states $\mathbb{H}$ is equipped with a system $\langle\cdots\rangle$ of *n-point functions*, that is, for every $n \in \mathbb{N}$ we have a map

$$\mathbb{H}^{\otimes n} \longrightarrow \mathrm{Maps}(\mathbb{C}^n \setminus \bigcup_{i\neq j} D_{i,j}, \mathbb{C}), \quad D_{i,j} := \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i = z_j\right\},$$

which is compatible with complex conjugation, and such that every function in the image is real analytic and allows an appropriate expansion about every partial diagonal $D_{i,j}$.

The following Property D, which along with Property E governs the behavior of the n-point functions, is immediate on the chiral algebras W and $\overline{W}$, by definition:

Property D. The n-point functions are *local*, that is, for every permutation $\sigma \in S_n$ and all $\phi_i \in \mathbb{H}$,

$$\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle = \langle\phi_{\sigma(1)}(z_{\sigma(1)})\cdots\phi_{\sigma(n)}(z_{\sigma(n)})\rangle.$$

Consider an n-point function $\langle\phi(z_1)\cdots\phi(z_n)\rangle$ with $\phi = \phi_1 = \cdots = \phi_n \in \mathbb{H}$ as a function of one complex variable $z = z_k$, while all other z_l, $l \neq k$, are fixed. The closure of the domain of definition of this function is the *worldsheet* on which the n-point function is defined. Therefore, Ingredient II yields n-point functions whose worldsheet is the Riemann sphere $\overline{\mathbb{C}}$. As a basic feature of conformal field theory, the n-point functions are assumed to transform covariantly under conformal maps between worldsheets. In particular,

Property E. The n-point functions are *Poincaré covariant*, that is, for all isometries and all dilations f of the Euclidean plane $\mathbb{C}$, and for all $\phi_i \in \mathbb{H}_{h_i,\overline{h}_i}$,

$$\langle\phi_1(f(z_1))\cdots\phi_n(f(z_n))\rangle = \prod_{i=1}^{n}\left[\left(f'(z_i)\right)^{-h_i}\,\overline{f'(z_i)}^{-\overline{h}_i}\right]\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle,$$

where $f'(z) = \partial_z f(z)$. Moreover, *infinitesimal translations* $\alpha L_1 + \overline{\alpha}\overline{L}_1$, $\alpha, \overline{\alpha} \in \mathbb{C}$, are represented by $\alpha\partial_z + \overline{\alpha}\partial_{\overline{z}}$, i.e. for arbitrary $\phi_i \in \mathbb{H}$,

$$\langle \phi_1(z_1) \cdots \phi_{n-1}(z_{n-1})(L_1\phi_n)(z_n)\rangle = \frac{\partial}{\partial z_n}\langle \phi_1(z_1) \cdots \phi_{n-1}(z_{n-1})\phi_n(z_n)\rangle,$$
$$\langle \phi_1(z_1) \cdots \phi_{n-1}(z_{n-1})(\overline{L}_1\phi_n)(z_n)\rangle = \frac{\partial}{\partial \overline{z}_n}\langle \phi_1(z_1) \cdots \phi_{n-1}(z_{n-1})\phi_n(z_n)\rangle.$$

The remaining requirements on the n-point functions, unfortunately, are rather more involved. Roughly, they firstly generalize Property E by ensuring that the representation of the two commuting copies of the Virasoro algebra on $\mathbb{H}$ (see Property A) induces an action by infinitesimal conformal transformations on the worldsheet. Furthermore, the operator product expansion of Definition 2 is generalized to induce the appropriate expansions of the n-point functions about partial diagonals, see Ingredient II. Finally, n-point functions must be defined on worldsheets with arbitrary genus. Since these additional properties are not needed explicitly in the remaining sections of the present exposition, here only the relevant keywords are listed in the final

Property F. The system $\langle \cdots \rangle$ of n-point functions is *conformally covariant*, and it *represents an operator product expansion* such that *reflection positivity* holds. Moreover, the *universality condition* holds, and sewing allows to define n-point functions on *worldsheets of arbitrary genus*.

As was mentioned at the beginning of this section, the ingredients of CFTs listed above yield *two-dimensional Euclidean unitary conformal field theories*. Indeed, these adjectives have been implemented in Properties A–F: According to the discussion that precedes Property E along with Property F, the worldsheets of our CFTs are *two-dimensional Euclidean* manifolds. *Conformality* is implemented by means of the two commuting copies of the Virasoro algebra, see the discussion of Eqs. (1) and (2), which act by infinitesimal conformal transformations on the worldsheets of the n-point functions by Properties E and F. On the space of states $\mathbb{H}$, Property A ensures that the representation of the infinitesimal conformal transformations is *unitary*.

Our approach to CFT is convenient, since it concretely implements the interplay between representation theory with the analytic properties of the n-point functions, which is characteristic of two-dimensional conformal quantum field theories. However, the relation to more general quantum field theories (QFTs) is not so evident. Let us briefly comment on this connection.

First, for the relevant QFTs we restrict to *Euclidean* quantum field theories according to a system of axioms that are based on the *Osterwalder-Schrader axioms* [78, 79], see [37, 82]. According to [78, 79], these axioms ensure that from such a QFT one can construct a Hilbert space $\widetilde{\mathbb{H}}$ of states ϕ and associated fields Y_ϕ, where each field Y_ϕ yields a densely defined linear operator $Y_\phi(h)$ on $\widetilde{\mathbb{H}}$ for every test function h. Moreover, there is a special state Ω which plays the role of the vacuum as in our Property C.

The Osterwalder-Schrader axioms require the existence of correlation functions associated to every n-tuple of states in $\widetilde{\mathbb{H}}$ which resemble the n-point functions of CFT according to our Ingredient II. To obtain the fields of CFT from those of the general QFT, one needs to perform a procedure called *localization*. Within the Hilbert space $\widetilde{\mathbb{H}}$ one restricts to the subspace $\mathbb{H}$ which is generated by those states that are created by the localized field operators from the vacuum, generalizing the vacuum axiom of our Definition 3. The Osterwalder-Schrader axioms then ensure that locality (Property D), Poincaré covariance under isometries (Property E) and reflection positivity (Property F) hold for the n-point functions obtained from the correlation functions of the QFT.

According to [37], conformal covariance can be implemented by means of three additional axioms, ensuring the covariance of the n-point functions under dilations (Property E), the existence of the Virasoro fields (Ingredient I) and of an OPE (Properties C and F) with all the necessary features. See [82, Sect. 9.3] for an excellent account.

If a CFT is obtained from a conformally covariant QFT by localization, then one often says that the CFT is the *short distance limit* of the QFT. For details on this mathematical procedure see [37, 43, 90]. To the author's knowledge, it is unknown whether a CFT in the sense of our approach can always be viewed as a short distance limit of a full-fledged QFT.

With the above, we do not claim to provide a *minimal* axiomatic approach to CFT. For example, the requirement of Property F that n-point functions are well-defined on worldsheets of arbitrary genus implies modular invariance of the partition function, which was assumed separately in Property B. Indeed, the partition function $Z(\tau)$ is the 0-point function on a worldsheet torus with modulus τ, where conformal invariance implies that $Z(\tau)$ indeed solely depends on the complex structure represented by $\tau \in \mathbb{C}$, $\Im(\tau) > 0$. Property B is stated separately for clarity, and because modular invariance plays a crucial role in the discussion of the elliptic genus in Sect. 2.4 which is also essential for the remaining sections of this exposition, while we refrain from a detailed discussion of Property F.

Mathematical implications of modular invariance for CFTs were first pointed out by Cardy [16]. He observed that for those theories that had been studied by Belavin, Polyakov and Zamolodchikov in their seminal paper [7], and that describe physical phenomena in statistical physics, modular invariance of the partition function poses constraints on the operator content. These constraints can be useful for the classification of CFTs.

In special cases, modular invariance can be proven from first principles, assuming only that the n-point functions are well-defined on the Riemann sphere. In [74], Nahm argues that the assumption that the n-point functions on the torus define thermal states of the field algebra, which in turn is of type I, suffices to deduce modular invariance.

Under an assumption known as *Condition C* or *Condition C_2*, which amounts to certain quotients of the chiral algebras being finite dimensional, Zhu proves in [102] that modular invariance follows, as well. This covers a large class of examples of CFTs, among them the ones studied by Belavin, Polyakov and Zamolodchikov.

An $N = (2, 2)$ *superconformal field theory* is a CFT as above, where the notion of locality is generalized according to what was said in the discussion of Definition 2, and the representations of the two commuting copies of a Virasoro algebra are extended to representations of $N = 2$ superconformal algebras, see Eqs. (9)–(10). As a first additional ingredient to these theories one therefore needs

Ingredient III. [Compatible $\mathbb{Z}_2$-grading of the space of states]
The space of states $\mathbb{H}$ carries a $\mathbb{Z}_2$-grading $\mathbb{H} = \mathbb{H}_b \oplus \mathbb{H}_f$ into *worldsheet bosons* $\mathbb{H}_b$ (even) and *worldsheet fermions* $\mathbb{H}_f$ (odd), which is compatible with Properties A–F.
In more detail, for compatibility with Property A, the decomposition $\mathbb{H} = \mathbb{H}_b \oplus \mathbb{H}_f$ must be orthogonal and invariant under the action of the two commuting copies of the Virasoro algebra. In Property B, the trace defining the partition function is taken over the bosonic subspace $\mathbb{H}_b$, only. The chiral algebras introduced in Property C must contain $N = 2$ superconformal vertex algebras as introduced in the discussion of Definition 3, whose modes act unitarily on $\mathbb{H}$. The notion of locality in Property D is generalized to *semi-locality*, meaning that

$$\langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle = (-1)^I \langle \phi_{\sigma(1)}(z_{\sigma(1)}) \cdots \phi_{\sigma(n)}(z_{\sigma(n)}) \rangle$$

if $\sigma \in S_n$ and all $\phi_i \in \mathbb{H}$ have definite parity. Here, I is the number of inversions of odd states in σ, that is, the number of pairs (i, j) of indices with $i < j$ and $\sigma(i) > \sigma(j)$ and such that $\phi_i, \phi_j \in \mathbb{H}_f$. Properties E and F remain unchanged.

The fields in the chiral algebras of the CFT that furnish the two commuting copies of $N = 2$ superconformal vertex algebras according to Property III are generally denoted $T(z), J(z), G^+(z), G^-(z)$ and $\overline{T}(\overline{z}), \overline{J}(\overline{z}), \overline{G}^+(\overline{z}), \overline{G}^-(\overline{z})$ with OPEs as in (9)–(10). The mode expansions for the even fields are denoted as

$$T(z) = \sum_n L_n z^{n-2}, \quad J(z) = \sum_n J_n z^{n-1}, \quad \overline{T}(\overline{z}) = \sum_n \overline{L}_n \overline{z}^{n-2}, \quad \overline{J}(\overline{z}) = \sum_n \overline{J}_n \overline{z}^{n-1}, \tag{11}$$

in accord with Definition 3. As mentioned in the discussion after Definition 3, the odd fields $G^\pm(z)$ can have mode expansions either in $\mathrm{End}_{\mathbb{C}}(\mathbb{H})[[z^{\pm 1}]]$ or in $z^{1/2} \cdot \mathrm{End}_{\mathbb{C}}(\mathbb{H})[[z^{\pm 1}]]$, and analogously for $\overline{G}^\pm(\overline{z})$. This induces another $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading of the space of states $\mathbb{H}$,

$$\mathbb{H} = \mathbb{H}^{NS,NS} \oplus \mathbb{H}^{R,R} \oplus \mathbb{H}^{NS,R} \oplus \mathbb{H}^{R,NS}, \tag{12}$$

where

$$G^{\pm}(z) \in \mathrm{End}_{\mathbb{C}}(\mathbb{H}^{NS,\bullet})[[z^{\pm 1}]], \quad \overline{G}^{\pm}(\overline{z}) \in \mathrm{End}_{\mathbb{C}}(\mathbb{H}^{\bullet,NS})[[\overline{z}^{\pm 1}]]$$

in the so-called *Neveu-Schwarz-* or *NS-sector*, while

$$G^{\pm}(z) \in z^{1/2} \cdot \mathrm{End}_{\mathbb{C}}(\mathbb{H}^{R,\bullet})[[z^{\pm 1}]], \quad \overline{G}^{\pm}(\overline{z}) \in \overline{z}^{1/2} \cdot \mathrm{End}_{\mathbb{C}}(\mathbb{H}^{\bullet,R})[[\overline{z}^{\pm 1}]]$$

in the so-called *Ramond-* or *R-sector*. That is, on $\mathbb{H}^{S,\overline{S}}$ the fields $G^{\pm}(z)$ and $\overline{G}^{\pm}(\overline{z})$ have mode expansions according to the S and the $\overline{S}$ sector, respectively, with $S, \overline{S} \in \{R, NS\}$.

In what follows, we restrict our attention to so-called *non–chiral* $N = (2, 2)$ *superconformal field theories with space-time supersymmetry*:

Ingredient IV. [Space-time supersymmetry]
The space of states $\mathbb{H}$ carries another compatible $\mathbb{Z}_2$-grading by means of the properties of the odd fields $G^{\pm}(z)$ and $\overline{G}^{\pm}(\overline{z})$ of the $N = 2$ superconformal vertex algebra into

$$\mathbb{H} = \mathbb{H}^{NS} \oplus \mathbb{H}^{R}.$$

Here, the decomposition (12) reduces to $\mathbb{H}^{NS} := \mathbb{H}^{NS,NS}$ and $\mathbb{H}^{R} := \mathbb{H}^{R,R}$, while the sectors $\mathbb{H}^{NS,R}$ and $\mathbb{H}^{R,NS}$ are trivial.
Moreover, as representations of the two commuting $N = 2$ superconformal algebras of Ingredient III, $\mathbb{H}^{NS}$ and $\mathbb{H}^{R}$ are equivalent under an isomorphism $\Theta\colon \mathbb{H} \to \mathbb{H}$ which interchanges $\mathbb{H}^{NS}$ and $\mathbb{H}^{R}$ and which obeys

$$\begin{aligned} [L_0, \Theta] &= \tfrac{c}{24}\Theta - \tfrac{1}{2}\Theta \circ J_0, \ \ [J_0, \Theta] = -\tfrac{c}{6}\Theta, \\ [\overline{L}_0, \Theta] &= \tfrac{\overline{c}}{24}\Theta - \tfrac{1}{2}\Theta \circ \overline{J}_0, \ [\overline{J}_0, \Theta] = -\tfrac{\overline{c}}{6}\Theta, \end{aligned} \tag{13}$$

where $L_0,\ J_0,\ \overline{L}_0,\ \overline{J}_0$ are the zero-modes of the fields $T(z),\ J(z),\ \overline{T}(\overline{z}),\ \overline{J}(\overline{z})$ obtained from the mode expansions (11). The isomorphism Θ is induced by a field of the theory called *spectral flow*, and it is also known as *space-time supersymmetry*.

With the above notion of CFT, a number of examples are known, like minimal models, both the bosonic [7, 55] and the supersymmetric ones [13, 24, 81, 101]. In string theory, so-called non-linear sigma model constructions are believed to provide a map from certain manifolds to CFTs. While this construction is well understood for the simplest manifolds, namely for tori, the mathematical details in general are far from known.

2.3 Example: Toroidal Conformal Field Theories

For illustration and for later reference, this section very briefly presents the class of so-called *toroidal conformal field theories*. These theories are characterized by the existence of “sufficiently many $U(1)$-currents” as in Example 1 of Sect. 2.1.

We say that the chiral algebra $W = \oplus_m W_m$ of a CFT (see Property C, Sect. 2.2) *contains a* $\mathfrak{u}(1)^d$*-current algebra*, if W_1 contains an orthogonal system $\left(a_1^k\Omega,\ k \in \{1, \ldots, d\}\right)$ of states, which under the state-field correspondence of Definition 3 have associated fields $j^k(z),\ k \in \{1, \ldots, d\}$, obeying the OPEs

$$\forall k, l \in \{1, \ldots, d\}: \qquad j^k(z) j^l(w) \sim \frac{\delta^{k,l}}{(z-w)^2}. \tag{14}$$

For (bosonic) CFTs we then have

Definition 4 A conformal field theory at central charges $c, \overline{c}$ is called TOROIDAL, if $c = \overline{c} = d$ with $d \in \mathbb{N}$, and if the chiral algebras $W = \oplus_m W_m$, $\overline{W} = \oplus_m \overline{W}_m$ of Property C, Sect. 2.2, each contain a $\mathfrak{u}(1)^d$-current algebra.

For our purposes, the *toroidal*$N = (2, 2)$ *superconformal field theories* are more relevant. They are characterized by the fact that their bosonic sector with space of states $\mathbb{H}_b$ contains a toroidal CFT at central charges $2D, 2D$ in the sense of Definition 4, and in addition, they contain D left- and D right-moving so-called *Dirac fermions with coupled spin structures*. By this we mean first of all that the subspace $W_{1/2} \subset W$ of the vector space underlying the chiral algebra contains an orthogonal system $\left((\psi_\pm^k)_{1/2}\Omega,\ k \in \{1, \ldots, D\}\right)$ of states, which under the state-field correspondence of Definition 3 have associated (odd) fields $\psi_k^\pm(z),\ k \in \{1, \ldots, D\}$, obeying the OPEs

$$\forall k, l \in \{1, \ldots, D\}: \qquad \psi_k^+(z)\psi_l^-(w) \sim \frac{\delta^{k,l}}{z-w}, \quad \psi_k^\pm(z)\psi_l^\pm(w) \sim 0, \tag{15}$$

and analogously for the subspace $\overline{W}_{1/2} \subset \overline{W}$ of the vector space underlying the second chiral algebra in Property C. In addition, all $\psi_k^\pm(z)$ are represented by formal power series in $\mathrm{End}_{\mathbb{C}}(\mathbb{H}^{NS})[[z^{\pm 1}]]$ on $\mathbb{H}^{NS}$, while on $\mathbb{H}^R$, they are represented in $z^{1/2} \cdot \mathrm{End}_{\mathbb{C}}(\mathbb{H}^R)[[z^{\pm 1}]]$, and analogously for the $\overline{\psi}_k^\pm(\overline{z})$. One shows that such a system of D left- and D right-moving Dirac fermions yields a well-defined CFT at central charges D, D (see e.g. [53, Sect. 8.2] or [91, Chap. 5]).

Definition 5 An $N = (2, 2)$ superconformal field theory at central charges $c, \overline{c}$ with space-time supersymmetry is toroidal, if $c = \overline{c} = 3D$ with $D \in \mathbb{N}$, and if this theory is the tensor product of a toroidal conformal field theory at central charges $2D, 2D$ according to Definition 4, and a system of D left- and D right-moving Dirac fermions with coupled spin structures. Moreover, the fields $\psi_k^\pm(z), k \in \{1, \ldots, D\}$, in (15) are the superpartners of the $U(1)$-currents $j^l(z)$, $l \in \{1, \ldots, 2D\}$ in (14), and analogously for the right-moving fields. By this we mean that for the fields $G^\pm(z)$, $\overline{G}^\pm(\overline{z})$ in the two commuting superconformal vertex algebras (9)–(10), and with notations as above, we have

$$\begin{aligned}\forall k \in \{1, \ldots, D\}:\quad & G^\pm_{-1/2} a_1^k \Omega = (\psi^k_\pm)_{1/2}\Omega,\ G^\pm_{-1/2} a_1^{k+D}\Omega = \mp i (\psi^k_\pm)_{1/2}\Omega,\\ & \overline{G}^\pm_{-1/2} \overline{a}_1^k \Omega = (\overline{\psi}^k_\pm)_{1/2}\Omega,\ \overline{G}^\pm_{-1/2} \overline{a}_1^{k+D}\Omega = \mp i (\overline{\psi}^k_\pm)_{1/2}\Omega.\end{aligned}$$

The toroidal conformal and superconformal field theories have been very well understood by string theorists since the mid eighties [17, 76], and these theories have also been reformulated in terms of the vertex algebras presented in Sect. 2.1 [38, 61, 63]. This includes the interpretation of the toroidal conformal field theories as non-linear sigma models on tori, their deformations, and thus the structure of the moduli space of toroidal CFTs:

Theorem 1 ([76]) *The moduli space $\mathcal{M}^{\text{tor}}_D$ of toroidal $N = (2, 2)$ superconformal field theories at central charges $c = \overline{c} = 3D$ with $D \in \mathbb{N}$ is a quotient of a $4D^2$-dimensional Grassmannian by an infinite discrete group,*

$$\begin{aligned}\mathcal{M}^{\text{tor}}_D = O^+(2D, 2D; \mathbb{Z})\backslash \mathcal{T}^{2D,2D},&\\ \textit{where } \mathcal{T}^{2D,2D} := O^+(2D, 2D; \mathbb{R})/SO(2D) \times O(2D).&\end{aligned}$$

Here, if $pq \neq 0$, then $O^+(p, q; \mathbb{R})$ denotes the group of those elements in $O(p, q; \mathbb{R}) = O(\mathbb{R}^{p,q})$ which preserve the orientation of maximal positive definite oriented subspaces in $\mathbb{R}^{p,q}$, and if $p \equiv q \bmod 8$, then $O^+(p, q; \mathbb{Z}) = O^+(p, q; \mathbb{R}) \cap O(\mathbb{Z}^{p,q})$ with $\mathbb{Z}^{p,q} \subset \mathbb{R}^{p,q}$ the standard even unimodular lattice of signature (p, q).

2.4 The Elliptic Genus

In this section, the conformal field theoretic elliptic genus is introduced and compared to the geometric elliptic genus that is known to topologists and geometers. This and the following section are completely expository with more details and proofs to be found in the literature as referenced.

Let us first consider an $N = (2, 2)$ superconformal field theory at central charges c, $\overline{c}$ with space-time supersymmetry according to the Ingredients I–IV of Sect. 2.2. For the zero-modes J_0, $\overline{J}_0$ of the fields $J(z)$, $\overline{J}(\overline{z})$ in the two commuting $N = 2$ superconformal algebras $T(z)$, $J(z)$, $G^+(z)$, $G^-(z)$, $\overline{T}(\overline{z})$, $\overline{J}(\overline{z})$, $\overline{G}^+(\overline{z})$, $\overline{G}^-(\overline{z})$ of Ingredient III according to (11) one finds: These linear operators are self-adjoint and simultaneously diagonalizable on the space of states $\mathbb{H} = \mathbb{H}^{NS} \oplus \mathbb{H}^R$. By Ingredient IV, the corresponding operator of spectral flow induces an equivalence of representations $\mathbb{H}^{NS} \cong \mathbb{H}^R$ of the two $N = 2$ superconformal algebras. This turns out to imply that the linear operator $J_0 - \overline{J}_0$ has only integral eigenvalues, which are even on $\mathbb{H}_b$ and odd on $\mathbb{H}_f$, see e.g. [92, Sect. 3.1]. Hence $(-1)^{J_0-\overline{J}_0}$ is an involution which yields the $\mathbb{Z}_2$-grading $\mathbb{H} = \mathbb{H}_b \oplus \mathbb{H}_f$ and an induced $\mathbb{Z}_2$-grading on $\mathbb{H}^R$. Following [30], this allows the definition of a supercharacter of the superconformal field theory, analogous to the partition function in Property B:

Definition 6 Consider an $N = (2, 2)$ superconformal field theory at central charges $c, \overline{c}$ with space-time supersymmetry. Set $q := \exp(2\pi i\tau)$ for $\tau \in \mathbb{C}$, $\Im(\tau) > 0$, and $y := \exp(2\pi i z)$ for $z \in \mathbb{C}$. Then

$$\begin{aligned}\mathcal{E}(\tau, z) &:= \mathrm{Str}_{\mathbb{H}^R}\left(y^{J_0} q^{L_0-c/24} \overline{q}^{\overline{L}_0-\overline{c}/24}\right) \\ &= \mathrm{Tr}_{\mathbb{H}^R}\left((-1)^{J_0-\overline{J}_0} y^{J_0} q^{L_0-c/24} \overline{q}^{\overline{L}_0-\overline{c}/24}\right)\end{aligned}$$

is the CONFORMAL FIELD THEORETIC ELLIPTIC GENUS of the theory.

Using known properties of the $N = 2$ superconformal algebra and of its irreducible unitary representations, one shows (see [23, 30, 100] for the original results and e.g. [92, Sect. 3.1] for a summary and proofs):

Proposition 1 *Consider the conformal field theoretic elliptic genus $\mathcal{E}(\tau, z)$ of an $N = (2, 2)$ superconformal field theory at central charges $c, \overline{c}$ with space-time supersymmetry.*
Then $\mathcal{E}(\tau, z)$ is holomorphic in τ and bounded when $\tau \to i\infty$.
It is invariant under smooth deformations of the underlying superconformal field theory to any other space-time supersymmetric $N = (2, 2)$ superconformal field theory with the same central charges.
Moreover, $\mathcal{E}(\tau, z)$ transforms covariantly under modular transformations,

$$\mathcal{E}(\tau + 1, z) = \mathcal{E}(\tau, z), \quad \mathcal{E}(-1/\tau, z/\tau) = e^{2\pi i \frac{c}{6} \cdot \frac{z^2}{\tau}} \mathcal{E}(\tau, z).$$

If in addition $c = \overline{c} \in 3\mathbb{N}$, and all eigenvalues of J_0 and $\overline{J}_0$ in the Ramond sector lie in $\frac{c}{6} + \mathbb{Z}$, then

$$\mathcal{E}(\tau, z + 1) = (-1)^{\frac{c}{3}} \mathcal{E}(\tau, z), \quad \mathcal{E}(\tau, z + \tau) = q^{-\frac{c}{6}} y^{-\frac{c}{3}} \mathcal{E}(\tau, z).$$

In other words, $\mathcal{E}(\tau, z)$ is a WEAK JACOBI FORM *(with a character, if $c/3$ is odd) of* WEIGHT 0 *and* INDEX $c/6$.

Note that the additional assumptions on the central charges and the eigenvalues of J_0 and $\overline{J}_0$ in the last statement of Proposition 1 are expected to hold for superconformal field theories that are obtained by a non-linear sigma model construction from some Calabi-Yau D-manifold.

On the other hand, following Hirzebruch's seminal work on multiplicative sequences and their genera [59], the elliptic genus is known to topologists as a ring homomorphism from the cobordism ring of smooth oriented compact manifolds into a ring of modular functions [60, 66]. For simplicity we assume that our underlying manifold X is a Calabi-Yau D-manifold. Then its associated geometric elliptic genus $\mathcal{E}_X(\tau, z)$ can be viewed as a modular function obeying the transformation properties of Proposition 1 with $c = 3D$ and interpolating between the standard topological invariants of X, namely its *Euler characteristic* $\chi(X)$, its *signature* $\sigma(X)$, and its *topological Euler characteristic* $\chi(\mathcal{O}_X)$.

To understand this in more detail, first recall the definition of the topological invariants mentioned above: For $y \in \mathbb{C}$ the Hirzebruch χ_y-genus [59] is defined by

$$\chi_y(X) := \sum_{p,q=0}^{D} (-1)^q y^p h^{p,q}(X),$$

where the $h^{p,q}(X)$ are the Hodge numbers of X. Then

$$\chi(X) := \chi_{-1}(X), \quad \sigma(X) := \chi_{+1}(X), \quad \chi(\mathcal{O}_X) := \chi_0(X). \tag{16}$$

Note that by the usual symmetries among the Hodge numbers $h^{p,q}(X)$ of a complex Kähler manifold X, the signature $\sigma(X) = \sum_{p,q} (-1)^q h^{p,q}(X)$ vanishes if the complex dimension D of X is odd; we have thus trivially extended the usual definition of the signature on oriented compact manifolds whose real dimension is divisible by 4 to all compact complex Kähler manifolds.

To motivate a standard formula for the specific elliptic genus which is of relevance to us, see Definition 7, we draw the analogy to the interpretation of the topological invariants (16) in terms of the Atiyah-Singer Index Theorem [5]. For any complex vector bundle E on X and a formal variable x, we introduce the shorthand notations

$$\Lambda_x E := \bigoplus_p x^p \Lambda^p E, \quad S_x E := \bigoplus_p x^p S^p E,$$

where $\Lambda^p E,\ S^p E$ denote the exterior and the symmetric powers of E, respectively, along with the *Chern character* on such formal power series in x whose coefficients are complex vector bundles F_p:

$$\mathrm{ch}(\bigoplus_p x^p F_p) := \sum_p x^p \mathrm{ch}(F_p).$$

Then by the Hirzebruch-Riemann-Roch formula [58], which can be viewed as a special case of the Atiyah-Singer Index Theorem, one finds

$$\chi_y(X) = \int_X \mathrm{Td}(X)\mathrm{ch}(\Lambda_y T^*), \tag{17}$$

where Td(X) denotes the *Todd genus* and $T := T^{1,0}X$ is the holomorphic tangent bundle of X. Generalizing the expression in Eq. (17) and following [60, 65, 97] we now have

Definition 7 Let X denote a compact complex D-manifold with holomorphic tangent bundle $T := T^{1,0}X$. Set

$$\mathbb{E}_{q,-y} := y^{-D/2} \bigotimes_{n=1}^{\infty} \left(\Lambda_{-yq^{n-1}} T^* \otimes \Lambda_{-y^{-1}q^n} T \otimes S_{q^n} T^* \otimes S_{q^n} T \right),$$

viewed as a formal power series with variables $y^{\pm 1/2}$, q, whose coefficients are holomorphic vector bundles on X.

Analogously to Definition 1, the integral $\int_X$ is extended linearly to the vector space of formal power series whose coefficients are characteristic classes on X. Then with $q := \exp(2\pi i \tau)$ and $y := \exp(2\pi i z)$, the holomorphic Euler characteristic of $\mathbb{E}_{q,-y}$,

$$\mathcal{E}_X(\tau, z) := \int_X \mathrm{Td}(X)\mathrm{ch}(\mathbb{E}_{q,-y}) \quad \in \quad y^{-D/2} \cdot \mathbb{Z}[[y^{\pm 1}, q]],$$

is the (geometric) ELLIPTIC GENUS of X.

By [60, 65, 97], the elliptic genus $\mathcal{E}_X(\tau, z)$ in fact yields a well-defined function in $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$ and in $z \in \mathbb{C}$. If X is a Calabi-Yau D-manifold, then $\mathcal{E}_X(\tau, z)$ is a weak Jacobi form (with a character, if D is odd) of weight 0 and index $D/2$ [11]. In other words, with $c := 3D$ the elliptic genus $\mathcal{E}_X(\tau, z)$ obeys the transformation properties stated for $\mathcal{E}(\tau, z)$ in Proposition 1, and it is bounded when $\tau \to i\infty$. One checks that by definition, the elliptic genus indeed is a topological invariant which interpolates between the standard topological invariants of Eq. (16), namely $\mathcal{E}_X(\tau, z) \stackrel{\tau \to i\infty}{\longrightarrow} y^{-D/2}\chi_{-y}(X)$ and

$$\begin{aligned} \mathcal{E}_X(\tau, z = 0) \;=\; \chi(X), \quad \mathcal{E}_X(\tau, z = 1/2) &= (-1)^{D/2}\sigma(X) + \mathcal{O}(q), \\ q^{D/4}\mathcal{E}_X(\tau, z = (\tau + 1)/2) &= (-1)^{D/2}\chi(\mathcal{O}_X) + \mathcal{O}(q). \end{aligned} \tag{18}$$

According to Witten [96, 97], the expression for the elliptic genus $\mathcal{E}_X(\tau, z)$ in Definition 7 can be interpreted as a regularized version of a $U(1)$-equivariant index of a Dirac-like operator on the loop space of X, see also [66]. This explains the notation chosen in Definition 7, and it also motivates why one expects that for CFTs which are obtained by a non-linear sigma model construction from some Calabi-Yau D-manifold X, the conformal field theoretic elliptic genus of Definition 6 agrees with the geometric elliptic genus of X as in Definition 7. Note that the resulting equation

$$\mathcal{E}_X(\tau, z) = \mathrm{Str}_{\mathbb{H}^R} \left(y^{J_0} q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - \overline{c}/24} \right) \tag{19}$$

would furnish a natural generalization of the *McKean-Singer Formula* [72]. While non-linear sigma model constructions are not understood sufficiently well to even attempt a general proof of this equation, there is some evidence for its truth. On the one hand, as was pointed out in Sect. 2.3, $N = (2, 2)$ superconformal field theories obtained from a non-linear sigma model on a complex torus are very well understood. One confirms that their conformal field theoretic elliptic genus vanishes, as does the geometric elliptic genus of a complex torus. Equation (19) is also compatible with the construction of symmetric powers of the manifold X [25]. Moreover, compatibility of the elliptic genus with orbifold constructions was proved in [12, 40]. Further evidence in favor of the expectation (19) arises from a discussion of the chiral de Rham complex, see Sect. 2.5.

2.5 The Chiral de Rham Complex

As was pointed out above, non-linear sigma model constructions of $N = (2, 2)$ superconformal field theories are in general not very well understood. Therefore, a direct proof of the expected equality (19) is out of reach. However, instead of a full-fledged superconformal field theory, in [70] the authors construct a *sheaf of superconformal vertex algebras*, known as the *chiral de Rham complex* Ω^{ch}_X, on any complex manifold X. The chiral de Rham complex of X is expected to be closely related to the non-linear sigma model on X, as we shall discuss in the present section.

Let us begin by summarizing the construction of the chiral de Rham complex Ω^{ch}_X for a complex D-dimensional manifold X, see [8, 56, 69, 70]. First, to any coordinate neighborhood $U \subset X$ with holomorphic coordinates $(z^1, \dots, z^D)$ one associates a $bc - \beta\gamma$ system $\Omega^{\mathrm{ch}}_X(U)$ as in Example 3, see Sect. 2.1. Here, the even fields a^j, b^j are interpreted as arising from quantizing the local sections $\partial/\partial z^j$, z^j of the sheaf of polyvector fields on X, while the odd fields ϕ^j, ψ^j correspond to the local sections dz^j, $\partial/\partial(dz^j)$ of the sheaf of differential operators on the de Rham algebra of differential forms. Indeed, by (3) the map

$$(\partial/\partial z^j,\ z^j,\ dz^j,\ \partial/\partial(dz^j)) \longmapsto (a^j_0,\ b^j_0,\ \phi^j_0,\ \psi^j_0)$$

induces a super-Lie algebra homomorphism.

According to [70], coordinate transforms on X induce corresponding transformation rules for the fields a^j, b^j, ϕ^j, ψ^j which are compatible with the structure of the $bc - \beta\gamma$-system as discussed in Example 3. This allows to glue the $\Omega^{\mathrm{ch}}_X(U)$ accordingly, and by localization, one indeed obtains a well-defined sheaf of vertex algebras over X, with a (non-associative) action of $\mathcal{O}_X$ on it.

A key result of [70] is the fact that under appropriate assumptions on X, there are well-defined global sections of the sheaf $\mathrm{End}_{\mathbb{C}}(\Omega^{\mathrm{ch}}_X)[[z^{\pm 1}]]$, which are locally given by the fields (4), (5) of the topological $N = 2$ superconformal algebra (6)–(7) discussed in Example 4 of Sect. 2.1:

Theorem 2 ([70]) *Let X denote a compact complex manifold of dimension D. As discussed above, there is an associated sheaf Ω_X^{ch} of vertex algebras on X. On every holomorphic coordinate chart $U \subset X$, let $T^{\mathrm{top}}(z)$, $J(z)$, $Q(z)$, $G(z)$ denote the local sections in* $\mathrm{End}_{\mathbb{C}}(\Omega_X^{\mathrm{ch}}(U))[[z^{\pm 1}]]$ *defined by* (4), (5)*, with mode expansions*

$$T^{\mathrm{top}}(z) = \sum_n L_n^{\mathrm{top}} z^{n-2}, \quad J(z) = \sum_n J_n z^{n-1},$$
$$Q(z) = \sum_n Q_n z^{n-1}, \quad G(z) = \sum_n G_n z^{n-2},$$

respectively. Then the following holds:

1. *The linear operators $F := J_0$ and $d_{\mathrm{dR}}^{\mathrm{ch}} := -Q_0$ are globally well-defined. Moreover, F defines a $\mathbb{Z}$-grading on Ω_X^{ch}, while $(d_{\mathrm{dR}}^{\mathrm{ch}})^2 = 0$, such that*
$$\forall p \in \mathbb{Z}: \quad \Omega_X^{\mathrm{ch},p}(U) := \left\{ \Phi \in \Omega_X^{\mathrm{ch}}(U) \mid F\Phi = p\Phi \right\}$$
yields a complex $(\Omega_X^{\mathrm{ch},\bullet}, d_{\mathrm{dR}}^{\mathrm{ch}})$, which is called the CHIRAL DE RHAM COMPLEX.
2. *The map $(z^j, dz^j) \mapsto (b_0^j, \phi_0^j)$ induces a quasi-isomorphism from the usual de Rham complex to the chiral de Rham complex of X.*
3. *The local fields $T^{\mathrm{top}}(z)$ given in* (4) *define a global field on the chiral de Rham complex, by which we mean a global section of the sheaf* $\mathrm{End}_{\mathbb{C}}(\Omega_X^{\mathrm{ch}})[[z^{\pm 1}]]$. *The chiral de Rham complex therefore is bigraded by F and L_0^{top}.*
4. *If X is a Calabi-Yau manifold, then the local fields $J(z)$, $Q(z)$, $G(z)$ given in* (5) *also define global fields on the chiral de Rham complex.*

As mentioned above, the sheaf Ω_X^{ch} is not quasi-coherent. However, it has a filtration which is compatible with the bigrading of Theorem 2 and such that the corresponding graded object yields a quasi-coherent sheaf isomorphic to (the sheaf of sections of) $(-y)^{D/2}\mathbb{E}_{q,y}$ as in Definition 7. This is used extensively in [10, 11] to study the *Čech cohomology* $H^*(X, \Omega_X^{ch})$. Note that this means classical Čech cohomology, ignoring the differential $d_{\mathrm{dR}}^{\mathrm{ch}}$ of the chiral de Rham complex. The authors find:

Theorem 3 ([10, 11]) *Consider a Calabi-Yau D-manifold X, and the Čech cohomology $H^*(X, \Omega_X^{ch})$ of its chiral de Rham complex Ω_X^{ch}. Equip it with the induced bigrading by the operators $F = J_0$ and L_0^{top} of Theorem* 2 *and the $\mathbb{Z}_2$-grading by $(-1)^{F+q}$ on $H^q(X, \Omega_X^{ch})$. Then $H^*(X, \Omega_X^{ch})$ carries a natural structure of a topological $N = 2$ superconformal vertex algebra* [10, Proposition 3.7 and Definition 4.1]. *Moreover* [11], *the* GRADED EULER CHARACTERISTIC *of the chiral de Rham complex, that is, the supertrace of the operator $y^{-D/2} \cdot (y^{J_0} q^{L_0^{\mathrm{top}}})$ on $H^*(X, \Omega_X^{ch})$, yields the elliptic genus $\mathcal{E}_X(\tau, z)$ of Definition* 7.

Thus Theorem 3 indicates a possible relationship between the chiral de Rham complex Ω_X^{ch} of a Calabi-Yau D-manifold X and a non-linear sigma model on X, since it recovers the (geometric!) elliptic genus $\mathcal{E}_X(\tau, z)$ by means of a supertrace which at least in spirit agrees with the expression on the right hand side of Eq. (19). Note that $L_0^{\text{top}} = L_0 - \frac{1}{2} J_0$ by (8) [using Definition 1 and (11)]. Therefore, using the fact that the elliptic genus is holomorphic, along with the spectral flow (13), the conformal field theoretic elliptic genus of Definition 6 can be expressed as

$$\begin{aligned}\mathcal{E}(\tau, z) &= \mathrm{Str}_{\mathbb{H}^R}\left(y^{J_0} q^{L_0 - c/24}\right)\\ &= y^{-c/6}\mathrm{Str}_{\mathbb{H}^{NS}}\left((yq^{-1/2})^{J_0} q^{L_0}\right) \;=\; y^{-c/6}\mathrm{Str}_{\mathbb{H}^{NS}}\left(y^{J_0} q^{L_0^{\text{top}}}\right).\end{aligned}$$

Hence recalling $c = 3D$ for a non-linear sigma model on a Calabi-Yau D-manifold, one is led to conjecture that one might be able to identify an appropriate cohomology of $\mathbb{H}^{NS}$ with $H^*(X, \Omega_X^{ch})$.

The details of such an identification are still more subtle, however. Indeed, by construction, the chiral de Rham complex depends only on the complex structure of X, while the non-linear sigma model, in addition, depends on the complexified Kähler structure of X. It is therefore natural to expect the vertex algebra of Theorem 3 to yield a truncated version of the non-linear sigma model by means of the topological twists mentioned in Example 5 of Sect. 2.1. Since the crucial bundle $\mathbb{E}_{q,-y}$ of Definition 7 resembles an infinite-dimensional Fock space, while the traditional topological A- and B-twists yield finite dimensional spaces of states, the so-called *half-twisted sigma model* according to Witten [99] is the most natural candidate. It still cannot yield the vertex algebra of Theorem 3, since it depends both on the complex and on the complexified Kähler structure of X. Moreover, the Čech resolution, which is implicit in $H^*(X, \Omega_X^{ch})$, does not resemble the standard features of non-linear sigma models on X. According to Kapustin, however, a large volume limit of Witten's half twisted sigma model on X yields the cohomology of Ω_X^{ch} with respect to yet another resolution of the complex, the so-called *Dolbeault resolution* [62] .

3 Conformal Field Theory on K3

As emphasized repeatedly, non-linear sigma model constructions are in general not well understood, except for the toroidal conformal field theories presented in Sect. 2.3. Recall however that there are only two topologically distinct types of Calabi-Yau 2-manifolds, namely the complex 2-tori and the K3 surfaces (see e.g. [6, Chap.VIII] for an excellent introduction to the geometry of K3 surfaces). By the Kummer construction, one obtains an example of a K3 surface by means of a $\mathbb{Z}_2$-orbifold procedure from every complex 2-torus. On the other hand, $\mathbb{Z}_2$-orbifolds of the toroidal CFTs are also reasonably well understood. One therefore expects to be able to construct examples of CFTs which allow a non-linear sigma model interpretation on some K3 surface. Compared to CFTs on higher-dimensional Calabi-Yau D-manifolds, those on K3 surfaces indeed provide a borderline case, in the sense that much more is known about these so-called *K3 theories*. Most importantly, we can give a mathematical definition of such theories without ever mentioning non-linear sigma model constructions. The current section presents this definition and summarizes some of the known properties of K3 theories.

To motivate the mathematical definition of K3 theories, let us recall the conformal field theoretic elliptic genus of Sect. 2.4. Here we assume that we are given an $N = (2,2)$ superconformal field theory that obeys the following conditions, which are necessary for the CFT to allow a non-linear sigma model interpretation on some Calabi-Yau 2-manifold: The theory is superconformal at central charges $c = 6$, $\overline{c} = 6$ with space-time supersymmetry, and such that all eigenvalues of J_0 and $\overline{J}_0$ are integral. This latter condition is equivalent to the assumption that in addition to the spectral flow operator of Ingredient IV in Sect. 2.2, the theory possesses a quartet of *two-fold left- and right-handed spectral flow operators* $\Theta^\pm$, $\overline{\Theta}^\pm$. By this we mean that these operators act analogously to $\Theta^{\pm 2}$ on the space of states, with Θ as in (13), namely

$$[L_0, \Theta^\pm] = \tfrac{c}{6}\Theta^\pm \mp \Theta^\pm \circ J_0, \;\; [J_0, \Theta^\pm] = \mp\tfrac{c}{3}\Theta^\pm,$$
$$[\overline{L}_0, \overline{\Theta}^\pm] = \tfrac{\overline{c}}{6}\overline{\Theta}^\pm \mp \overline{\Theta}^\pm \circ \overline{J}_0, \; [\overline{J}_0, \overline{\Theta}^\pm] = \mp\tfrac{\overline{c}}{3}\overline{\Theta}^\pm,$$

but with all other commutators vanishing. The fields associated to $\Theta^\pm\Omega$, $\overline{\Theta}^\pm\Omega$ by the state-field correspondence (Definition 3) are denoted $J^\pm(z)$ and $\overline{J}^\pm(\overline{z})$, respectively. By Proposition 1, the conformal field theoretic elliptic genus $\mathcal{E}(\tau, z)$ of such a CFT is a weak Jacobi form of weight 0 and index 1. However, the space of such Jacobi forms is one-dimensional, as follows from the methods introduced in [36] (see [11] or [92, Theorem. 3.1.12] for direct proofs). According to the discussion that follows Definition 7, the (geometric) elliptic genus $\mathcal{E}_{\mathrm{K3}}(\tau, z)$ of a K3 surface is a weak Jacobi form of weight 0 and index 1 as well, which by (18) is non-zero, since $\mathcal{E}_{\mathrm{K3}}(\tau, z = 0) = \chi(\mathrm{K3}) = 24$. The precise form of the function $\mathcal{E}_{\mathrm{K3}}(\tau, z)$ is well-known, and we obtain

$$\mathcal{E}(\tau, z) = a \cdot \mathcal{E}_{\mathrm{K3}}(\tau, z) = a \cdot \Big(2y + 20 + 2y^{-1} + \mathcal{O}(q)\Big) \tag{20}$$

for some constant a. In fact,

Proposition 2 ([92, Sect. 7.1]) *Consider an* $N = (2,2)$ *superconformal field theory at central charges* $c = 6$, $\overline{c} = 6$ *with space-time supersymmetry and such that all the eigenvalues of* J_0 *and of* $\overline{J}_0$ *are integral.*

1. *The elliptic genus of this CFT either vanishes, or it agrees with the geometric elliptic genus* $\mathcal{E}_{\mathrm{K3}}(\tau, z)$ *of a K3 surface.*
2. *The conformal field theoretic elliptic genus vanishes if and only if the theory is a toroidal* $N = (2,2)$ *superconformal field theory according to Definition 5.*

This result is mentioned in [75] and proved in [92, Sect. 7.1], where the proof however contains a few typos. The sketch of a corrected proof is banned to the

Appendix, since it uses a number of properties of superconformal field theories with space-time supersymmetry which are well-known to the experts, but which we have not derived in this exposition.

While as mentioned before, the toroidal $N = (2, 2)$ superconformal field theories are well understood, it is also not hard to find examples of theories whose conformal field theoretic elliptic genus is $\mathcal{E}_{\mathrm{K3}}(\tau, z)$, see [30]. In particular, the authors of [30] prove that the standard $\mathbb{Z}_2$-orbifold of every toroidal $N = (2, 2)$ superconformal field theory at central charges $c = 6$, $\overline{c} = 6$ yields such an example. By the above this is in accord with the expectations based on the Kummer construction, hence our

Definition 8 A superconformal field theory is called a K3 THEORY, if the following conditions hold: The CFT is an $N = (2, 2)$ superconformal field theory at central charges $c = 6$, $\overline{c} = 6$ with space-time supersymmetry, all the eigenvalues of J_0 and of $\overline{J}_0$ are integral, and the conformal field theoretic elliptic genus of the theory is

$$\mathcal{E}(\tau, z) = \mathcal{E}_{\mathrm{K3}}(\tau, z).$$

Possibly, every K3 theory allows a non-linear sigma model interpretation on some K3 surface, however a proof is far out of reach. Nevertheless, under standard assumptions on the deformation theory of such theories it is possible to determine the form of every connected component of the moduli space of K3 theories. Namely, one assumes that all deformations by so-called marginal operators are integrable for these theories, an assumption which can be justified in string theory and which is demonstrated to all orders of perturbation theory in [26]. Then, based on the previous results [18, 83], one obtains

Theorem 4 ([3, 75]) *With the notations introduced in Theorem 1, let $\mathcal{T}^{4,20}$ denote the Grassmannian of maximal positive definite oriented subspaces of $\mathbb{R}^{4,20}$,*

$$\mathcal{T}^{4,20} := O^+(4, 20; \mathbb{R})/SO(4) \times O(20).$$

By $\mathcal{T}_0^{4,20} \subset \mathcal{T}^{4,20}$ we denote the set of all those maximal positive definite oriented subspaces $x \subset \mathbb{R}^{4,20}$ which have the property that $x^\perp$ does not contain any roots, that is, all $\alpha \in x^\perp \cap \mathbb{Z}^{4,20}$ obey $\langle \alpha, \alpha \rangle \neq -2$.

If the above-mentioned assumptions on deformations of K3 theories hold, namely that all deformations by so-called marginal operators are integrable, then each connected component $\mathcal{M}_s^{\mathrm{K3}}$ of the moduli space of K3 theories has the following form:

$$\mathcal{M}_s^{\mathrm{K3}} = O^+(4, 20; \mathbb{Z}) \backslash \mathcal{T}_0^{4,20}.$$

This result reinforces the expectation that one connected component $\mathcal{M}_\sigma^{\mathrm{K3}}$ of the moduli space of K3 theories can be identified with the space of non-linear sigma models on K3 surfaces, since in addition, we have

Proposition 3 ([3]) *The partial completion $\mathcal{T}^{4,20}$ of the smooth universal covering space $\mathcal{T}_0^{4,20}$ of $\mathcal{M}_s^{\mathrm{K3}}$ can be isometrically identified with the* PARAMETER SPACE OF

NON- LINEAR SIGMA MODELS ON $K3$. *Namely, denoting by X the diffeomorphism type of a* $K3$ *surface,* $\mathcal{T}^{4,20}$ *is a cover of the space of triples* (Σ, V, B) *where* Σ *denotes a* HYPERKÄHLER STRUCTURE *on X,* $V \in \mathbb{R}^+$ *is interpreted as the* VOLUME *of X, and B is the de Rham cohomology class of a real closed two-form on X, a so-called* B- FIELD.

If a K3 theory in $\mathcal{M}_s^{\mathrm{K3}}$ *lifts to a point in* $\mathcal{T}^{4,20}$ *which is mapped to the triple* (Σ, V, B)*, then* (Σ, V, B) *is called a* GEOMETRIC INTERPRETATION *of the K3 theory.*

In [75, 93] it is shown that the expectation that non-linear sigma models on K3 yield K3 theories indeed is compatible with orbifold constructions, more precisely with every orbifold construction of a K3 surface from a complex two-torus by means of a discrete subgroup of SU(2). As mentioned above, one might conversely expect that every K3 theory with geometric interpretation (Σ, V, B) can be constructed as a non-linear sigma model on a K3 surface, specified by the data (Σ, V, B) – at least the existence of a non-linear sigma model interpretation has not been disproved for any K3 theory, so far.

The statement of Proposition 3 makes use of the fact that every K3 surface is a hyperkähler manifold. The analogous statement for K3 theories is the observation that the two commuting copies of $N = 2$ superconformal algebras (9)–(10) are each extended to an $N = 4$ *superconformal algebra* in these theories. This is a direct consequence of our Definition 8 of K3 theories. Indeed, as mentioned at the beginning of this section, the assumption of space-time supersymmetry together with the integrality of the eigenvalues of J_0 and $\overline{J}_0$ imply that the fields $J^\pm(z)$, $\overline{J}^\pm(\overline{z})$ corresponding to two-fold left- and right-handed spectral flow are fields of the CFT. One checks that at central charges $c = 6$, $\overline{c} = 6$, these fields create states in the subspaces W_1 and $\overline{W}_1$ of the vector spaces underlying the chiral algebras of Property C (see the vacuum axiom in Definition 3), whose J_0- (respectively $\overline{J}_0$-) eigenvalues are ± 2. Moreover, with the $U(1)$-currents $J(z)$, $\overline{J}(\overline{z})$ of the two commuting copies of $N = 2$ superconformal vertex algebras, the fields $J^\pm(z)$, $\overline{J}^\pm(\overline{z})$ generate two commuting copies of a so-called $\mathfrak{su}(2)_1$-*current algebra*, which in turn is known to extend the $N = 2$ superconformal algebra to an $N = 4$ superconformal algebra [1].

The characters of the irreducible unitary representations of the $N = 4$ superconformal algebra at arbitrary central charges have been determined in [31–34, 84]. Their transformation properties under modular transforms in general are *not* modular, in contrast to the situation at lower supersymmetry, where an infinite class of characters of irreducible unitary representations does enjoy modularity. Instead, the $N = 4$ characters exhibit a so-called *Mock modular* behavior, see e.g. [22] for a recent account. Since in the context of non-linear sigma models, $N = 4$ supersymmetry is linked to the geometric concept of hyperkähler manifolds [2], this seems to point towards a connection between Mock modularity and hyperkähler geometry. The nature of this connection however, to date, is completely mysterious.

4 The Elliptic Genus of K3

Recall that the elliptic genus $\mathcal{E}_{K3}(\tau, z)$ of K3 plays center stage in our Definition 8 of K3 theories. Though this function is explicitly known and well understood, recent years have uncovered a number of mysteries around it. In the present section, some of these mysteries are discussed. This involves more open than solved problems, and as a reminder, the titles of all the following subsections are questions instead of statements.

4.1 A Non-geometric Decomposition of the Elliptic Genus?

As was mentioned at the end of Sect. 3, our very Definition 8 ensures that every K3 theory enjoys $N = (4, 4)$ supersymmetry. The current section summarizes how this induces a decomposition of the function $\mathcal{E}_{K3}(\tau, z)$, which is a priori not motivated geometrically and which turns out to bear some intriguing surprises.

In what follows, assume that we are given a K3 theory according to Definition 8 with space of states $\mathbb{H} = \mathbb{H}^{NS} \oplus \mathbb{H}^{R}$. Both $\mathbb{H}^{NS}$ and $\mathbb{H}^{R}$ can be decomposed into direct sums of irreducible unitary representations with respect to the $N = (4, 4)$ superconformal symmetry. According to [31, 32], there are three types of irreducible unitary representations of the $N = 4$ superconformal algebra at central charge $c = 6$, namely the *vacuum representation*, the *massless matter representation*, and finally the *massive matter representations* which form a one-parameter family indexed by $h \in \mathbb{R}_{>0}$. For later convenience we focus on the Ramond-sector $\mathbb{H}^{R}$ of our theory and denote the respective irreducible unitary representations by $\mathcal{H}_0, \mathcal{H}_{\text{mm}}, \mathcal{H}_h$ ($h \in \mathbb{R}_{>0}$). This notation alludes to the properties of the corresponding representations in the Neveu-Schwarz sector $\mathbb{H}^{NS}$, which are related to the representations in $\mathbb{H}^{R}$ by spectral flow Θ according to (13). Indeed, the vacuum representation in the NS-sector has the vacuum Ω as its ground state. The massive matter representations are characterized by the spontaneous breaking of supersymmetry at every mass level, including the ground state [95].

Setting $y = \exp(2\pi i z)$ and $q = \exp(2\pi i \tau)$ for $z, \tau \in \mathbb{C}$ with $\Im(\tau) > 0$ as before and using $c/24 = 1/4$, the characters of the irreducible unitary $N = 4$ representations that are relevant to our discussion are denoted by

$$\chi_a(\tau, z) := \mathrm{Str}_{\mathcal{H}_a}\left(y^{J_0} q^{L_0 - 1/4}\right) = \mathrm{Tr}_{\mathcal{H}_a}\left((-1)^{J_0} y^{J_0} q^{L_0 - 1/4}\right), \quad a \in \mathbb{R}_{\geq 0} \cup \{\text{mm}\}.$$

These functions have been determined explicitly in [32]. For our purposes, only the following properties are relevant,

$$\begin{aligned}
&\chi_0(\tau, z=0) = -2, && \chi_{\text{mm}}(\tau, z=0) = 1,\\
\forall h>0\colon\quad &\chi_h(\tau,z) = q^h\widehat{\chi}(\tau,z) && \text{with}\quad \widehat{\chi}(\tau,z) = \chi_0(\tau,z) + 2\chi_{\text{mm}}(\tau,z),\\
& && \text{hence}\quad \chi_h(\tau,z=0) = \widehat{\chi}(\tau,z=0) \;=\; 0.
\end{aligned} \tag{21}$$

The constant $\chi_a(\tau, z=0)$ yields the so-called *Witten index* [95] of the respective representation.

The most general ansatz for a decomposition of $\mathbb{H}^R$ into irreducible representations of the two commuting $N=4$ superconformal algebras therefore reads

$$\mathbb{H}^R = \bigoplus_{a,\,\overline{a}\in\mathbb{R}_{\geq 0}\cup\{\text{mm}\}} m_{a,\overline{a}}\mathcal{H}_a \otimes \overline{\mathcal{H}_{\overline{a}}}$$

with appropriate non-negative integers $m_{a,\overline{a}}$. Then

$$\mathrm{Tr}_{\mathbb{H}^R}\left((-1)^{J_0-\overline{J}_0}y^{J_0}\overline{y}^{\overline{J}_0}q^{L_0-1/4}\overline{q}^{\overline{L}_0-1/4}\right) = \sum_{a,\,\overline{a}\in\mathbb{R}_{\geq 0}\cup\{\text{mm}\}} m_{a,\overline{a}}\cdot\chi_a(\tau,z)\cdot\overline{\chi_{\overline{a}}(\tau,z)},$$

together with Definition 6 yields the conformal field theoretic elliptic genus of our CFT as

$$\mathcal{E}(\tau,z) = \sum_{a,\,\overline{a}\in\mathbb{R}_{\geq 0}\cup\{\text{mm}\}} m_{a,\overline{a}}\cdot\chi_a(\tau,z)\cdot\overline{\chi_{\overline{a}}(\tau,z=0)}. \tag{22}$$

This expression simplifies dramatically on insertion of (21). In addition, the known properties of K3 theories impose a number of constraints on the coefficients $m_{a,\overline{a}}$. First, since under spectral flow, $\mathcal{H}_0$ is mapped to the irreducible representation of the $N=4$ superconformal algebra whose ground state is the vacuum Ω, the uniqueness of the vacuum (see Property C) implies $m_{0,0}=1$. Moreover, from the proof of Proposition 2 (see the Appendix) or from the known explicit form (20) of $\mathcal{E}(\tau,z)$, we deduce that in every K3 theory, $m_{0,\text{mm}} = m_{\text{mm},0} = 0$. Finally, according to the discussion of Property B in Sect. 2.2, $\mathbb{H}_{h,\overline{h}}\cap\mathbb{H}_b$ can only be non-trivial if $h-\overline{h}\in\mathbb{Z}$, which on $\mathbb{H}_{h,\overline{h}}\cap\mathbb{H}_f\cap\mathbb{H}^{NS}$ generalizes to $h-\overline{h}\in\frac{1}{2}+\mathbb{Z}$. Since the groundstates of $\mathcal{H}_0$, $\mathcal{H}_{\text{mm}}$, $\mathcal{H}_h$ under spectral flow yield states with L_0-eigenvalues 0, $\frac{1}{2}$, h, and J_0-eigenvalues 0, ± 1, 0, respectively [32], this implies that $m_{0,\overline{h}}$, $m_{\text{mm},\overline{h}}$, $m_{h,0}$, $m_{h,\text{mm}}$ with $h,\overline{h}>0$ can only be non-zero if $h,\overline{h}\in\mathbb{N}$. In conclusion, we obtain a refined ansatz for the $N=(4,4)$ decomposition of $\mathbb{H}^R$,

$$\begin{aligned}
\mathbb{H}^R = \mathcal{H}_0\otimes\overline{\mathcal{H}_0} \oplus h^{1,1}\mathcal{H}_{\text{mm}}\otimes\overline{\mathcal{H}_{\text{mm}}} \oplus \bigoplus_{h,\,\overline{h}\in\mathbb{R}_{>0}} k_{h,\overline{h}}\mathcal{H}_h\otimes\overline{\mathcal{H}_{\overline{h}}}&\\
\oplus\bigoplus_{n=1}^{\infty}\left[f_n\mathcal{H}_n\otimes\overline{\mathcal{H}_0}\oplus\overline{f_n}\mathcal{H}_0\otimes\overline{\mathcal{H}_n}\right]&\\
\oplus\bigoplus_{n=1}^{\infty}\left[g_n\mathcal{H}_n\otimes\overline{\mathcal{H}_{\text{mm}}}\oplus\overline{g_n}\mathcal{H}_{\text{mm}}\otimes\overline{\mathcal{H}_n}\right].&
\end{aligned} \tag{23}$$

Here, all the coefficients $h^{1,1}$, $k_{h,\overline{h}}$, f_n, $\overline{f}_n$, g_n, $\overline{g}_n$ are non-negative integers, whose precise values depend on the specific K3 theory under inspection.

By (22), and inserting (21) and the refined ansatz (23), we obtain

$$\begin{aligned}\mathcal{E}(\tau, z) &= -2\chi_0(\tau, z) + h^{1,1}\chi_{\mathrm{mm}}(\tau, z) + \sum_{n=1}^{\infty}[-2f_n + g_n]\,\chi_n(\tau, z)\\ &= -2\chi_0(\tau, z) + h^{1,1}\chi_{\mathrm{mm}}(\tau, z) + e(\tau)\,\widehat{\chi}(\tau, z) \text{ with } e(\tau) := \sum_{n=1}^{\infty}[g_n - 2f_n]\,q^n.\end{aligned}$$

Now recall from Definition 8 that $\mathcal{E}(\tau, z) = \mathcal{E}_{\mathrm{K3}}(\tau, z)$, where by the discussion preceding (20) we have $\mathcal{E}_{\mathrm{K3}}(\tau, z = 0) = 24$. Using (21), this implies $h^{1,1} = 20$. Since the geometric elliptic genus $\mathcal{E}_{\mathrm{K3}}(\tau, z)$ is a topological invariant of all K3 surfaces, we conclude

Proposition 4 *The elliptic genus $\mathcal{E}_{\mathrm{K3}}(\tau, z)$ of K3 decomposes into the characters of irreducible unitary representations of the $N = 4$ superconformal algebra in the Ramond sector according to*

$$\mathcal{E}_{\mathrm{K3}}(\tau, z) = -2\chi_0(\tau, z) + 20\chi_{\mathrm{mm}}(\tau, z) + e(\tau)\,\widehat{\chi}(\tau, z),$$
$$\textit{where}\ \ e(\tau) := \sum_{n=1}^{\infty}[g_n - 2f_n]\,q^n,$$

and the coefficients g_n, f_n give the respective multiplicities of representations in the decomposition (23)*. While the values of g_n, f_n vary within the moduli space of K3 theories, the coefficients $g_n - 2f_n$ of $e(\tau)$ are invariant.*

A decomposition of $\mathcal{E}_{\mathrm{K3}}(\tau, z)$ in the spirit of Proposition 4 was already given in [30]. In [77] and independently in [92, Conjecture7.2.2] it was conjectured that all coefficients of the function $e(\tau)$ are non-negative, for the following reason: Recall that under spectral flow, the irreducible representation $\mathcal{H}_0$ is mapped to the representation of the $N = 4$ superconformal algebra whose ground state is the vacuum Ω. Therefore, in (23), the coefficients f_n determine those contributions to the subspace $W_n \subset W$ of the vector space underlying the chiral algebra of Property C that do *not* belong to the vacuum representation under the $N = (4, 4)$ supersymmetry. For any fixed value of $n \in \mathbb{N}$ with $n > 0$, we generically expect no such additional contributions to W_n. In other words, we expect that generically $f_n = 0$ and thus that the n^{th} coefficient of $e(\tau)$ agrees with $g_n \geq 0$. Since these coefficients are invariant on the moduli space of K3 theories, they should always be non-negative.

The conjectured positivity of the coefficients $g_n - 2f_n$ is proved in [27, 29] in the context of an intriguing observation. Namely, in [29], Eguchi, Ooguri and Tachikawa observe that each of these coefficients seems to give the dimension of a representation of a certain sporadic group, namely of the *Mathieu group M_{24}*. For small values of n, they find dimensions of irreducible representations, while at higher order, more work is required to arrive at a well-defined conjecture. The quest for understanding this observation, which is often referred to as *Mathieu Moonshine*, has sparked enormous interest in the mathematical physics community. Building on results of [19, 28, 44, 45], the observation has been recently verified by Gannon in the following form:

Theorem 5 ([50]) *There are virtual representations of the Mathieu group M_{24} on spaces $\mathcal{R}_0$, $\mathcal{R}_{\mathrm{mm}}$, and true representations on spaces $\mathcal{R}_n$, $n \in \mathbb{N}_{>0}$, such that*

$$\mathcal{R} := \mathcal{H}_0 \otimes \mathcal{R}_0 \oplus \mathcal{H}_{\mathrm{mm}} \otimes \mathcal{R}_{\mathrm{mm}} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_n \otimes \mathcal{R}_n$$

has the following properties: With the $N = 4$ superconformal algebra acting non-trivially only on the first factor in each summand of $\mathcal{R}$, and the Mathieu group M_{24} acting non-trivially only on the second factor, one obtains functions

$$\forall g \in M_{24}: \quad \mathcal{E}_g(\tau, z) := Tr_{\mathcal{R}} \left(g y^{J_0} q^{L_0 - 1/4} \right)$$

which under modular transformations generate a collection of M_{24}-TWISTED ELLIPTIC GENERA *of K3. In particular, $\mathcal{E}_{\mathrm{id}}(\tau, z) = \mathcal{E}_{\mathrm{K3}}(\tau, z)$.*

4.2 A Geometric Mathieu Moonshine Phenomenon?

While Theorem 5 beautifully specifies a well-defined formulation of the Mathieu Moonshine observation and proves it, the proof does not offer any insight into the role of the Mathieu group M_{24} in the context of K3 theories. The present section summarizes some ideas for a possible interpretation that is based in geometry.

Indeed, the relevance of the group M_{24} for the geometry of K3 surfaces had been discovered much earlier by Mukai:

Theorem 6 ([73]) *Let G denote a finite group of* SYMPLECTIC AUTOMORPHISMS *of a K3 surface X. By this we mean that X denotes a K3 surface whose complex structure has been fixed, and that G is a finite group of biholomorphic maps on X whose induced action on the holomorphic volume form is trivial.*

Then G is a subgroup of the Mathieu group M_{24}. More precisely, G is a subgroup of one out of a list of 11 *subgroups of $M_{23} \subset M_{24}$, the largest of which has order* 960.

Hence although M_{24} does play a crucial role in describing symplectic automorphisms of K3 surfaces, Theorem 6 cannot immediately explain Mathieu Moonshine. Indeed, Mathieu Moonshine suggests that there is an action of the entire group M_{24} on some mathematical object which underlies the elliptic genus of K3, while Theorem 6 implies that no K3 surface allows M_{24} as its symplectic automorphism group.

Namely, the theorem states that the maximal order of a symplectic automorphism group of any K3 surface is 960, which is smaller by orders of magnitude than the order 244.823.040 of M_{24}.

Since the *non-geometric* decomposition of $\mathcal{E}_{K3}(\tau, z)$ by means of $N = 4$ supersymmetry presented in Sect. 4.1 led to the discovery of Mathieu Moonshine, one may suspect that rather than the properties of K3 surfaces, the properties of K3 theories should explain the Mathieu Moonshine phenomena. However, symmetry groups of K3 theories, in general, need not be subgroups of M_{24}, as apparently was first noted independently by the authors of [29, 85]. Conversely, no K3 theory can have M_{24} as its symmetry group, as follows from [46], where Gaberdiel, Hohenegger and Volpato generalize a very enlightening second proof of Theorem 6 due to Kondo [64] to a classification result for symmetries of K3 theories.

Because by the above, the symmetries of K3 theories seem not to explain Mathieu Moonshine, in a series of papers [85–87] it has been argued that possibly, the action of M_{24} arises as a *combined action* of all finite symplectic symmetry groups of K3 surfaces. This idea can be motivated by the mathematical properties of the elliptic genus which were presented in Sects. 2.4 and 2.5:

By Theorem 3, the geometric elliptic genus $\mathcal{E}_{K3}(\tau, z)$ is recovered from the chiral de Rham complex Ω_X^{ch} of a K3 surface X as its graded Euler characteristic, that is, as the supertrace of the appropriate operator on the Čech cohomology $H^*(X, \Omega_X^{\mathrm{ch}})$. In accord with [56, (2.1.3)], one can expect that every symplectic automorphism of a K3 surface X induces an action on $H^*(X, \Omega_X^{\mathrm{ch}})$. Therefore, the Čech cohomology $H^*(\mathrm{K3}, \Omega_{\mathrm{K3}}^{\mathrm{ch}})$ of the chiral de Rham complex appears to be an excellent candidate for the desired mathematical object which both underlies the elliptic genus, and which carries actions of all finite symplectic automorphism groups of K3 surfaces, thus combining them to the action of a possibly larger group.

Note that according to Theorem 3, there exists a natural structure of a vertex algebra on $H^*(\mathrm{K3}, \Omega_{\mathrm{K3}}^{\mathrm{ch}})$. This additional structure on the mathematical object which underlies the elliptic genus is in complete accord with the implications of Theorem 5. Indeed, it was already conjectured in [47, 48], that Mathieu Moonshine is governed by some vertex algebra which carries an M_{24}-action, whose properties would immediately induce the modular transformation properties of the twisted elliptic genera of Theorem 5. As was argued in the discussion of Theorem 3, $H^*(X, \Omega_X^{\mathrm{ch}})$ is moreover expected to be related to a non-linear sigma model on X, at least in a large volume limit, providing the desired link to K3 theories. Indeed, the required compatibility with a large volume limit might also explain the restriction to those symmetries of K3 theories which can be induced by some symplectic automorphism of a K3 surface, as seems to be the case for the generators of M_{24} in Mathieu Moonshine.

Unfortunately, despite all its convincing properties promoting it to an excellent candidate to resolve Mathieu Moonshine, the vertex algebra structure of $H^*(X, \Omega_X^{\mathrm{ch}})$ is notoriously hard to calculate, as are the precise properties of general non-linear sigma models on K3, even in a large volume limit. Therefore, these ideas remain conjectural, so far. Sadly, known alternative constructions for vertex algebras that underlie the elliptic genus and that are easier to calculate seem not to explain Mathieu Moonshine [21].

A possible mechanism of combining symplectic automorphism groups of distinct K3 surfaces to larger groups is presented in [85, 87], and the following result can be seen as evidence in favor of these ideas:

Proposition 5 ([86]) *Consider the "smallest massive" representation of M_{24} that occurs in Theorem 5, that is, the representation on $\mathcal{R}_1$.*

The space $\mathcal{R}_1$ is isomorphic to a certain vector space V^{CFT} of states which is common to all K3 theories that are obtained by a standard $\mathbb{Z}_2$-orbifold construction from a toroidal $N = (2, 2)$ superconformal field theory. Moreover, on V^{CFT}, the combined action of all finite symplectic automorphism groups of Kummer surfaces induces a faithful action of the maximal subgroup $\mathbb{Z}_2^4 \rtimes A_8$ of order 322.560 *in M_{24}. The resulting representation on V^{CFT} is equivalent to the representation of $\mathbb{Z}_2^4 \rtimes A_8$ on $\mathcal{R}_1$ which is induced by restriction from M_{24} to this subgroup.*

This result is the first piece of evidence in the literature for any trace of an M_{24}-action on a space of states of a K3 theory. It is remarkable that the CFT techniques produce precisely the representation of a maximal subgroup of M_{24} which is predicted by Mathieu Moonshine according to the idea of "combining symplectic automorphism groups". Note that the group $\mathbb{Z}_2^4 \rtimes A_8$ is not a subgroup of M_{23}, indicating that indeed M_{24} rather than M_{23} should be expected to be responsible for Mathieu Moonshine, despite Theorem 6, by which all finite symplectic automorphism groups of K3 surfaces are subgroups of M_{23}. This preference for M_{24} to M_{23} is in accord with the findings of [50].

Encouraged by Proposition 5, one may hope that in a large volume limit, V^{CFT} can be identified with a subspace of $H^*(\mathrm{K3}, \Omega^{\mathrm{ch}}_{\mathrm{K3}})$, inducing an equivalence of vertex algebras. Furthermore, by combining the action of $\mathbb{Z}_2^4 \rtimes A_8$ with the action of finite symplectic automorphism groups of K3 surfaces which are not Kummer, one may hope to generate an action of the entire group M_{24}. Finally, one may hope that this result generalizes to the remaining representations on $\mathcal{R}_n$, $n > 1$, found in Theorem 5. In conclusion, there is certainly much work left.

4.3 A Geometric Decomposition of the Elliptic Genus?

Even if the ideas presented in Sect. 4.2 prove successful, then so far, they give no indication for the reason for M_{24} -of all groups- to arise from the combined action of finite symplectic automorphism groups of K3 surfaces. Circumventing this intrinsic problem, the current section presents a simpler conjecture which can be formulated independently of Mathieu Moonshine. If true, however, it could serve as a step towards understanding Mathieu Moonshine.

Taking the idea seriously that there should be a purely geometric explanation for Mathieu Moonshine, one main obstacle to unraveling its mysteries is the lack of geometric interpretation for the non-geometric decomposition of $\mathcal{E}_{\mathrm{K3}}(\tau, z)$ stated in Proposition 4, which is at the heart of the discovery of Mathieu Moonshine. Recall that the derivation of Proposition 4 rests on the identification (19) of the geometric elliptic genus of a Calabi-Yau D-manifold X with the conformal field theoretic elliptic genus of a CFT that is obtained from X by a non-linear sigma model construction. We have incorporated this identification into our Definition 8, and it is the motivation for decomposing the geometric elliptic genus of K3 into the characters of irreducible unitary representations of the $N = 4$ superconformal algebra at central charge $c = 6$. While the conformal field theoretic elliptic genus by Definition 6 is obtained as a trace over the space of states $\mathbb{H}^R$, the geometric elliptic genus by Definition 7 is an analytic trace over a formal power series $\mathbb{E}_{q,-y}$ whose coefficients are holomorphic vector bundles on our K3 surface. The decomposition of the space of states $\mathbb{H}^R$ of every K3 theory by $N = (4, 4)$ supersymmetry which was performed in Sect. 4.1 to derive Proposition 4 should accordingly be counterfeited by a decomposition of $\mathbb{E}_{q,-y}$. We thus expect

Conjecture 1 *Let X denote a K3 surface with holomorphic tangent bundle $T := T^{1,0}X$, and consider $\mathbb{E}_{q,-y}$ as in Definition 7. Furthermore, let $e(\tau)$ denote the function defined in Proposition 4. Then there are polynomials p_n, $n \in \mathbb{N}_{>0}$, such that*

$$\mathbb{E}_{q,-y} = -\mathcal{O}_X \cdot \chi_0(\tau, z) + T \cdot \chi_{\text{mm}}(\tau, z) + \sum_{n=1}^{\infty} p_n(T) \cdot q^n \widehat{\chi}(\tau, z),$$

$$\text{and} \quad e(\tau) = \sum_{n=1}^{\infty} \left(\int_X Td(X) p_n(T) \right) \cdot q^n,$$

where $p_k(T) = \sum_{n=0}^{N_k} a_n T^{\otimes n}$ if $p_k(x) = \sum_{n=0}^{N_k} a_n x^n$, and where $T^{\otimes 0} = \mathcal{O}_X$ is understood.

If (19) is interpreted as a generalization of the McKean-Singer Formula, as indicated in the discussion of that equation, then Conjecture 1 can be viewed as a generalization of a *local index theorem* [4, 51, 52, 80]. Note that the conjecture is formulated without even alluding to Mathieu Moonshine, so it may be of independent interest. If true, then for each $n \in \mathbb{N}_{>0}$, every finite symplectic automorphism group of a K3 surface X naturally acts on $p_n(T)$, and one may hope that this will yield insight into the descent of this action to the representation of M_{24} on $\mathcal{R}_n$ which was found in Theorem 5.

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Appendix—Proof of Proposition 2 in Sect. 3

The entire proof of Proposition 2 rests on the study of the $+1$-eigenspace of the linear operator $\overline{J}_0$ on the subspace $\overline{W}_{1/2}$ of the vector space $\overline{W}$ underlying the chiral algebra. First, one shows that this eigenspace is either trivial or two-dimensional, and from this one deduces claim 1. of the proposition. One direction of claim 2. is checked by direct calculation, using the defining properties of toroidal $N = (2, 2)$ superconformal field theories. To obtain the converse, one shows that $\mathcal{E}(\tau, z) \equiv 0$ implies that an antiholomorphic counterpart of the conformal field theoretic elliptic genus must vanish as well, from which claim 2 is shown to follow.

1. Assume that the space $\overline{W}_{1/2}$ contains an eigenvector of $\overline{J}_0$ with eigenvalue $+1$. We denote the field associated to this state by $\overline{\psi}_1^+(\overline{z})$. The properties of the real structure on the space of states $\mathbb{H}$ of our CFT imply that there is a complex conjugate state with $\overline{J}_0$-eigenvalue -1 whose associated field we denote by $\overline{\psi}_1^-(\overline{z})$. The properties of unitary irreducible representations of the Virasoro algebra imply that these fields form a *Dirac fermion* [see the discussion around (15)], and that therefore $\overline{J}_1(\overline{z}) := \frac{1}{2} :\overline{\psi}_1^+\overline{\psi}_1^-:(\overline{z})$ is a $U(1)$-current as in Example 1 in Sect. 2.1. By a procedure known as *GKO-construction* [54], one obtains $\overline{J}(\overline{z}) = \overline{J}_1(\overline{z}) + \overline{J}_2(\overline{z})$ for the field $\overline{J}(\overline{z})$ in the $N = 2$ superconformal algebra (9)–(10), and $\overline{J}_k(\overline{z}) = i\partial\overline{H}_k(\overline{z})$ with $\overline{\psi}_1^\pm(\overline{z}) =: e^{\pm i\overline{H}_1} : (\overline{z})$. The fields of twofold right-handed spectral flow, which by assumption are fields of the theory, are moreover given by $\overline{J}^\pm(\overline{z}) =: e^{\pm i(\overline{H}_1+\overline{H}_2)} : (\overline{z})$. Their OPEs with the $\overline{\psi}_1^\pm(\overline{z})$ yield an additional Dirac-fermion, with fields $\overline{\psi}_2^\pm(\overline{z}) :=: e^{\pm i\overline{H}_2}:(\overline{z})$ in the CFT. This proves that the ±1-eigenspaces of $\overline{J}_0$ on $\overline{W}_{1/2}$ each are precisely two-dimensional, since by the same argument no further Dirac fermions can be fields of the theory. Note that by definition, the corresponding states belong to the sector $\mathbb{H}_f \cap \mathbb{H}^{NS} \subset \mathbb{H}$ of the space of states of our theory.
   In summary, the $+1$-eigenspace of the linear operator $\overline{J}_0$ on $\overline{W}_{1/2}$ is either trivial or two-dimensional.

   We now study the leading order contributions in the conformal field theoretic elliptic genus $\mathcal{E}(\tau, z)$ of our theory. From (20) and by the very Definition 6 we deduce that $2ay^{-1}$ counts states in the subspace $V \subset \mathbb{H}^R$ where L_0, $\overline{L}_0$ both take eigenvalue $\frac{c}{24} = \frac{1}{4} = \frac{\overline{c}}{24}$ and J_0 takes eigenvalue -1. More precisely,

$$2a = \mathrm{Tr}_V\left((-1)^{J_0-\overline{J}_0}\right) = -\mathrm{Tr}_V\left((-1)^{\overline{J}_0}\right).$$

As follows from properties of the so-called *chiral ring*, see e.g. [92, Sect. 3.1.1], a basis of V is obtained by spectral flow Θ (see Ingredient IV in Sect. 2.2) from (i) the vacuum Ω, (ii) the state whose corresponding field is $\overline{J}^+(\overline{z})$, and (iii) a basis of the $+1$-eigenspace of the linear operator $\overline{J}_0$ on $\overline{W}_{1/2}$. Since according to (13), the eigenvalues of $\overline{J}_0$ after spectral flow to V are (i) -1, (ii) $+1$, (iii) 0, the above trace vanishes if the $+1$-eigenspace of the linear operator $\overline{J}_0$ on $\overline{W}_{1/2}$ is two-dimensional, implying $2a = 0$, and if this eigenspace is trivial, then we obtain $2a = 2$.

In conclusion, the conformal field theoretic elliptic genus of our theory either vanishes, in which case the $+1$-eigenspace of the linear operator $\overline{J}_0$ on $\overline{W}_{1/2}$ is created by two Dirac fermions, or $\mathcal{E}(\tau, z) = \mathcal{E}_{\mathrm{K3}}(\tau, z)$. □

2. a. Using the details of toroidal $N = (2, 2)$ superconformal field theories that are summarized in Sect. 2.3, one checks by a direct calculation that the conformal field theoretic elliptic genus of all such theories vanishes. □

   b. To show the converse, first observe that in our discussion of $N = (2, 2)$ superconformal field theories, the two commuting copies of a superconformal algebra are mostly treated on an equal level. However, the Definition 6 breaks this symmetry, and

$$\overline{\mathcal{E}}(\overline{\tau}, \overline{z}) := \mathrm{Tr}_{\mathbb{H}^R}\left((-1)^{J_0-\overline{J}_0}\overline{y}^{\overline{J}_0} q^{L_0-c/24}\overline{q}^{\overline{L}_0-\overline{c}/24}\right)$$

should define an equally important antiholomorphic counterpart of the conformal field theoretic elliptic genus. In our case by the same reasoning as for $\mathcal{E}(\tau, z)$, it must yield zero or $\overline{\mathcal{E}_{\mathrm{K3}}(\tau, z)}$. Note that Proposition 1 implies that

$$\mathcal{E}(\tau, z = 0) = \overline{\mathcal{E}}(\overline{\tau}, \overline{z} = 0)$$

is a constant, which in fact is known as the *Witten index* [95–97]. In particular, by (18) we have $\mathcal{E}_{\mathrm{K3}}(\tau, z = 0) = 24$, hence $\mathcal{E}(\tau, z) \equiv 0$ implies $\overline{\mathcal{E}}(\overline{\tau}, \overline{z}) \equiv 0$. It remains to be shown that our theory is a toroidal theory according to Definition 5 in this case.

But Step 1. of our proof then implies that the $+1$-eigenspace of the linear operator $\overline{J}_0$ on $\overline{W}_{1/2}$ is created by two Dirac fermions and that the analogous statement holds for the $+1$-eigenspace of the linear operator J_0 on $W_{1/2}$. Hence we have Dirac fermions $\psi_k^{\pm}(z)$ and $\overline{\psi}_k^{\pm}(\overline{z})$, $k \in \{1, 2\}$, with OPEs as in (15). Compatibility with supersymmetry then implies that the superpartners of these fields yield the two $\mathfrak{u}(1)^4$-current algebras, as is required in order to identify our theory as a toroidal one. □

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Part II
Chern–Simons Theory

The foundational work of Witten has led to tremendously fruitful interactions between topological field theory and low-dimensional topology. After his major breakthrough (Witten observed that one could recover link and 3-manifold invariants *via* the path integral quantization of the Chern–Simons classical action functional) Reshetikhin and Turaev proposed a rigorous mathematical construction of (non-perturbative!) quantizations of Chern–Simons theory in terms of quantum groups and modular tensor categories. Despite this great achievement, there are many questions that remain open, both of computational and theoretical nature.

Part II begins with "Faddeev's Quantum Dilogarithm and State-integrals on Shaped Triangulations" (written by Jørgen Ellegaard Andersen and Rinat Kashaev) on a construction of $SL(2,\mathbb{C})$ quantum Chern–Simons theory by means of Teichmüller theory and the quantum dilogarithm of Faddeev. Explicit calculations of the partition function and a variant of the volume conjecture are provided.

It continues with "A Higher Stacky Perspective on Chern–Simons Theory" (written by Domenico Fiorenza, Hisham Sati and Urs Schreiber) discussing the appearance of higher gerbes in classical Chern–Simons theories (which do not only exist in dimension 3). Many important concepts briefly discussed in the introductory chapter make a first appearance here: fully extended TFTs, higher structures, and mapping stacks.

"Factorization Homology in 3-Dimensional Topology" consists of two short contributions (by Nikita Markarian and Hiro Lee Tanaka) on the relation between Chern–Simons theory and factorization homology. Both seem to rely on the fact that, at the perturbative level, Chern–Simons theory is a fully extended TFT.

Part II ends with "Deligne-Beilinson Cohomology in U(1) Chern–Simons Theories" (written by Frank Thuillier) that one could view as a variation on "A Higher Stacky Perspective on Chern–Simons Theory." Namely, it reviews in detail the use of Deligne–Beilinson cohomology in $U(1)$ Chern–Simons theory, and characteristic classes of higher gerbes with connection precisely take values in Deligne–Beilinson cohomology.

Faddeev's Quantum Dilogarithm and State-Integrals on Shaped Triangulations

Jørgen Ellegaard Andersen and Rinat Kashaev

Abstract Using Faddeev's quantum dilogarithm function, we review our description of a one parameter family of state-integrals on shaped triangulated pseudo 3-manifolds. This invariant is part of a certain TQFT, which we have constructed previously in a number of papers on the subject.

1 Introduction

In this article we review few key points of our paper [10, 12] dedicated to the construction of a certain kind of TQFT which we believe is related to the exact quantum partition functions of Chern–Simons theory with gauge group $PSL(2, \mathbb{C})$. See also [11], where we provide a generalisation of the construction reviewed here.

Recall that for a given finite-dimensional simple Lie group G and a 3-manifold M, the Chern–Simons action functional is defined as follows:

$$CS_M(A) := \int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \tag{1}$$

where $A \in \mathscr{A} := \Omega^1(M, Lie\, G)$ is the Lie algebra valued gauge field. This action functional is gauge invariant with respect to (small) gauge transformations given by elements of the connected component of the identity $\mathscr{G}_0$ in the group $\mathscr{G} :=$

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J.E. Andersen (✉)
Center for Quantum Geometry of Moduli Spaces, University of Aarhus, 8000 Aarhus, Denmark
e-mail: andersen@qgm.au.dk

R. Kashaev
University of Geneva, 2-4 Rue du Lièvre, Case Postale 64, 1211 Genève 4, Switzerland
e-mail: rinat.kashaev@unige.ch

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$\mathscr{C}^\infty(M, G)$ of G-valued smooth functions on M, the action being given by the formula

$$\mathscr{A} \times \mathscr{G}_0 \to \mathscr{A}, \quad (A, g) \mapsto A^g := g^{-1}Ag + g^{-1}dg. \tag{2}$$

The critical points of the action functional (1) correspond to flat connections, and together with the invariance under the gauge transformations (2), the phase space of the field-theoretical model can be identified with the moduli space of flat G-connections

$$H^1(M, G) := \hom(\pi_1(M), G)/G. \tag{3}$$

Following Witten's proposal [53], the object of interest in quantum theory is the quantum partition function defined formally by the following path integral

$$Z_\hbar(M) = \int_{\mathscr{A}/\mathscr{G}_0} e^{\frac{i}{\hbar} CS_M(A)} \mathscr{D}A \tag{4}$$

which, unfortunately, mathematically is not well defined. In this form, it can only be used for asymptotical quasi-classical expansions or for writing some other path integrals by performing formal changes of variables. Using the fact that the phase space (3) is finite-dimensional, the problem of defining quantum Chern–Simons theory is eventually much simpler than the quantization problem of any non-linear field theoretical model with infinite-dimensional phase space. This fact justifies the hope that one can give an alternative and mathematically rigorous definition for the partition function (4). In the case of compact Lie groups this hope was indeed materialized in the works of Reshetikhin and Turaev [20, 51, 54], and the resulting theory has been a source of numerous new topological invariants of 3-manifolds. Further more the geometric quantisation of the moduli spaces (3) was performed first by Axelrod, Della Pietra and Witten [18] and then subsequently by Hitchin [41]. For a purely differential geometric account of the construction of this connection see [4, 6, 7]. By combining the work of Laszlo [47] with the work of Ueno and the first author of this paper [13–16], it has now been confirmed that one can use the geometric quantisation of the moduli space of flat connections as an alternative construction of the Witten–Reshetikhin–Turaev TQFT. This has been exploited in the works [1–3, 5, 8, 9, 17].

Nonetheless, the case of non-compact Lie groups still lacks a mathematically rigorous formulation despite some progress in developing direct field-theoretical approaches originally done in the works of Witten [55, 56]. For the latest mathematical developments please see [11] and the so called index by Garoufalidis and Dimofte [27, 35], which should be related to the level $k = 0$ theory. In the physics literature, the complex quantum Chern–Simons theory has been discussed from a path integral point of view in a number of papers [19, 24, 26, 28, 29, 36, 37, 39, 40, 56] and latest by Dimofte [25] using the more advanced 3d-3d correspondence.

Quantum theory of Teichmüller spaces of [34, 43] can be considered as a part of quantum Chern–Simons theory with gauge group $PSL(2, \mathbb{R})$. Its main ingredient, given by the operator realizing the diagonal flip in ideal triangulations of punctured surfaces, can naturally be interpreted as the partition function of a tetrahedron seen as a 3-manifold with boundary. Based on this interpretation and the quasi-classical expansion of the quantum dilogarithm [31, 32], formal quantum partition functions of triangulated 3-manifolds in the form of finite-dimensional integrals were suggested in [22, 23, 29, 39, 40] but without analyzing the convergence or topological invariance. The partition function which we describe in this article also uses the diagonal flip operator of quantum Teichmüller theory, but unlike the previous attempts, our definition is mathematically rigorous. In the case of particular examples of knot complements, the quasi-classical behavior reveals the hyperbolic volume which makes it possible that our partition functions are related with the partition function (4) in the case of the gauge group $G = PSL(2, \mathbb{C})$, the group of orientation preserving isometries of the 3-dimensional hyperbolic space.

2 Faddeev's Quantum Dilogarithm

For $\hbar \in \mathbb{R}_{>0}$, Faddeev's quantum dilogarithm function is defined by the formula [31]

$$\Phi_\hbar(z) = (\bar{\Phi}_\hbar(z))^{-1} = \exp\left(\int_{\mathbb{R}+i0} \frac{e^{-i2xz}}{4\sinh(x\mathsf{b})\sinh(x\mathsf{b}^{-1})x}dx\right) \tag{5}$$

in the strip $|\Im z| < \frac{1}{2\sqrt{\hbar}}$, where

$$(\mathsf{b} + \mathsf{b}^{-1})^2 = \hbar^{-1}, \tag{6}$$

and extended to the whole complex plane through the functional equations

$$\Phi_\hbar(z - i\mathsf{b}^{\pm 1}/2) = (1 + e^{2\pi\mathsf{b}^{\pm 1}z})\Phi_\hbar(z + i\mathsf{b}^{\pm 1}/2). \tag{7}$$

There is an alternative integral formula due to Woronowicz [57]:

$$\Phi_\hbar(z) = \exp\left(\frac{i}{2\pi}\int_{\mathbb{R}} \log\left(1 + e^{\mathsf{b}^2 t}\right)\frac{dt}{1 + e^{t-2\pi\mathsf{b}^{-1}z}}\right). \tag{8}$$

Among the set of solutions of the Eq. (6) for the parameter b, there is a unique choice with $\Re\mathsf{b} > 0$ and $\Im\mathsf{b} \geq 0$. If $\Im\mathsf{b} > 0$ (i.e. $\hbar > 1/4$), then one can show that

$$\Phi_\hbar(z) = \frac{(-qe^{2\pi \mathsf{b} z}; q^2)_\infty}{(-\bar{q}e^{2\pi \mathsf{b}^{-1} z}; \bar{q}^2)_\infty} \tag{9}$$

where $q := e^{i\pi \mathsf{b}^2}$, $\bar{q} := e^{-i\pi \mathsf{b}^{-2}}$, and$(x; y)_\infty := (1-x)(1-xy)(1-xy^2)\dots$

2.1 Analytical Properties

Faddeev's quantum dilogarithm is a meromorphic function with the following zeros and poles

$$(\Phi_\hbar(z))^{\pm 1} = 0 \;\Leftrightarrow\; z = \mp\left(\frac{i}{2\sqrt{\hbar}} + mi\mathsf{b} + ni\mathsf{b}^{-1}\right),\; m, n \in \mathbb{Z}_{\geq 0}, \tag{10}$$

and the essential singularity at infinity. Its precise behavior at infinity depends on the direction along which one goes:

$$\Phi_\hbar(z)\Big|_{|z|\to\infty} \approx \begin{cases} 1 & |\arg z| > \frac{\pi}{2} + \arg \mathsf{b} \\ \zeta_{inv}^{-1} e^{i\pi z^2} & |\arg z| < \frac{\pi}{2} - \arg \mathsf{b} \\ \frac{(\bar{q}^2;\bar{q}^2)_\infty}{\Theta(i\mathsf{b}^{-1}z;-\mathsf{b}^{-2})} & |\arg z - \frac{\pi}{2}| < \arg \mathsf{b} \\ \frac{\Theta(i\mathsf{b}z;\mathsf{b}^2)}{(q^2;q^2)_\infty} & |\arg z + \frac{\pi}{2}| < \arg \mathsf{b} \end{cases} \tag{11}$$

where

$$\zeta_{inv} := e^{\pi i(2-\hbar^{-1})/12}, \tag{12}$$

and

$$\Theta(z;\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i \tau n^2 + 2\pi i z n}, \quad \Im\tau > 0. \tag{13}$$

It has simple behavior with respect to negation of the argument given by the inversion relation:

$$\Phi_\hbar(z)\Phi_\hbar(-z) = \zeta_{inv}^{-1} e^{i\pi z^2}. \tag{14}$$

It also behaves nicely with respect to complex conjugation:

$$\overline{\Phi_\hbar(z)}\Phi_\hbar(\bar{z}) = 1. \tag{15}$$

2.2 Five Term Quantum Identity

In terms of specifically normalized self adjoint Heisenberg's momentum and position operators in $L^2(\mathbb{R})$ defined by the formulae

$$\mathbf{p}f(x) = \frac{1}{2\pi i} f'(x), \quad \mathbf{q}f(x) = xf(x), \tag{16}$$

the following five term or pentagon quantum identity for unitary operators is satisfied [30, 33, 57]

$$\Phi_\hbar(\mathbf{p})\Phi_\hbar(\mathbf{q}) = \Phi_\hbar(\mathbf{q})\Phi_\hbar(\mathbf{p}+\mathbf{q})\Phi_\hbar(\mathbf{p}). \tag{17}$$

2.3 Fourier Transformation Formulae

The pentagon identity is equivalent to the following Fourier transformation formula [33]:

$$\int_{\mathbb{R}} \frac{\Phi_\hbar(x+u)}{\Phi_\hbar\left(x - \frac{i}{2\sqrt{\hbar}} + i0\right)} e^{-2\pi i w x}\, dx = \zeta_o \frac{\Phi_\hbar(u)\, \Phi_\hbar\left(\frac{i}{2\sqrt{\hbar}} - w\right)}{\Phi_\hbar(u-w)} \tag{18}$$

where $0 < \Im w < \Im u < \frac{1}{2\sqrt{\hbar}}$, and

$$\zeta_o := e^{\frac{\pi i}{12}\left(1+\frac{1}{\hbar}\right)}. \tag{19}$$

A particular case of this formula is the following Fourier transformation formulae for Faddeev's quantum dilogarithm:

$$\int_{\mathbb{R}} \bar{\Phi}_\hbar(x) e^{-2\pi i w x}\, dx = \zeta_o e^{-\frac{\pi w}{\sqrt{\hbar}}} \Phi_\hbar\left(\frac{i}{2\sqrt{\hbar}} - w\right) \tag{20}$$

and

$$\int_{\mathbb{R}} \Phi_\hbar(x) e^{-2\pi i w x}\, dx = \zeta_o e^{-\pi i w^2} \Phi_\hbar\left(\frac{i}{2\sqrt{\hbar}} - w\right). \tag{21}$$

2.4 Quasi-Classical Asymptotics

For $\mathsf{b} \to 0$ and fixed x, one has the following asymptotic expansion

$$\ln \Phi_\hbar\left(\frac{x}{2\pi\mathsf{b}}\right) = \sum_{n=0}^{\infty}\left(2\pi i\mathsf{b}^2\right)^{2n-1}\frac{B_{2n}(1/2)}{(2n)!}\frac{\partial^{2n}\mathrm{Li}_2(-e^x)}{\partial x^{2n}} \tag{22}$$

where $B_{2n}(1/2)$ are the Bernoulli polynomials evaluated at $1/2$. In particular, we have

$$\Phi_\hbar\left(\frac{x}{\sqrt{\hbar}}\right)\bigg|_{\hbar\to 0} \sim e^{\frac{1}{2\pi i\hbar}\mathrm{Li}_2(-e^{2\pi x})}. \tag{23}$$

3 The Tetrahedral Operator of Quantum Teichmüller Theory

In this section, we recall the main algebraic ingredients of quantum Teichmüller theory, following the approach of [43–45]. First, we consider the usual canonical quantization of $T^*(\mathbb{R}^n)$ with the standard symplectic structure in the position representation, i.e. with respect to the vertical real polarization. The Hilbert space we get is of course just $L^2(\mathbb{R}^n)$. The specifically normalized position coordinates q_i and momentum coordinates p_i on $T^*(\mathbb{R}^n)$ upon quantization become selfadjoint unbounded operators $\mathbf{q}_i$ and $\mathbf{p}_i$ acting on $L^2(\mathbb{R}^n)$ via the formulae

$$\mathbf{q}_j(f)(t) = t_j f(t), \quad \mathbf{p}_j(f)(t) = \frac{1}{2\pi i}\frac{\partial}{\partial t_j} f(t), \quad \forall t \in \mathbb{R}^n,$$

satisfying the Heisenberg commutation relations

$$[\mathbf{p}_j, \mathbf{p}_k] = [\mathbf{q}_j, \mathbf{q}_k] = 0, \quad [\mathbf{p}_j, \mathbf{q}_k] = (2\pi i)^{-1}\delta_{j,k}. \tag{24}$$

By the spectral theorem, we can define operators

$$\mathbf{u}_i = e^{2\pi\mathsf{b}\mathbf{q}_i}, \quad \mathbf{v}_i = e^{2\pi\mathsf{b}\mathbf{p}_i}.$$

The corresponding commutation relations between $\mathbf{u}_i$ and $\mathbf{v}_j$ take the form

$$[\mathbf{u}_j, \mathbf{u}_k] = [\mathbf{v}_j, \mathbf{v}_k] = 0, \quad \mathbf{u}_j\mathbf{v}_k = e^{2\pi i\mathsf{b}^2\delta_{j,k}}\mathbf{v}_k\mathbf{u}_j.$$

Following [43], we consider the operations for $\mathbf{w}_j = (\mathbf{u}_j, \mathbf{v}_j)$, $j \in \{1, 2\}$,

$$\mathbf{w}_1 \cdot \mathbf{w}_2 := (\mathbf{u}_1\mathbf{u}_2, \mathbf{u}_1\mathbf{v}_2 + \mathbf{v}_1) \tag{25}$$

$$\mathbf{w}_1 * \mathbf{w}_2 := (\mathbf{v}_1\mathbf{u}_2(\mathbf{u}_1\mathbf{v}_2 + \mathbf{v}_1)^{-1}, \mathbf{v}_2(\mathbf{u}_1\mathbf{v}_2 + \mathbf{v}_1)^{-1}) \tag{26}$$

Proposition 1 ([43]) *Let $\psi(z)$ be some solution of the functional equation*

$$\psi(z + i\mathsf{b}/2) = \psi(z - i\mathsf{b}/2)(1 + e^{2\pi \mathsf{b} z}), \quad z \in \mathbb{C}, \tag{27}$$

bounded along the real axis. Then the operator

$$\mathbf{T} = \mathbf{T}_{12} := e^{2\pi i \mathbf{p}_1 \mathbf{q}_2}\psi(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2) = \psi(\mathbf{q}_1 - \mathbf{p}_1 + \mathbf{p}_2)e^{2\pi i \mathbf{p}_1 \mathbf{q}_2} \tag{28}$$

is bounded and satisfies the equations

$$\mathbf{w}_1 \cdot \mathbf{w}_2 \mathbf{T} = \mathbf{T}\mathbf{w}_1, \quad \mathbf{w}_1 * \mathbf{w}_2 \mathbf{T} = \mathbf{T}\mathbf{w}_2. \tag{29}$$

Proof Boundedness of $\mathbf{T}$ follows from the fact that it is a product of unitary and bounded operators. Equation (29) is equivalent to the following system of equations:

$$\mathbf{T}\mathbf{q}_1 = (\mathbf{q}_1 + \mathbf{q}_2)\mathbf{T}, \tag{30}$$

$$\mathbf{T}(\mathbf{p}_1 + \mathbf{p}_2) = \mathbf{p}_2\mathbf{T}, \tag{31}$$

$$\mathbf{T}(\mathbf{p}_1 + \mathbf{q}_2) = (\mathbf{p}_1 + \mathbf{q}_2)\mathbf{T}, \tag{32}$$

$$\mathbf{T}e^{2\pi \mathsf{b}\mathbf{p}_1} = (e^{2\pi \mathsf{b}(\mathbf{q}_1 + \mathbf{p}_2)} + e^{2\pi \mathsf{b}\mathbf{p}_1})\mathbf{T}. \tag{33}$$

Under substitution of (28), the first three equations become identities while the forth becomes the functional equation (27).

One particular solution of (27) is given by the inverse of Faddeev's quantum dilogarithm

$$\psi(z) = \bar{\Phi}_\hbar(z) = 1/\Phi_\hbar(z) \tag{34}$$

which corresponds to a unitary operator $\mathbf{T}$. The most important property of the operator (28) with ψ given by (34) is the pentagon identity

$$\mathbf{T}_{12}\mathbf{T}_{13}\mathbf{T}_{23} = \mathbf{T}_{23}\mathbf{T}_{12} \tag{35}$$

which follows from the five-term identity (17) satisfied by $\Phi_\hbar(z)$. The indices in (35) have the standard meaning, for example, $\mathbf{T}_{13}$ is obtained from $\mathbf{T}_{12}$ by replacing $\mathbf{p}_2$ and $\mathbf{q}_2$ by $\mathbf{p}_3$ and $\mathbf{q}_3$ respectively, and so on.

4 The Pentagon Identity and the 2–3 Pachner Moves

4.1 Δ-Complexes

Following Hatcher's book on algebraic topology [38], a Δ-*complex* is a cellular complex where all cells are standard simplices, and all characteristic maps are consistent with the boundary maps, i.e. if

$$\sigma\colon \Delta^n \to X \tag{36}$$

is the characteristic map of a n-dimensional cell in a Δ-complex X, then for any boundary inclusion

$$\partial_i\colon \Delta^{n-1} \to \Delta^n, \quad i \in \{0, 1, \ldots, n\}, \tag{37}$$

$\sigma \circ \partial_i$ is the characteristic map of a $(n-1)$-dimensional cell. A manifold with a Δ-complex structure will simply be called *triangulated manifold*.

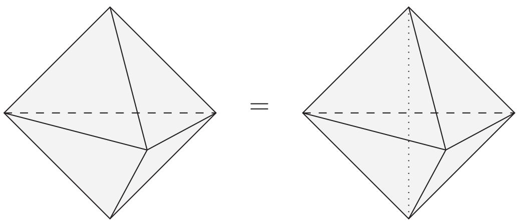
4.2 Notation

For any Δ-complex X, we denote by $\Delta_i(X)$ the set of i-dimensional cells of X and

$$\Delta_i^j(X) := \{(a, b) |\ a \in \Delta_i(X),\ b \in \Delta_j(\bar{a})\}. \tag{38}$$

4.3 2–3 Pachner Moves

Topological applications of the pentagon identity are based on its interpretation in terms of 2–3 Pachner moves between triangulated 3-manifolds which correspond to two different tetrahedral decompositions of the suspension of a triangle:



(39)

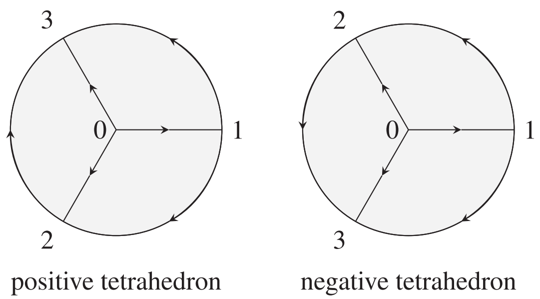
where, in the left hand side, we have two tetrahedra glued along one common face, while in the right hand side we have three tetrahedra glued so that any pair of them shares a common interior face and all three share one common interior edge which connects the two suspension points.

4.4 Branchings

To make a precise correspondence between the pentagon identity (35) and the 2–3 Pachner moves, we use the *branching* associated with any Δ-complex, i.e. the arrangement of arrows on edges without cycles. Indeed, the integers 0, 1, 2, 3 associated with the vertices of the standard tetrahedron induce an arrow on each edge which points from the smaller to the bigger end-point so that each integer expresses the number of incoming arrows at the associated vertex.

4.5 Positive and Negative Tetrahedra

In an oriented triangulated 3-manifold there are two possible relative orientations of each tetrahedron as is seen in these pictures:



(40)

4.6 Tetrahedral Weights

Let T be a positive tetrahedron seen as an oriented Δ-complex. To each coloring of faces of T by real numbers, which corresponds to a map

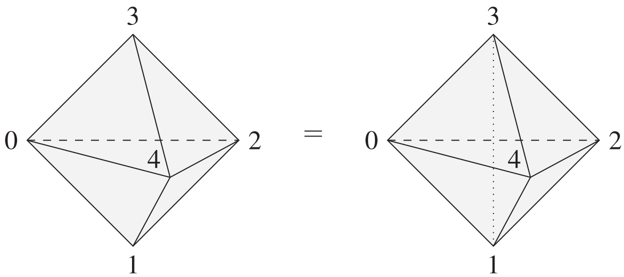
$$x\colon \Delta_2(T) \to \mathbb{R}, \tag{41}$$

we associate a tempered distribution valued weight function given by the integral kernel of the diagonal flip operator $\mathbf{T}$ in quantum Teichmüller theory:

$$\langle x_0, x_2 | \mathbf{T} | x_1, x_3 \rangle, \quad x_i := x(\partial_i T). \tag{42}$$

4.7 The Pentagon Identity and the 2–3 Pachner Moves

In order to relate the pentagon identity (35) to the 2–3 Pachner move (39), we choose the branching corresponding to the order of the five vertices of the suspension of a triangle given by the arrangement of the integers 0, 1, 2, 3, 4 so that the two odd integers 1 and 3 get associated with the two suspension points (the south and the north poles respectively) while the three even integers 0, 2, 4 get associated with the three vertices at the equator:



(43)

Let $x_{ijk} \in \mathbb{R}$ be arbitrary real numbers associated with triples of integers $0 \leq i < j < k \leq 4$ corresponding to triangular cells of the polyhedra entering the geometric equality (43). Now, taking the products of tetrahedral weight functions (42) according to two sides of (43) and integrating over the variables associated with the internal faces, we obtain the equality

$$\int_{\mathbb{R}} \langle x_{124}, x_{014}|\mathbf{T}|x_{024}, x_{012}\rangle \langle x_{234}, x_{024}|\mathbf{T}|x_{034}, x_{023}\rangle dx_{024}$$
$$= \int_{\mathbb{R}^3} \langle x_{234}, x_{124}|\mathbf{T}|x_{134}, x_{123}\rangle \langle x_{134}, x_{014}|\mathbf{T}|x_{034}, x_{013}\rangle$$
$$\times \langle x_{123}, x_{013}|\mathbf{T}|x_{023}, x_{012}\rangle dx_{134}dx_{123}dx_{013} \tag{44}$$

which is exactly the pentagon identity (35) written in the form of an integral identity for operator kernels. It is this realization of the 2–3 Pachner moves in terms of the pentagon identity of quantum Teichmülcer theory which has been used in the works [22, 23, 29, 39, 40] for defining formal state-integral partition functions of triangulated pseudo 3-manifolds. The reason for those partition functions to be formal is that the problems of convergence, independence of the branching, and (or) independence of the Δ-complex structure has not beed verified. As has been shown in [12], the notion of *shape structure* in oriented triangulated pseudo 3-manifolds remarkably permits to handle all these problems simultaneously.

5 Shapes and States

5.1 Shapes

Let T be a tetrahedron seen as an oriented triangulated 3-manifold. A *shape* on T is a map $\alpha\colon \Delta_1(T) \to]0, \pi[$ such that

$$\sum_{i=0}^{2} \alpha(\partial_i \partial_j T) = \pi, \quad \forall j \in \{0, 1, 2, 3\}. \tag{45}$$

It is easily verified that

$$\alpha(e) = \alpha(e^{\mathrm{op}}), \tag{46}$$

where e^{op} is the edge opposite to e, and the sum of three values of α at any vertex is equal to π. This exactly corresponds to dihedral angles of an ideal hyperbolic tetrahedron. For this reason, the values of α will be called *dihedral angles*.

The space of all shapes in an oriented tetrahedron is naturally a symplectic space with the Neumann–Zagier symplectic structure $\omega_{NZ} = d\alpha_0 \wedge d\alpha_2$ [50].

More generally, a *shape structure* on an oriented triangulated pseudo 3-manifold X is a map

$$\alpha\colon \Delta_3^1(X) \to]0, \pi[\tag{47}$$

such that for any tetrahedron T of X the restriction $\alpha|_{(T,.)}$ is a shape for T. In other words, a shape structure in X is a choice of shape structure in each tetrahedron of X.

The *total dihedral angle* function associated with a shape structure α is the map

$$w_\alpha\colon \Delta_1(X) \to \mathbb{R}_{>0} \tag{48}$$

defined by

$$w_\alpha(e) = \sum_{(T,e)\in\Delta_3^1(X)} \alpha(T, e) \tag{49}$$

An internal edge is *balanced* if the total dihedral angle around it is 2π. A shape structure on a closed oriented pseudo 3-manifold where all edges are balanced is known as *angle structure* [21, 46, 52].

5.2 Shaped 2–3 Pachner Moves

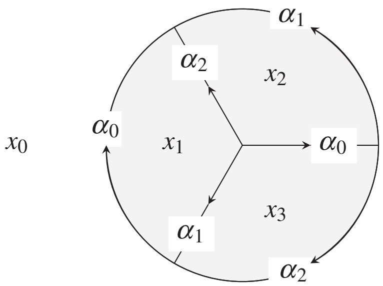
A *shaped* 2–3 *Pachner move* corresponds to same combinatorial setting as in (39) but where all tetrahedra are shaped and the total dihedral angles on all boundary edges are the same in both sides of the equality. This is possible only if the internal edge in the right hand side of (39) is balanced.

5.3 Shape Gauge Transformations

There is a gauge group action in the space of shape structures generated by total dihedral angles around internal edges acting through the Neumann–Zagier Poisson bracket. The *gauge reduced shape structure* is the Hamiltonian reduction of a shape structure over fixed values of the total dihedral angles around internal edges with respect to this gauge group action. The gauge reduced angle structure is invariant with respect to shaped 2–3 Pachner moves, see [12] for more details.

5.4 States

A *state* of a tetrahedron T is a map $x\colon \Delta_2(T) \to \mathbb{R}$. Pictorially, a positive tetrahedron T with shape α and in state x looks as follows:



$$\alpha_i := \alpha(\partial_i\partial_0 T)$$
$$x_i := x(\partial_i T) \tag{50}$$

More generally, a *state* of a triangulated pseudo 3-manifold X is a map

$$x\colon \Delta_2(X) \to \mathbb{R}. \tag{51}$$

5.5 Weight Functions

Define a function taking values in the space of tempered distributions

$$W_\hbar(s,t,x,y,u,v) := \delta(x+u-y)\phi_{s,t}(v-u)e^{i2\pi x(v-u)}, \tag{52}$$

where

$$\phi_{s,t}(z) := \bar{\Phi}_\hbar\left(z + \frac{\pi - s}{2\pi i\sqrt{\hbar}}\right) e^{tz/\sqrt{\hbar}}. \tag{53}$$

Here we assume that $(x,y,u,v) \in \mathbb{R}^4$ and $(s,t) \in]0,\pi[^2$ with the condition $s+t < \pi$. To a tetrahedron T with shape α and in state x, we associate the weight function

$$Z_\hbar(T,\alpha,x) = W_\hbar(\alpha_0,\alpha_2,x_0,x_1,x_2,x_3) \tag{54}$$

if T is positive and the complex conjugate weight function otherwise, i.e.

$$Z_\hbar(T,\alpha,x) = \overline{W_\hbar(\alpha_0,\alpha_2,x_0,x_1,x_2,x_3)} \tag{55}$$

if T is negative.

In the case of a positive flat tetrahedron with dihedral angles $\alpha_0 = \alpha_2 = 0$, $\alpha_1 = \pi$, the weight function $Z_\hbar(T,\alpha,x)$, up to an overall phase factor, is given by the integral kernel of the operator $\mathbf{T}$ in coordinate representation:

$$Z_\hbar(T,\alpha,x) = e^{\frac{\pi i}{12}(1+\hbar^{-1})}\langle x_0, x_2|\mathbf{T}|x_1, x_3\rangle. \tag{56}$$

This fact is consistent with the following fundamental properties of the weight function $Z_\hbar(T,\alpha,x)$: it realizes the projectivized shaped 2–3 Pachner moves, and carries the complete tetrahedral symmetry group (up to multiplication by overall phase factors) with respect to all reorderings of the vertices, see [12] for further details.

5.6 Partition Functions

For a closed oriented triangulated pseudo 3-manifold X with shape structure α, we associate the partition function

$$Z_\hbar(X,\alpha) := \int_{x\in\mathbb{R}^{\Delta_2(X)}} \prod_{T\in\Delta_3(X)} Z_\hbar(T,\alpha,x)dx. \tag{57}$$

Theorem 1 ([12]) *If $H_2(X \setminus \Delta_0(X), \mathbb{Z}) = 0$, then the quantity $|Z_\hbar(X, \alpha)|$ is well defined in the sense that the integral is absolutely convergent, and it*

1. *depends on only the gauge reduced class of α;*
2. *is invariant under shaped 2–3 Pachner moves.*

In the particular case of closed 3-manifolds, rather than pseudo 3-maifolds, the total dihedral angles w_α characterize completely the gauge reduced class of α, so that the dependence of the partition function (57) on α in this case factors through w_α.

The definition of the partition function (57) can be easily extended to manifolds with boundary eventually giving rize to a sort of TQFT, see [12].

5.7 One-Vertex H-Triangulations of Knots in 3-Manifolds

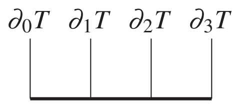
Let $K \subset M$ be a knot in an oriented closed compact 3-manifold. Let X be a one vertex *H-triangulation* of the pair (M, K), i.e. a one vertex triangulation of M where K is represented by an edge e_0 of X. Fix another edge e_1, and for any small $\varepsilon > 0$, consider a shape structure α_ε such that the total dihedral angle is ε around e_0, $2\pi - \varepsilon$ around e_1, and 2π around any other edge. We claim that the following *renormalized partition function*

$$\tilde{Z}_\hbar(X) := \left| \lim_{\varepsilon \to 0} Z_\hbar(X, \alpha_\varepsilon) \Phi_\hbar \left(\frac{\pi - \varepsilon}{2\pi i \sqrt{\hbar}} \right) \right| \tag{58}$$

is finite and is invariant under shaped 2–3 Pachner moves of triangulated pairs (M, K).

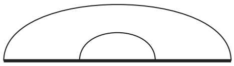
6 Examples of Calculation

We use the following graphical notation for a tetrahedron T with totally ordered vertices:

$T \quad =$  (59)

6.1 One-Vertex Triangulation of S^3

Let X be represented by the diagram



(60)

Choosing an orientation, it consists of one (positive) tetrahedron T with two identifications

$$\partial_i T \simeq \partial_{3-i} T, \quad i \in \{0, 1\}, \tag{61}$$

so that $\partial X = \varnothing$, and as a topological space, X is homeomorphic to 3-sphere. Combinatorially, we have

$$\Delta_0(X) = \{*\}, \quad \Delta_1(X) = \{e_0, e_1\}, \quad \Delta_2(X) = \{f_0, f_1\}, \quad \Delta_3(X) = \{T\} \tag{62}$$

with the boundary maps

$$f_i = \partial_i T = \partial_{3-i} T, \quad i \in \{0, 1\}, \tag{63}$$

$$\partial_i f_j = \begin{cases} e_0, \text{ if } i = j = 1; \\ e_1, \text{ otherwise}, \end{cases} \tag{64}$$

$$\partial_i e_j = *, \quad i, j \in \{0, 1\}. \tag{65}$$

The set $\Delta_3^1(X)$ consists of elements $(T, e_{j,k})$ for $0 \le j < k \le 3$. We fix a shape structure

$$\alpha \colon \Delta_3^1(X) \to \mathbb{R}_{>0} \tag{66}$$

by the formulae

$$\alpha(T, e_{0,j}) =: \alpha_j, \quad j \in \{1, 2, 3\}, \tag{67}$$

where $\sum_{j=1}^3 \alpha_j = \pi$. The total dihedral angle function

$$w_\alpha \colon \Delta_1(X) \to \mathbb{R}_{>0} \tag{68}$$

takes the values

$$w_\alpha(e_0) = \alpha_3, \quad w_\alpha(e_1) = 2\pi - \alpha_3, \tag{69}$$

so that the gauge equivalence class of α is determined by only one real variable $\alpha_3 \in]0, \pi[$. Geometrically, an interesting case corresponds to the value $\alpha_3 = 0$ with balanced edge e_1 and non-balanced e_0 knotted as the trefoil knot. This point, being singular, can nonetheless be approached arbitrarily closely, and it corresponds to a one-vertex H-triangulation of the pair (3-sphere, the trefoil knot).

We calculate the absolute value of the partition function:

$$\begin{aligned}|Z_\hbar(X)| &= \left| \int_{\mathbb{R}^2} W_\hbar(\alpha_1, \alpha_3, x_0, x_1, x_1, x_0) dx_0 dx_1 \right| = \left| \int_{\mathbb{R}} \phi_{\alpha_1,\alpha_3}(x) dx \right| \\ &= \left| \int_{\mathbb{R}} \bar{\Phi}_\hbar \left(x + \frac{\pi - \alpha_1}{2\pi i \sqrt{\hbar}} \right) e^{\frac{\alpha_3 x}{\sqrt{\hbar}}} dx \right| = \left| \int_{\mathbb{R}} \bar{\Phi}_\hbar(x)\, e^{\frac{\alpha_3 x}{\sqrt{\hbar}}} dx \right| \\ &= \left| \Phi_\hbar \left(\frac{\alpha_3 - \pi}{2\pi i \sqrt{\hbar}} \right) \right|. \end{aligned} \tag{70}$$

The latter formula also gives the following result for the renormalized partition function (58):

$$\tilde{Z}_\hbar(X) = \lim_{\alpha_3 \to 0} \left| Z_\hbar(X, \alpha) \Phi_\hbar \left(\frac{\pi - \alpha_3}{2\pi i \sqrt{\hbar}} \right) \right| = \lim_{\alpha_3 \to 0} 1 = 1, \tag{71}$$

which, as was remarked above, corresponds to the pair $(S^3, 3_1)$.

By using the quasi-classical formula (23), we have

$$|Z_\hbar(X)||_{\hbar \to 0} \sim \exp \left(\frac{1}{2\pi\hbar} \Im \mathrm{Li}_2 \left(e^{-i\alpha_3} \right) \right) = \exp \left(-\frac{1}{\pi\hbar} \Lambda \left(\frac{\alpha_3}{2} \right) \right), \tag{72}$$

where

$$\Lambda(x) := -\int_0^x \log |2 \sin(t)| dt \tag{73}$$

is π-periodic and anti-symmetric Lobachevsky's function.

On the other hand, Milnor's formula [48] for the volume of an ideal hyperbolic tetrahedron with dihedral angles α_j, $j \in \{1, 2, 3\}$, has the form

$$V(\alpha_1, \alpha_2, \alpha_3) = \sum_{j=1}^{3} \Lambda(\alpha_j), \tag{74}$$

which, when maximized for a fixed value of α_3, gives the hyperbolic volume of the conical 3-sphere (X, w_α) where the conical singularities are located along the

1-skeleton of X and are determined by the total dihedral angles w_α, see Eq. (69):

$$\operatorname{vol}(X, w_\alpha) = \max_{\alpha_1} V(\alpha_1, \alpha_2, \alpha_3) = V\left(\frac{\pi - \alpha_3}{2}, \frac{\pi - \alpha_3}{2}, \alpha_3\right) = 2\Lambda\left(\frac{\alpha_3}{2}\right), \tag{75}$$

where we have used the formula

$$\Lambda(x) = 2\Lambda\left(\frac{x}{2}\right) - 2\Lambda\left(\frac{\pi - x}{2}\right), \quad \forall x \in \mathbb{R}. \tag{76}$$

Comparing (75) with (72), we come to the following quasi-classical behavior of geometrical nature:

$$|Z_\hbar(X, \alpha)||_{\hbar \to 0} \sim \exp\left(-\frac{1}{2\pi\hbar}\operatorname{vol}(X, w_\alpha)\right). \tag{77}$$

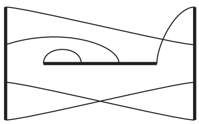
This calculation supports the following conjecture.

Conjecture 1 For any oriented shaped triangulated closed 3-manifold (X, α), the following formula holds true:

$$\lim_{\hbar \to 0} 2\pi\hbar \log |Z_\hbar(X, \alpha)| = -\operatorname{vol}(X, w_\alpha). \tag{78}$$

6.2 An *H*-Triangulation of the Pair $(S^3, 4_1)$ (Figure-Eight Knot)

There exists a one vertex H-triangulation X_{4_1} of the pair $(S^3, 4_1)$ composed of three tetrahedra which in our notation is of the form:



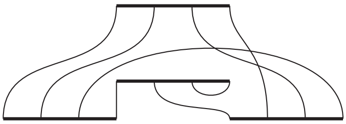
(79)

with the result of calculation of the renormalized partition function

$$\tilde{Z}_\hbar(X_{4_1}) = \int_{\mathbb{R} - i\varepsilon} \frac{e^{i\pi z^2}}{\Phi_\hbar(z)^2} dz. \tag{80}$$

6.3 An H-Triangulation of the Pair $(S^3, 5_2)$

There exists a one vertex H-triangulation X_{5_2} of the pair $(S^3, 5_2)$ composed of four tetrahedra:



(81)

with the result of calculation of the renormalized partition function

$$\tilde{Z}_\hbar(X_{5_2}) = \left| \int_{\mathbb{R}-i\varepsilon} \frac{e^{i\pi z^2}}{\Phi_\hbar(z)^3} dz \right|. \tag{82}$$

6.4 A Version of the Volume Conjecture

We still do not know how to relate our partition function to the colored Jones polynomials, but the following version of the volume conjecture can be a guiding principle in the search of such relation.

Conjecture 2 (Volume Conjecture for $\tilde{Z}_\hbar$) Let X be a one vertex H-triangulation of a pair (M, K), where $K \subset M$ is a hyperbolic knot in an oriented 3-manifold M. Then one has

$$\lim_{\hbar \to 0} 2\pi\hbar \log \tilde{Z}_\hbar(X) = -\mathrm{vol}(M \setminus K) \tag{83}$$

Unlike the volume conjecture for the colored Jones polynomials [42, 49], the quantity $\tilde{Z}_\hbar$ exponentially decays rather than grows.

Theorem 2 ([43]) *The volume conjecture for $\tilde{Z}_\hbar$ holds true in the case of H-triangulations X_{4_1} and X_{5_2}, see formulae* (80) *and* (82).

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A Higher Stacky Perspective on Chern–Simons Theory

Domenico Fiorenza, Hisham Sati and Urs Schreiber

Abstract The first part of this text is a gentle exposition of some basic constructions and results in the extended prequantum theory of Chern–Simons-type gauge field theories. We explain in some detail how the action functional of ordinary 3d Chern–Simons theory is naturally localized ("extended", "multi-tiered") to a map on the universal moduli stack of principal connections, a map that itself modulates a circle-principal 3-connection on that moduli stack, and how the iterated transgressions of this extended Lagrangian unify the action functional with its prequantum bundle and with the WZW-functional. In the second part we provide a brief review and outlook of the higher prequantum field theory of which this is a first example. This includes a higher geometric description of supersymmetric Chern–Simons theory, Wilson loops and other defects, generalized geometry, higher Spin-structures, anomaly cancellation, and various other aspects of quantum field theory.

1 Introduction

One of the fundamental examples of quantum field theory is 3-dimensional Chern–Simons gauge field theory as introduced in [88]. We give a pedagogical exposition of this from a new, natural, perspective of *higher geometry* formulated using *higher stacks* in *higher toposes* along the lines of [30] and references given there. Then we indicate how this opens the door to a more general understanding of *extended prequantum* (topological) field theory, constituting a pre-quantum analog of the extended quantum field theory as in [60], in the sense of higher geometric quantization [67].

D. Fiorenza (✉)
Università degli Studi di Roma "La Sapienza", Piazzale Aldo Moro 2, 00185 Rome, Italy
e-mail: fiorenza@mat.uniroma1.it

H. Sati
University of Pittsburgh, 4200 Fifth Avenue, Pittsburgh, PA 15260, USA
e-mail: hsati@pitt.edu

U. Schreiber
Radboud University Nijmegen, PO Box 9010, 6500 GL Nijmegen, The Netherlands
e-mail: urs.schreiber@gmail.com

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The aim of this text is twofold. On the one hand, we will attempt to dissipate the false belief that higher toposes are an esoteric discipline whose secret rites are reserved to initiates. To do this we will present a familiar example from differential topology, namely *Chern–Simons theory*, from the perspective of higher stacks, to show how this is a completely natural and powerful language in differential geometry. Furthermore, since any language is best appreciated by listening to it rather than by studying its grammar, in this presentation we will omit most of the rigorous definitions, leaving the reader the task to imagine and reconstruct them from the context. Clearly this does not mean that such definitions are not available: we refer the interested reader to [59] for the general theory of higher toposes and to [79] for general theory and applications of *differential cohesive* higher toposes that can express differential geometry, differential cohomology and prequantum gauge field theory; the reader interested in the formal mathematical aspects of the theory might enjoy looking at [81].

On the other hand, the purpose of this note is not purely pedagogical: we show how the stacky approach unifies in a natural way all the basic constructions in classical Chern–Simons theory (e.g., the action functional, the Wess-Zumino-Witten bundle gerbe, the symplectic structure on the moduli space of flat G-bundles as well as its prequantization), clarifies the relations of these with differential cohomology, and clearly points towards "higher Chern–Simons theories" and their higher and extended geometric prequantum theory. A brief survey and outlook of this more encompassing theory is given in the last sections. This is based on our series of articles including [28–31] and [74–76]. A set of lecture notes explaining this theory is [80].

We assume the reader has a basic knowledge of characteristic classes and of Chern–Simons theory. Friendly, complete and detailed introductions to these two topics can be found in [63] and [20, 32–34], respectively.

In this article we focus on the (extended) *geometric quantization* of Chern–Simons theory. Another important approach is the (extended) *perturbative quantization* based on path integrals in the BV-BRST formalism, as discussed notably in [1], based on the general program of extended perturbative BV-quantization laid out in [18, 19]. The BV-BRST formalism—a description of phase spaces/critical loci in higher ("derived") geometry—is also naturally formulated in terms of the higher cohesive geometry of higher stacks that we consider here, but further discussion of this point goes beyond the scope of this article. The interested reader can find more discussion in Sect. 1.2.15.2 and 3.10.8 of [79].

2 A Toy Example: 1-Dimensional $U(n)$-Chern–Simons Theory

Before describing the archetypical 3-dimensional Chern–Simons theory with a compact simply connected gauge group[1] from a stacky perspective, here we first look from this point of view at 1-dimensional Chern–Simons theory with gauge group

[1] We are using the term "gauge group" to refer to the structure group of the theory. This is not to be confused with the group of gauge transformations.

$U(n)$. Although this is a very simplified version, still it will show in an embryonic way all the features of the higher dimensional theory.[2] Moreover, a slight variant of this 1-dimensional CS theory shows up as a component of 3d Chern–Simons theory *with Wilson line defects*, this we discuss at the end of the exposition part in Sect. 3.4.5.

2.1 The Basic Definition

Let A be a $\mathfrak{u}_n$-valued differential 1-form on the circle S^1. Then $\frac{1}{2\pi i}\mathrm{tr}(A)$ is a real-valued 1-form, which we can integrate over S^1 to get a real number. This construction can be geometrically interpreted as a map

$$\{\text{trivialized } U(n)\text{-bundles with connections on } S^1\} \xrightarrow{\frac{1}{2\pi i}\int_{S^1}\mathrm{tr}} \mathbb{R}.$$

Since the Lie group $U(n)$ is connected, the classifying space $BU(n)$ of principal $U(n)$-bundles is simply connected, and so the set of homotopy classes of maps from S^1 to $BU(n)$ is trivial. By the characterizing property of the classifying space, this set is the set of isomorphism classes of principal $U(n)$-bundles on S^1, and so every principal $U(n)$-bundle over S^1 is trivializable. Using a chosen trivialization to pull-back the connection, we see that an arbitrary $U(n)$-principal bundle with connection (P, ∇) is (noncanonically) isomorphic to a trivialized bundle with connection, and so our picture enlarges to

$$\begin{array}{ccc} \{\text{trivialized } U(n)\text{-bundles with connections on } S^1\} & \xrightarrow{\frac{1}{2\pi i}\int_{S^1}\mathrm{tr}} & \mathbb{R} \\ \downarrow & & \\ \{U(n)\text{-bundles with connections on } S^1\}/\text{iso} & & \end{array}$$

and it is tempting to fill the square by placing a suitable quotient of $\mathbb{R}$ in the right bottom corner. To see that this is indeed possible, we have to check what happens when we choose two different trivializations for the same bundle, i.e., we have to compute the quantity

$$\frac{1}{2\pi i}\int_{S^1} \mathrm{tr}(A') - \mathrm{tr}(A),$$

where A and A' are two 1-form incarnations of the same connection ∇ under different trivializations of the underlying bundle. What one finds is that this quantity is always an integer, thus giving a commutative diagram

[2] Even 1-dimensional Chern–Simons theory exhibits a rich structure once we pass to *derived* higher gauge groups as in [46]. This goes beyond the present exposition, but see Sect. 5.1 for an outlook and Sect. 5.7.10 of [79] for more details.

$$\begin{array}{ccc}\{\text{trivialized } U(n)\text{-bundles with connections on } S^1\} & \xrightarrow{\frac{1}{2\pi i}\int_{S^1}\mathrm{tr}} & \mathbb{R} \\ \downarrow & & \downarrow \\ \{U(n)\text{-bundles with connections on } S^1\}/\mathrm{iso} & \xrightarrow{\exp\int_{S^1}\mathrm{tr}} & U(1)\,. \end{array}$$

The bottom line in this diagram is the *1-dimensional Chern–Simons action for $U(n)$-gauge theory*. An elegant way of proving that $\frac{1}{2\pi i}\int_{S^1}\mathrm{tr}(A) - \mathrm{tr}(A')$ is always an integer is as follows. Once a trivialization has been chosen, one can extend a principal $U(n)$-bundle with connection (P, ∇) on S^1 to a trivialized principal $U(n)$-bundle with connection over the disk D^2. Denoting by the same symbol ∇ the extended connection and by A the 1-form representing it, then by Stokes' theorem we have

$$\frac{1}{2\pi i}\int_{S^1}\mathrm{tr}(A) = \frac{1}{2\pi i}\int_{\partial D^2}\mathrm{tr}(A) = \frac{1}{2\pi i}\int_{D^2} d\mathrm{tr}(A) = \frac{1}{2\pi i}\int_{D^2}\mathrm{tr}(F_\nabla),$$

where F_∇ is the curvature of ∇. If we choose two distinct trivializations, what we get are two trivialized principal $U(n)$-bundles with connection over D^2 together with an isomorphism of their boundary data. Using this isomorphism to glue together the two bundles, we get a (generally nontrivial) $U(n)$-bundle with connection $(\tilde{P}, \tilde{\nabla})$ on $S^2 = D^2 \coprod_{S^1} D^2$, the disjoint union of the upper and lower hemisphere glued along the equator, and

$$\frac{1}{2\pi i}\int_{S^1}\mathrm{tr}(A') - \mathrm{tr}(A) = \frac{1}{2\pi i}\int_{S^2}\mathrm{tr}(\tilde{\nabla}) = \langle c_1(\tilde{P}), [S^2]\rangle,$$

the first Chern number of the bundle $\tilde{P}$. Note how the generator c_1 of the second integral cohomology group $H^2(BU(n), \mathbb{Z}) \cong \mathbb{Z}$ has come into play.

Despite its elegance, the argument above has a serious drawback: it relies on the fact that S^1 is a boundary. And, although this is something obvious, still it is something nontrivial and indicates that generalizing 1-dimensional Chern–Simons theory to higher dimensional Chern–Simons theory along the above lines will force limiting the construction to those manifolds which are boundaries. For standard 3-dimensional Chern–Simons theory with a compact simply connected gauge group, this will actually be no limitation, since the oriented cobordism ring is trivial in dimension 3, but one sees that this is a much less trivial statement than saying that S^1 is a boundary. However, in any case, that would definitely not be true in general for higher dimensions, as well as for topological structures on manifolds beyond orientations.

2.2 A Lie Algebra Cohomology Approach

A way of avoiding the cobordism argument used in the previous section is to focus on the fact that

$$\frac{1}{2\pi i}\mathrm{tr} : \mathfrak{u}_n \to \mathbb{R}$$

is a Lie algebra morphism, i.e., it is a real-valued 1-cocycle on the Lie algebra $\mathfrak{u}_n$ of the group $U(n)$. A change of trivialization for a principal $U(n)$-bundle $P \to S^1$ is given by a gauge transformation $\boldsymbol{g} : S^1 \to U(n)$. If A is the $\mathfrak{u}_n$-valued 1-form corresponding to the connection ∇ in the first trivialization, the gauge-transformed 1-form A' is given by

$$A' = \boldsymbol{g}^{-1}A\boldsymbol{g} + \boldsymbol{g}^{-1}d\boldsymbol{g},$$

where $\boldsymbol{g}^{-1}d\boldsymbol{g} = \boldsymbol{g}^*\theta_{U(n)}$ is the pullback of the Maurer–Cartan form $\theta_{U(n)}$ of $U(n)$ via $\boldsymbol{g}$. Since $\frac{1}{2\pi i}\mathrm{tr}$ is an invariant polynomial (i.e., it is invariant under the adjoint action of $U(n)$ on $\mathfrak{u}_n$), it follows that

$$\frac{1}{2\pi i}\int_{S^1} \mathrm{tr}(A') - \mathrm{tr}(A) = \frac{1}{2\pi i}\int_{S^1} \boldsymbol{g}^*\mathrm{tr}(\theta_{U(n)}),$$

and our task is reduced to showing that the right-hand term is a "quantized" quantity, i.e., that it always assumes integer values. Since the Maurer–Cartan form satisfies the Maurer–Cartan equation

$$d\theta_{U(n)} + \tfrac{1}{2}[\theta_{U(n)}, \theta_{U(n)}] = 0,$$

we see that

$$d\mathrm{tr}(\theta_{U(n)}) = -\tfrac{1}{2}\mathrm{tr}\big([\theta_{U(n)}, \theta_{U(n)}]\big) = 0,$$

i.e., $\mathrm{tr}(\theta_{U(n)})$ is a closed 1-form on $U(n)$. As an immediate consequence,

$$\frac{1}{2\pi i}\int_{S^1} \boldsymbol{g}^*\mathrm{tr}(\theta_{U(n)}) = \langle \boldsymbol{g}^*[\tfrac{1}{2\pi i}\mathrm{tr}(\theta_{U(n)})], [S^1]\rangle$$

only depends on the homotopy class of $\boldsymbol{g} : S^1 \to U(n)$, and these homotopy classes are parametrized by the additive group $\mathbb{Z}$ of the integers. Notice how the generator $[\frac{1}{2\pi i}\mathrm{tr}(\theta_{U(n)})]$ of $H^1(U(n);\mathbb{Z})$ has appeared. This shows how this proof is related to the one in the previous section via the transgression isomorphism $H^1(U(n);\mathbb{Z}) \to H^2(BU(n);\mathbb{Z})$.

It is useful to read the transgression isomorphism in terms of differential forms by passing to real coefficients and pretending that $BU(n)$ is a finite dimensional smooth manifold. This can be made completely rigorous in various ways, e.g., by looking at $BU(n)$ as an inductive limit of finite dimensional Grasmannians. Then a connection on the universal $U(n)$-bundle $EU(n) \to BU(n)$ is described à la Ehresmann by a $\mathfrak{u}_n$-valued $U(n)$-equivariant 1-form A on $EU(n)$ which gives the Maurer–Cartan form when restricted to the fibers. The $\mathbb{R}$-valued 1-form $\frac{1}{2\pi i}\mathrm{tr}(A)$ restricted to the fibers gives the closed 1-form $\frac{1}{2\pi i}\mathrm{tr}(\theta_{U(n)})$ which is the generator of $H^1(U(n), \mathbb{R})$; the differential $d\frac{1}{2\pi i}\mathrm{tr}(A) = \frac{1}{2\pi i}\mathrm{tr}(F_A)$ is an exact 2-form on $EU(n)$ which is $U(n)$-invariant and so is the pullback of a closed 2-form on $BU(n)$ which, since it represents the first Chern class, is the generator of $H^2(U(n), \mathbb{R})$.

One sees that $\frac{1}{2\pi i}\mathrm{tr}$ plays a triple role in the above description, which might be initially confusing. To get a better understanding of what is going on, let us consider more generally an arbitrary compact connected Lie group G. Then the transgression isomorphism between $H^1(G, \mathbb{R})$ and $H^2(BG; \mathbb{R})$ is realized by a Chern–Simons element CS_1 for the Lie algebra $\mathfrak{g}$. This element is characterized by the following property: for $A \in \Omega^1(EG; \mathfrak{g})$ the connection 1-form of a principal G-connection on $EG \to BG$, we have the following transgression diagram

$$\langle F_A \rangle \xleftarrow{\ d\ } \mathrm{CS}_1(A) \xmapsto{A=\theta_G} \mu_1(\theta_G)\ ,$$

where on the left hand side $\langle - \rangle$ is a degree 2 invariant polynomial on $\mathfrak{g}$, and on the right hand side μ_1 is 1-cocycle on $\mathfrak{g}$. One says that CS_1 transgresses μ_1 to $\langle - \rangle$. Via the identification of $H^1(G; \mathbb{R})$ with the degree one Lie algebra cohomology $H^1_{\mathrm{Lie}}(\mathfrak{g}; \mathbb{R})$ and of $H^2(BG; \mathbb{R})$ with the vector space of degree 2 elements in the graded algebra $\mathrm{inv}(\mathfrak{g})$ (with elements of $\mathfrak{g}^*$ placed in degree 2), one sees that this indeed realizes the transgression isomorphism.

2.3 The First Chern Class as a Morphism of Stacks

Note that, by the end of the previous section, the base manifold S^1 has completely disappeared. This suggests that one should be able to describe 1-dimensional Chern–Simons theory with gauge group $U(n)$ more generally as a map

$$\{U(n)\text{-bundles with connections on } X\}/\mathrm{iso} \to\ ??,$$

where now X is an arbitrary manifold, and "??" is some natural target to be determined. To try to figure out what this natural target could be, let us look at something simpler and forget the connection. Then we know that the first Chern class gives a morphism of sets

$$c_1 : \{U(n)\text{-bundles on } X\}/\mathrm{iso} \to H^2(X; \mathbb{Z}).$$

Here the right hand side is much closer to the left hand side than it might appear at first sight. Indeed, the second integral cohomology group of X precisely classifies principal $U(1)$-bundles on X up to isomorphism, so that the first Chern class is actually a map

$$c_1 : \{U(n)\text{-bundles on } X\}/\text{iso} \to \{U(1)\text{-bundles over } X\}/\text{iso}.$$

Writing $\mathbf{B}U(n)(X)$ and $\mathbf{B}U(1)(X)$ for the groupoids of principal $U(n)$- and $U(1)$-bundles over X, respectively,[3] one can further rewrite c_1 as a function

$$c_1 : \pi_0\mathbf{B}U(n)(X) \to \pi_0\mathbf{B}U(1)(X)$$

between the connected components of these groupoids. This immediately leads one to suspect that c_1 could actually be $\pi_0(\mathbf{c}_1(X))$ for some morphism of groupoids $\mathbf{c}_1(X) : \mathbf{B}U(n)(X) \to \mathbf{B}U(1)(X)$. Moreover, naturality of the first Chern class suggests that, independently of X, there should actually be a morphism of stacks

$$\mathbf{c}_1 : \mathbf{B}U(n) \to \mathbf{B}U(1)$$

over the site of smooth manifolds.[4] Since a smooth manifold is built by patching together, in a smooth way, open balls of $\mathbb{R}^n$ for some n, this in turn is equivalent to saying that $\mathbf{c}_1 : \mathbf{B}U(n) \to \mathbf{B}U(1)$ is a morphism of stacks over the full sub-site of Cartesian spaces, where by definition a Cartesian space is a smooth manifold diffeomorphic to $\mathbb{R}^n$ for some n. To see that c_1 is indeed induced by a morphism of stacks, notice that $\mathbf{B}U(n)$ can be obtained by stackification from the simplicial presheaf which to a Cartesian space U associates the nerve of the action groupoid $*//C^\infty(U; U(n))$. This is nothing but saying, in a very compact way, that to give a principal $U(n)$-bundle on a smooth manifold X one picks a good open cover $\mathcal{U} = \{U_\alpha\}$ of X and local data given by smooth functions on the double intersections

$$g_{\alpha\beta} : U_{\alpha\beta} \to U(n)$$

such that $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ on the triple intersections $U_{\alpha\beta\gamma}$. The group homomorphism

$$\det : U(n) \to U(1)$$

maps local data $\{g_{\alpha\beta}\}$ for a principal $U(n)$ bundle to local data $\{h_{\alpha\beta} = \det(g_{\alpha\beta})\}$ for a principal $U(1)$-bundle and, by the basic properties of the first Chern class, one sees that

$$\mathbf{B}\det : \mathbf{B}U(n) \to \mathbf{B}U(1)$$

induces c_1 at the level of isomorphism classes, i.e., one can take $\mathbf{c}_1 = \mathbf{B}\det$.

Note that there is a canonical notion of *geometric realization* of stacks on smooth manifolds by topological spaces (see Sect. 4.3.4.1 of [79]). Under this realization the morphism of stacks $\mathbf{B}\det$ becomes a continuous function of classifying spaces $BU(n) \to K(\mathbb{Z}, 2)$ which represents the universal first Chern class.

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[3] That is, for the collections of all such bundles, with gauge transformations as morphisms.

[4] The reader unfamiliar with the language of higher stacks and simplicial presheaves in differential geometry can find an introduction in [31].

2.4 Adding Connections to the Picture

The above discussion suggests that what should really lie behind 1-dimensional Chern–Simons theory with gauge group $U(n)$ is a morphism of stacks

$$\hat{\mathbf{c}}_1 : \mathbf{B}U(n)_{\mathrm{conn}} \to \mathbf{B}U(1)_{\mathrm{conn}}$$

from the stack of $U(n)$-principal bundles with connection to the stack of $U(1)$-principal bundles with connection, lifting the first Chern class. This morphism is easily described, as follows. Local data for a $U(n)$-principal bundle with connection on a smooth manifold X are

- smooth $\mathfrak{u}_n$-valued 1-forms A_α on U_α;
- smooth functions $\boldsymbol{g}_{\alpha\beta} \colon U_{\alpha\beta} \to U(n)$,

such that

- $A_\beta = \boldsymbol{g}_{\alpha\beta}^{-1} A_\alpha \boldsymbol{g}_{\alpha\beta} + \boldsymbol{g}_{\alpha\beta}^{-1} d\boldsymbol{g}_{\alpha\beta}$ on $U_{\alpha\beta}$;
- $\boldsymbol{g}_{\alpha\beta}\boldsymbol{g}_{\beta\gamma}\boldsymbol{g}_{\gamma\alpha} = 1$ on $U_{\alpha\beta\gamma}$,

and this is equivalent to saying that $\mathbf{B}U(n)_{\mathrm{conn}}$ is the stack of simplicial sets[5] which to a Cartesian space U assigns the nerve of the action groupoid

$$\Omega^1(U; \mathfrak{u}_n)//C^\infty(U; U(n)),$$

where the action is given by $\boldsymbol{g} : A \mapsto \boldsymbol{g}^{-1} A \boldsymbol{g} + \boldsymbol{g}^{-1} d\boldsymbol{g}$. To give a morphism $\hat{\mathbf{c}}_1 : \mathbf{B}U(n)_{\mathrm{conn}} \to \mathbf{B}U(1)_{\mathrm{conn}}$ we therefore just need to give a morphism of simplicial prestacks

$$\mathcal{N}(\Omega^1(-; \mathfrak{u}_n)//C^\infty(-; U(n))) \longrightarrow \mathcal{N}(\Omega^1(-; \mathfrak{u}_1)//C^\infty(-; U(1)))$$

lifting

$$\mathbf{B}\det : \mathcal{N}(*//C^\infty(-; U(n))) \longrightarrow \mathcal{N}(*//C^\infty(-; U(1))),$$

where $\mathcal{N}$ is the nerve of the indicated groupoid. In more explicit terms, we have to give a natural linear morphism

$$\varphi : \Omega^1(U; \mathfrak{u}_n) \to \Omega^1(U; \mathfrak{u}_1),$$

such that

$$\varphi(\boldsymbol{g}^{-1} A \boldsymbol{g} + \boldsymbol{g}^{-1} d\boldsymbol{g}) = \varphi(A) + \det(\boldsymbol{g})^{-1} d \det(\boldsymbol{g}),$$

and it is immediate to check that the linear map

$$\mathrm{tr} : \mathfrak{u}_n \to \mathfrak{u}_1$$

[5] It is noteworthy that this indeed is a stack on the site CartSp. On the larger but equivalent site of all smooth manifolds it is just a prestack that needs to be further stackified.

does indeed induce such a morphism φ. In the end we get a commutative diagram of stacks

$$\begin{array}{ccc} \mathbf{B}U(n)_{\mathrm{conn}} & \xrightarrow{\hat{\mathbf{c}}_1} & \mathbf{B}U(1)_{\mathrm{conn}} \\ \downarrow & & \downarrow \\ \mathbf{B}U(n) & \xrightarrow{\mathbf{c}_1} & \mathbf{B}U(1) \end{array} ,$$

where the vertical arrows forget the connections.

2.5 Degree 2 Differential Cohomology

If we now fix a base manifold X and look at isomorphism classes of principal $U(n)$-bundles (with connection) on X, we get a commutative diagram of sets

$$\begin{array}{ccc} \{U(n)\text{-bundles with connection on } X\}/\mathrm{iso} & \xrightarrow{\hat{c}} & \hat{H}^2(X;\mathbb{Z}) \\ \downarrow & & \downarrow \\ \{U(n)\text{-bundles on } X\}/\mathrm{iso} & \xrightarrow{c} & H^2(X;\mathbb{Z}) \end{array} ,$$

where $\hat{H}^2(X;\mathbb{Z})$ is the second differential cohomology group of X. This is defined as the degree 0 hypercohomology group of X with coefficients in the two-term Deligne complex, i.e., in the sheaf of complexes

$$C^\infty(-;U(1)) \xrightarrow{\frac{1}{2\pi i}d\log} \Omega^1(-;\mathbb{R}),$$

with $\Omega^1(-;\mathbb{R})$ in degree zero [8, 39]. That $\hat{H}^2(X;\mathbb{Z})$ classifies principal $U(1)$-bundles with connection is manifest by this description: via the Dold–Kan correspondence, the sheaf of complexes indicated above precisely gives a simplicial presheaf which produces $\mathbf{B}U(1)_{\mathrm{conn}}$ via stackification. Note that we have two natural morphisms of complexes of sheaves

$$\begin{array}{ccc} C^\infty(-;U(1)) & \xrightarrow{\frac{1}{2\pi i}d\log} & \Omega^1(-;\mathbb{R}) \\ \downarrow & & \downarrow \\ C^\infty(-;U(1)) & \longrightarrow & 0 \end{array} \quad \text{and} \quad \begin{array}{ccc} C^\infty(-;U(1)) & \xrightarrow{\frac{1}{2\pi i}d\log} & \Omega^1(-;\mathbb{R}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega^2(-;\mathbb{R})_{\mathrm{cl}} \end{array} .$$

The first one induces the forgetful morphism $\mathbf{B}U(1)_{\mathrm{conn}} \to \mathbf{B}U(1)$, while the second one induces the curvature morphism $F_{(-)} : \mathbf{B}U(1)_{\mathrm{conn}} \to \Omega^2(-;\mathbb{R})_{\mathrm{cl}}$ mapping a $U(1)$-bundle with connection to its curvature 2-form. From this one sees that degree 2 differential cohomology implements in a natural geometric way the simple idea of having an integral cohomology class together with a closed 2-form representing it in de Rham cohomology.

The last step that we need to recover the 1-dimensional Chern–Simons action functional from Sect. 2.1 is to give a natural morphism

$$\mathrm{hol} : \hat{H}^2(S^1;\mathbb{Z}) \to U(1)$$

so as to realize the 1-dimensional Chern–Simons action functional as the composition

$$\mathrm{CS}_1 : \{U(n)\text{-bundles with connection on } S^1\}/\mathrm{iso} \xrightarrow{\hat{c}} \hat{H}^2(S^1;\mathbb{Z}) \xrightarrow{\mathrm{hol}} U(1)\ .$$

As the notation "hol" suggests, this morphism is nothing but the holonomy morphism mapping a principal $U(1)$-bundle with connection on S^1 to its holonomy.

An enlightening perspective from which to look at this situation is in terms of fiber integration and moduli stacks of principal $U(1)$-bundles with connections over a base manifold X. Namely, for a fixed X we can consider the *mapping stack*

$$\mathbf{Maps}(X, \mathbf{B}U(1)_{\mathrm{conn}}),$$

which is presented by the simplicial presheaf that sends a Cartesian space U to the nerve of the groupoid of principal $U(1)$-bundles with connection on $U \times X$. In other words, $\mathbf{Maps}(X, \mathbf{B}U(1)_{\mathrm{conn}})$ is the *internal hom* space between X and $\mathbf{B}U(1)_{\mathrm{conn}}$ in the category of simplicial sheaves over the site of smooth manifolds. Then, if X is an oriented compact manifold of dimension one, the fiber integration formula from [44, 45] can be naturally interpreted as a morphism of simplicial sheaves

$$\mathrm{hol}_X : \mathbf{Maps}(X, \mathbf{B}U(1)_{\mathrm{conn}}) \to \underline{U}(1),$$

where on the right one has the sheaf of smooth functions with values in $U(1)$. Taking global sections over the point one gets the morphism of simplicial sets

$$\mathrm{hol}_X : \mathbf{H}(X, \mathbf{B}U(1)_{\mathrm{conn}}) \to U(1)_{\mathrm{discr}},$$

where on the right the Lie group $U(1)$ is seen as a 0-truncated simplicial object and where $\mathbf{H}(X, \mathbf{B}U(1)_{\mathrm{conn}})$ is a simplicial model for (the nerve of) the groupoid of principal $U(1)$-bundles with connection on X. Finally, passing to isomorphism classes/connected components one gets the morphism

$$\hat{H}^2(X;\mathbb{Z}) \to U(1).$$

This morphism can also be described in purely algebraic terms by noticing that for any 1-dimensional oriented compact manifold X the short exact sequence of complexes of sheaves

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & C^\infty(-;U(1)) & \longrightarrow & C^\infty(-;U(1)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \frac{1}{2\pi i} d\log} & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(-;\mathbb{R}) & \longrightarrow & \Omega^1(-;\mathbb{R}) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

induces an isomorphism

$$\Omega^1(X)/\Omega^1_{\mathrm{cl},\mathbb{Z}}(X) \xrightarrow{\sim} \hat{H}^2(X;\mathbb{Z})$$

in hypercohomology, where $\Omega^1(X)/\Omega^1_{\mathrm{cl},\mathbb{Z}}(X)$ is the group of differential 1-forms on X modulo those 1-forms which are closed and have integral periods. In terms of this isomorphism, the holonomy map is realized as the composition

$$\hat{H}^2(X;\mathbb{Z}) \xrightarrow{\sim} \Omega^1(X)/\Omega^1_{\mathrm{cl},\mathbb{Z}}(X) \xrightarrow{\exp(2\pi i \int_X -)} U(1).$$

2.6 The Brylinski–McLaughlin 2-Cocycle

It is natural to expect that the lift of the universal first Chern class c_1 to a morphism of stacks $\mathbf{c}_1 : \mathbf{B}U(n)_{\mathrm{conn}} \to \mathbf{B}U(1)_{\mathrm{conn}}$ is a particular case of a more general construction that holds for the generator c of the second integral cohomology group of an arbitrary compact connected Lie group G with $\pi_1(G) \cong \mathbb{Z}$. Namely, if $\langle - \rangle$ is the degree 2 invariant polynomial on $\mathfrak{g}[2]$ corresponding to the characteristic class c, then for any G-connection ∇ on a principal G-bundle $P \to X$ one has that $\langle F_\nabla \rangle$ is a closed 2-form on X representing the integral class c. This precisely suggests that (P, ∇) defines an element in degree 2 differential cohomology, giving a map

$$\{G\text{-bundles with connection on } X\}/\mathrm{iso} \to \hat{H}^2(X;\mathbb{Z}).$$

That this is indeed so can be seen following Brylinski and McLaughlin [12] (see [9] for an exposition an [10, 11] for related discussion). Let $\{A_\alpha, \boldsymbol{g}_{\alpha\beta}\}$ the local data for a G-connection on $P \to X$, relative to a trivializing good open cover $\mathcal{U}$ of X. Then, since G is connected and the open sets $U_{\alpha\beta}$ are contractible, we can smoothly extend the transition functions $\boldsymbol{g}_{\alpha\beta} : U_{\alpha\beta} \to G$ to functions $\hat{\boldsymbol{g}}_{\alpha\beta} : [0,1] \times U_{\alpha\beta} \to G$ with $\hat{\boldsymbol{g}}_{\alpha\beta}(0) = e$, the identity element of G, and $\hat{\boldsymbol{g}}_{\alpha\beta}(1) = \boldsymbol{g}_{\alpha\beta}$. Using the functions $\hat{\boldsymbol{g}}_{\alpha\beta}$ one can interpolate from $A_\alpha\big|_{U_{\alpha\beta}}$ to $A_\beta|_{U_{\alpha\beta}}$ by defining the $\mathfrak{g}$-valued 1-form

$$\hat{A}_{\alpha\beta} = \hat{\boldsymbol{g}}_{\alpha\beta}^{-1} A_\alpha|_{U_{\alpha\beta}} \hat{\boldsymbol{g}}_{\alpha\beta} + \hat{\boldsymbol{g}}_{\alpha\beta}^{-1} d\hat{\boldsymbol{g}}_{\alpha\beta}$$

on $U_{\alpha\beta}$. Now pick a real-valued 1-cocycle μ_1 on the Lie algebra $\mathfrak{g}$ representing the cohomology class c and a Chern–Simons element CS_1 realizing the transgression from μ_1 to $\langle - \rangle$. Then the element

$$(\mathrm{CS}_1(A_\alpha), \quad \int_{\Delta^1} \mathrm{CS}_1(\hat{A}_{\alpha\beta}) \mod \mathbb{Z})$$

is a degree 2 cocycle in the Čech–Deligne total complex lifting the cohomology class $c \in H^2(BG, \mathbb{Z})$ to a differential cohomology class $\hat{c}$. Notice how modding out by $\mathbb{Z}$ in the integral $\int_{\Delta^1} \mathrm{CS}_1(\hat{A}_{\alpha\beta})$ precisely takes care of G being connected but not simply connected, with $H^1(G;\mathbb{Z}) \cong \pi_1(G) \cong \mathbb{Z}$. That is, choosing two different extensions $\hat{\boldsymbol{g}}_{\alpha\beta}$ of $\boldsymbol{g}_{\alpha\beta}$ will produce two different values for that integral, but their difference will lie in the rank 1 lattice of 1-dimensional periods of G, and with the correct normalization this will be a copy of $\mathbb{Z}$.

A close look at the construction of Brylinski and McLaughlin, see [31], reveals that it actually provides a refinement of the characteristic class $c \in H^2(BG; \mathbb{Z})$ to a commutative diagram of stacks

$$\begin{array}{ccc} \mathbf{B}G_{\mathrm{conn}} & \xrightarrow{\hat{\mathbf{c}}} & \mathbf{B}U(1)_{\mathrm{conn}} \\ \downarrow & & \downarrow \\ \mathbf{B}G & \xrightarrow{\mathbf{c}} & \mathbf{B}U(1)\ . \end{array}$$

2.7 The Presymplectic Form on $\mathbf{B}U(n)_{\mathrm{conn}}$

In geometric quantization it is customary to call *pre-quantization* of a symplectic manifold (M, ω) the datum of a $U(1)$-principal bundle with connection on M whose curvature form is ω.[6] Furthermore, it is shown that most of the good features of symplectic manifolds continue to hold under the weaker hypothesis that the 2-form ω is only closed; this leads to introducing the term *pre-symplectic manifold* to denote a smooth manifold equipped with a closed 2-form ω and to speak of *prequantum line bundles* for these. In terms of the morphisms of stacks described in the previous sections, a prequantization of a presymplectic manifold is a lift of the morphism $\omega : M \to \Omega^2(-\mathbb{R})_{\mathrm{cl}}$ to a map ∇ fitting into a commuting diagram

$$\begin{array}{ccc} & & \mathbf{B}U(1)_{\mathrm{conn}} \\ & \nearrow^{\nabla} & \downarrow{\scriptstyle F_{(-)}} \\ M & \xrightarrow{\omega} & \Omega^2(-;\mathbb{R})_{\mathrm{cl}}\ , \end{array}$$

where the vertical arrow is the curvature morphism. From this perspective there is no reason to restrict M to being a manifold. By taking M to be the universal moduli stack $\mathbf{B}U(n)_{\mathrm{conn}}$, we see that the morphism $\hat{\mathbf{c}}_1$ can be naturally interpreted as giving a canonical prequantum line bundle over $\mathbf{B}U(n)_{\mathrm{conn}}$, whose curvature 2-form

$$\omega_{\mathbf{B}U(n)_{\mathrm{conn}}} : \mathbf{B}U(n)_{\mathrm{conn}} \xrightarrow{\hat{\mathbf{c}}_1} \mathbf{B}U(1)_{\mathrm{conn}} \xrightarrow{F} \Omega^2(-;\mathbb{R})_{\mathrm{cl}}$$

is the natural presymplectic 2-form on the stack $\mathbf{B}U(n)_{\mathrm{conn}}$: the invariant polynomial $\langle - \rangle$ viewed in the context of stacks. The datum of a principal $U(n)$-bundle with connection (P, ∇) on a manifold X is equivalent to the datum of a morphism $\varphi : X \to \mathbf{B}U(n)_{\mathrm{conn}}$, and the pullback $\varphi^*\omega_{\mathbf{B}U(n)_{\mathrm{conn}}}$ of the canonical 2-form on $\mathbf{B}U(n)_{\mathrm{conn}}$ is the curvature 2-form $\frac{1}{2\pi i}\mathrm{tr}(F_\nabla)$ on X. If (P, ∇) is a principal $U(n)$-bundle with connection over a compact closed oriented 1-dimensional manifold Σ_1

[6] See for instance [54] for an original reference on geometric quantization and see [67] for further pointers.

and the morphism $\varphi : \Sigma_1 \to \mathbf{B}U(n)_{\mathrm{conn}}$ defining it can be extended to a morphism $\tilde{\varphi} : \Sigma_2 \to \mathbf{B}U(n)_{\mathrm{conn}}$ for some 2-dimensional oriented manifold Σ_2 with $\partial \Sigma_2 = \Sigma_1$, then

$$CS_1(\nabla) = \exp \int_{\Sigma_2} \tilde{\varphi}^* \omega_{\mathbf{B}U(n)_{\mathrm{conn}}},$$

and the right hand side is independent of the extension $\tilde{\varphi}$. In other words,

$$CS_1(\nabla) = \exp \int_{\Sigma_2} \mathrm{tr}(F_{\tilde{\nabla}}),$$

for any extension $(\tilde{P}, \tilde{\nabla})$ of (P, ∇) to Σ_2. This way we recover the definition of the Chern–Simons action functional for $U(n)$-principal connections on S^1 given in Sect. 2.1.

More generally, the differential refinement $\hat{\mathbf{c}}$ of a characteristic class c of a compact connected Lie group G with $H^1(G; \mathbb{Z}) \cong \mathbb{Z}$, endows the stack $\mathbf{B}G_{\mathrm{conn}}$ with a canonical presymplectic structure with a prequantum line bundle given by $\hat{\mathbf{c}}$ itself, and the same considerations apply.

2.8 The Determinant as a Holonomy Map

We have so far met two natural maps with target the sheaf $\underline{U}(1)$ of smooth functions with values in the group $U(1)$. The first one was the determinant

$$\det : \underline{U}(n) \to \underline{U}(1),$$

and the second one was the holonomy map

$$\mathrm{hol}_X : \mathbf{Maps}(X; \mathbf{B}U(1)_{\mathrm{conn}}) \to \underline{U}(1),$$

defined on the moduli stack of principal $U(1)$-bundles with connection on a 1-dimensional compact oriented manifold X. To see how these two are related, take $X = S^1$ and notice that, by definition, a morphism from a smooth manifold M to the stack $\mathbf{Maps}(S^1; \mathbf{B}U(n)_{\mathrm{conn}})$ is the datum of a principal $U(n)$-bundle with connection over the product manifold $M \times S^1$. Taking the holonomy of the $U(n)$-connection along the fibers of $M \times S^1 \to M$ locally defines a smooth $U(n)$-valued function on M which is well defined up to conjugation. In other words, holonomy along S^1 defines a morphism from M to the stack $\underline{U}(n)//_{\mathrm{Ad}}\underline{U}(n)$, where Ad indicates the adjoint action. Since this construction is natural in M we have defined a natural $U(n)$-holonomy morphism

$$\mathrm{hol}^{U(n)} : \mathbf{Maps}(S^1; \mathbf{B}U(n)_{\mathrm{conn}}) \to \underline{U}(n)//_{\mathrm{Ad}}\underline{U}(n).$$

For $n = 1$, due to the fact that $U(1)$ is abelian, we also have a natural morphism $U(1)//_{\mathrm{Ad}}U(1) \to U(1)$, and the holonomy map hol_{S^1} factors as

$$\mathrm{hol}_{S^1} : \mathbf{Maps}(S^1; \mathbf{B}U(1)_{\mathrm{conn}}) \xrightarrow{\mathrm{hol}^{U(1)}} \underline{U}(1)//_{\mathrm{Ad}}\underline{U}(1) \to \underline{U}(1).$$

Therefore, by naturality of **Maps** we obtain the following commutative diagram

$$\begin{array}{ccccccc} & & \mathbf{Maps}(S^1; \mathbf{B}U(n)_{\mathrm{conn}}) & \xrightarrow{\mathbf{Maps}(S^1, \hat{\mathbf{c}}_1)} & \mathbf{Maps}(S^1; \mathbf{B}U(1)_{\mathrm{conn}}) & & \\ & & \downarrow{\scriptstyle \mathrm{hol}^{U(n)}} & & \downarrow{\scriptstyle \mathrm{hol}^{U(1)}} & & \\ \underline{U}(n) & \longrightarrow & \underline{U}(n)//_{\mathrm{Ad}}\underline{U}(n) & \xrightarrow{\det} & \underline{U}(1)//_{\mathrm{Ad}}\underline{U}(1) & \longrightarrow & \underline{U}(1)\,, \end{array}$$

where the leftmost bottom arrow is the natural quotient projection $\underline{U}(n) \to \underline{U}(n)//_{\mathrm{Ad}} \underline{U}(n)$. In the language of [79] (3.9.6.4) one says that the determinant map is the "concretification" of the morphism $\mathbf{Maps}(S^1, \hat{\mathbf{c}}_1)$, we come back to this in Sect. 5.3. This construction immediately generalizes to the case of an arbitrary compact connected Lie group G with $H^1(G; \mathbb{Z}) \cong \mathbb{Z}$: the Lie group morphism $\rho : G \to U(1)$ integrating the Lie algebra cocycle μ_1 corresponding to the characteristic class $c \in H^2(BG; \mathbb{Z})$ is the concretification of $\mathbf{Maps}(S^1, \hat{\mathbf{c}})$.

2.9 Killing the First Chern Class: SU(n)-bundles

Recall from the theory of characteristic classes (see [63]) that the first Chern class is the obstruction to reducing the structure group of a principal $U(n)$-bundle to $SU(n)$. In the stacky perspective that we have been adopting so far this amounts to saying that the stack $\mathbf{B}SU(n)$ of principal $SU(n)$-bundles is the *homotopy fiber* of $\mathbf{c}_1$, hence the object fitting into the homotopy pullback diagram of stacks of the form

$$\begin{array}{ccc} \mathbf{B}SU(n) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}U(n) & \xrightarrow{\mathbf{c}_1} & \mathbf{B}U(1)\,. \end{array}$$

By the universal property of the homotopy pullback, this says that an $SU(n)$-principal bundle over a smooth manifold X is equivalently a $U(n)$-principal bundle P, together with a choice of trivialization of the associated determinant $U(1)$-principal bundle. Moreover, the whole groupoid of $SU(n)$-principal bundles on X is equivalent to the groupoid of $U(n)$-principal bundles on X equipped with a trivialization of their associated determinant bundle. To explicitly see this equivalence, let us write the local data for a morphism from a smooth manifold X to the homotopy pullback above. In terms of a fixed good open cover $\mathcal{U}$ of X, these are:

- smooth functions $\rho_\alpha : U_\alpha \to U(1)$;

- smooth functions $g_{\alpha\beta} : U_{\alpha\beta} \to U(n)$,

subject to the constraints

- $\det(g_{\alpha\beta})\rho_\beta = \rho_\alpha$ on $U_{\alpha\beta}$;
- $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ on $U_{\alpha\beta\gamma}$.

Morphisms between $\{\rho_\alpha, g_{\alpha\beta}\}$ and $\{\rho'_\alpha, g'_{\alpha\beta}\}$ are the gauge transformations locally given by $U(n)$-valued functions h_α on U_α such that $h_\alpha g_{\alpha\beta} = g'_{\alpha\beta}h_\beta$ and $\rho_\alpha \det(h_\alpha) = \rho'_\alpha$. The classical description of objects in $\mathbf{B}SU(n)$ corresponds to the gauge fixing $\rho_\alpha \equiv 1$; at the level of morphisms, imposing this gauge fixing constrains the gauge transformation h_α to satisfy $\det(h_\alpha) = 1$, i.e. to take values in $SU(n)$. From a categorical point of view, this amounts to saying that the embedding of the groupoid of $SU(n)$-principal bundles over X into the groupoid of morphisms from X to the homotopy fiber of $\mathbf{c}_1$ given by $\{g_{\alpha\beta}\} \mapsto \{1, g_{\alpha\beta}\}$ is fully faithful. It is also essentially surjective: use the embedding $U(1) \to U(n)$ given by $e^{it} \mapsto (e^{it}, 1, 1, \ldots, 1)$ to lift ρ_α^{-1} to a $U(n)$-valued function h_α with $\det(h_\alpha) = {\rho_\alpha}^{-1}$; then $\{h_\alpha\}$ is an isomorphism between $\{\rho_\alpha, g_{\alpha\beta}\}$ and $\{1, h_\alpha g_{\alpha\beta}{h_\beta}^{-1}\}$.

Similarly, the stack of $SU(n)$-principal bundles with $\mathfrak{su}_n$-connections is the homotopy pullback

$$\begin{array}{ccc} \mathbf{B}SU(n)_{\mathrm{conn}} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}U(n)_{\mathrm{conn}} & \xrightarrow{\hat{\mathbf{c}}_1} & \mathbf{B}U(1)_{\mathrm{conn}} \end{array} .$$

Details on this homotopy pullback description of $\mathbf{B}SU(n)_{\mathrm{conn}}$ can be found in [28].

In summary, what we have discussed means that the map $\hat{\mathbf{c}}_1$ between universal moduli stacks equivalently plays the following different roles:

1. it is a smooth and differential refinement of the universal first Chern class;
2. it induces a 1-dimensional Chern–Simons action functional by *transgression* to maps from the circle;
3. it represents the obstruction to lifting a smooth unitary structure to a smooth special unitary structure.

In the following we will consider higher analogs of $\hat{\mathbf{c}}_1$ and will see these different but equivalent roles of universal differential characteristic maps amplified further.

3 The Archetypical Example: 3d Chern–Simons Theory

We now pass from the toy example of 1-dimensional Chern–Simons theory to the archetypical example of 3-dimensional Chern–Simons theory, and in fact to its extended (or "multi-tiered") geometric prequantization.

While this is a big step as far as the content of the theory goes, a pleasant consequence of the higher geometric formulation of the 1d theory above is that *conceptually* essentially nothing new happens when we move from 1-dimensional theory to 3-dimensional theory (and further). For the 3d theory we only need to restrict our attention to simply connected compact simple Lie groups, so as to have $\pi_3(G) \cong \mathbb{Z}$ as the first nontrivial homotopy group, and to move from stacks to higher stacks, or more precisely, to 3-stacks. (For non-simply connected groups one needs a little bit more structure, as we briefly indicate in Sect. 4.)

3.1 Higher $U(1)$-bundles with Connections and Differential Cohomology

The basic 3-stack naturally appearing in ordinary 3d Chern–Simons theory is the 3-stack $\mathbf{B}^3U(1)_{\text{conn}}$ of principal $U(1)$-3-bundles with connection (also known as $U(1)$-bundle-2-gerbes with connection). It is convenient to introduce in general the n-stack $\mathbf{B}^nU(1)_{\text{conn}}$ and to describe its relation to differential cohomology.

By definition, $\mathbf{B}^nU(1)_{\text{conn}}$ is the n-stack obtained by stackifying the prestack on Cartesian spaces which corresponds, via the Dold–Kan correspondence, to the $(n+1)$-term Deligne complex

$$\underline{U}(1)[n]^\infty_D = \left(\underline{U}(1) \xrightarrow{\frac{1}{2\pi i}d\log} \Omega^1(-;\mathbb{R}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(-;\mathbb{R})\right),$$

where $\underline{U}(1)$ is the sheaf of smooth functions with values in $U(1)$, and with $\Omega^n(-;\mathbb{R})$ in degree zero. It is immediate from the definition that the equivalence classes of $U(1)$-n-bundles with connection on a smooth manifold X are classified by the $(n+1)$-st differential cohomology group of X,

$$\hat{H}^{n+1}(X;\mathbb{Z}) \cong \mathbb{H}^0(X;\underline{U}(1)[n]^\infty_D) \cong \pi_0\mathbf{H}(X;\mathbf{B}^nU(1)_{\text{conn}}),$$

where in the middle we have degree zero hypercohomology of X with coefficients in $\underline{U}(1)[n]^\infty_D$. Similarly, the n-stack of $U(1)$-n-bundles (without connection) $\mathbf{B}^nU(1)$ is obtained via Dold–Kan and stackification from the sheaf of chain complexes

$$\underline{U}(1)[n] = \left(\underline{U}(1) \to 0 \to \cdots \to 0\right),$$

with $C^\infty(-;U(1))$ in degree n. Equivalence classes of $U(1)$-n-bundles on X are in natural bijection with

$$H^{n+1}(X;\mathbb{Z}) \cong H^n(X;\underline{U}(1)) \cong \mathbb{H}^0(X;\underline{U}(1)[n]) \cong \pi_0\mathbf{H}(X;\mathbf{B}^nU(1)).$$

The obvious morphism of chain complexes of sheaves $\underline{U}(1)[n]^\infty_D \to \underline{U}(1)[n]$ induces the "forget the connection" morphism $\mathbf{B}^n U(1)_{\mathrm{conn}} \to \mathbf{B}^n U(1)$ and, at the level of equivalence classes, the natural morphism

$$\hat{H}^{n+1}(X;\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z})$$

from differential cohomology to integral cohomology. If we denote by $\Omega^{n+1}(-;\mathbb{R})_{\mathrm{cl}}$ the sheaf (a 0-stack) of closed n-forms, then the morphism of complexes $\underline{U}(1)[n]^\infty_D \to \Omega^{n+1}(-;\mathbb{R})_{\mathrm{cl}}$ given by

$$\begin{array}{ccccccc}
\underline{U}(1) & \xrightarrow{\frac{1}{2\pi i}d\log} & \Omega^1(-;\mathbb{R}) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^n(-;\mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle d} \\
0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & \Omega^{n+1}(-;\mathbb{R})_{\mathrm{cl}}
\end{array}$$

induces the morphism of stacks $\mathbf{B}^n U(1)_{\mathrm{conn}} \xrightarrow{F_{(-)}} \Omega^{n+1}(-;\mathbb{R})_{\mathrm{cl}}$ mapping a circle n-bundle ($(n-1)$-bundle gerbe) with connection to the curvature $(n+1)$-form of its connection. At the level of differential cohomology, this is the morphism

$$\hat{H}^{n+1}(X;\mathbb{Z}) \to \Omega^{n+1}(X;\mathbb{R})_{\mathrm{cl}}.$$

The last n-stack we need to introduce to complete this sketchy picture of differential cohomology formulated on universal moduli stacks is the n-stack $\flat\mathbf{B}^{n+1}\mathbb{R}$ associated with the chain complex of sheaves

$$\flat\underline{\mathbb{R}}[n+1]^\infty = \left(\Omega^1(-;\mathbb{R}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(-;\mathbb{R}) \xrightarrow{d} \Omega^{n+1}(-;\mathbb{R})_{\mathrm{cl}}\right),$$

with $\Omega^{n+1}(-;\mathbb{R})_{\mathrm{cl}}$ in degree zero. The obvious morphism of complexes of sheaves $\Omega^{n+1}(-;\mathbb{R})_{\mathrm{cl}} \to \flat\underline{\mathbb{R}}[n+1]^\infty$ induces a morphism of stacks $\Omega^{n+1}(-;\mathbb{R})_{\mathrm{cl}} \to \flat\mathbf{B}^{n+1}\mathbb{R}$. Moreover one can show (see, e.g., [31, 79]) that there is a "universal curvature characteristic" morphism $\mathrm{curv} : \mathbf{B}^n U(1) \to \flat\mathbf{B}^{n+1}\mathbb{R}$ and a homotopy pullback diagram

$$\begin{array}{ccc}
\mathbf{B}^n U(1)_{\mathrm{conn}} & \xrightarrow{F} & \Omega^{n+1}(-;\mathbb{R})_{\mathrm{cl}} \\
\downarrow & & \downarrow \\
\mathbf{B}^n U(1) & \xrightarrow{\mathrm{curv}} & \flat\mathbf{B}^{n+1}\mathbb{R}\,,
\end{array}$$

of higher moduli stacks in $\mathbf{H}$, which induces in cohomology the commutative diagram

$$\begin{array}{ccc}
\hat{H}^{n+1}(X;\mathbb{Z}) & \xrightarrow{F} & \Omega^{n+1}(X;\mathbb{R})_{\mathrm{cl}} \\
\downarrow & & \downarrow \\
H^{n+1}(X;\mathbb{Z}) & \longrightarrow & H^{n+1}_{\mathrm{dR}}(X;\mathbb{R})\,.
\end{array}$$

This generalizes to any degree $n \geq 1$ what we remarked in Sect. 2.5 for the degree 2 case: differential cohomology encodes in a systematic and geometric way the simple idea of having an integral cohomology class together with a closed differential form representing it in de Rham cohomology. For $n = 0$ we have $\hat{H}^1(X;\mathbb{Z}) \equiv H^0(X;\underline{U}(1)) = C^\infty(X;U(1))$ and the map $\hat{H}^1(X;\mathbb{Z}) \to H^1(X;\mathbb{Z})$ is the morphism induced in cohomology by the short exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{R}} \to \underline{U}(1) \to 1.$$

At the level of stacks, this corresponds to the morphism

$$U(1) \to \mathbf{B}\mathbb{Z}$$

induced by the canonical principal $\mathbb{Z}$-bundle $\mathbb{R} \to U(1)$.

3.2 Compact Simple and Simply Connected Lie Groups

From a cohomological point of view, a compact simple and simply connected Lie group G is the degree 3 analogue of the group $U(n)$ considered in our 1-dimensional toy model. That is, the homotopy (hence the homology) of G is trivial up to degree 3, and $\pi_3(G) \cong H^3(G;\mathbb{Z}) \cong \mathbb{Z}$, by the Hurewicz isomorphism. Passing from G to its classifying space BG we find $H^4(BG;\mathbb{Z}) \cong \mathbb{Z}$, so that the fourth integral cohomology group of BG is generated by a fundamental characteristic class $c \in H^4(BG;\mathbb{Z})$. All other elements in $H^4(BG;\mathbb{Z})$ are of the form kc for some integer k, usually called the "level" in the physics literature. For P a G-principal bundle over a smooth manifold X, we will write $c(P)$ for the cohomology class $f^*c \in H^4(X,\mathbb{Z})$, where $f : X \to BG$ is any classifying map for P. This way we realize c as a map

$$c : \{\text{principal } G\text{-bundles on } X\}/\text{iso} \to H^4(X;\mathbb{Z}).$$

Moving to real coefficients, the fundamental characteristic class c is represented, via the isomorphism $H^4(BG;\mathbb{R}) \cong H^3(G;\mathbb{R}) \cong H^3_{\mathrm{Lie}}(\mathfrak{g},\mathbb{R})$ by the canonical 3-cocycle μ_3 on the Lie algebra $\mathfrak{g}$ of G, i.e., up to normalization, by the 3-cocycle $\langle[-,-],-\rangle$, where $\langle -,-\rangle$ is the Killing form of $\mathfrak{g}$ and $[-,-]$ is the Lie bracket. On the other hand, via the Chern-Weil isomorphism

$$H^*(BG;\mathbb{R}) \cong \mathrm{inv}(\mathfrak{g}[2]),$$

the characteristic class c corresponds to the Killing form, seen as a degree four invariant polynomial on $\mathfrak{g}$ (with elements of $\mathfrak{g}^*$ placed in degree 2). The transgression between μ_3 and $\langle -,-\rangle$ is witnessed by the canonical degree 3 Chern–Simons element CS_3 of $\mathfrak{g}$. That is, for a $\mathfrak{g}$-valued 1-form A on some manifold, let

$$\mathrm{CS}_3(A) = \langle A, dA\rangle + \tfrac{1}{3}\langle A,[A,A]\rangle.$$

Then, for $A \in \Omega^1(EG; \mathfrak{g})$ the connection 1-form of a principal G-connection on $EG \to BG$, we have the following transgression diagram

$$\langle F_A, F_A \rangle \xleftarrow{\ d\ } \mathrm{CS}_3(A) \xmapsto{A=\theta_G} \mu_3(\theta_G, \theta_G, \theta_G)\ ,$$

where θ_G is the Maurer–Cartan form of G (i.e., the restriction of A to the fibers of $EG \to BG$) and $F_A = dA + \frac{1}{2}[A, A]$ is the curvature 2-form of A. Notice how both the invariance of the Killing form and the Maurer–Cartan equation $d\theta_G + \frac{1}{2}[\theta_G, \theta_G] = 0$ play a rôle in the above transgression diagram.

3.3 *The Differential Refinement of Degree 4 Characteristic Classes*

The description of the Brylinski–McLaughlin 2-cocycle from Sect. 2.6 has an evident generalization to degree four. Indeed, let $\{A_\alpha, g_{\alpha\beta}\}$ be the local data for a G-connection ∇ on $P \to X$, relative to a trivializing good open cover $\mathcal{U}$ of X, with G a compact simple and simply connected Lie group. Then, since G is connected and the open sets $U_{\alpha\beta}$ are contractible, we can smoothly extend the transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ to functions $\hat{g}_{\alpha\beta} : [0,1] \times U_{\alpha\beta} \to G$ with $\hat{g}_{\alpha\beta}(0) = e$, the identity element of G, and $\hat{g}_{\alpha\beta}(1) = g_{\alpha\beta}$, and using the functions $\hat{g}_{\alpha\beta}$ one can interpolate from $A_\alpha\big|_{U_{\alpha\beta}}$ to $A_\beta\big|_{U_{\alpha\beta}}$ as in Sect. 2.6, defining a $\mathfrak{g}$-valued 1-form $\hat{A}_{\alpha\beta} = \hat{g}_{\alpha\beta}^{-1} A_\alpha\big|_{U_{\alpha\beta}} \hat{g}_{\alpha\beta} + \hat{g}_{\alpha\beta}^{-1} d\hat{g}_{\alpha\beta}$. On the triple intersection $U_{\alpha\beta\gamma}$ we have the paths in G

$$\begin{array}{ccccc} & & g_{\alpha\beta} & & \\ & {\scriptstyle \hat{g}_{\alpha\beta}} \nearrow & & \searrow {\scriptstyle \hat{g}_{\alpha\beta}\cdot\hat{g}_{\beta\gamma}} & \\ e & & \xrightarrow[\hat{g}_{\alpha\gamma}]{} & & g_{\alpha\gamma}\ . \end{array}$$

Since G is simply connected we can find smooth functions

$$\hat{g}_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \times \Delta^2 \to G$$

filling these 2-simplices, and we can use these to extend the interpolation between $\hat{A}_{\alpha\beta}$, $\hat{A}_{\beta\gamma}$ and $\hat{A}_{\gamma\alpha}$ over the 2-simplex. Let us denote this interpolation by $\hat{A}_{\alpha\beta\gamma}$. Finally, since G is 2-connected, on the quadruple intersections we can find smooth functions

$$\hat{g}_{\alpha\beta\gamma\delta} : U_{\alpha\beta\gamma\delta} \times \Delta^3 \to G$$

cobounding the union of the 2-simplices corresponding to the $\hat{g}_{\alpha\beta\gamma}$'s on the triple intersections. We can again use the $\hat{g}_{\alpha\beta\gamma\delta}$'s to interpolate between the $\hat{A}_{\alpha\beta\gamma}$'s over the 3-simplex. Finally, one considers the degree zero Čech–Deligne cochain with coefficients in $\underline{U}(1)[3]^\infty_D$

$$\left(\mathrm{CS}_3(A_\alpha), \int_{\Delta^1} \mathrm{CS}_3(\hat{A}_{\alpha\beta}), \int_{\Delta^2} \mathrm{CS}_3(\hat{A}_{\alpha\beta\gamma}), \int_{\Delta^3} \mathrm{CS}_3(\hat{A}_{\alpha\beta\gamma\delta}) \mod \mathbb{Z} \right). \tag{1}$$

Brylinski and McLaughlin [12] show (see also [9] for an exposition and [10, 11] for related discussion) that this is indeed a degree zero Čech–Deligne cocycle, and thus defines an element in $\hat{H}^4(X;\mathbb{Z})$. Moreover, they show that this cohomology class only depends on the isomorphism class of (P, ∇), inducing therefore a well-defined map

$$\hat{c} : \{G\text{-bundles with connection on } X\}/\text{iso} \to \hat{H}^4(X;\mathbb{Z}).$$

Notice how modding out by $\mathbb{Z}$ in the rightmost integral in the above cochain precisely takes care of $\pi_3(G) \cong H^3(G;\mathbb{Z}) \cong \mathbb{Z}$. Notice also that, by construction,

$$\int_{\Delta^3} \mathrm{CS}_3(\hat{A}_{\alpha\beta\gamma\delta}) = \int_{\Delta^3} \hat{\boldsymbol{g}}_{\alpha\beta\gamma\delta}{}^* \, \mu_3(\theta_G \wedge \theta_G \wedge \theta_G),$$

where θ_G is the Maurer–Cartan form of G. Hence the Brylinski–McLaughlin cocycle lifts the degree 3 cocycle with coefficients in $\underline{U}(1)$

$$\int_{\Delta^3} \hat{\boldsymbol{g}}_{\alpha\beta\gamma\delta}{}^* \, \mu_3(\theta_G \wedge \theta_G \wedge \theta_G) \quad \mod \mathbb{Z},$$

which represents the characteristic class $c(P)$ in $H^3(X;\underline{U}(1)) \cong H^4(X;\mathbb{Z})$. As a result, the differential characteristic class $\hat{c}$ lifts the characteristic class c, i.e., we have a natural commutative diagram

$$\begin{array}{ccc} \{G\text{-bundles with connection on } X\}/\text{iso} & \xrightarrow{\hat{c}} & \hat{H}^4(X;\mathbb{Z}) \\ \downarrow & & \downarrow \\ \{G\text{-bundles on } X\}/\text{iso} & \xrightarrow{c} & H^4(X;\mathbb{Z}) . \end{array}$$

By looking at the Brylinski–McLaughlin construction through the eyes of simplicial integration of ∞-Lie algebras one sees [31] that the above commutative diagram is naturally enhanced to a commutative diagram of stacks

$$\begin{array}{ccc} \mathbf{B}G_{\mathrm{conn}} & \xrightarrow{\hat{\mathbf{c}}} & \mathbf{B}^3U(1)_{\mathrm{conn}} \\ \downarrow & & \downarrow \\ \mathbf{B}G & \xrightarrow{\mathbf{c}} & \mathbf{B}^3U(1) . \end{array}$$

As we are going to show, the morphism $\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3U(1)_{\mathrm{conn}}$ that refines the characteristic class c to a morphism of stacks is the morphism secretly governing all basic features of level 1 three-dimensional Chern–Simons theory with gauge group G. Similarly, for any $k \in \mathbb{Z}$, one has a morphism of stacks

$$k\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

governing level k 3d Chern–Simons theory with gauge group G. Indeed, this map may be regarded as the very *Lagrangian* of 3d Chern–Simons theory *extended* ("localized", "multi-tiered") to codimension 3. We discuss this next.

3.4 Prequantum n-bundles on Moduli Stacks of G-connections on a Fixed Manifold

We discuss now how the differential refinement $\hat{\mathbf{c}}$ of the universal characteristic map c constructed above serves as the *extended* Lagrangian for 3d Chern–Simons theory in that its *transgression* to mapping stacks out of k-dimensional manifolds yields all the "geometric prequantum" data of Chern–Simons theory in the corresponding dimension, in the sense of geometric quantization. For the purpose of this exposition we use terms such as "prequantum n-bundle" freely without formal definition. We expect the reader can naturally see at least vaguely the higher prequantum picture alluded to here. A more formal survey of these notions is in Sect. 5.4.

If X is a compact oriented manifold without boundary, then there is a fiber integration in differential cohomology lifting fiber integration in integral cohomology [48]:

$$\begin{array}{ccc} \hat{H}^{n+\dim X}(X \times Y; \mathbb{Z}) & \xrightarrow{\int_X} & \hat{H}^n(Y; \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^{n+\dim X}(X \times Y; \mathbb{Z}) & \xrightarrow{\int_X} & H^n(Y; \mathbb{Z}) \,. \end{array}$$

In [44] Gomi and Terashima describe an explicit lift of this to the level of Čech–Deligne cocycles; see also [25]. One observes [30] that such a lift has a natural interpretation as a morphism of moduli stacks

$$\mathrm{hol}_X : \mathbf{Maps}(X, \mathbf{B}^{n+\dim X} U(1)_{\mathrm{conn}}) \to \mathbf{B}^n U(1)_{\mathrm{conn}}$$

from the $(n + \dim X)$-stack of moduli of $U(1)$-$(n + \dim X)$-bundles with connection over X to the n-stack of $U(1)$-n-bundles with connection (Sect. 2.4 of [30]). Therefore, if Σ_k is a compact oriented manifold of dimension k with $0 \le k \le 3$, we have a composition

$$\mathbf{Maps}(\Sigma_k, \mathbf{B}G_{\mathrm{conn}}) \xrightarrow{\mathbf{Maps}(\Sigma_k, \hat{\mathbf{c}})} \mathbf{Maps}(\Sigma_k, \mathbf{B}^3 U(1)_{\mathrm{conn}}) \xrightarrow{\mathrm{hol}_{\Sigma_k}} \mathbf{B}^{3-k} U(1)_{\mathrm{conn}}.$$

This is the canonical $U(1)$-$(3-k)$-bundle with connection over the moduli space of principal G-bundles with connection over Σ_k induced by $\hat{\mathbf{c}}$: the *transgression* of $\hat{\mathbf{c}}$ to the mapping space. Composing on the right with the curvature morphism we get the underlying canonical closed $(4-k)$-form

$$\mathbf{Maps}(\Sigma_k, \mathbf{B}G_{\mathrm{conn}}) \to \Omega^{4-k}(-; \mathbb{R})_{\mathrm{cl}}$$

on this moduli space. In other words, the moduli stack of principal G-bundles with connection over Σ_k carries a canonical *pre*-$(3-k)$-*plectic structure* (the higher order generalization of a symplectic structure, [67]) and, moreover, this is equipped with a canonical geometric prequantization: the above $U(1)$-$(3-k)$-bundle with connection.

Let us now investigate in more detail the cases $k = 0, 1, 2, 3$.

3.4.1 $k = 0$: The Universal Chern–Simons 3-Connection $\hat{\mathbf{c}}$

The connected 0-manifold Σ_0 is the point and, by definition of **Maps**, one has a canonical identification

$$\mathbf{Maps}(*, \mathbf{S}) \cong \mathbf{S}$$

for any (higher) stack **S**. Hence the morphism

$$\mathbf{Maps}(*, \mathbf{B}G_{\mathrm{conn}}) \xrightarrow{\mathbf{Maps}(*,\hat{\mathbf{c}})} \mathbf{Maps}(*, \mathbf{B}^3U(1)_{\mathrm{conn}})$$

is nothing but the universal differential characteristic map $\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3U(1)_{\mathrm{conn}}$ that refines the universal characteristic class c. This map modulates a circle 3-bundle with connection (bundle 2-gerbe) on the universal moduli stack of G-principal connections. For $\nabla : X \longrightarrow \mathbf{B}G_{\mathrm{conn}}$ any given G-principal connection on some X, the pullback

$$\hat{\mathbf{c}}(\nabla) : \; X \xrightarrow{\nabla} \mathbf{B}G_{\mathrm{conn}} \xrightarrow{\hat{\mathbf{c}}} \mathbf{B}^3U(1)_{\mathrm{conn}}$$

is a 3-bundle (bundle 2-gerbe) on X which is sometimes in the literature called the *Chern–Simons 2-gerbe* of the given connection ∇. Accordingly, $\hat{\mathbf{c}}$ modulates the *universal* Chern–Simons bundle 2-gerbe with universal 3-connection. From the point of view of higher geometric quantization, this is the *prequantum 3-bundle* of extended prequantum Chern–Simons theory.

This means that the prequantum $U(1)$-$(3-k)$-bundles associated with k- dimensional manifolds are all determined by the prequantum $U(1)$-3-bundle associated with the point, in agreement with the formulation of fully extended topological field theories [36]. We will denote by the symbol $\omega^{(4)}_{\mathbf{B}G_{\mathrm{conn}}}$ the pre-3-plectic 4-form induced on $\mathbf{B}G_{\mathrm{conn}}$ by the curvature morphism.

3.4.2 $k = 1$: The Wess-Zumino-Witten Bundle Gerbe

We now come to the transgression of the extended Chern–Simons Lagrangian to the closed connected 1-manifold, the circle $\Sigma_1 = S^1$. Here we find a higher analog of the construction described in Sect. 2.8. Notice that, on the one hand, we can think of the mapping stack $\mathbf{Maps}(\Sigma_1, \mathbf{B}G_{\mathrm{conn}}) \simeq \mathbf{Maps}(S^1, \mathbf{B}G_{\mathrm{conn}})$ as a kind of moduli stack of G-connections on the circle—up to a slight subtlety, which we explain in more detail below in Sect. 5.3. On the other hand, we can think of that mapping stack as the *free loop space* of the universal moduli stack $\mathbf{B}G_{\mathrm{conn}}$.

The subtlety here is related to the differential refinement, so it is instructive to first discard the differential refinement and consider just the smooth characteristic map $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^3U(1)$ which underlies the extended Chern–Simons Lagrangian and which modulates the universal circle 3-bundle on $\mathbf{B}G$ (without connection). Now, for every pointed stack $* \to \mathbf{S}$ we have the corresponding (categorical) *loop space* $\Omega\mathbf{S} := * \times_{\mathbf{S}} *$, which is the homotopy pullback of the point inclusion along itself. Applied to the moduli stack $\mathbf{B}G$ this recovers the Lie group G, identified with the sheaf (i.e., the 0-stack) of smooth functions with target G: $\Omega\mathbf{B}G \simeq \underline{G}$. This kind of looping/delooping equivalence is familiar from the homotopy theory of classifying spaces; but notice that since we are working with smooth (higher) stacks, the loop space $\Omega\mathbf{B}G$ also knows the smooth structure of the group G, i.e. it knows G as a Lie group. Similarly, we have

$$\Omega\mathbf{B}^3U(1) \simeq \mathbf{B}^2U(1)$$

and so forth in higher degrees. Since the looping operation is functorial, we may also apply it to the characteristic map $\mathbf{c}$ itself to obtain a map

$$\Omega\mathbf{c} : \underline{G} \to \mathbf{B}^2U(1)$$

which modulates a $\mathbf{B}U(1)$-principal 2-bundle on the Lie group G. This is also known as the *WZW-bundle gerbe*; see [41, 83]. The reason, as discussed there and as we will see in a moment, is that this is the 2-bundle that underlies the 2-connection with surface holonomy over a worldsheet given by the Wess-Zumino-Witten action functional. However, notice first that there is more structure implied here: for any pointed stack $\mathbf{S}$ there is a natural equivalence $\Omega\mathbf{S} \simeq \mathbf{Maps}_*(\Pi(S^1), \mathbf{S})$, between the loop space object $\Omega\mathbf{S}$ and the moduli stack of *pointed maps* from the categorical circle $\Pi(S^1) \simeq \mathbf{B}\mathbb{Z}$ to $\mathbf{S}$. Here Π denotes the *path ∞-groupoid* of a given (higher) stack.[7] On the other hand, if we do not fix the base point then we obtain the *free loop space object* $\mathcal{L}\mathbf{S} \simeq \mathbf{Maps}(\Pi(S^1), \mathbf{S})$. Since a map $\Pi(\Sigma) \to \mathbf{B}G$ is equivalently a map $\Sigma \to \flat\mathbf{B}G$, i.e., a flat G-principal connection on Σ, the free loop space $\mathcal{L}\mathbf{B}G$ is equivalently the moduli stack of flat G-principal connections on S^1. We will come back to this perspective in Sect. 5.3. The homotopies that do not fix the base point act by conjugation on loops, hence we have, for any smooth (higher) group, that

$$\mathcal{L}\mathbf{B}G \simeq \underline{G}//_{\mathrm{Ad}}\underline{G}$$

is the (homotopy) quotient of the adjoint action of G on itself; see [64] for details on homotopy actions of smooth higher groups. For G a Lie group this is the familiar

[7] The existence and functoriality of the path ∞-groupoids is one of the features characterizing the higher topos of higher smooth stacks as being *cohesive*, see [79].

adjoint action quotient stack. But the expression holds fully generally. Notably, we also have

$$\mathcal{L}\mathbf{B}^3U(1) \simeq \mathbf{B}^2U(1)//_{\mathrm{Ad}}\mathbf{B}^2U(1)$$

and so forth in higher degrees. However, in this case, since the smooth 3-group $\mathbf{B}^2U(1)$ is abelian (it is a groupal E_∞-algebra) the adjoint action splits off in a direct factor and we have a projection

$$\mathcal{L}\mathbf{B}^3U(1) \simeq \mathbf{B}^2U(1) \times (*//\mathbf{B}^2U(1)) \xrightarrow{p_1} \mathbf{B}^2U(1) \ .$$

In summary, this means that the map $\Omega\mathbf{c}$ modulating the WZW 2-bundle over G descends to the adjoint quotient to the map

$$p_1 \circ \mathcal{L}\mathbf{c} : \underline{G}//_{\mathrm{Ad}}\underline{G} \to \mathbf{B}^2U(1),$$

and this means that the WZW 2-bundle is canonically equipped with the structure of an ad_G-*equivariant* bundle gerbe, a crucial feature of the WZW bundle gerbe [41, 42].

We emphasize that the derivation here is fully general and holds for any smooth (higher) group G and any smooth characteristic map $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^nU(1)$. Each such pair induces a WZW-type $(n-1)$-bundle on the smooth (higher) group G modulated by $\Omega\mathbf{c}$ and equipped with G-equivariant structure exhibited by $p_1 \circ \mathcal{L}\mathbf{c}$. We discuss such higher examples of higher Chern–Simons-type theories with their higher WZW-type functionals further below in Sect. 4.

We now turn to the differential refinement of this situation. In analogy to the above construction, but taking care of the connection data in the extended Lagrangian $\hat{\mathbf{c}}$, we find a homotopy commutative diagram in $\mathbf{H}$ of the form

$$\begin{array}{ccccccc}
 & & \mathbf{Maps}(S^1;\mathbf{B}G_{\mathrm{conn}}) & \xrightarrow{\mathbf{Maps}(S^1,\hat{\mathbf{c}})} & \mathbf{Maps}(S^1;\mathbf{B}^3U(1)_{\mathrm{conn}}) & & \\
 & & \downarrow{\scriptstyle \mathrm{hol}} & & \downarrow{\scriptstyle \mathrm{hol}} & & \\
\underline{G} & \longrightarrow & \underline{G}//_{\mathrm{Ad}}\underline{G} & \xrightarrow{\mathbf{wzw}} & \mathbf{B}^2U(1)_{\mathrm{conn}}//_{\mathrm{Ad}}\mathbf{B}^2U(1)_{\mathrm{conn}} & \longrightarrow & \mathbf{B}^2U(1)_{\mathrm{conn}}\ ,
\end{array}$$

where the vertical maps are obtained by forming holonomies of (higher) connections along the circle. The lower horizontal row is the differential refinement of $\Omega\mathbf{c}$: it modulates the Wess-Zumino-Witten $U(1)$-bundle gerbe with connection

$$\mathbf{wzw} : \underline{G} \to \mathbf{B}^2U(1)_{\mathrm{conn}}.$$

That $\mathbf{wzw}$ is indeed the correct differential refinement can be seen, for instance, by interpreting the construction by Carey et al. [15] in terms of the above diagram. That is, choosing a basepoint x_0 in S^1 one obtains a canonical lift of the leftmost vertical arrow:

$$\begin{array}{ccc}
 & & \mathbf{Maps}(S^1;\mathbf{B}G_{\mathrm{conn}}) \\
 & \nearrow^{(P_{x_0},\nabla_{x_0})} & \downarrow{\scriptstyle \mathrm{hol}} \\
\underline{G} & \longrightarrow & \underline{G}//_{\mathrm{Ad}}\underline{G}\ ,
\end{array}$$

where $(P_{x_0}\nabla_{x_0})$ is the principal G-bundle with connection on the product $G \times S^1$ characterized by the property that the holonomy of ∇_{x_0} along $\{\boldsymbol{g}\} \times S^1$ with starting point $(\boldsymbol{g}, x_0)$ is the element $\boldsymbol{g}$ of G. Correspondingly, we have a homotopy commutative diagram

$$\begin{array}{ccccccc} & & \mathbf{Maps}(S^1;\mathbf{B}G_{\mathrm{conn}}) & \xrightarrow{\mathbf{Maps}(S^1,\hat{\mathbf{c}})} & \mathbf{Maps}(S^1;\mathbf{B}^3U(1)_{\mathrm{conn}}) & & \\ & \nearrow^{(P_{x_0},\nabla_{x_0})} & \downarrow{\scriptstyle \mathrm{hol}} & & \downarrow{\scriptstyle \mathrm{hol}} & \searrow^{\mathrm{hol}_{S^1}} & \\ \underline{G} & \longrightarrow & \underline{G}/\!/_{\mathrm{Ad}}\underline{G} & \xrightarrow{\mathbf{wzw}} & \mathbf{B}^2U(1)_{\mathrm{conn}}/\!/_{\mathrm{Ad}}\mathbf{B}^2U(1)_{\mathrm{conn}} & \longrightarrow & \mathbf{B}^2U(1)_{\mathrm{conn}}\,. \end{array}$$

Then Proposition 3.4 from [15] identifies the upper path (hence also the lower path) from $\underline{G}$ to $\mathbf{B}^2U(1)_{\mathrm{conn}}$ with the Wess-Zumino-Witten bundle gerbe.

Passing to equivalence classes of global sections, we see that **wzw** induces, for any smooth manifold X, a natural map $C^\infty(X; G) \to \hat{H}^2(X; \mathbb{Z})$. In particular, if $X = \Sigma_2$ is a compact Riemann surface, we can further integrate over X to get

$$\boldsymbol{wzw} : C^\infty(\Sigma_2; G) \to \hat{H}^2(X; \mathbb{Z}) \xrightarrow{\int_{\Sigma_2}} U(1).$$

This is the *topological term* in the Wess-Zumino-Witten model; see [14, 38, 40]. Notice how the fact that **wzw** factors through $\underline{G}/\!/_{\mathrm{Ad}}\underline{G}$ gives the conjugation invariance of the Wess-Zumino-Witten bundle gerbe, hence of the topological term in the Wess-Zumino-Witten model.

3.4.3 $k = 2$: The Symplectic Structure on the Moduli Space of Flat Connections on Riemann Surfaces

For Σ_2 a compact Riemann surface, the transgression of the extended Lagrangian $\hat{\mathbf{c}}$ yields a map

$$\mathbf{Maps}(\Sigma_2; \mathbf{B}G_{\mathrm{conn}}) \xrightarrow{\mathbf{Maps}(\Sigma_2,\hat{\mathbf{c}})} \mathbf{Maps}(\Sigma_2; \mathbf{B}^3U(1)_{\mathrm{conn}}) \xrightarrow{\mathrm{hol}_{\Sigma_2}} \mathbf{B}U(1)_{\mathrm{conn}},$$

modulating a circle-bundle with connection on the moduli space of gauge fields on Σ_2. The underlying curvature of this connection is the map obtained by composing this with

$$\mathbf{B}U(1)_{\mathrm{conn}} \xrightarrow{F_{(-)}} \Omega^2(-;\mathbb{R})_{\mathrm{cl}} \quad ,$$

which gives the canonical presymplectic 2-form

$$\omega : \ \mathbf{Maps}(\Sigma_2; \mathbf{B}G_{\mathrm{conn}}) \longrightarrow \Omega^2(-;\mathbb{R})_{\mathrm{cl}}$$

on the moduli stack of principal G-bundles with connection on Σ_2. Equivalently, this is the transgression of the invariant polynomial

$$\langle - \rangle : \ \mathbf{B}G_{\mathrm{conn}} \longrightarrow \Omega^4_{\mathrm{cl}}$$

to the mapping stack out of Σ_2. The restriction of this 2-form to the moduli stack $\mathbf{Maps}(\Sigma_2; \flat\mathbf{B}G_{\mathrm{conn}})$ of flat principal G-bundles on Σ_2 induces a canonical symplectic structure on the moduli space

$$\mathrm{Hom}(\pi_1(\Sigma_2), G)/_{\mathrm{Ad}}G$$

of flat G-bundles on Σ_2. Such a symplectic structure seems to have been first made explicit in [3] and then identified as the phase space structure of Chern–Simons theory in [88]. Observing that differential forms on the moduli stack, and hence de Rham cocycles $\mathbf{B}G \to \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$, may equivalently be expressed by simplicial forms on the bar complex of G, one recognizes in the above transgression construction a stacky refinement of the construction of [87].

To see more explicitly what this form ω is, consider any test manifold $U \in \mathrm{CartSp}$. Over this the map of stacks ω is a function which sends a G-principal connection $A \in \Omega^1(U \times \Sigma_2)$ (using that every G-principal bundle over $U \times \Sigma_2$ is trivializable) to the 2-form

$$\int_{\Sigma_2} \langle F_A \wedge F_A \rangle \in \Omega^2(U).$$

Now if A represents a field in the phase space, hence an element in the concretification of the mapping stack, then it has no "leg"[8] along U, and so it is a 1-form on Σ_2 that depends smoothly on the parameter U: it is a U-parameterized *variation* of such a 1-form. Accordingly, its curvature 2-form splits as

$$F_A = F_A^{\Sigma_2} + d_U A,$$

where $F_A^{\Sigma_2} := d_{\Sigma_2} A + \frac{1}{2}[A \wedge A]$ is the U-parameterized collection of curvature forms on Σ_2. The other term is the *variational differential* of the U-collection of forms. Since the fiber integration map $\int_{\Sigma_2} : \Omega^4(U \times \Sigma_2) \to \Omega^2(U)$ picks out the component of $\langle F_A \wedge F_A \rangle$ with two legs along Σ_2 and two along U, integrating over the former we have that

$$\omega|_U = \int_{\Sigma_2} \langle F_A \wedge F_A \rangle = \int_{\Sigma_2} \langle d_U A \wedge d_U A \rangle \in \Omega^2_{\mathrm{cl}}(U).$$

In particular if we consider, without loss of generality, $(U = \mathbb{R}^2)$-parameterized variations and expand

$$d_U A = (\delta_1 A) du^1 + (\delta_2 A) du^2 \in \Omega^2(\Sigma_2 \times U),$$

then

[8] That is, when written in local coordinates (u, σ) on $U \times \Sigma_2$, then $A = A_i(u,\sigma)du^i + A_j(u,\sigma)d\sigma^j$ reduces to the second summand.

$$\omega|_U = \int_{\Sigma_2} \langle \delta_1 A, \delta_2 A \rangle.$$

In this form the symplectic structure appears, for instance, in prop. 3.17 of [32] (in [88] this corresponds to (3.2)).

In summary, this means that the circle bundle with connection obtained by transgression of the extended Lagrangian $\hat{\mathbf{c}}$ is a *geometric prequantization* of the phase space of 3d Chern–Simons theory. Observe that traditionally prequantization involves an arbitrary *choice*: the choice of prequantum bundle with connection whose curvature is the given symplectic form. Here we see that in *extended* prequantization this choice is eliminated, or at least reduced: while there may be many differential cocycles lifting a given curvature form, only few of them arise by transgression from a higher differential cocycles in top codimension. In other words, the restrictive choice of the single geometric prequantization of the invariant polynomial $\langle -, - \rangle : \mathbf{B}G_{\mathrm{conn}} \to \Omega^4_{\mathrm{cl}}$ by $\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$ down in top codimension induces canonical choices of prequantization over all Σ_k in all lower codimensions $(n-k)$.

3.4.4 $k = 3$: The Chern–Simons Action Functional

Finally, for Σ_3 a compact oriented 3-manifold without boundary, transgression of the extended Lagrangian $\hat{\mathbf{c}}$ produces the morphism

$$\mathbf{Maps}(\Sigma_3; \mathbf{B}G_{\mathrm{conn}}) \xrightarrow{\mathbf{Maps}(\Sigma_3, \hat{\mathbf{c}})} \mathbf{Maps}(\Sigma_3; \mathbf{B}^3 U(1)_{\mathrm{conn}}) \xrightarrow{\mathrm{hol}_{\Sigma_3}} \underline{U}(1).$$

Since the morphisms in $\mathbf{Maps}(\Sigma_3; \mathbf{B}G_{\mathrm{conn}})$ are *gauge transformations* between field configurations, while $\underline{U}(1)$ has no non-trivial morphisms, this map necessarily gives a *gauge invariant* $U(1)$-valued function on field configurations. Indeed, evaluating over the point and passing to isomorphism classes (hence to gauge equivalence classes), this induces the *Chern–Simons action functional*

$$S_{\hat{\mathbf{c}}} : \{G\text{-bundles with connection on } \Sigma_3\}/\mathrm{iso} \to U(1).$$

It follows from the description of $\hat{\mathbf{c}}$ given in Sect. 3.3 that if the principal G-bundle $P \to \Sigma_3$ is trivializable then

$$S_{\hat{\mathbf{c}}}(P, \nabla) = \exp 2\pi i \int_{\Sigma_3} \mathrm{CS}_3(A),$$

where $A \in \Omega^1(\Sigma_3, \mathfrak{g})$ is the $\mathfrak{g}$-valued 1-form on Σ_3 representing the connection ∇ in a chosen trivialization of P. This is actually always the case, but notice two things: first, in the stacky description one does not need to know a priori that every principal

G-bundle on a 3-manifold is trivializable; second, the independence of $S_{\hat{\mathbf{c}}}(P, \nabla)$ on the trivialization chosen is automatic from the fact that $S_{\hat{\mathbf{c}}}$ is a morphism of stacks read at the level of equivalence classes.

Furthermore, if (P, ∇) can be extended to a principal G-bundle with connection $(\tilde{P}, \tilde{\nabla})$ over a compact 4-manifold Σ_4 bounding Σ_3, one has

$$S_{\hat{\mathbf{c}}}(P, \nabla) = \exp 2\pi i \int_{\Sigma_4} \tilde{\varphi}^* \omega^{(4)}_{\mathbf{B}G_{\mathrm{conn}}} = \exp 2\pi i \int_{\Sigma_4} \langle F_{\tilde{\nabla}}, F_{\tilde{\nabla}} \rangle,$$

where $\tilde{\varphi} : \Sigma_4 \to \mathbf{B}G_{\mathrm{conn}}$ is the morphism corresponding to the extended bundle $(\tilde{P}, \tilde{\nabla})$. Notice that the right hand side is independent of the extension chosen. Again, this is always the case, so one can actually take the above equation as a definition of the Chern–Simons action functional, see, e.g., [32, 33]. However, notice how in the stacky approach we do not need a priori to know that the oriented cobordism ring is trivial in dimension 3. Even more remarkably, the stacky point of view tells us that there would be a natural and well-defined 3d Chern–Simons action functional even if the oriented cobordism ring were nontrivial in dimension 3 or even if not every G-principal bundle on a 3-manifold were trivializable. An instance of checking that a nontrivial higher cobordism group vanishes can be found in [57], allowing for the application of the construction of Hopkins–Singer [48].

3.4.5 The Chern–Simons Action Functional with Wilson Loops

To conclude our exposition of the examples of 1d and 3d Chern–Simons theory in higher geometry, we now briefly discuss how both unify into the theory of 3d Chern–Simons gauge fields with Wilson line defects. Namely, for every embedded knot

$$\iota : S^1 \hookrightarrow \Sigma_3$$

in the closed 3d worldvolume and every complex linear representation $R : G \to \mathrm{Aut}(V)$ one can consider the *Wilson loop observable* $W_{\iota,R}$ mapping a gauge field $A : \Sigma \to \mathbf{B}G_{\mathrm{conn}}$, to the corresponding "Wilson loop holonomy"

$$W_{\iota,R} : A \mapsto \mathrm{tr}_R(\mathrm{hol}(\iota^* A)) \in \mathbb{C}.$$

This is the trace, in the given representation, of the parallel transport defined by the connection A around the loop ι (for any choice of base point). It is an old observation[9] that this Wilson loop $W(C, A, R)$ is itself the *partition function* of a 1-dimensional topological σ-model quantum field theory that describes the topological sector of a particle charged under the nonabelian background gauge field A. In Sect. 3.3 of [88] it was therefore emphasized that Chern–Simons theory with Wilson loops should really

[9] This can be traced back to [4]; a nice modern review can be found in Sect. 4 of [6].

be thought of as given by a single Lagrangian which is the sum of the 3d Chern–Simons Lagrangian for the gauge field as above, plus that for this topologically charged particle.

We now briefly indicate how this picture is naturally captured by higher geometry and refined to a single *extended* Lagrangian for coupled 1d and 3d Chern–Simons theory, given by maps on higher moduli stacks. In doing this, we will also see how the ingredients of Kirillov's orbit method and the Borel-Weil-Bott theorem find a natural rephrasing in the context of smooth differential moduli stacks. The key observation is that for $\langle\lambda, -\rangle$ an integral weight for our simple, connected, simply connected and compact Lie group G, the contraction of $\mathfrak{g}$-valued differential forms with λ extends to a morphism of smooth moduli stacks of the form

$$\langle\lambda, -\rangle \ : \ \Omega^1(-, \mathfrak{g})//\underline{T}_\lambda \to \mathbf{B}U(1)_{\text{conn}},$$

where $T_\lambda \hookrightarrow G$ is the maximal torus of G which is the stabilizer subgroup of $\langle\lambda, -\rangle$ under the coadjoint action of G on $\mathfrak{g}^*$. Indeed, this is just the classical statement that exponentiation of $\langle\lambda, -\rangle$ induces an isomorphism between the integral weight lattice $\Gamma_{\text{wt}}(\lambda)$ relative to the maximal torus T_λ and the $\mathbb{Z}$-module $\text{Hom}_{\text{Grp}}(T_\lambda, U(1))$ and that under this isomorphism a gauge transformation of a $\mathfrak{g}$-valued 1-form A turns into that of the $\mathfrak{u}(1)$-valued 1-form $\langle\lambda, A\rangle$.

Comparison with the discussion in Sect. 2 shows that this is the extended Lagrangian of a 1-dimensional Chern–Simons theory. In fact it is just a slight variant of the trace-theory discussed there: if we realize $\mathfrak{g}$ as a matrix Lie algebra and write $\langle\alpha, \beta\rangle = \text{tr}(\alpha \cdot \beta)$ as the matrix trace, then the above Chern–Simons 1-form is given by the "λ-shifted trace"

$$\text{CS}_\lambda(A) := \text{tr}(\lambda \cdot A) \in \Omega^1(-; \mathbb{R}).$$

Then, clearly, while the "plain" trace is invariant under the adjoint action of all of G, the λ-shifted trace is invariant only under the subgroup T_λ of G that fixes λ.

Notice that the domain of $\langle\lambda, -\rangle$ naturally sits inside $\mathbf{B}G_{\text{conn}}$ by the canonical map

$$\Omega^1(-, \mathfrak{g})//\underline{T}_\lambda \to \Omega^1(-, \mathfrak{g})//\underline{G} \simeq \mathbf{B}G_{\text{conn}}.$$

One sees that the homotopy fiber of this map is the *coadjoint orbit* $\mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$ of $\langle\lambda, -\rangle$, equipped with the map of stacks

$$\theta : \mathcal{O}_\lambda \simeq \underline{G}//\underline{T}_\lambda \to \Omega^1(-, \mathfrak{g})//\underline{T}_\lambda$$

which over a test manifold U sends $\boldsymbol{g} \in C^\infty(U, G)$ to the pullback $\boldsymbol{g}^*\theta_G$ of the Maurer–Cartan form. Composing this with the above extended Lagrangian $\langle\lambda, -\rangle$ yields a map

$$\langle\lambda, \theta\rangle : \ \mathcal{O}_\lambda \xrightarrow{\theta} \Omega^1(-, \mathfrak{g})//\underline{T}_\lambda \xrightarrow{\langle\lambda, -\rangle} \mathbf{B}U(1)_{\text{conn}}$$

which modulates a canonical $U(1)$-principal bundle with connection on the coadjoint orbit. One finds that this is the canonical prequantum bundle used in the orbit method [53]. In particular its curvature is the canonical symplectic form on the coadjoint orbit.

So far this shows how the ingredients of the orbit method are incarnated in smooth moduli stacks. This now immediately induces Chern–Simons theory with Wilson loops by considering the map $\Omega^1(-,\mathfrak{g})//\underline{T}_\lambda \to \mathbf{B}G_{\mathrm{conn}}$ itself as the target[10] for a field theory defined on knot inclusions $\iota : S^1 \hookrightarrow \Sigma_3$. This means that a field configuration is a diagram of smooth stacks of the form

$$\begin{array}{ccc} S^1 & \xrightarrow{(\iota^* A)^g} & \Omega^1(-,\mathfrak{g})//\underline{T}_\lambda \\ {\scriptstyle \iota}\downarrow & \swarrow\!\!\!\swarrow_{g} & \downarrow \\ \Sigma_3 & \xrightarrow[A]{} & \mathbf{B}G_{\mathrm{conn}} \end{array},$$

i.e., that a field configuration consists of

- a gauge field A in the "bulk" Σ_3;
- a G-valued function $\boldsymbol{g}$ on the embedded knot

such that the restriction of the ambient gauge field A to the knot is equivalent, via the gauge transformation $\boldsymbol{g}$, to a $\mathfrak{g}$-valued connection on S^1 whose local $\mathfrak{g}$-valued 1-forms are related each other by local gauge transformations taking values in the torus T_λ. Moreover, a gauge transformation between two such field configurations $(A, \boldsymbol{g})$ and $(A', \boldsymbol{g}')$ is a pair (t_{Σ_3}, t_{S^1}) consisting of a G-gauge transformation t_{Σ_3} on Σ_3 and a T_λ-gauge transformation t_{S^1} on S^1, intertwining the gauge transformations $\boldsymbol{g}$ and $\boldsymbol{g}'$. In particular if the bulk gauge field on Σ_3 is held fixed, i.e., if $A = A'$, then t_{S^1} satisfies the equation $\boldsymbol{g}' = \boldsymbol{g}\, t_{S^1}$. This means that the Wilson-line components of gauge-equivalence classes of field configurations are naturally identified with smooth functions $S^1 \to G/T_\lambda$, i.e., with smooth functions on the Wilson loop with values in the coadjoint orbit. This is essentially a rephrasing of the above statement that G/T_λ is the homotopy fiber of the inclusion of the moduli stack of Wilson line field configurations into the moduli stack of bulk field configurations.

We may postcompose the two horizontal maps in this square with our two extended Lagrangians, that for 1d and that for 3d Chern–Simons theory, to get the diagram

$$\begin{array}{ccccc} S^1 & \xrightarrow{(\iota^* A)^g} & \Omega^1(-,\mathfrak{g})//T & \xrightarrow{\langle\lambda,-\rangle} & \mathbf{B}U(1)_{\mathrm{conn}} \\ {\scriptstyle \iota}\downarrow & \swarrow\!\!\!\swarrow^{g} & \downarrow & & \\ \Sigma_3 & \xrightarrow{A} & \mathbf{B}G_{\mathrm{conn}} & \xrightarrow{\hat{\mathbf{c}}} & \mathbf{B}^3U(1)_{\mathrm{conn}} \end{array}.$$

Therefore, writing $\mathbf{Fields}_{\mathrm{CS+W}}\left(S^1 \overset{\iota}{\hookrightarrow} \Sigma_3\right)$ for the moduli stack of field configurations for Chern–Simons theory with Wilson lines, we find two action functionals as the composite top and left morphisms in the diagram

[10] This means that here we are secretly moving from the topos of (higher) stacks on smooth manifolds to its *arrow topos*, see Sect. 5.2.

$$
\begin{array}{ccccc}
\mathbf{Fields}_{\mathrm{CS+W}}\left(S^1 \overset{\iota}{\hookrightarrow} \Sigma_3\right) & \longrightarrow & \mathbf{Maps}(\Sigma_3, \mathbf{B}G_{\mathrm{conn}}) & \xrightarrow{\mathrm{hol}_{\Sigma_3}\mathbf{Maps}(\Sigma_3, \hat{\mathbf{c}})} & \underline{U}(1) \\
\downarrow & & \downarrow & & \\
\mathbf{Maps}(S^1, \Omega^1(-,\mathfrak{g})//T_\lambda) & \longrightarrow & \mathbf{Maps}(S^1, \mathbf{B}G_{\mathrm{con}}) & & \\
\Big\downarrow{\scriptstyle \mathrm{hol}_{S^1}\mathbf{Maps}(S^1, \langle\lambda, -\rangle)} & & & & \\
\underline{U}(1) & & & &
\end{array}
$$

in $\mathbf{H}$, where the top left square is the homotopy pullback that characterizes maps in $\mathbf{H}^{(\Delta^1)}$ in terms of maps in $\mathbf{H}$. The product of these is the action functional

$$
\begin{array}{ccccc}
\mathbf{Fields}_{\mathrm{CS+W}}\left(S^1 \overset{\iota}{\hookrightarrow} \Sigma_3\right) & \longrightarrow & \mathbf{Maps}(\Sigma_3, \mathbf{B}^3U(1)_{\mathrm{conn}}) \times \mathbf{Maps}(S^1, \mathbf{B}U(1)_{\mathrm{conn}}) & & \\
 & & \downarrow & & \\
 & & \underline{U}(1) \times \underline{U}(1) & \xrightarrow{\;\cdot\;} & \underline{U}(1) \,.
\end{array}
$$

where the rightmost arrow is the multiplication in $U(1)$. Evaluated on a field configuration with components $(A, \boldsymbol{g})$ as just discussed, this is

$$
\exp\left(2\pi i\left(\int_{\Sigma_3} \mathrm{CS}_3(A) + \int_{S^1} \langle\lambda, (\iota^* A)^{\boldsymbol{g}}\rangle\right)\right).
$$

This is indeed the action functional for Chern–Simons theory with Wilson loop ι in the representation R corresponding to the integral weight $\langle\lambda, -\rangle$ by the Borel-Weil-Bott theorem, as reviewed for instance in Sect. 4 of [6].

Apart from being an elegant and concise repackaging of this well-known action functional and the quantization conditions that go into it, the above reformulation in terms of stacks immediately leads to prequantum line bundles in Chern–Simons theory with Wilson loops. Namely, by considering the codimension 1 case, one finds the symplectic structure and the canonical prequantization for the moduli stack of field configurations on surfaces with specified singularities at specified punctures [88]. Moreover, this is just the first example in a general mechanism of (extended) action functionals with defect and/or boundary insertions. Another example of the same mechanism is the gauge coupling action functional of the open string. This we discuss in Sect. 5.4.2.

4 Extension to More General Examples

The way we presented the two examples of the previous sections indicates that they are clearly just the beginning of a rather general pattern of extended prequantized higher gauge theories of Chern–Simons type: for every smooth higher group G with universal differential higher moduli stack $\mathbf{B}G_{\text{conn}}$ (and in fact for any higher moduli stack at all, as further discussed in Sect. 5.1) every differentially refined universal characteristic map of stacks

$$\mathbf{L}: \mathbf{B}G_{\text{conn}} \longrightarrow \mathbf{B}^n U(1)_{\text{conn}}$$

constitutes an extended Lagrangian—hence, by iterated transgression, the action functional, prequantum theory and WZW-type action functional—of an n-dimensional Chern–Simons type gauge field theory with (higher) gauge group G. Moreover, just moving from higher stacks on the site of smooth manifolds to higher stacks on the site of smooth supermanifolds one has an immediate and natural generalization to super-Chern–Simons theories. Here we briefly survey some examples of interest, which were introduced in detail in [76] and [30]. Further examples and further details can be found in Sect. 5.7 of [79].

4.1 String Connections and Twisted String structures

Notice how we have moved from the 1d Chern–Simons theory of Sect. 2 to the 3d Chern-Simon theory of Sect. 3 by replacing the connected but not 1-connected compact Lie group $U(n)$ with a compact 2-connected but not 3-connected Lie group G. The natural further step towards a higher dimensional Chern–Simons theory would then be to consider a compact Lie group which is (at least) 3-connected. Unfortunately, there exists no such Lie group: if G is compact and simply connected then its third homotopy group will be nontrivial, see e.g. [62]. However, a solution to this problem does exist if we move from compact Lie groups to the more general context of smooth higher groups, i.e. if we focus on the stacks of principal bundles rather than on their gauge groups. As a basic example, think of how we obtained the stacks $\mathbf{B}SU(n)$ and $\mathbf{B}SU(n)_{\text{conn}}$ out of $\mathbf{B}U(n)$ and $\mathbf{B}U(n)_{\text{conn}}$ in Sect. 2.9. There we first obtained these stacks as homotopy fibers of the morphisms of stacks

$$\mathbf{c}_1 : \mathbf{B}U(n) \to \mathbf{B}U(1); \quad \hat{\mathbf{c}}_1 : \mathbf{B}U(n)_{\text{conn}} \to \mathbf{B}U(1)_{\text{conn}}$$

refining the first Chern class. Then, in a second step, we identified these homotopy fibers with the stack of principal bundles (with and without connection) for a certain compact Lie group, which turned out to be $SU(n)$. However, the homotopy fiber definition would have been meaningful even in case we would have been unable to show that there was a compact Lie group behind it, or even in case there would have been no such. This may seem too far a generalization, but actually Milnor's theorem [61] would have assured us in any case that there existed a *topological* group $SU(n)$ whose classifying space is homotopy equivalent to the topological realization of the homotopy fiber $\mathbf{B}SU(n)$, that is, equivalently, to the homotopy fiber of the topological realization of the morphism $\mathbf{c}_1$. This is nothing but the topological characteristic map

$$c_1 : BU(n) \to BU(1) \simeq K(\mathbb{Z}, 2)$$

defining the first Chern class. In other words, one defines the space $BSU(n)$ as the homotopy pullback

$$\begin{array}{ccc} BSU(n) & \longrightarrow & * \\ \downarrow & & \downarrow \\ BU(n) & \xrightarrow{c_1} & K(\mathbb{Z},2) \end{array} ;$$

the based loop space $\Omega BSU(n)$ has a natural structure of topological group "up to homotopy", and Milnor's theorem precisely tells us that we can strictify it, i.e. we can find a topological group $SU(n)$ (unique up to homotopy) such that $SU(n) \simeq \Omega BSU(n)$. Moreover, $BSU(n)$, defined as a homotopy fiber, will be a classifying space for this "homotopy-$SU(n)$" group. From this perspective, we see that having a model for the homotopy-$SU(n)$ which is a compact Lie group is surely something nice to have, but that we would have nevertheless been able to speak in a rigorous and well-defined way of the groupoid of smooth $SU(n)$-bundles over a smooth manifold X even in case such a compact Lie model did not exist. The same considerations apply to the stack of principal $SU(n)$-bundles with connections.

These considerations may look redundant, since one is well aware that there is indeed a compact Lie group $SU(n)$ with all the required features. However, this way of reasoning becomes prominent and indeed essential when we move to higher characteristic classes. The fundamental example is probably the following. For $n \geq 3$ the spin group $\mathrm{Spin}(n)$ is compact and simply connected with $\pi_3(\mathrm{Spin}(n)) \cong \mathbb{Z}$. The generator of $H^4(B\mathrm{Spin}(n); \mathbb{Z})$ is the first factional Pontrjagin class $\frac{1}{2}p_1$, which can be equivalently seen as a characteristic map

$$\tfrac{1}{2}p_1 \colon B\mathrm{Spin}(n) \to K(\mathbb{Z}; 4).$$

The String group $\mathrm{String}(n)$ is then defined as the topological group whose classifying space is the homotopy fiber of $\frac{1}{2}p_1$, i.e., the homotopy pullback

$$\begin{array}{ccc} B\mathrm{String}(n) & \longrightarrow & * \\ \downarrow & & \downarrow \\ B\mathrm{Spin}(n) & \xrightarrow{\frac{1}{2}p_1} & K(\mathbb{Z},4) \end{array} ;$$

this defines $\mathrm{String}(n)$ uniquely up to homotopy. The topological group $\mathrm{String}(n)$ is 6-connected with $\pi_7(\mathrm{String}(n)) \cong \mathbb{Z}$. The generator of $H^8(B\mathrm{String}(n); \mathbb{Z})$ is the second fractional Pontrjagin class $\frac{1}{6}p_2$, see [75]. One can then define the 3-stack of smooth $\mathrm{String}(n)$-principal bundles as the homotopy pullback

$$\begin{array}{ccc} \mathbf{B}\mathrm{String}(n) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}\mathrm{Spin}(n) & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3U(1) \end{array} ,$$

where $\frac{1}{2}\mathbf{p}_1$ is the morphism of stacks whose topological realization is $\frac{1}{2}p_1$. In other words, a $\mathrm{String}(n)$-principal bundle over a smooth manifold X is the datum of a $\mathrm{Spin}(n)$-principal bundle over X together with a trivialization of the associated $B^2U(1)$-principal 3-bundle. The characteristic map

$$\tfrac{1}{6}p_2 : B\mathrm{String}(n) \to K(\mathbb{Z}; 8)$$

is the topological realization of a morphism of stacks

$$\tfrac{1}{6}\mathbf{p}_2 : \mathbf{B}\mathrm{String}(n) \to \mathbf{B}^7 U(1),$$

see [31, 76]. Similarly, one can define the 3-stack of smooth String bundles with connections as the homotopy pullback

$$\begin{array}{ccc} \mathbf{B}\mathrm{String}(n)_{\mathrm{conn}} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}\mathrm{Spin}(n)_{\mathrm{conn}} & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1} & \mathbf{B}^3 U(1)_{\mathrm{conn}} \end{array},$$

where $\frac{1}{2}\hat{\mathbf{p}}_1$ is the lift of $\frac{1}{2}\mathbf{p}_1$ to the stack of $\mathrm{Spin}(n)$-bundles with connections. Again, this means that a $\mathrm{String}(n)$-bundle with connection over a smooth manifold X is the datum of a $\mathrm{Spin}(n)$-bundle with connection over X together with a trivialization of the associated $U(1)$-3-bundle with connection. The morphism $\frac{1}{6}\mathbf{p}_2$ lifts to a morphism

$$\tfrac{1}{6}\hat{\mathbf{p}}_2 : \mathbf{B}\mathrm{String}(n)_{\mathrm{conn}} \to \mathbf{B}^7 U(1)_{\mathrm{conn}},$$

see [31], and this defines a 7d Chern–Simons theory with gauge group the $\mathrm{String}(n)$-group.

In the physics literature one usually considers also a more flexible notion of String connection, in which one requires that the underlying $U(1)$-3-bundle of a $\mathrm{Spin}(n)$-bundle with connection is trivialized, but does not require the underlying 3-connection to be trivialized. In terms of stacks, this corresponds to considering the homotopy pullback

$$\begin{array}{ccc} \mathbf{B}\mathrm{String}(n)_{\mathrm{conn}'} & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}\mathrm{Spin}(n)_{\mathrm{conn}} & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3 U(1) \end{array};$$

see, e.g., [84]. Furthermore, it is customary to consider not only the case where the underlying $U(1)$-3-bundle (with or without connection) is trivial, but also the case when it is equivalent to a fixed *background* $U(1)$-3-bundle (again, eventually with connection). Notably, the connection 3-form of this fixed background is the C-field of the M-theory literature (cf. [70, 71]). The moduli stacks of $\mathrm{Spin}(n)$-bundles on a smooth manifold X with possibly nontrivial fixed $U(1)$-3-bundle background are called [76] moduli stacks of *twisted* String bundles on X. A particular interesting case is when the twist is independent of X, hence is itself given by a universal characteristic class, hence by a twisting morphism

$$\mathbf{c} : \mathbf{S} \longrightarrow \mathbf{B}^3 U(1),$$

where $\mathbf{S}$ is some (higher moduli) stack. In this case, indeed, one can define the stack $\mathbf{B}\mathrm{String}(n)^{\mathbf{c}}$ of $\mathbf{c}$-twisted $\mathrm{String}(n)$-structures as the homotopy pullback

$$\begin{array}{ccc} \mathbf{B}\mathrm{String}(n)^{\mathbf{c}} & \longrightarrow & \mathbf{S} \\ \downarrow & & \downarrow{\scriptstyle \mathbf{c}} \\ \mathbf{B}\mathrm{Spin}(n) & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1} & \mathbf{B}^3U(1) \end{array} ,$$

and similarly for the stack of $\mathbf{c}$-twisted String(n)-connections. This is a higher analog of Spin^c-structures, whose universal moduli stack sits in the analogous homotopy pullback diagram

$$\begin{array}{ccc} \mathbf{B}\mathrm{Spin}^c(n) & \longrightarrow & \mathbf{B}U(1) \\ \downarrow & & \downarrow{\scriptstyle \mathbf{c}_1 \bmod 2} \\ \mathbf{B}SO(n) & \xrightarrow{\mathbf{w}_1} & \mathbf{B}^2\mathbb{Z}_2 \end{array} .$$

(For more on higher Spin^c-structures see also [72, 73] and Sect. 5.2 of [79].). By a little abuse of terminology, when the twisting morphism $\mathbf{a}$ is the refinement of a characteristic class for a compact simply connected simple Lie group G to a morphism of stacks $\mathbf{a} : \mathbf{B}G \to \mathbf{B}^3U(1)$, one may speak of *G-twisted structures* rather than of $\mathbf{a}$-twisted structures.

By the discussion in Sect. 3 the differential twisting maps $\frac{1}{2}\hat{\mathbf{p}}_1$ and $\hat{\mathbf{a}}$ appearing here are at the same time extended Lagrangians of Chern–Simons theories. Together with the nature of homotopy pullback, it follows [31] that a field $\phi : X \to \mathbf{B}\mathrm{String}^{\mathbf{a}}{}_{\mathrm{conn}}$ consists of a pair of gauge fields and a homotopy between their Chern–Simons data, namely of

1. a Spin-connection $\nabla_{\mathfrak{so}}$;
2. a G-connection $\nabla_{\mathfrak{g}}$;
3. a twisted 2-form connection B whose curvature 3-form H is locally given by $H = dB + \mathrm{CS}(\nabla_{\mathfrak{so}}) - \mathrm{CS}(\nabla_{\mathfrak{g}})$.

These are the data for (Green–Schwarz-) anomaly-free background gauge fields (gravity, gauge field, Kalb–Ramond field) for the heterotic string [76]. A further refinement of this construction yields the universal moduli stack for the *supergravity C-field* configurations in terms of E_8-twisted String connections [29]. Here the presence of the differential characteristic maps $\hat{\mathbf{c}}$ induces the Chern–Simons gauge-coupling piece of the supergravity 2-brane (the *M2-brane*) action functional.

4.2 Cup-Product Chern–Simons Theories

In Sect. 3 we had restricted attention to 3d Chern–Simons theory with simply connected gauge groups. Another important special case of 3d Chern–Simons theory is that with gauge group the circle group $U(1)$, which is of course not simply connected. In this case the universal characteristic map that controls the theory is the differential refinement of the *cup product class* $c_1 \cup c_1$. Here we briefly indicate this case and the analogous higher dimensional Chern–Simons theories obtained from cup products of higher classes and from higher order cup products.

The cup product $\cup$ in integral cohomology can be lifted to a cup product $\hat{\cup}$ in differential cohomology, i.e., for any smooth manifold X we have a natural commutative diagram

$$\begin{array}{ccc} \hat{H}^p(X;\mathbb{Z}) \otimes \hat{H}^q(X;\mathbb{Z}) & \xrightarrow{\hat{\cup}} & \hat{H}^{p+q}(X;\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^p(X;\mathbb{Z}) \otimes H^q(X;\mathbb{Z}) & \xrightarrow{\cup} & H^{p+q}(X;\mathbb{Z}) \end{array},$$

for any $p, q \geq 0$. Moreover, this cup product is induced by a cup product defined at the level of Čech–Deligne cocycles, the so called *Beilinson-Drinfeld cup product*, see [8]. This, in turn, may be seen [30] to come from a morphism of higher universal moduli stacks

$$\widehat{\cup} : \mathbf{B}^{n_1}U(1)_{\text{conn}} \times \mathbf{B}^{n_2}U(1)_{\text{conn}} \to \mathbf{B}^{n_1+n_2+1}U(1)_{\text{conn}}.$$

Moreover, since the Beilinson–Deligne cup product is associative up to homotopy, this induces a well-defined morphism

$$\mathbf{B}^{n_1}U(1)_{\text{conn}} \times \mathbf{B}^{n_2}U(1)_{\text{conn}} \times \cdots \times \mathbf{B}^{n_{k+1}}U(1)_{\text{conn}} \to \mathbf{B}^{n_1+\cdots+n_{k+1}+k}U(1)_{\text{conn}}.$$

In particular, for $n_1 = \cdots = n_{k+1} = 3$, one finds a cup product morphism

$$\left(\mathbf{B}^3U(1)_{\text{conn}}\right)^{k+1} \to \mathbf{B}^{4k+3}U(1)_{\text{conn}}.$$

Furthermore, one sees from the explicit expression of the Beilinson–Deligne cup product that, on a local chart U_α, if the 3-form datum of a connection on a $U(1)$-3-bundle is the 3-form C_α, then the $(4k+3)$-form local datum for the corresponding connection on the associated $U(1)$-$(4k+3)$-bundle is

$$C_\alpha \wedge \underbrace{dC_\alpha \wedge \cdots \wedge dC_\alpha}_{k \text{ times}}.$$

Now let G be a compact and simply connected simple Lie group and let $\hat{\mathbf{c}} : \mathbf{B}G_{\text{conn}} \to \mathbf{B}^3U(1)_{\text{conn}}$ be the morphism of stacks underlying the fundamental characteristic class $c \in H^4(BG, \mathbb{Z})$. Then we can consider the $(k+1)$-fold product of $\hat{\mathbf{c}}$ with itself:

$$\hat{\mathbf{c}} \mathbin{\hat{\cup}} \hat{\mathbf{c}} \mathbin{\hat{\cup}} \cdots \mathbin{\hat{\cup}} \hat{\mathbf{c}} : \mathbf{B}G_{\text{conn}} \xrightarrow{(\hat{\mathbf{c}},\ldots,\hat{\mathbf{c}})} \left(\mathbf{B}^3U(1)_{\text{conn}}\right)^{k+1} \xrightarrow{\hat{\cup}} \mathbf{B}^{4k+3}U(1)_{\text{conn}}.$$

If X is a compact oriented smooth manifold, fiber integration along X gives the morphism

$$\mathbf{Maps}(X, \mathbf{B}G_{\text{conn}}) \to \mathbf{Maps}(X, \mathbf{B}^{4k+3}U(1)_{\text{conn}}) \xrightarrow{\text{hol}_X} \mathbf{B}^{4k+3-\dim X}U(1)_{\text{conn}}.$$

In particular, if $\dim X = 4k+3$, by evaluating over the point and taking equivalence classes we get a canonical morphism

$$\{G\text{-bundles with connections on } X\}/\text{iso} \to U(1).$$

This is the action functional of the $(k+1)$-fold *cup product Chern–Simons theory* induced by the $(k+1)$-fold cup product of c with itself [30]. This way one obtains, for every $k \geq 0$, a $(4k+3)$-dimensional theory starting with a 3d Chern–Simons theory. Moreover, in the special case that the principal G-bundle on X is topologically trivial, this action functional has a particularly simple expression: it is given by

$$\exp 2\pi i \int_X \mathrm{CS}_3(A) \wedge \langle F_A, F_A \rangle \wedge \cdots \wedge \langle F_A, F_A \rangle,$$

where $A \in \Omega^1(X; \mathfrak{g})$ is the $\mathfrak{g}$-valued 1-form on X representing the connection in the chosen trivialization of the G-bundle. But notice that in this more general situation now not *every* gauge field configuration will have an underlying trivializable (higher) bundle anymore, the way it was true for the 3d Chern–Simons theory of a simply connected Lie group in Sect. 3.

More generally, one can consider an arbitrary smooth (higher) group G, e.g. $U(n) \times \mathrm{Spin}(m) \times \mathrm{String}(l)$, together with $k+1$ characteristic maps $\hat{\mathbf{c}}_i : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^{n_i} U(1)_{\mathrm{conn}}$ and one can form the $(k+1)$-fold product

$$\hat{\mathbf{c}}_1 \,\hat{\cup} \cdots \hat{\cup}\, \hat{\mathbf{c}}_{k+1} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^{n_1 + \cdots n_{k+1} + k} U(1)_{\mathrm{conn}},$$

inducing a $(n_1 + \cdots n_{k+1} + k)$-dimensional Chern–Simons-type theory. For instance, if G_1 and G_2 are two compact simply connected simple Lie groups, then we have a 7d cup product Chern–Simons theory associated with the cup product $\hat{\mathbf{c}}_1 \,\hat{\cup}\, \hat{\mathbf{c}}_2$. If (P_1, ∇_1) and (P_2, ∇_2) are a pair of topologically trivial principal G_1- and G_2-bundles with connections over a 7-dimensional oriented compact manifold without boundary X, the action functional of this Chern–Simons theory on this pair is given by

$$\exp 2\pi i \int_X \mathrm{CS}_3(A_1) \wedge \langle F_{A_2}, F_{A_2} \rangle = \exp 2\pi i \int_X \mathrm{CS}_3(A_2) \wedge \langle F_{A_1}, F_{A_1} \rangle,$$

where A_i is the connection 1-forms of ∇_i, for $i = 1, 2$. Notice how in general a G_i-principal bundle on a 7-dimensional manifold is not topologically trivial, but still we have a well defined cup-product Chern–Simons action $S_{\hat{\mathbf{c}}_1 \hat{\cup} \hat{\mathbf{c}}_2}$. In the topologically nontrivial situation, however, there will not be such a simple global expression for the action.

Let us briefly mention a few representative important examples from string theory and M-theory which admit a natural interpretation as cup-product Chern–Simons theories, the details of which can be found in [30]. For all examples presented below we write the Chern–Simons action for the topologically trivial sector.

- *Abelian higher dimensional CS theory and self-dual higher gauge theory.* For every $k \in \mathbb{N}$ the differential cup product yields the extended Lagrangian

$$\mathbf{L}: \mathbf{B}^{2k+1}U(1)_{\text{conn}} \longrightarrow \mathbf{B}^{2k+1}U(1)_{\text{conn}} \times \mathbf{B}^{2k+1}U(1)_{\text{conn}} \xrightarrow{\hat{\cup}} \mathbf{B}^{4k+3}U(1)_{\text{conn}}$$

for a $4k+3$-dimensional Chern–Simons theory of $(2k+1)$-form connections on higher circle bundles (higher bundle gerbes). Over a 3-dimensional manifold Σ the corresponding action functional applied to gauge fields A whose underlying bundle is trivial is given by

$$\exp 2\pi i \int_{\Sigma} \mathrm{CS}_1(A) \cup d\mathrm{CS}_1(A) = \exp 2\pi i \int_{\Sigma} A \wedge F_A,$$

where $F_A = dA$ is the curvature of a $U(1)$-connection A. Similarly, the transgression of $\mathbf{L}$ to codimension 1 over a manifold Σ of dimension $4k+2$ yields the prequantization of a symplectic form on $(2k+1)$-form connections which, by a derivation analogous to that in Sect. 3.4.3, is given by

$$\omega(\delta A_1, \delta A_1) = \int_{\Sigma} \delta A_1 \wedge \delta A_1.$$

A complex polarization of this symplectic structure is given by a choice of conformal metric on Σ and the corresponding canonical coordinates are complex Hodge self-dual forms on Σ. This yields the famous holographic relation between higher abelian Chern–Simons theory and self-dual higher abelian gauge theory in one dimension lower.

- *The M5-brane self-dual theory.* In particular, for $k=1$ it was argued in [89] that the 7-dimensional Chern–Simons theory which we refine to an extended prequantum theory by the extended Lagrangian

$$\mathbf{L}: \mathbf{B}^{3}U(1)_{\text{conn}} \longrightarrow \mathbf{B}^{3}U(1)_{\text{conn}} \times \mathbf{B}^{3}U(1)_{\text{conn}} \xrightarrow{\hat{\cup}} \mathbf{B}^{7}U(1)_{\text{conn}}$$

describes, in this holographic manner, the quantum theory of the self-dual 2-form in the 6-dimensional worldvolume theory of a single M5-brane. Since moreover in [90] it was argued that this abelian 7-dimensional Chern–Simons theory is to be thought of as the abelian piece in the Chern–Simons term of 11-dimensional supergravity compactified on a 4-sphere, and since this term in general receives non-abelian corrections from "flux quantization" (see [29] for a review of these and for discussion in the present context of higher moduli stacks), we discussed in [28] the appropriate non-abelian refinement of this 7d Chern–Simons term, which contains also cup product terms of the form $\hat{\mathbf{a}}_1 \,\widehat{\cup}\, \hat{\mathbf{a}}_2$ as well we the term $\frac{1}{6}\widehat{\mathbf{p}}_2$ from Sect. 4.1.
- *Five-dimensional and eleven-dimensional supergravity.* The topological part of the five-dimensional supergravity action is $\exp 2\pi i \int_{Y^5} A \wedge F_A \wedge F_A$, where A is a $U(1)$-connection. Writing the action as $\exp 2\pi i \int_{Y^5} \mathrm{CS}_1(A) \cup d\mathrm{CS}_1(A) \cup d\mathrm{CS}_1(A)$, one sees this is a 3-fold Chern–Simons theory. Next, in eleven dimensions, the C-field C_3 with can be viewed as a 3-connection on a 2-gerbe with 4-curvature G_4. By identifying the C-field with the Chern–Simons 3-form $\mathrm{CS}_3(A)$

of a $U(1)$-3-connection A, the topological action $\exp 2\pi i \int_{Y^{11}} C_3 \wedge G_4 \wedge G_4$, is seen to be of the form $\exp 2\pi i \int_{Y^{11}} \mathrm{CS}_3(A) \cup d\mathrm{CS}_3(A) \cup d\mathrm{CS}_3(A)$. This realizes the 11d supergravity C-field action as the action for a 3-tier cup-product abelian Chern–Simons theory induced by a morphism of 3-stacks [29].

4.3 Super-Chern–Simons Theories

The (higher) topos **H** of (higher) stacks on the smooth site of manifolds which we have been considering for most of this paper has an important property common to various similar toposes such as that on supermanifolds: it satisfies a small set of axioms called (differential) *cohesion*, see [79]. Moreover, essentially every construction described in the above sections makes sense in an arbitrary cohesive topos. For constructions like homotopy pullbacks, mapping spaces, adjoint actions etc., this is true for every topos, while the differential cohesion in addition guarantees the existence of differential geometric structures such as de Rham coefficients, connections, differential cohomology, etc. This setting allows to transport all considerations based on the cohesion axioms across various kinds of geometries. Notably, one can speak of higher *supergeometry*, and hence of fermionic quantum fields, simply by declaring the site of definition to be that of supermanifolds: indeed, the higher topos of (higher) stacks on supermanifolds is differentially cohesive ([79], Sect. 4.6). This leads to a natural notion of *super-Chern–Simons theories*.

In order to introduce these notions, we need a digression on higher complex line bundles. Namely, we have been using the n-stacks $\mathbf{B}^n U(1)$, but without any substantial change in the theory we could also use the n-stacks $\mathbf{B}^n\mathbb{C}^\times$ with the multiplicative group $U(1)$ of norm 1 complex numbers replaced by the full multiplicative group of non-zero complex numbers. Since we have a fiber sequence

$$\mathbb{R}_{>0} \to \mathbb{C}^\times \to U(1)$$

with topologically contractible fiber, under geometric realization $|-|$ the canonical map $\mathbf{B}^n U(1) \to \mathbf{B}^n\mathbb{C}^\times$ becomes an equivalence. Nevertheless, some constructions are more naturally expressed in terms of $U(1)$-principal n-bundles, while others are more naturally expressed in terms of $\mathbb{C}^\times$-principal n-bundles, and so it is useful to be able to switch from one description to the other. For $n = 1$ this is the familiar fact that the classifying space of principal $U(1)$-bundles is homotopy equivalent to the classifying space of complex line bundles. For $n = 2$ we still have a noteworthy (higher) linear algebra interpretation: $\mathbf{B}^2\mathbb{C}^\times$ is naturally identified with the 2-stack $2\mathbf{Line}_\mathbb{C}$ of *complex line 2-bundles*. Namely, for R a commutative ring (or more generally an E_∞-ring), one considers the 2-category of R-algebras, bimodules and bimodule homomorphisms (e.g. [22]). We may think of this as the 2-category of *2-vector spaces* over R (appendix A of [78], Sect. 4.4 of [82], Sect. 7 of [36]). Notice that this 2-category is naturally braided monoidal. We then write

$$2\mathrm{Line}_R \hookrightarrow 2\mathrm{Vect}_R$$

for the full sub-2-groupoid on those objects which are invertible under this tensor product: the *2-lines* over R. This is the *Picard 2-groupoid* over R, and with the inherited monoidal structure it is a 3-group, the *Brauer 3-group* of R. Its homotopy groups have a familiar algebraic interpretation:

- $\pi_0(2\mathrm{Line}_R)$ is the *Brauer group* of R;
- $\pi_1(2\mathrm{Line}_R)$ is the ordinary *Picard group* of R (of ordinary R-lines);
- $\pi_2(2\mathrm{Line}_R) \simeq R^\times$ is the *group of units*.

(This is the generalization to $n = 2$ of the familiar Picard 1-groupoid $1\mathrm{Line}_R$ of invertible R-modules.) Since the construction is natural in R and naturality respects 2-lines, by taking R to be a sheaf of k-algebras, with k a fixed field, one defines the 2-stacks $2\mathbf{Vect}_k$ of *k-2-vector bundles* and $2\mathbf{Line}_k$ of 2-line bundles over k. If k is algebraically closed, then there is, up to equivalence, only a single 2-line and only a single invertible bimodule, hence $2\mathbf{Line}_k \simeq \mathbf{B}^2 k^\times$. In particular, we have that

$$2\mathbf{Line}_{\mathbb{C}} \simeq \mathbf{B}^2\mathbb{C}^\times .$$

The background B-field of the bosonic string has a natural interpretation as a section of the differential refinement $\mathbf{B}^2\mathbb{C}^\times_{\mathrm{conn}}$ of the 2-stack $\mathbf{B}^2\mathbb{C}^\times$. Hence, by the above discussion, it is identified with a 2-connection on a complex 2-line bundle. However, a careful analysis, due to [23] and made more explicit in [35], shows that for the superstring the background B-field is more refined. Expressed in the language of higher stacks the statement is that the superstring B-field is a connection on a complex *super*-2-line bundle. This means that one has to move from the (higher) topos of (higher) stacks on the site of smooth manifolds to that of stacks on the site of smooth supermanifolds (Sect. 4.6 of [79]). The 2-stack of complex 2-line bundles is then replaced by the 2-stack $2\mathbf{sLine}_{\mathbb{C}}$ of super-2-line bundles, whose global points are complex Azumaya superalgebras. Of these there are, up to equivalence, not just one but two: the canonical super 2-line and its "superpartner" [85]. Moreover, there are now, up to equivalence, two different invertible 2-linear maps from each of these super-lines to itself. In summary, the homotopy sheaves of the super 2-stack of super line 2-bundles are

- $\pi_0(2\mathbf{sLine}_{\mathbb{C}}) \simeq \mathbb{Z}_2$,
- $\pi_1(2\mathbf{sLine}_{\mathbb{C}}) \simeq \mathbb{Z}_2$,
- $\pi_2(2\mathbf{sLine}_{\mathbb{C}}) \simeq \mathbb{C}^\times$.

Since the homotopy groups of the group $\mathbb{C}^\times$ are $\pi_0(\mathbb{C}^\times) = 0$ and $\pi_1(\mathbb{C}^\times) = \mathbb{Z}$, it follows that the geometric realization of this 2-stack has homotopy groups

- $\pi_0(|2\mathbf{sLine}_{\mathbb{C}}|) \simeq \mathbb{Z}_2$,
- $\pi_1(|2\mathbf{sLine}_{\mathbb{C}}|) \simeq \mathbb{Z}_2$,
- $\pi_2(|2\mathbf{sLine}_{\mathbb{C}}|) \simeq 0$,
- $\pi_3(|2\mathbf{sLine}_{\mathbb{C}}|) \simeq \mathbb{Z}$.

These are precisely the correct coefficients for the twists of complex K-theory [24], witnessing the fact that the B-field background of the superstring twists the Chan-Paton bundles on the D-branes [23, 35].

The braided monoidal structure of the 2-category of complex super-vector spaces induces on $2\mathbf{sLine}_{\mathbb{C}}$ the structure of a *braided 3-group*. Therefore, one has a naturally defined 3-stack $\mathbf{B}(2\mathbf{sLine}_{\mathbb{C}})_{\mathrm{conn}}$ which is the supergeometric refinement of the coefficient object $\mathbf{B}^3\mathbb{C}^\times_{\mathrm{conn}}$ for the extended Lagrangian of bosonic 3-dimensional Chern–Simons theory. Therefore, for G a super-Lie group a super-Chern–Simons theory, inducing a super-WZW action functional on G, is naturally given by an extended Lagrangian which is a map of higher moduli stacks of the form

$$\mathbf{L} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}(2\mathbf{sLine}_{\mathbb{C}})_{\mathrm{conn}}.$$

Notice that, by the canonical inclusion $\mathbf{B}^3\mathbb{C}^\times_{\mathrm{conn}} \to \mathbf{B}(2\mathbf{sLine}_{\mathbb{C}})_{\mathrm{conn}}$, every bosonic extended Lagrangian of 3d Chern–Simons type induces such a supergeometric theory with trivial super-grading part.

5 Outlook: Higher Prequantum Theory

The discussion in Sects. 2 and 3 of low dimensional Chern–Simons theories and the survey on higher dimensional Chern–Simons theories in Sect. 4, formulated and extended in terms of higher stacks, is a first indication of a fairly comprehensive theory of higher and extended prequantum gauge field theory that is naturally incarnated in a suitable context of higher stacks. In this last section we give a brief glimpse of some further aspects. Additional, more comprehensive expositions and further pointers are collected for instance in [79, 80].

5.1 σ-models

The Chern–Simons theories presented in the previous sections are manifestly special examples of the following general construction: one has a universal (higher) stack **Fields** of field configurations for a certain field theory, equipped with an *extended* Lagrangian, namely with a map of higher stacks

$$\mathbf{L} \ : \ \mathbf{Fields} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$$

to the n-stack of $U(1)$-principal n-bundles with connections. The Lagrangian $\mathbf{L}$ induces Lagrangian data in arbitrary codimension: for every closed oriented world-volume Σ_k of dimension $k \leq n$ there is a *transgressed* Lagrangian

$$\mathbf{Maps}(\Sigma_k; \mathbf{Fields}) \xrightarrow{\mathbf{Maps}(\Sigma_k;\mathbf{L})} \mathbf{Maps}(\Sigma_k; \mathbf{B}^n U(1)_{\mathrm{conn}}) \xrightarrow{\mathrm{hol}_{\Sigma_k}} \mathbf{B}^{n-k}U(1)_{\mathrm{conn}}$$

defining the (off-shell) prequantum $U(1)$-$(n-k)$-bundle of the given field theory. In particular, the curvature forms of these bundles induce the canonical pre-$(n-k)$-plectic structure on the moduli stack of field configurations on Σ_k.

In codimension 0, i.e., for $k = n$ one has the morphism of stacks

$$\exp(2\pi i \int_{\Sigma_n} -) : \mathbf{Maps}(\Sigma_n; \mathbf{Fields}) \to \underline{U}(1)$$

and so taking global sections over the point and passing to equivalence classes one finds the *action functional*

$$\exp(2\pi i \int_{\Sigma_n} -) : \{\text{Field configurations}\}/\text{equiv} \to U(1).$$

Notice how the stacky origin of the action functional automatically implies that its value only depends on the gauge equivalence class of a given field configuration. Moreover, the action functional of an extended Lagrangian field theory as above is manifestly a σ-model action functional: the target "space" is the universal moduli stack of field configurations itself. Furthermore, the composition

$$\omega : \mathbf{Fields} \xrightarrow{\mathbf{L}} \mathbf{B}^n U(1)_{\text{conn}} \xrightarrow{F_{(-)}} \Omega^{n+1}(-;\mathbb{R})_{\text{cl}}$$

shows that the stack of field configurations is naturally equipped with a pre-n-plectic structure [67], which means that actions of extended Lagrangian field theories in the above sense are examples of σ-models with (pre)-n-plectic targets. For *binary* dependence of the n-plectic form on the fields this includes the AKSZ σ-models [2, 16–19, 26, 50, 51, 55, 56, 69]. For instance, from this perspective, the action functional of classical 3d Chern–Simons theory is the σ-model action functional with target the stack $\mathbf{B}G_{\text{conn}}$ equipped with the pre-3-plectic form $\langle -, - \rangle : \mathbf{B}G_{\text{conn}} \to \Omega^4_{\text{cl}}$ (the Killing form invariant polynomial) as discussed in 3. If we consider binary invariant polynomials in *derived* geometry, hence on objects with components also in negative degree, then also closed bosonic string field theory as in [91] is an example (see 5.7.10 of [79]) as are constructions such as [21]. Examples of n-plectic structures of higher arity on moduli stacks of higher gauge fields are in [28, 30].

More generally, we have transgression of the extended Lagrangian over manifolds Σ_k with boundary $\partial\Sigma_k$. Again by inspection of the constructions in [44] in terms of Deligne complexes, one finds that under the Dold–Kan correspondence these induce the corresponding constructions on higher moduli stacks: the *higher parallel transport* of $\mathbf{L}$ over Σ_k yields a *section* of the $(n-k+1)$-bundle which is modulated over the boundary by $\mathbf{Maps}(\partial\Sigma_k, \mathbf{B}G_{\text{conn}}) \to \mathbf{B}^{n-k+1}U(1)_{\text{conn}}$. This is the incarnation at the prequantum level of the *propagator* of the full extended TQFT in the sense of [60] over Σ_k, as indicated in [58]. Further discussion of this full prequantum field theory obtained this way is well beyond the scope of the present article. However, below in Sect. 5.4 we indicate how familiar *anomaly cancellation* constructions in open string theory naturally arise as examples of such transgression of extended Lagrangians over worldvolumes with boundary.

5.2 Fields in Slices: Twisted Differential Structures

Our discussion of σ-model-type actions in the previous section might seem to suggest that all the fields that one encounters in field theory have moduli that form (higher) stacks on the site of smooth manifolds. However, this is actually not the case and one need not look too far in order to find a counterexample: the field of gravity in general relativity is a (pseudo-)Riemannian metric on spacetime, and there is no such thing as a stack of (pseudo-)Riemannian metrics on the smooth site. This is nothing but the elementary fact that a (pseudo-)Riemannian metric cannot be pulled back along an arbitrary smooth morphism between manifolds, but only along local diffeomorphisms. Translated into the language of stacks, this tells us that (pseudo-)Riemannian metrics is a stack on the étale site of smooth manifolds, but not on the smooth site.[11] Yet we can still look at (pseudo-)Riemannian metrics on a smooth n-dimensional manifold X from the perspective of the topos $\mathbf{H}$ of stacks over the smooth site, and indeed this is the more comprehensive point of view. Namely, working in $\mathbf{H}$ also means to work with all its *slice toposes* (or *over-toposes*) $\mathbf{H}_{/\mathbf{S}}$ over the various objects $\mathbf{S}$ in $\mathbf{H}$. For the field of gravity this means working in the slice $\mathbf{H}_{/\mathbf{B}GL(n;\mathbb{R})}$ over the stack $\mathbf{B}GL(n;\mathbb{R})$.[12]

Once again, this seemingly frightening terminology is just a concise and rigorous way of expressing a familiar fact from Riemannian geometry: endowing a smooth n-manifold X with a pseudo-Riemannian metric of signature $(p, n-p)$ is equivalent to performing a reduction of the structure group of the tangent bundle of X to $O(p, n-p)$. Indeed, one can look at the tangent bundle as a morphism $\tau_X : X \to \mathbf{B}GL(n;\mathbb{R})$.

Example: Orthogonal Structures.

The above reduction is then the datum of a homotopy lift of τ_X

$$\begin{array}{ccc} & & \mathbf{B}O(n) \\ & \overset{o_X}{\nearrow} \; \Downarrow e & \downarrow \\ X & \xrightarrow[\tau_X]{} & \mathbf{B}\mathrm{GL}(n;\mathbb{R}) \end{array} ,$$

where the vertical arrow

$$\mathbf{OrthStruc}_n : \; \mathbf{B}O(n) \longrightarrow \mathbf{B}\mathrm{GL}(n;\mathbb{R})$$

is induced by the inclusion of groups $O(n) \hookrightarrow GL(n;\mathbb{R})$. Such a commutative diagram is precisely a map

$$(o_X, e) : \; \tau_X \longrightarrow \mathbf{OrthStruc}_n$$

in the slice $\mathbf{H}_{/\mathbf{B}GL(n;\mathbb{R})}$. The homotopy e appearing in the above diagram is precisely the *vielbein field* (frame field) which exhibits the reduction, hence which induces the

[11] See [13] for a comprehensive treatment of the étale site of smooth manifolds and of the higher topos of higher stacks over it.

[12] More detailed discussion of how (quantum) fields generally are maps in slices of cohesive toposes has been given in the lecture notes [80] and in Sects. 1.2.16, 5.4 of [79].

Riemannian metric. So the moduli stack of Riemannian metrics in n dimensions is $\mathbf{OrthStruc}_n$, not as an object of the ambient cohesive topos $\mathbf{H}$, but of the slice $\mathbf{H}_{/\mathbf{B}\mathrm{GL}(n)}$. Indeed, a map between manifolds regarded in this slice, namely a map $(\phi, \eta) : \tau_Y \to \tau_X$, is equivalently a smooth map $\phi : Y \to X$ in $\mathbf{H}$, but equipped with an equivalence $\eta : \phi^*\tau_X \to \tau_Y$. This precisely exhibits ϕ as a local diffeomorphism. In this way the slicing formalism automatically knows along which kinds of maps metrics may be pulled back.

Example: (Exceptional) Generalized Geometry.

If we replace in the above example the map $\mathbf{OrthStruc}_n$ with inclusions of other maximal compact subgroups, we similarly obtain the moduli stacks for *generalized* geometry (metric and B-field) as appearing in type II superstring backgrounds (see, e.g., [47]), given by

$$\mathbf{typeII} : \mathbf{B}(O(n) \times O(n)) \longrightarrow \mathbf{B}O(n,n) \quad \in \mathbf{H}_{/\mathbf{B}O(n,n)}$$

and of *exceptional generalized geometry* appearing in compactifications of 11-dimensional supergravity [49], given by

$$\mathbf{ExcSugra}_n : \mathbf{B}K_n \longrightarrow \mathbf{B}E_{n(n)} \quad \in \mathbf{H}_{/\mathbf{B}E_{n(n)}}.$$

For instance, a manifold X in type II-geometry is represented by $\tau_X^{\mathrm{gen}} : X \to \mathbf{B}O(n,n)$ in the slice $\mathbf{H}_{/\mathbf{B}O(n,n)}$, which is the map modulating what is called the *generalized tangent bundle*, and a field of generalized type II gravity is a map $(o_X^{\mathrm{gen}}, e) : \tau_X^{\mathrm{gen}} \to \mathbf{typeII}$ to the moduli stack in the slice. One checks that the homotopy e is now precisely what is called the *generalized vielbein field* in type II geometry. We read off the kind of maps along which such fields may be pulled back: a map $(\phi, \eta) : \tau_Y^{\mathrm{gen}} \to \tau_X^{\mathrm{gen}}$ is a *generalized* local diffeomorphism: a smooth map $\phi : Y \to X$ equipped with an equivalence of generalized tangent bundles $\eta : \phi^*\tau_X^{\mathrm{gen}} \to \tau_Y^{\mathrm{gen}}$. A directly analogous discussion applies to the exceptional generalized geometry.

Furthermore, various topological structures are generalized fields in this sense, and become fields in the more traditional sense after differential refinement.

Example: Spin Structures.

The map $\mathbf{SpinStruc} : \mathbf{B}\mathrm{Spin} \to \mathbf{B}\mathrm{GL}$ is, when regarded as an object of $\mathbf{H}_{/\mathbf{B}\mathrm{GL}}$, the moduli stack of spin structures. Its differential refinement $\mathbf{SpinStruc}_{\mathrm{conn}} : \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \to \mathbf{B}\mathrm{GL}_{\mathrm{conn}}$ is such that a domain object $\tau_X^{\nabla} \in \mathbf{H}_{/\mathbf{GL}_{\mathrm{conn}}}$ is given by an affine connection, and a map $(\nabla_{\mathrm{Spin}}, e) : \tau_X^{\nabla} \to \mathbf{SpinStruc}_{\mathrm{conn}}$ is precisely a *Spin connection* and a Lorentz frame/vielbein which identifies ∇ with the corresponding Levi-Civita connection.

This example is the first in a whole tower of *higher Spin structure* fields [74–76], each of which is directly related to a corresponding higher Chern–Simons theory. The next higher example in this tower is the following.

Example: Heterotic Fields.

For $n \geq 3$, let **Heterotic** be the map

$$\mathbf{Heterotic}: \ \mathbf{B}\mathrm{Spin}(n) \xrightarrow{(p,\frac{1}{2}\mathbf{p}_1)} \mathbf{B}\mathrm{GL}(n;\mathbb{R}) \times \mathbf{B}^3 U(1)$$

regarded as an object in the slice $\mathbf{H}_{/\mathbf{B}\mathrm{GL}(n;\mathbb{R})\times\mathbf{B}^3U(1)}$. Here p is the morphism induced by

$$\mathrm{Spin}(n) \to O(n) \hookrightarrow GL(n;\mathbb{R})$$

while $\frac{1}{2}\mathbf{p}_1 : \mathbf{B}\mathrm{Spin}(n) \to \mathbf{B}^3U(1)$ is the morphism of stacks underlying the first fractional Pontrjagin class which we met in Sect. 4.1. To regard a smooth manifold X as an object in the slice $\mathbf{H}_{/\mathbf{B}\mathrm{GL}(n;\mathbb{R})\times\mathbf{B}^3U(1)}$ means to equip it with a $U(1)$-3-bundle $\mathbf{a}_X : X \to \mathbf{B}^3U(1)$ in addition to the tangent bundle $\tau_X : X \to \mathbf{B}GL(n;\mathbb{R})$. A Green–Schwarz anomaly-free background field configuration in heterotic string theory is (the differential refinement of) a map $(s_X, \phi) : (\tau_X, \mathbf{a}_X) \to$ **Heterotic**, i.e., a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{s_X} & \mathbf{B}\mathrm{Spin} \\ & {\scriptstyle (\tau_X,\mathbf{a}_X)}\searrow \ \Swarrow_{\phi} \ \swarrow{\scriptstyle \mathbf{Heterotic}} & \\ & \mathbf{B}\mathrm{GL}(n)\times\mathbf{B}^3U(1)\,. & \end{array}$$

The 3-bundle $\mathbf{a}_X$ serves as a twist: when $\mathbf{a}_X$ is trivial then we are in presence of a String structure on X; so it is customary to refer to (s_X, ϕ) as to an $\mathbf{a}_X$*-twisted String structure* on X, in the sense of [76, 86]. The Green–Schwarz anomaly cancellation condition is then imposed by requiring that $\mathbf{a}_X$ (or rather its differential refinement) factors as

$$X \longrightarrow \mathbf{B}SU \xrightarrow{\mathbf{c}_2} \mathbf{B}^3U(1) \ ,$$

where $\mathbf{c}_2(E)$ is the morphism of stacks underlying the second Chern class. Notice that this says that the extended Lagrangians of Spin- and SU-Chern–Simons theory in 3-dimensions, as discussed above, at the same time serve as the twists that control the higher background gauge field structure in heterotic supergravity backgrounds.

Example: Dual Heterotic Fields.

Similarly, the morphism

$$\mathbf{DualHeterotic}: \ \mathbf{B}\mathrm{String}(n) \xrightarrow{(p,\frac{1}{6}\mathbf{p}_2)} \mathbf{B}\mathrm{GL}(n;\mathbb{R}) \times \mathbf{B}^7 U(1)$$

governs field configurations for the dual heterotic string. These examples, in their differentially refined version, have been discussed in [76]. The last example above is governed by the extended Lagrangian of the 7-dimensional Chern–Simons-type higher gauge field theory of String-2-connections. This has been discussed in [28].

There are many more examples of (quantum) fields modulated by objects in slices of a cohesive higher topos. To close this brief discussion, notice that the twisted String structure example has an evident analog in one lower degree: a central extension of Lie groups $A \to \hat{G} \to G$ induces a long fiber sequence

$$\underline{A} \longrightarrow \underline{\hat{G}} \longrightarrow \underline{G} \longrightarrow \mathbf{B}A \longrightarrow \mathbf{B}\hat{G} \longrightarrow \mathbf{B}G \xrightarrow{\mathbf{c}} \mathbf{B}^2A$$

in $\mathbf{H}$, where $\mathbf{c}$ is the group 2-cocycle that classifies the extension. If we regard this as a coefficient object in the slice $\mathbf{H}_{/\mathbf{B}^2 A}$, then regarding a manifold X in this slice means to equip it with an $(\mathbf{B}A)$-principal 2-bundle (an A-bundle gerbe) modulated by a map $\tau_X^A : X \to \mathbf{B}^2 A$; and a field $(\phi, \eta) : \tau_X^A \to \mathbf{c}$ is equivalently a G-principal bundle $P \to X$ equipped with an equivalence $\eta : \mathbf{c}(E) \simeq \tau_X^A$ with the 2-bundle which obstructs its lift to a $\hat{G}$-principal bundle (the "lifting gerbe"). The differential refinement of this setup similarly yields G-gauge fields equipped with such an equivalence. A concrete example for this is discussed below in Sect. 5.4.

This special case of fields in a slice is called a *twisted (differential)* $\hat{G}$*-structure* in [76] and a *relative field* in [37]. In more generality, the terminology *twisted (differential)* $\mathbf{c}$*-structures* is used in [76] to denote spaces of fields of the form $\mathbf{H}/_{\mathbf{S}}(\sigma_X, \mathbf{c})$ for some slice topos $\mathbf{H}/_{\mathbf{S}}$ and some coefficient object (or "twisting object") $\mathbf{c}$; see also the exposition in [80]. In fact in full generality (quantum) fields in slice toposes are equivalent to cocycles in (generalized and parameterized and possibly non-abelian and differential) *twisted cohomology*. The constructions on which the above discussion is built is given in some generality in [64].

In many examples of twisted (differential) structures/fields in slices the twist is constrained to have a certain factorization. For instance the twist of the (differential) String-structure in a heterotic background is constrained to be the (differential) second Chern-class of a (differential) $E_8 \times E_8$-cocycle, as mentioned above; or for instance the gauging of the 1d Chern–Simons fields on a knot in a 3d Chern–Simons theory bulk is constrained to be the restriction of the bulk gauge field, as discussed in Sect. 3.4.5. Another example is the twist of the Chan-Paton bundles on D-branes, discussed below in Sect. 5.4, which is constrained to be the restriction of the ambient Kalb–Ramond field to the D-brane. In all these cases the fields may be thought of as being maps in the slice topos that arise from maps in the *arrow topos* $\mathbf{H}^{\Delta^1}$. A moduli stack here is a map of moduli stacks

$$\mathbf{Fields}_{\mathrm{bulk+def}} : \mathbf{Fields}_{\mathrm{def}} \longrightarrow \mathbf{Fields}_{\mathrm{bulk}}$$

in $\mathbf{H}$; and a domain on which such fields may be defined is an object $\Sigma_{\mathrm{bulk}} \in \mathbf{H}$ equipped with a map (often, but not necessarily, an inclusion) $\Sigma_{\mathrm{def}} \to \Sigma_{\mathrm{bulk}}$, and a field configuration is a square of the form

$$\begin{array}{ccc} \Sigma_{\mathrm{def}} & \xrightarrow{\phi_{\mathrm{def}}} & \mathbf{Fields}_{\mathrm{def}} \\ \downarrow & \swArrow_{\simeq} & \downarrow^{\mathbf{Fields}} \\ \Sigma_{\mathrm{bulk}} & \xrightarrow[\phi_{\mathrm{bulk}}]{} & \mathbf{Fields}_{\mathrm{bulk}} \end{array}$$

in $\mathbf{H}$. If we now fix ϕ_{bulk} then $(\phi_{\mathrm{bulk}})|_{\Sigma_{\mathrm{def}}}$ serves as the twist, in the above sense, for ϕ_{def}. If $\mathbf{Fields}_{\mathrm{def}}$ is trivial (the point/terminal object), then such a field is a cocycle in *relative cohomology*: a cocycle ϕ_{bulk} on Σ_{bulk} equipped with a trivialization $(\phi_{\mathrm{bulk}})|_{\Sigma_{\mathrm{def}}}$ of its restriction to Σ_{def}.

The fields in Chern–Simons theory with Wilson loops displayed in Sect. 3.4.5 clearly constitute an example of this phenomenon. Another example is the field content of type II string theory on a 10-dimensional spacetime X with D-brane $Q \hookrightarrow X$, for which the above diagram reads

$$\begin{array}{ccc} Q & \longrightarrow & \mathbf{B}\mathrm{PU}_{\mathrm{conn}} \\ \big\downarrow & \swarrow\!\!\!\swarrow & \big\downarrow{\scriptstyle \mathbf{dd}_{\mathrm{conn}}} \\ X & \xrightarrow{\;B\;} & \mathbf{B}^2 U(1)_{\mathrm{conn}} \end{array} ,$$

discussed further below in Sect. 5.4. In [29] we discussed how the supergravity C-field over an 11-dimensional Hořava-Witten background with 10-dimensional boundary $X \hookrightarrow Y$ is similarly a relative cocyle, with the coefficients controlled, once more, by the extended Chern–Simons Lagrangian

$$\hat{\mathbf{c}} : \mathbf{B}(E_8 \times E_8)_{\mathrm{conn}} \longrightarrow \mathbf{B}^3 U(1)_{\mathrm{conn}} ,$$

now regarded in $\mathbf{H}^{(\Delta^1)}$.

5.3 *Differential Moduli Stacks*

In the exposition in Sects. 2 and 3 above we referred, for ease of discussion, to the mapping stacks of the form $\mathbf{Maps}(\Sigma_k, \mathbf{B}G_{\mathrm{conn}})$ as moduli stacks of G-gauge fields on Σ_k. From a more refined perspective this is not quite true. While certainly the global points of these mapping stacks are equivalently the G-gauge field configurations on Σ_k, for U a parameter space, the U-parameterized collections in the mapping stack are not quite those of the intended moduli stack: for the former these are gauge fields and gauge transformations on $U \times \Sigma_k$, while for the latter these are genuine cohesively U-parameterized collections of gauge fields on Σ_k.

In the exposition above we saw this difference briefly in Sect. 3.4.3, where we constrained a 1-form $A \in \Omega^1(U \times \Sigma, \mathfrak{g})$ (a U-plot of the mapping stack) to vanish on vector fields tangent to U; this makes it a smooth function on U with values in connections on Σ. More precisely, for G a Lie group and Σ a smooth manifold, let

$$G\mathbf{Conn}(\Sigma) \in \mathbf{H}$$

be the stack which assigns to any $U \in \mathrm{CartSp}$ the groupoid of smoothly U-parameterized collections of smooth G-principal connections on Σ, and of smoothly U-parameterized collections of smooth gauge transformations between these connections. This is the actual moduli stack of G-connections. In this form, but over a different site of definition, it appears for instance in geometric Langlands duality. In physics this stack is best known in the guise of its infinitesimal approximation: the corresponding Lie algebroid is dually the (off-shell) *BRST-complex* of the gauge theory, and the BRST ghosts are the cotangents to the morphisms in $G\mathbf{Conn}(\Sigma)$ at the identity.

Notice that while the mapping stack is itself not quite the right answer, there is a canonical map that comes to the rescue

$$\mathbf{Maps}(\Sigma, \mathbf{B}G_{\mathrm{conn}}) \longrightarrow G\mathbf{Conn}(\Sigma) .$$

We call this the *concretification* map. We secretly already saw an example of this in Sect. 3.4.2, where this was the map $\mathbf{Maps}(S^1, \mathbf{B}G_{\mathrm{conn}}) \longrightarrow \underline{G}//_{\mathrm{Ad}}\underline{G}$.

In more complicated examples, such as for higher groups G and base spaces Σ which are not plain manifolds, it is in general less evident what $G\mathbf{Conn}(\Sigma)$ should be. But if the ambient higher topos is cohesive, then there is a general abstract procedure that produces the differential moduli stack. This is discussed in Sects. 3.9.6.4 and 4.4.15.3 of [79] and in [65].

5.4 Prequantum Geometry in Higher Codimension

We had indicated in Sect. 3.4 how a single extended Lagrangian, given by a map of universal higher moduli stacks $\mathbf{L} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$, induces, by transgression, circle $(n-k)$-bundles with connection

$$\mathrm{hol}_{\Sigma_k}\mathbf{Maps}(\Sigma_k, \mathbf{L}) \;:\; \mathbf{Maps}(\Sigma_k, \mathbf{B}G_{\mathrm{conn}}) \longrightarrow \mathbf{B}^{n-k}U(1)_{\mathrm{conn}}$$

on moduli stacks of field configurations over each closed k-manifold Σ_k. In codimension 1, hence for $k = n-1$, this reproduces the ordinary *prequantum circle bundle* of the n-dimensional Chern–Simons type theory, as discussed in Sect. 3.4.3. The space of sections of the associated line bundle is the space of *prequantum states* of the theory. This becomes the space of genuine quantum states after choosing a *polarization* (i.e., a decomposition of the moduli space of fields into *canonical coordinates* and *canonical momenta*) and restricting to polarized sections (i.e., those depending only on the canonical coordinates). But moreover, for each Σ_k we may regard $\mathrm{hol}_{\Sigma_k}\mathbf{Maps}(\Sigma_k, \mathbf{L})$ as a *higher prequantum bundle* of the theory in higher codimension hence consider its prequantum geometry in higher codimension.

We discuss now some generalities of such a higher geometric prequantum theory and then show how this perspective sheds a useful light on the gauge coupling of the open string, as part of the transgression of prequantum 2-states of Chern–Simons theory in codimension 2 to prequantum states in codimension 1.

5.4.1 Higher Prequantum States and Prequantum Operators

We indicate here the basic concepts of higher extended prequantum theory and how they reproduce traditional prequantum theory.[13]

Consider a (pre)-n-plectic form, given by a map

$$\omega : X \longrightarrow \Omega^{n+1}(-;\mathbb{R})_{\mathrm{cl}}$$

[13] A discussion of this and the following can be found in Sects. 3.9.13 and 4.4.19 of [79]; see also [27].

in **H**. A *n-plectomorphism* of (X, ω) is an auto-equivalence of ω regarded as an object in the slice $\mathbf{H}_{/\Omega^{n+1}_{\mathrm{cl}}}$, hence a diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & X \\ & {\scriptstyle\omega}\searrow \quad \swarrow{\scriptstyle\omega} & \\ & \Omega^{n+1}(-;\mathbb{R})_{\mathrm{cl}} & \end{array} .$$

A *prequantization* of (X, ω) is a choice of prequantum line bundle, hence a choice of lift ∇ in

$$\begin{array}{ccc} & & \mathbf{B}^n U(1)_{\mathrm{conn}} \\ & \nearrow^{\nabla} & \downarrow^{F_{(-)}} \\ X & \xrightarrow{\omega} & \Omega^{n+1}_{\mathrm{cl}} \end{array} ,$$

modulating a circle n-bundle with connection on X. We write $\mathbf{c}(\nabla) : X \xrightarrow{\nabla} \mathbf{B}^n U(1)_{\mathrm{conn}} \to \mathbf{B}^n U(1)$ for the underlying $(\mathbf{B}^{n-1}U(1))$-principal n-bundle. An autoequivalence

$$\hat{O} : \nabla \xrightarrow{\simeq} \nabla$$

of the prequantum n-bundle regarded as an object in the slice $\mathbf{H}_{/\mathbf{B}^n U(1)_{\mathrm{conn}}}$, hence a diagram in **H** of the form

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & X \\ & {\scriptstyle\nabla}\searrow \;\Swarrow_{O}\; \swarrow{\scriptstyle\nabla} & \\ & \mathbf{B}^n U(1)_{\mathrm{conn}} & \end{array}$$

is an (exponentiated) *prequantum operator* or *quantomorphism* or *regular contact transformation* of the prequantum geometry (X, ∇). These form an ∞-group in **H**. The L_∞-algebra of this *quantomorphism ∞-group* is the higher *Poisson bracket* Lie algebra of the system. If X is equipped with group structure then the quantomorphisms covering the action of X on itself form the *Heisenberg ∞-group*. The homotopy labeled O in the above diagram is the *Hamiltonian* of the prequantum operator. The image of the quantomorphisms in the symplectomorphisms (given by composition the above diagram with the curvature morphism $F_{(-)} : \mathbf{B}^n U(1)_{\mathrm{conn}} \to \Omega^{n+1}_{\mathrm{cl}}$) is the group of *Hamiltonian n-plectomorphisms*. A lift of an ∞-group action $G \to \mathbf{Aut}(X)$ on X from automorphisms of X (diffeomorphism) to quantomorphisms is a *Hamiltonian action*, infinitesimally (and dually) a *momentum map*.

To define higher prequantum states we fix a representation (V, ρ) of the circle n-group $\mathbf{B}^{n-1}U(1)$. By the general results in [64] this is equivalent to fixing a homotopy fiber sequence of the form

$$\begin{array}{ccc} \underline{V} & \longrightarrow & \underline{V}/\!/\mathbf{B}^{n-1}U(1) \\ & & \downarrow^{\rho} \\ & & \mathbf{B}^n U(1) \end{array}$$

in **H**. The vertical morphism here is the *universal ρ-associated V-fiber ∞-bundle* and characterizes ρ itself. Given such, a section of the V-fiber bundle which is ρ-associated to $\mathbf{c}(\nabla)$ is equivalently a map

$$\Psi : \mathbf{c}(\nabla) \longrightarrow \rho$$

in the slice $\mathbf{H}_{/\mathbf{B}^n U(1)}$. This is a higher *prequantum state* of the prequantum geometry (X, ∇). Since every prequantum operator $\hat{O}$ as above in particular is an auto-equivalence of the underlying prequantum bundle $\hat{O} : \mathbf{c}(\nabla) \xrightarrow{\simeq} \mathbf{c}(\nabla)$ it canonically acts on prequantum states given by maps as above simply by precomposition

$$\Psi \mapsto \hat{O} \circ \Psi.$$

Notice also that from the perspective of Sect. 5.2 all this has an equivalent interpretation in terms of twisted cohomology: a preqantum state is a cocycle in twisted V-cohomology, with the twist being the prequantum bundle. And a prequantum operator/quantomorphism is equivalently a twist automorphism (or "generalized local diffeomorphism").

For instance if $n = 1$ then ω is an ordinary (pre)symplectic form and ∇ is the connection on a circle bundle. In this case the above notions of prequantum operators, quantomorphism group, Heisenberg group and Poisson bracket Lie algebra reproduce exactly all the traditional notions if X is a smooth manifold, and generalize them to the case that X is for instance an orbifold or even itself a higher moduli stack, as we have seen. The canonical representation of the circle group $U(1)$ on the complex numbers yields a homotopy fiber sequence

$$\begin{array}{ccc} \underline{\mathbb{C}} & \longrightarrow & \underline{\mathbb{C}}/\!/\underline{U}(1) \\ & & \downarrow{\scriptstyle \rho} \\ & & \mathbf{B}U(1) \end{array} ,$$

where $\underline{\mathbb{C}}/\!/\underline{U}(1)$ is the stack corresponding to the ordinary action groupoid of the action of $U(1)$ on $\mathbb{C}$, and where the vertical map is the canonical functor forgetting the data of the local $\mathbb{C}$-valued functions. This is the *universal complex line bundle* associated to the universal $U(1)$-principal bundle. One readily checks that a prequantum state $\Psi : \mathbf{c}(\nabla) \to \rho$, hence a diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{\quad \sigma \quad} & \underline{\mathbb{C}}/\!/\underline{U}(1) \\ & {\scriptstyle \mathbf{c}(\nabla)} \searrow \quad \swarrow {\scriptstyle \rho} & \\ & \mathbf{B}U(1) & \end{array}$$

in **H** is indeed equivalently a section of the complex line bundle canonically associated to $\mathbf{c}(\nabla)$ and that under this equivalence the pasting composite

$$\begin{array}{ccccc} X & \xrightarrow{\simeq} & X & \longrightarrow & \underline{\mathbb{C}}/\!/\underline{U}(1) \\ & {\scriptstyle \mathbf{c}(\nabla)} \searrow \;\; \swarrow\!\!\swarrow {\scriptstyle O} & \downarrow {\scriptstyle \mathbf{c}(\nabla)} & \swarrow {\scriptstyle \rho} & \\ & & \mathbf{B}U(1) & & \end{array}$$

is the result of the traditional formula for the action of the prequantum operator $\hat{O}$ on Ψ.

Instead of forgetting the connection on the prequantum bundle in the above composite, one can equivalently equip the prequantum state with a differential refinement, namely with its *covariant derivative* and then exhibit the prequantum operator action directly. Explicitly, let $\mathbb{C}//U(1)_{\mathrm{conn}}$ denote the quotient stack $(\underline{\mathbb{C}} \times \Omega^1(-, \mathbb{R}))//\underline{U}(1)$, with $U(1)$ acting diagonally. This sits in a homotopy fiber sequence

$$\begin{array}{ccc} \underline{\mathbb{C}} & \longrightarrow & \underline{\mathbb{C}}//\underline{U}(1)_{\mathrm{conn}} \\ & & \downarrow{\scriptstyle \rho_{\mathrm{conn}}} \\ & & \mathbf{B}U(1)_{\mathrm{conn}} \end{array}$$

which may be thought of as the differential refinement of the above fiber sequence $\underline{\mathbb{C}} \to \underline{\mathbb{C}}//\underline{U}(1) \to \mathbf{B}U(1)$. (Compare this to Sect. 3.4.5, where we had similarly seen the differential refinement of the fiber sequence $\underline{G}/\underline{T}_\lambda \to \mathbf{B}T_\lambda \to \mathbf{B}G$, which analogously characterizes the canonical action of G on the coset space G/T_λ.) Prequantum states are now equivalently maps

$$\widehat{\boldsymbol{\Psi}} : \nabla \longrightarrow \rho_{\mathrm{conn}}$$

in $\mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}}$. This formulation realizes a section of an associated line bundle equivalently as a connection on what is sometimes called a groupoid bundle. As such, $\widehat{\boldsymbol{\Psi}}$ has not just a 2-form curvature (which is that of the prequantum bundle) but also a 1-form curvature: this is the covariant derivative $\nabla\sigma$ of the section.

Such a relation between sections of higher associated bundles and higher covariant derivatives holds more generally. In the next degree for $n = 2$ one finds that the quantomorphism 2-group is the Lie 2-group which integrates the *Poisson bracket Lie 2-algebra* of the underlying 2-plectic geometry as introduced in [67]. In the next section we look at an example for $n = 2$ in more detail and show how it interplays with the above example under transgression.

The above higher prequantum theory becomes a genuine quantum theory after a suitable higher analog of a choice of *polarization*. In particular, for $\mathbf{L} : X \to \mathbf{B}^n U(1)_{\mathrm{conn}}$ an extended Lagrangian of an n-dimensional quantum field theory as discussed in all our examples here, and for Σ_k any closed manifold, the polarized prequantum states of the transgressed prequantum bundle $\mathrm{hol}_{\Sigma_k}\mathbf{Maps}(\Sigma_k, \mathbf{L})$ should form the $(n-k)$-vector spaces of higher quantum states in codimension k. These states would be assigned to Σ_k by the *extended quantum field theory*, in the sense of [60], obtained from the extended Lagrangian $\mathbf{L}$ by extended geometric quantization. There is an equivalent reformulation of this last step for $n = 1$ given simply by the push-forward of the prequantum line bundle in K-theory (see Sect. 6.8 of [43]) and so one would expect that accordingly the last step of higher geometric quantization involves similarly a push-forward of the associated V-fiber ∞-bundles above in some higher generalized cohomology theory. But this remains to be investigated.

5.4.2 Example: The Anomaly-Free Gauge Coupling of the Open String

As an example of these general phenomena, we close by briefly indicating how the higher prequantum states of 3d Chern–Simons theory in codimension 2 reproduce the *twisted Chan-Paton gauge bundles* of open string backgrounds, and how their transgression to codimension 1 reproduces the cancellation of the Freed-Witten-Kapustin anomaly of the open string.

By the above, the Wess-Zumino-Witten gerbe $\mathbf{wzw} : G \to \mathbf{B}^2 U(1)_{\mathrm{conn}}$ as discussed in Sect. 3.4.2 may be regarded as the *prequantum 2-bundle* of Chern–Simons theory in codimension 2 over the circle. Equivalently, if we consider the WZW σ-model for the string on G and take the limiting TQFT case obtained by sending the kinetic term to 0 while keeping only the gauge coupling term in the action, then it is the extended Lagrangian of the string σ-model: its transgression to the mapping space out of a *closed* worldvolume Σ_2 of the string is the topological piece of the exponentiated WZW σ-model action. For Σ_2 with boundary the situation is more interesting, and this we discuss now.

The *Heisenberg 2-group* of the prequantum geometry $(G, \mathbf{wzw})$ is[14] the *String 2-group* (see the appendix of [28] for a review), the smooth 2-group $\mathrm{String}(G)$ which is, up to equivalence, the loop space object of the homotopy fiber of the smooth universal class $\mathbf{c}$

$$\mathbf{B}\mathrm{String}(G) \longrightarrow \mathbf{B}G \xrightarrow{\mathbf{c}} \mathbf{B}^3 U(1).$$

The canonical representation of the 2-group $BU(1)$ is on the complex K-theory spectrum, whose smooth (stacky) refinement is given by $\mathbf{B}U := \varinjlim_n \mathbf{B}U(n)$ in $\mathbf{H}$ (see Sect. 5.4.3 of [79] for more details). On any component for fixed n the action of the smooth 2-group $\mathbf{B}U(1)$ is exhibited by the long homotopy fiber sequence

$$U(1) \longrightarrow U(n) \to \mathrm{PU}(n) \longrightarrow \mathbf{B}U(1) \longrightarrow \mathbf{B}U(n) \longrightarrow \mathbf{B}\mathrm{PU}(n) \xrightarrow{\mathbf{dd}_n} \mathbf{B}^2 U(1)$$

in $\mathbf{H}$, in that $\mathbf{dd}_n$ is the universal $(\mathbf{B}U(n))$-fiber 2-bundle which is associated by this action to the universal $(\mathbf{B}U(1))$-2-bundle.[15] Using the general higher representation theory in $\mathbf{H}$ as developed in [64], a local section of the $(\mathbf{B}U(n))$-fiber prequantum 2-bundle which is $\mathbf{dd}_n$-associated to the prequantum 2-bundle $\mathbf{wzw}$, hence a local prequantum 2-state, is, equivalently, a map

$$\Psi : \mathbf{wzw}|_Q \longrightarrow \mathbf{dd}_n$$

in the slice $\mathbf{H}_{/\mathbf{B}^2 U(1)}$, where $\iota_Q : Q \hookrightarrow G$ is some subspace. Equivalently (compare with the general discussion in Sect. 5.2), this is a map

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[14] This follows for instance as the Lie integration of the result in [5] that the Heisenberg Lie 2-algebra here is the $\mathfrak{string}(\mathfrak{g})$ Lie 2-algebra; see also [27].

[15] The notion of $(\mathbf{B}U(n))$-fiber 2-bundle is equivalently that of nonabelian $U(n)$-*gerbes* in the original sense of Giraud, see [64]. Notice that for $n = 1$ this is more general than then notion of $U(1)$-bundle gerbe: a G-gerbe has structure 2-group $\mathbf{Aut}(\mathbf{B}G)$, but a $U(1)$-bundle gerbe has structure 2-group only in the left inclusion of the fiber sequence $\mathbf{B}U(1) \hookrightarrow \mathbf{Aut}(\mathbf{B}U(1)) \to \mathbb{Z}_2$.

$$(\Psi, \mathbf{wzw}) : \iota_Q \longrightarrow \mathbf{dd}_n$$

in $\mathbf{H}^{(\Delta^1)}$, hence a diagram in $\mathbf{H}$ of the form

$$\begin{array}{ccc} Q & \xrightarrow{\Psi} & \mathbf{B}\mathrm{PU}(n) \\ {\scriptstyle \iota_Q}\downarrow & \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle \mathbf{dd}_n} \\ G & \xrightarrow[\mathbf{wzw}]{} & \mathbf{B}^2 U(1)\,. \end{array}$$

One finds (Sect. 5.4.3 of [79]) that this equivalently modulates a unitary bundle on Q which is *twisted* by the restriction of **wzw** to Q as in twisted K-theory (such a twisted bundle is also called a *gerbe module* if **wzw** is thought of in terms of bundle gerbes [7]). So

$$\mathbf{dd}_n \in \mathbf{H}_{/\mathbf{B}^2 U(1)}$$

is the moduli stack for twisted rank-n unitary bundles. As with the other moduli stacks before, one finds a differential refinement of this moduli stack, which we write

$$(\mathbf{dd}_n)_{\mathrm{conn}} : (\mathbf{B}U(n)//\mathbf{B}U(1))_{\mathrm{conn}} \to \mathbf{B}^2 U(1)_{\mathrm{conn}},$$

and which modulates twisted unitary bundles with twisted connections (bundle gerbe modules with connection). Hence a differentially refined state is a map $\widehat{\Psi} : \mathbf{wzw}|_Q \to (\mathbf{dd}_n)_{\mathrm{conn}}$ in $\mathbf{H}_{/\mathbf{B}^2 U(1)_{\mathrm{conn}}}$; and this is precisely a twisted gauge field on a D-brane Q on which open strings in G may end. Hence these are the *prequantum 2-states* of Chern–Simons theory in codimension 2. Precursors of this perspective of Chan-Paton bundles over D-branes as extended prequantum 2-states can be found in [68, 77].

Notice that by the above discussion, together the discussion in Sect. 5.2, an equivalence

$$\hat{O} : \mathbf{wzw} \xrightarrow{\simeq} \mathbf{wzw}$$

in $\mathbf{H}_{/\mathbf{B}^2 U(1)_{\mathrm{conn}}}$ has two different, but equivalent, important interpretations:

1. it is an element of the *quantomorphism 2-group* (i.e. the possibly non-linear generalization of the Heisenberg 2-group) of 2-prequantum operators;
2. it is a twist automorphism analogous to the generalized diffeomorphisms for the fields in gravity.

Moreover, such a transformation is locally a structure well familiar from the literature on D-branes: it is locally (on some cover) given by a transformation of the B-field of the form $B \mapsto B + d_{\mathrm{dR}} a$ for a local 1-form a (this is the *Hamiltonian 1-form* in the interpretation of this transformation in higher prequantum geometry) and its prequantum operator action on prequantum 2-states, hence on Chan-Paton gauge fields

$$\widehat{\mathbf{\Psi}} : \mathbf{wzw} \longrightarrow (\mathbf{dd}_n)$$

(by precomposition) is given by shifting the connection on a twisted Chan-Paton bundle (locally) by this local 1-form a. This local gauge transformation data

$$B \mapsto B + da, \quad A \mapsto A + a,$$

is familiar from string theory and D-brane gauge theory (see e.g. [66]). The 2-prequantum operator action $\Psi \mapsto \hat{O}\Psi$ which we see here is the fully globalized refinement of this transformation.

Surface Transport and the Twisted Bundle Part of Freed-Witten-Kapustin Anomalies.

The map $\widehat{\Psi} : (\iota_Q, \mathbf{wzw}) \to (\mathbf{dd}_n)_{\mathrm{conn}}$ above is the gauge-coupling part of the extended Lagrangian of the *open* string on G in the presence of a D-brane $Q \hookrightarrow G$. We indicate what this means and how it works. Note that for all of the following the target space G and background gauge field $\mathbf{wzw}$ could be replaced by any target space with any circle 2-bundle with connection on it.

The object ι_Q in $\mathbf{H}^{(\Delta^1)}$ is the target space for the open string. The worldvolume of that string is a smooth compact manifold Σ with boundary inclusion $\iota_{\partial\Sigma} : \partial\Sigma \to \Sigma$, also regarded as an object in $\mathbf{H}^{(\Delta^1)}$. A field configuration of the string σ-model is then a map

$$\phi : \iota_\Sigma \to \iota_Q$$

in $\mathbf{H}^{(\Delta^1)}$, hence a diagram

$$\begin{array}{ccc} \partial\Sigma & \longrightarrow & Q \\ {\scriptstyle \iota_{\partial\Sigma}} \big\downarrow & & \big\downarrow {\scriptstyle \iota_Q} \\ \Sigma & \xrightarrow{\ \phi\ } & G \end{array}$$

in $\mathbf{H}$, hence a smooth function $\phi : \Sigma \to G$ subject to the constraint that the boundary of Σ lands on the D-brane Q. Postcomposition with the background gauge field $\widehat{\Psi}$ yields the diagram

$$\begin{array}{ccccc} \partial\Sigma & \longrightarrow & Q & \xrightarrow{\ \widehat{\Psi}\ } & (\mathbf{B}U(n)/\!/U(1))_{\mathrm{conn}} \\ {\scriptstyle \iota_{\partial\Sigma}} \big\downarrow & & \big\downarrow {\scriptstyle \iota_Q} & & \\ \Sigma & \xrightarrow{\ \phi\ } & G & \xrightarrow[\mathbf{wzw}]{} & \mathbf{B}^2U(1)_{\mathrm{conn}} \end{array} .$$

Comparison with the situation of Chern–Simons theory with Wilson lines in Sect. 3.4.5 shows that the total action functional for the open string should be the product of the fiber integration of the top composite morphism with that of the bottom composite morphisms. Hence that functional is the product of the surface parallel transport of the $\mathbf{wzw}$ B-field over Σ with the line holonomy of the twisted Chan-Paton bundle over $\partial\Sigma$.

This is indeed again true, but for more subtle reasons this time, since the fiber integrations here are *twisted*. For the surface parallel transport we mentioned this already at the end of Sect. 5.1: since Σ has a boundary, parallel transport over Σ

does not yield a function on the mapping space out of Σ, but rather a section of the line bundle on the mapping space out of $\partial\Sigma$, pulled back to this larger mapping space.

Furthermore, the connection on a twisted unitary bundle does not quite have a well-defined traced holonomy in $\mathbb{C}$, but rather a well defined traced holonomy up to a coherent twist. More precisely, the transgression of the WZW 2-connection to maps out of the circle as in Sect. 3.4 fits into a diagram of moduli stacks in $\mathbf{H}$ of the form

$$\begin{array}{ccc}
\mathbf{Maps}(S^1, (\mathbf{B}U(n)/\!/\mathbf{B}U(1))_{\mathrm{conn}}) & \xrightarrow{\mathrm{tr\,hol}_{S^1}} & \underline{\mathbb{C}}/\!/\underline{U}(1)_{\mathrm{conn}} \\
\Big\downarrow{\scriptstyle \mathbf{Maps}(S^1,(\mathbf{dd}_n)_{\mathrm{conn}})} & & \Big\downarrow \\
\mathbf{Maps}(S^1, \mathbf{B}^2U(1)_{\mathrm{conn}}) & \xrightarrow{\mathrm{hol}_{S^1}} & \mathbf{B}U(1)_{\mathrm{conn}}\,.
\end{array}$$

This is a transgression-compatibility of the form that we have already seen in Sect. 3.4.2.

In summary, we obtain the transgression of the extended Lagrangian of the open string in the background of B-field and Chan-Paton bundles as the following pasting diagram of moduli stacks in $\mathbf{H}$ (all squares are filled with homotopy 2-cells, which are notationally suppressed for readability)

$$\begin{array}{ccccc}
\mathbf{Fields}_{\mathrm{OpenString}}(\iota_{\partial\Sigma}) & \longrightarrow & \mathbf{Maps}(\Sigma, G) & \xrightarrow{\exp(2\pi i \int_\Sigma [\Sigma, \mathbf{wzw}])} & \underline{\mathbb{C}}/\!/\underline{U}(1)_{\mathrm{conn}} \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \mathbf{Maps}(\iota_{\partial\Sigma}, G)} & & \Big\downarrow \\
\mathbf{Maps}(S^1, Q) & \xrightarrow{\mathbf{Maps}(S^1, \iota_Q)} & \mathbf{Maps}(S^1, G) & & \\
\Big\downarrow{\scriptstyle \mathbf{Maps}(S^1, \widehat{\boldsymbol{\Psi}})} & & \searrow{\scriptstyle \mathbf{Maps}(S^1, \mathbf{wzw})} & & \\
\mathbf{Maps}(S^1, (\mathbf{B}U(n)/\!/\mathbf{B}U(1))_{\mathrm{conn}}) & \xrightarrow{\mathbf{Maps}(S^1,(\mathbf{dd}_n)_{\mathrm{conn}})} & & \mathbf{Maps}(S^1, \mathbf{B}^2U(1)_{\mathrm{conn}}) & \\
\Big\downarrow{\scriptstyle \mathrm{tr\,hol}_{S^1}} & & & \searrow{\scriptstyle \mathrm{hol}_{S^1}} & \\
\underline{\mathbb{C}}/\!/\underline{U}(1)_{\mathrm{conn}} & \longrightarrow & & & \mathbf{B}U(1)_{\mathrm{conn}}
\end{array}$$

Here

- the top left square is the homotopy pullback square that computes the mapping stack $\mathbf{Maps}(\iota_{\partial\Sigma}, \iota_Q)$ in $\mathbf{H}^{(\Delta^1)}$, which here is simply the smooth space of string configurations $\Sigma \to G$ which are such that the string boundary lands on the D-brane Q;
- the top right square is the twisted fiber integration of the **wzw** background 2-bundle with connection: this exhibits the parallel transport of the 2-form connection over the worldvolume Σ with boundary S^1 as a section of the pullback of the transgression line bundle on loop space to the space of maps out of Σ;
- the bottom square is the above compatibility between the twisted traced holonomy of twisted unitary bundles and the transgression of their twisting 2-bundles.

The total diagram obtained this way exhibits a difference between two section of a single complex line bundle on $\mathbf{Fields}_{\mathrm{OpenString}}(\iota_{\partial\Sigma})$ (at least one of them non-vanishing), hence a map

$$\exp\left(2\pi i \int_{\Sigma} [\Sigma, \mathbf{wzw}]\right) \cdot \mathrm{tr}\,\mathrm{hol}_{S^1}([S^1, \widehat{\Psi}]) : \mathbf{Fields}_{\mathrm{OpenString}}(\iota_{\partial\Sigma}) \longrightarrow \underline{\mathbb{C}}.$$

This is the well-defined action functional of the open string with endpoints on the D-brane $Q \hookrightarrow G$, charged under the background **wzw** B-field and under the twisted Chan-Paton gauge bundle $\widehat{\Psi}$.

Unwinding the definitions, one finds that this phenomenon is precisely the twisted-bundle-part, due to Kapustin [52], of the Freed-Witten anomaly cancellation for open strings on D-branes, hence is the Freed-Witten-Kapustin anomaly cancellation mechanism either for the open bosonic string or else for the open type II superstring on Spin^c-branes. Notice how in the traditional discussion the existence of twisted bundles on the D-brane is identified just as *some* construction that happens to cancel the B-field anomaly. Here, in the perspective of extended quantization, we see that this choice follows uniquely from the general theory of extended prequantization, once we recognize that $\mathbf{dd}_n$ above is (the universal associated 2-bundle induced by) the canonical representation of the circle 2-group $\mathbf{B}U(1)$, just as in one codimension up $\mathbb{C}$ is the canonical representation of the circle 1-group $U(1)$.

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Factorization Homology in 3-Dimensional Topology

Nikita Markarian and Hiro Lee Tanaka

Abstract This Chapter consists of two contributions about the relevance of *factorization homology* (a.k.a. *manifoldic homology* or *topological chiral homology*) in three dimensional topology: 1. Manifoldic Homology and Chern-Simons Formalism, by Nikita Markarian; 2. Factorization Homology and Links Invariants, by Hiro Lee Tanaka.

1 Manifoldic Homology and Chern–Simons Formalism (by Nikita Markarian)

Abstract The aim of this note is to define for any e_n-algebra A and a compact parallelizable n-manifold M without boundary a morphism from the homology of homotopy Lie algebra $A[n-1]$ to the topological chiral homology of M with coefficients in A. This map plays a crucial role in the perturbative Chern-Simons theory.

1.1 Introduction

Manifoldic homology (we suggest this term instead of "topological chiral homology with constant coefficients" from [13]) is a far-reaching generalization of Hochschild homology. In the theory of Hochschild and cyclic homology the additive Dennis

N. Markarian (✉)
Department of Mathematics, National Research University Higher School of Economics, 20 Myasnitskaya Str., 101000 Moscow, Russia
e-mail: nikita.markarian@gmail.com

H.L. Tanaka
Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138, USA
e-mail: hirolee@math.harvard.edu

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trace map (e.g. [12, 8.4.16]) plays an important role. Let A be an associative algebra. Denote by $L(A)$ the underlying Lie algebra of A. Then the additive Dennis trace gives a map from $H_*(L(A))$ to $HH_*(A)$. The aim of the present note is to generalize this morphism to any e_n-algebra A.

Let e_n be the operad of rational chains of the operad of little discs and A be an algebra over it. The complex $A[n-1]$ is equipped with homotopy Lie algebra structure, denote it by $L(A)$. Fix a compact oriented n-manifold M without boundary. For simplicity we restrict ourselves to parallelizable manifolds, this restriction may be removed by introducing framed little discs as in [17]. Denote by $HM_*(A)$ the manifoldic homology of A on M introduced in Definition 2, and by $H_*(L(A))$ the Lie algebra homology. In the central Proposition 3 we give a morphism $H_*(L(A)) \to HM_*(A)$ explicitly in terms of the Fulton–MacPherson operad.

For $n = 1$ and $M = S^1$, that is for homotopy associative algebras and Hochschild homology, the above morphism may be presented as the composition of natural morphisms $H_*(L(A)) \to HH_*(U(L(A)) \to HH_*(A)$, where $U(-)$ is the universal enveloping algebra. For $n > 1$ the analogous statement holds, with the universal enveloping algebra replaced by the universal enveloping e_n-algebra. The definition of the latter notion naturally appears in the context of Koszul duality for e_n-algebras, which is still under construction, see nevertheless e.g. [6] and references therein. We need even more, than Koszul duality. The description of application of our construction to manifold invariants requires the Koszul duality for e_n-algebras with curvature. These subjects are briefly discussed in the last section.

Our main construction is exemplary and may be generalized in many ways. For example, one may take some modules over A and put them into some points of M. Then one get a map from homology of $L(A)$ with coefficients in an appropriate module to the manifoldic homology with coefficients in these modules. In particular, if one take copies of A itself as modules, then manifoldic homology with coefficient in them equals to the usual manifoldic homology; thus one get a map from homology of $L(A)$ with coefficients in the tensor product of adjoint modules to $HM_*(A)$. One needs this generalization to build a working theory of invariants of 3-manifolds, I hope to treat this subject elsewhere.

Remark 1 The present note is partially initiated by the work of K. Costello and O. Gwilliam on factorization algebras in perturbative quantum field theory [4], although it is hard to point at exact relations.

1.2 Trees and L_∞

1.2.1 Trees

A tree is an oriented connected graph with three type of vertices: *root* has one incoming edge and no outgoing ones, *leaves* have one outgoing edge and no incoming ones and *internal vertexes* have one outgoing edge and more than one incoming ones.

Edges incident to leaves will be called *inputs*, the edge incident to the root will be called the *output* and all other edges will be called *internal edges*. The degenerate tree has one edge and no internal vertexes. Denote by $T_k(S)$ the set of non-degenerate trees with k internal edges and leaves labeled by a set S.

For two trees $t_1 \in T_{k_1}(S_1)$ and $t_2 \in T_{k_2}(S_2)$ and an element $s \in S_1$ the composition of trees $t_1 \circ_s t_2 \in T_{k_1+k_2+1}$ is obtained by identification of the input of t_1 corresponding to s and the output of t_2. Composition of trees is associative and the degenerate tree is the unit. The set of trees with respect to the composition forms an operad.

Call the tree with only one internal vertex the *star*. Any non-degenerate tree with k internal edges may be uniquely presented as a composition of $k+1$ stars.

The operation of *edge splitting* is the following: take a non-degenerate tree, present it as a composition of stars and replace one star with a tree that is a product of two stars and has the same set of inputs . The operation of an edge splitting depends on a internal vertex and a subset of more than one incoming edges.

1.2.2 L_∞

For a non-degenerate tree t denote by $\mathrm{Det}(t)$ the one-dimensional $\mathbb{Q}$-vector space that is the determinant of the vector space generated by internal edges. For $s > 1$ consider the complex

$$L(s)\colon \bigoplus_{t\in T_0([\mathbf{s}])} \mathrm{Det}(t) \to \bigoplus_{t\in T_1([\mathbf{s}])} \mathrm{Det}(t) \to \bigoplus_{t\in T_2([\mathbf{s}])} \mathrm{Det}(t) \to \cdots, \tag{1}$$

where $[\mathbf{s}]$ is the set of s elements, the cohomological degree of a tree $t \in T_k([\mathbf{s}])$ is $2-s+k$ and the differential is given by all possible splitting of an edge (see e.g. [9]). The composition of trees equips the sequence $L(i) \otimes \mathrm{sgn}$ with the structure of a dg-operad, here sgn is the sign representation of the symmetric group.

This operad is called L_∞ *operad*. Denote by $L_\infty[n]$ the dg-operad given by the complex $L(s)[n(s-1)] \otimes (\mathrm{sgn})^n$ and refer to it as n-shifted L_∞ operad.

1.3 Fulton–MacPherson Operad

1.3.1 Fulton–MacPherson Compactification

The Fulton–MacPherson compactification is introduced in [8, 14], see also [1, 17]. We cite here its properties that are essential for our purposes.

For a finite set S denote by $(\mathbb{R}^n)^S$ the set of ordered S-tuples in $\mathbb{R}^n$. For a finite set S denote by $\Delta_S \colon \mathbb{R}^n \to (\mathbb{R}^n)^S$ the diagonal embedding. We will denote by $[\mathbf{n}]$ the set of n elements.

Let $\mathscr{C}^0_S(\mathbb{R}^n) \subset (\mathbb{R}^n)^S$ be the space of ordered pairwise distinct points in $\mathbb{R}^n$ labeled by S. The *Fulton–MacPherson compactification* $\mathscr{C}_S(\mathbb{R}^n)$ is a manifold with corners with interior $\mathscr{C}^0_S(\mathbb{R}^n)$. The projection $\mathscr{C}_S(\mathbb{R}^n) \xrightarrow{\pi} (\mathbb{R}^n)^S$ is defined, which is an isomorphism on $\mathscr{C}^0_S(\mathbb{R}^n) \subset \mathscr{C}_S(\mathbb{R}^n)$. Moreover there is a sequence of manifolds with corners $F_n(S)$ labeled by finite sets and maps $\phi_{S_1,\ldots,S_k}$ that fit in the diagram

$$\begin{array}{ccc} F_n(S_1)\times\cdots\times F_n(S_k)\times\mathscr{C}_{[\mathbf{k}]}(\mathbb{R}^n) & \xrightarrow{\phi_{S_1,\ldots,S_k}} & \mathscr{C}_{(S_1\cup\cdots\cup S_k)}(\mathbb{R}^n) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ (\mathbb{R}^n)^k & \xrightarrow{\Delta_{S_1}\times\cdots\times\Delta_{S_k}} & (\mathbb{R}^n)^{(S_1\cup\cdots\cup S_k)} \end{array}$$

where the left arrow is the projection to the point on the first factors and π on the last one. Restrictions of $\phi_{S_1,\ldots,S_k}$ to $F_n(S_1)\times\cdots\times F_n(S_k)\times\mathscr{C}^0_{[\mathbf{k}]}(\mathbb{R}^n)$ are isomorphisms onto the image. It follows that $F_n(S) = \pi^{-1}\mathbf{0}$, where $\mathbf{0}\in(\mathbb{R}^n)^S$ is S-tuple sitting at the origin. Being restricted on $F_n(S)\subset\mathscr{C}_S(\mathbb{R}^n)$, maps ϕ equip $F_n(S)$ with an operad structure:

$$\phi_{[\mathbf{s_1}],\ldots,[\mathbf{s_k}]}\colon F_n([\mathbf{s_1}])\times\cdots\times F_n([\mathbf{s_k}])\times F_n([\mathbf{k}])\to F_n([\mathbf{s_1}+\cdots+\mathbf{s_k}]).$$

Manifolds $\mathscr{C}_{[\mathbf{k}]}(\mathbb{R}^n)$ and $F_n([\mathbf{k}])$ are equipped with a k-th symmetric group action consistent with its natural action on $\mathscr{C}^0_{[\mathbf{k}]}(\mathbb{R}^n)$ and all maps are compatible with this action.

Definition 1 [8, 14, 17] The sequence of spaces $F_n([\mathbf{k}])$ with the symmetric group action and composition morphisms as above is called the *Fulton–MacPherson operad.*

1.3.2 Strata, Trees and L_∞

There is a map of sets $F_n(S)\xrightarrow{\mu} T(S)$ that subdivides $F_n(S)$ into smooth strata. This map is totally defined by the following properties. Firstly, μ is consistent with the operad structure in the sense that the preimage of a composition is the composition of preimages. Secondly, the zero codimension stratum corresponding to a star tree is the intersection of $\pi^{-1}\mathbf{0}$ and the stratum of $\mathscr{C}_S(\mathbb{R}^n)$ that is the blow-up of the small diagonal minus pull backs of other diagonals. These latter strata freely generate the Fulton–MacPherson operad as a set.

Denote by $C_*(F_n)$ the dg-operad of rational chains of the Fulton–MacPherson operad. For a tree $t\in T(S)$ let $[\mu^{-1}(t)]\in C_*(F_n(S))$ be the chain presented by its preimage under μ.

Proposition 1 *Map* $[\mu^{-1}(\cdot)]$ *gives a morphism from shifted* L_∞ *operad* $L(s)[s(1-n)]$ *to the* dg*-operad* $C_*(F_n([\mathbf{s}]))$ *of rational chains of the Fulton–MacPherson operad.*

Proof To see that the map commutes with the differential note, that two strata given by μ with dimensions differing by 1 are incident if and only if one of the corresponding trees is obtained from another by edge splitting. In this way we get a basis in the conormal bundle to a stratum labeled by the internal edges. It follows the consistency of the map from the statement with signs.

It follows that there is a morphism of dg-operads

$$L_\infty[1-n] \to C_*(F_n) \tag{2}$$

Let e_n be the dg-operad of rational chains of the operad of little n-discs.

Proposition 2 *Operad $C_*(F_n)$ is weakly homotopy equivalent to e_n.*

Proof See [17, Proposition 3.9].

Thus there is a homotopy morphism of operads $L_\infty[1-n] \to e_n$.

1.4 Manifoldic and Lie Algebra Homology

1.4.1 Manifoldic Homology

Let M be an n-dimensional parallelized compact manifold without boundary. In the same way as for $\mathbb{R}^n$ there is the Fulton-MacPherson compactification $\mathscr{C}_S(M)$ of the space $\mathscr{C}_S^0(M)$ of ordered pairwise distinct points in M labeled by S; inclusion $\mathscr{C}_S^0(M) \hookrightarrow \mathscr{C}_S(M)$ is a homotopy equivalence. There is a projection $\mathscr{C}_S(M) \xrightarrow{\pi} M^S$ and maps $\phi_{S_1,\dots,S_k}$ that fit in the diagram

$$\begin{array}{ccc}
F_n(S_1)\times\cdots\times F_n(S_k)\times\mathscr{C}_{[\mathbf{k}]}(M) & \xrightarrow{\phi_{S_1,\dots,S_k}} & \mathscr{C}_{(S_1\cup\cdots\cup S_k)}(M) \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
M^k & \xrightarrow{\Delta_{S_1}\times\cdots\times\Delta_{S_k}} & M^{(S_1\cup\cdots\cup S_k)}
\end{array}$$

and are isomorphisms on $F_n(S_1)\times\cdots\times F_n(S_k)\times\mathscr{C}^0_{[\mathbf{k}]}(M)$, where $\Delta_S\colon M\to M^S$ are the diagonal maps. It follows that spaces $\mathscr{C}_*(M)$ form a right module over the PROP generated by the Fulton-MacPherson operad

$$P(F_n)(m,l) = \bigcup_{\sum m_i = m} F_n(m_1)\times\cdots\times F_n(m_l).$$

This module as a set is freely generated by $\mathscr{C}_*^0(M)$. The stratification on F_n defines a stratification on $\mathscr{C}_*(M)$.

Denote by $C_*(\mathscr{C}_{[\mathbf{k}]}(M))$ the complex of rational chains of the Fulton-MacPherson compactification.

Definition 2 For a $C_*(F_n)$-algebra A and a compact parallelized n-manifold without boundary M call the complex $CM_*(A) = C_*(\mathscr{C}_*(M))\otimes_{C_*(P(F_n))} A$ the *manifoldic* *chain complex* of A on M. Call the homology of the manifoldic chain complex the *manifoldic homology* of A on M.

This definition is based on Definition 4.14 from [17]. By Proposition 2 one may pass from a $C_*(F_n)$-algebra to an e_n-algebra. As it is shown in [13], the manifoldic homology is the same as the topological chiral homology with constant coefficients introduced in *loc. cit* of this e_n-algebra.

1.4.2 Morphism

Let $(\mathfrak{g}, d)$ be a L_∞-algebra. Let $l_{i>1} \colon \Lambda^i \mathfrak{g}[i-2] \to \mathfrak{g}$ be its higher brackets, that is, the operations in complex (1) corresponding to the star trees. The structure of L_∞-algebra may be encoded in a derivation $D = D_1 + D_2 + \cdots$ on the free super-commutative algebra generated by $\mathfrak{g}^\vee[1]$, where D_1 is dual to d and D_i is dual to l_i on generators and are continued on the whole algebra by the Leibniz rule. The *Chevalley–Eilenberg chain complex* $CE_*(\mathfrak{g})$ is the super-symmetric power $S^*(\mathfrak{g}[-1])$ with the differential $\mathrm{d_{tot}} = d + \theta_2 + \theta_3 + \cdots$, where θ_i is dual to D_i.

Denote by $[\mathscr{C}^0_{[\mathbf{k}]}] \in C_*(\mathscr{C}_{[\mathbf{k}]}(M))$ the chain given by the submanifold $\mathscr{C}^0_{[\mathbf{k}]}(M)$ in $\mathscr{C}_{[\mathbf{k}]}(M)$. For a $C_*(F_n)$-algebra A and a cycle $c \in C_*(\mathscr{C}_{[\mathbf{k}]}(M))$ denote by $(a_1 \otimes \cdots \otimes a_k) \otimes_{\Sigma_k} c \in CM_*(A)$ the chain given by the tensor product over the symmetric group. Recall that by (2) for any $C_*(F_n)$-algebra A the complex $A[n-1]$ is equipped with a L_∞ structure. Denote this L_∞-algebra by $L(A)$. Denote by $\mathrm{Alt}(a_1 \otimes \cdots \otimes a_k)$ the sum $\sum_\sigma \pm a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)}$ by all permutations, where signs are sign given by the sign of the permutation and the Koszul sign rule.

Proposition 3 *For a $C_*(F_n)$-algebra A and a parallelized compact manifold without boundary M the map* $\mathrm{T} \colon a_1 \wedge \cdots \wedge a_k \mapsto \mathrm{Alt}(a_1 \otimes \cdots \otimes a_k) \otimes_{\Sigma_k} [\mathscr{C}^0_{[\mathbf{k}]}]$ *defines a morphism from Chevalley–Eilenberg complex $CE_*(L(A))$ to the manifoldic chain complex $CM_*(A)$.*

Proof Denote the total differentials on both complexes $CE_*(L(A))$ and $CM_*(A)$ by $\mathrm{d_{tot}}$. One needs to show that $\mathrm{d_{tot}} \circ \mathrm{T} = \mathrm{T} \circ \mathrm{d_{tot}}$.

The boundary of $[\mathscr{C}^0_{[\mathbf{k}]}]$ in $C_*(\mathscr{C}_{[\mathbf{k}]}(M))$ is the sum of all codimension one strata: $\partial[\mathscr{C}^0_{[\mathbf{k}]}] = \sum_i \theta_i [\mathscr{C}^0_{[\mathbf{k-i+1}]}]$. Here θ_i is the symmetrization in $C_*(P(F_n))$ of the operation in $C_*(F_n)$ that corresponds by Proposition 1 to the star with i inputs. This means that

$$\mathrm{d_{tot}} \circ \mathrm{T}\,(a_1 \wedge \cdots \wedge a_k) = (d\,\mathrm{Alt}(a_1 \otimes \cdots \otimes a_k)) \underset{\Sigma_k}{\otimes} [\mathscr{C}^0_{[\mathbf{k}]}]$$

$$+ \mathrm{Alt}(a_1 \otimes \cdots \otimes a_k) \underset{\Sigma_k}{\otimes} \sum_{i>1} \theta_i [\mathscr{C}^0_{[\mathbf{k-i+1}]}]$$

One may carry θ's from one factor of $\otimes_{\Sigma_k}$ to another by the very definition of the tensor product over $C_*(P(F_n))$. And the action of θ's on the alternating sum again by definition is given by the higher brackets of the L_∞-algebra. After summing with d it gives the differential on the Chevalley–Eilenberg complex. It follows that $\mathrm{d}_{\mathrm{tot}} \circ \mathrm{T} = \mathrm{T} \circ \mathrm{d}_{\mathrm{tot}}$.

1.5 Sketch: Invariants of a Parallelized Manifold and Koszul Duality

1.5.1 Invariant of a Parallelized Manifold

The idea how to apply manifoldic homology to manifolds invariant is the following. Below (Definition 3) we sketch a construction of a e_n-algebra $\mathfrak{D}^n(V)$ such that for any n-dimensional parallelized compact manifold without boundary M manifoldic homology $HM_*(\mathfrak{D}^n(V))$ is one-dimensional (Proposition 4). Then,

$$H_*(L(\mathfrak{D}^n(V))) \to HM_*(\mathfrak{D}^n(V)) \tag{3}$$

given by Proposition 3 supplies us with a cocycle in the Lie algebra cohomology of $L(\mathfrak{D}^n(V))$.

In this way we obtain an invariant that is conjecturally related to the universal Chern–Simons invariant (see [1, 3]) which takes value in "graph cohomology", with the Chevalley–Eilenberg cochain complex of $L(\mathfrak{D}^n(V))$ representing the "graph complex".

1.5.2 Koszul Duality

Quillen duality [11, 16] gives an equivalence between homotopy categories of Lie algebras Lie and connected cocommutative coalgebras $coCom$. Koszul duality [6, 13] is an analogous equivalence between the categories of augmented e_n- algebras and coaugmented e_n-coalgebras satisfying certain conditions analogous to connectedness. I hope to elaborate on these conditions elsewhere. Denote the above mentioned categories by $e_n - alg$ and $e_n - coalg$. The relationship between Quillen and Koszul dualities is displayed in the diagram

$$\begin{array}{rcccl} L: & e_n - alg & \rightleftarrows & Lie & : U^n \\ & \text{Koszul duality} \updownarrow & & \updownarrow \text{Quillen duality} & \\ \mathrm{Ab}: & e_n - coalg & \rightleftarrows & coComm & : \imath \end{array}$$

Here, the functor L is given by (2), U^n is the derived universal enveloping e_n- algebra the functor that is derived left adjoint to L, $\imath$ is the embedding of cocommutative coalgebras in e_n-coalgebras and Ab is its derived right adjoint.

The linear dual of a e_n-coalgebra is a e_n-algebra. If some e_n-algebra and e_n-coalgebra are related by Koszul duality, then the first one and the linear dual of the second one are called *Koszul dual e_n-algebras*.

The following statement generalizes the well-known fact that Hochschild homologies of Koszul dual algebras are dual to each other (see e.g. [18, Appendix D]).

Claim (Poincaré–Koszul duality) For a n-dimensional parallelized compact manifold without boundary M, the manifoldic homologies on M of Koszul dual e_n-algebras are linear dual to each other.

1.5.3 *n*-Weyl Algebra

We say that an element c of a e_n-algebra A is *central*, if the product map

$$e_n(k+1) \otimes \underbrace{c \otimes A \otimes \cdots \otimes A}_{k+1} \to A$$

factors through

$$e_n(k+1) \otimes \underbrace{c \otimes A \otimes \cdots \otimes A}_{k+1} \to c \otimes e_n(k) \otimes \underbrace{A \otimes \cdots \otimes A}_{k}.$$

The latter map is induced by the natural projection from $k+1$-ary operations to k-ary ones.

By a *e_n-algebra with curvature* we mean a e_n-algebra with a central element c of degree $n+1$. The condition on c may be relaxed by analogy with [15]. The new condition may be formulated in terms of the deformation complex of an e_n-algebra.

Conjecturally, one may define Koszul duality for e_n-algebras with curvature in such a way, that for $n = 1$, one recovers Koszul duality for algebras with curvature, see [15].

Let V be a graded vector space with a non-degenerate symmetric in the graded sense bilinear form q of degree $-(n+1)$. Let $S^*(V^\vee)$ be the free graded commutative algebra generated by the vector space dual to V. Denote by $S^*(V^\vee)^\vee$ the restricted dual coalgebra. By means of inclusion ι from (4) consider the pair $(S^*(V^\vee)^\vee, q)$ as a e_n-coalgebra with curvature.

Definition 3 For a graded vector space V with a non-degenerate symmetric in the graded sense bilinear form q of degree $-(n+1)$ we denote by $\mathfrak{D}^n(V)$ the e_n-algebra Koszul dual to $(S^*(V^\vee)^\vee, q)$ and refer to it as *n-Weyl algebra*.

Proposition 4 *For a n-dimensional parallelized compact manifold without boundary M, the manifoldic homology $HM_*(\mathfrak{D}^n(V))$ is one-dimensional.*

Proof By Statement 1.5, $HM_*(\mathfrak{D}^n(V))$ is linear dual to the manifoldic homology of the e_n-algebra that is Koszul dual to $\mathfrak{D}^n(V)$. Thus, one needs to prove that the latter homology is one-dimensional. The Koszul dual e_n-algebra is e_n-algebra with curvature $(S^*(V^\vee), q)$. The manifoldic homology of $S^*(V^\vee)$ is the free commutative algebra generated by $V^\vee \otimes H_*(M)$, where $H_*(M)$ is homology of M negatively graded. The curvature equips the underlying space of this algebra with a differential given by multiplication by an element of cohomological degree 1 and of homogeneous degree 2. This element represents the pairing induced by the tensor product of q on V and the Poincaré paring on $H^*(M)$. The cohomology of this differential, that is of the de Rham complex of a graded vector space, is manifoldic homology of the e_n-algebra with curvature. As the cohomology of the de Rham complex is one-dimensional, this implies the proposition.

Example 1 Let $n = 1$ and V is concentrated in degree 1. Then $\mathfrak{D}^1(V)$ is the usual Weyl algebra, that is the symplectic Clifford algebra generated by vector space $V[-1]$ with the skew-symmetric form on it. For $M = S^1$ the manifoldic homology is the Hochschild homology and Proposition 4 matches with the well-known fact about Weyl algebra:

$$\dim HH_i(\mathfrak{D}^1(V)) = \begin{cases} 1, & i = \dim V, \\ 0, & \text{otherwise.} \end{cases}$$

Note that this fact is crucially used in [5] and classes like (3) and (6) below restricted to the Lie algebra of vector fields are exploited there to present the Todd class.

1.5.4 Concluding Remarks

Finally, let us look at the morphism from Proposition 3 from the Koszul duality viewpoint.

For a commutative algebra C there is a canonical morphism $HH_*(C) \to C$. It may be generalized to manifoldic homology as follows.

Claim For a homotopy commutative algebra (= e_∞-algebra) C and for a n-dimensional parallelized compact manifold without boundary M there is a canonical map from manifoldic chain complex of C to C itself:

$$\pi\colon CM_*(\iota(C)) \to C. \tag{4}$$

Morphism π may be constructed by means of manifoldic homology of non-compact manifolds: every manifold may be embedded $\mathbb{R}^N$ and as commutative algebra may be considered as e_N-algebra, the embedding induces a morphism of manifoldic homologies, and manifoldic homology of C on $\mathbb{R}^N$ is C.

Diagram (4) shows that for a Lie algebra $\mathfrak{g}$ the e_n-algebras $U^n(\mathfrak{g})$ and $\iota(CE^*(\mathfrak{g}))$ are Koszul dual, where CE^* is the Chevalley–Eilenberg cochain complex. By Poincaré–Koszul duality (Statement 1.5) for a n-dimensional parallelized compact manifold without boundary M homologies $HM_*(U^n(\mathfrak{g}))$ and $HM_*(\iota(CE^*(\mathfrak{g})))$ are dual to each other. Morphism (4) gives $\pi\colon CM_*(\iota(CE^*(\mathfrak{g})) \to CE^*(\mathfrak{g})$ and composing with Poincaré–Koszul duality we obtain the map

$$H_*(\mathfrak{g}) \to HM_*(U^n(\mathfrak{g})). \tag{5}$$

Functors U^n and L from (4) being adjoint, there is as canonical morphism $U^n(L(A)) \to A$ for any e_n-algebra A. It induces a map on manifoldic homologies:

$$HM_*(U^n(L(A))) \to HM_*(A). \tag{6}$$

Claim The effect of the morphism from Proposition 3 on homologies is the composition of (5) for $\mathfrak{g} = L(A)$ and (6).

This morphism may be described even simpler in Koszul dual terms. The Koszul dual morphism is the composition

$$CM_*(A^!) \to CM_*(\iota(\mathrm{Ab}(A^!))) \to A^!, \tag{7}$$

where the first arrow is induced by the canonical morphism for a pair of adjoint functors ι and Ab and the second arrow is given by (4).

For our main example $A = \mathfrak{D}^n$ the Koszul dual e_n-algebra $A^!$ is a e_n-algebra with curvature and the formula (7) is not applicable directly. It is not clear, what the functor Ab means for such algebras. The question is interesting even for $n = 1$, where Ab is the derived quotient by the ideal generated by commutators.

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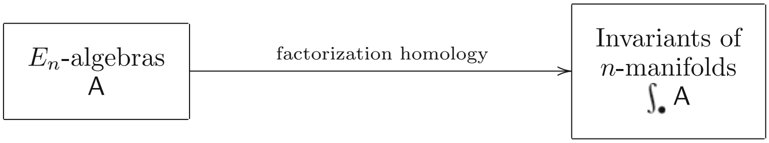
2 Factorization Homology and Link Invariants (by Hiro Lee Tanaka)

Abstract We define the notions of E_n-algebras and factorization homology, sketching how one can construct link invariants using a version of factorization homology for stratified manifolds. The work on factorization homology for stratified manifolds is joint with David Ayala and John Francis.

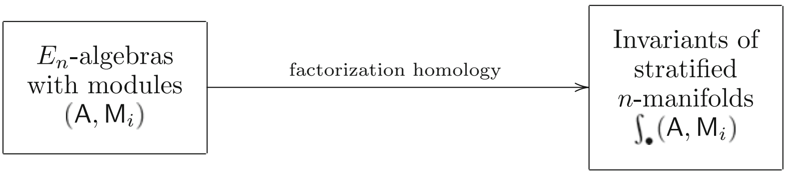
2.1 Overview

Factorization homology is a way to construct invariants of n-manifolds from one piece of algebraic data. This algebraic data is an *E_n-algebra* A, and the invariant associated to an n-manifold X is called the *factorization homology of X with coefficients in* A. We write this as $\int_X \mathsf{A}$.



The idea of using E_n-algebras to create invariants of n-manifolds has been in the air for some time, but ongoing work with Ayala and Francis [2] generalizes factorization homology to define invariants of *stratified manifolds*. For instance, one can define invariants for manifolds with boundary, for singular manifolds (such as graphs or cones), for singular manifolds with decorations (such as colored graphs), and for manifolds stratified by the image of an embedding. (As in the title, this includes the case of a link inside S^3.)

To define such an invariant we need more algebraic data than just an E_n-algebra. Roughly speaking, we need the data of E_n-algebras and modules over them.



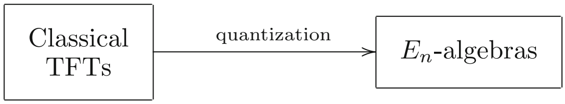
There is also a physical motivation for factorization homology. Factorization homology is also called *topological chiral homology* (for instance, by Jacob Lurie in [13]) and this terminology is no accident. 'Chiral homology' is a concept familiar from conformal field theories—in studying conformal field theories, one inputs a chiral algebra, and chiral homology (i.e., the space of conformal blocks) is what one assigns to a Riemann surface.

Topological chiral homology is the topologist's analogue of this invariant—instead of a chiral algebra we input an E_n-algebra, and we produce an invariant sensitive to the diffeomorphism type of a manifold.[1] In other words, topological chiral homology should be the output of a topological field theory, instead of a conformal one.

More precisely, when one has a classical field theory defined on a space-time manifold X, the observables of the quantized field theory form a *factorization algebra*

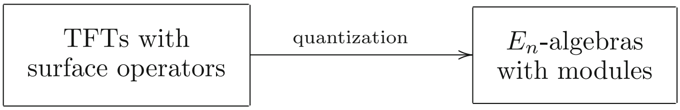
[1] As will be mentioned later, in this talk we create an invariant of manifolds *with a framing*. If we are interested in studying manifolds with some structure B (such as an orientation), there is a class of algebras (such as an E_B algebra) which defines invariants for all manifolds with B-structures.

on X. This is explained in Chap. 3. If the field theory is topological, a factorization algebra on a space-time is locally the same thing as an E_n-algebra.



The global observables of the field theory on X is precisely the factorization homology of X with coefficients in this E_n-algebra.

Finally, in a theory with surface operators (as in the work of Gukov and Witten in [10]), or Wilson loops (such as Chern-Simons theory) or 't Hooft lines, one expects the quantization to see the structure of embedded surfaces and embedded curves. In the case these field theories are topological, one can broadly call such theories "TQFTS with surface operators," and we expect to produce E_n algebras with modules from such field theories:



One goal is to understand the E_3-algebras with modules that should arise from Chern-Simons theory.

2.2 E_n-algebras

In what follows, D^n denotes the open n-disk of unit radius. We first define a category Disk_n^{fr}, enriched over topological spaces.

Definition 4 An object of Disk_n^{fr} is a (possibly empty) finite set S. The set of morphisms $\mathsf{Disk}_n^{fr}(S, T)$ is given by embeddings $f : (D^n)^{\sqcup S} \to (D^n)^{\sqcup T}$ such that, on each connected component, f is of the form

$$f(\mathbf{x}) = \lambda \mathbf{x} + C$$

for some fixed $\lambda > 0, C \in D^n$. The set $\mathsf{Disk}_n^{fr}(S, T)$ inherits a topology as a subspace of all continuous maps, which we topologize by the compact-open topology.

Note that Disk_n^{fr} has a symmetric monoidal structure given by disjoint union. Also, the fr stands for 'framed'—see the remark after Definition 3.1.

Definition 5 (E_n-*algebra*) Let Chain_k be the category of chain complexes over some base field k. An E_n-algebra A is a symmetric monoidal functor

$$\mathsf{A} : \mathsf{Disk}_n^{fr} \to \mathsf{Chain}_k.$$

of categories enriched in topological spaces.

Remark 2 Chain_k admits an enrichment over topological spaces in a standard way, for instance by using the Dold-Kan correspondence on morphisms. Heuristically, the condition that A be a symmetric monoidal functor of categories enriched in topological spaces means:

1. $\mathsf{A}(\emptyset) = k$
2. A sends disjoint unions to tensor products
3. A sends (higher) isotopies of embeddings to (higher) chain homotopies,

where the last condition means A is a continuous map on morphism spaces.

Remark 3 One can obviously define what an E_n-algebra is for any target category $\mathcal{C}$ whose morphisms sets are spaces, and who has a symmetric monoidal structure.

Remark 4 Throughout, we write A for the functor, and we will write A for $\mathsf{A}(D^n)$.

Example 2 ($n = 1$) As we've heard before during the winter school, the $n = 1$ case recovers the notion of an associative algebra. Let me explain how.

First, we note that the inclusion of two disjoint intervals into a single interval gives a map

$$m : A \otimes A \to A$$

and the inclusion of the empty set into D^1 yields a unit map

$$1 : k \to A.$$

Composition of embeddings and the tensor product property (2) shows that 1 is indeed a unit for the multiplication m. Finally, factoring the inclusion

$$D^1 \sqcup D^1 \sqcup D^1 \to D^1$$

in two different ways yields the associativity condition on m.

However, there is a subtlety—because one can wiggle embeddings by isotopies, what we really find is that m ought to be an associative multiplication *up to higher homotopies*. So the correct statement is that any E_1-algebra is in fact an A_∞ algebra. We state this result for the record:

Proposition 5 *The category of E_1-algebras is equivalent to the category of unital A_∞-algebras in Chain_k.*

Remark 5 On a first pass, no real intuition is lost by simply thinking of A_∞ algebras as associative algebras. However, as the next example shows, for $n > 1$ we shouldn't be so cavalier.

Example 3 ($n = 2$) Let A be an E_2-algebra. Given a configuration of two disks inside the unit disk, we get a multiplication

$$m : A \otimes A \to A.$$

However, we see that there is an isotopy of embeddings taking this configuration to one in which the embedded rectangles have flipped labeles—i.e., there is a homotopy between

$$m(x_1, x_2) \qquad \text{and} \qquad m(x_2, x_1)$$

where $x_i \in A$ are elements of the algebra. This shows that m is in fact a commutative multiplication if you only remember m up to homotopy. However, as anybody who's studied the configuration space of points in $\mathbb{R}^2$ knows, there is a braid group hiding in this picture—namely, if you have j embedded disks in D^2, you can reconfigure them in ways that are homotopic, but not canonically so. (i.e., the configuration space of rectangles is connected, but has non-trivial topology.)

So remembering m only up to homotopy would discard the information of the braid group. (In fact, if the target category were vector spaces, rather than chain complexes, an E_2-algebra is the same things as a commutative algebra.) The conclusion is that E_2-algebras in fact encode a delicate system of multiplications, and sees the geometry of the configuration space of disks in D^2. This is precisely the reason that an E_n algebra should be expected to yield invariants of n-manifolds.

2.3 Factorization Homology

Definition 6 Let Mfld_n^{fr} be the topologically enriched category whose objects are smooth n-manifolds X together with a *framing*, i.e., a choice of isomorphism $\phi_X : TX \cong X \times \mathbb{R}^n$. A morphism in $\mathsf{Mfld}_n^{fr}(X, Y)$ is a pair (f, h) where $f : X \to Y$ is an embedding, and h is a choice of homotopy from $f^*\phi_Y$ to ϕ_X.

Note that Disk_n^{fr} can be viewed as the full subcategory of Mfld_n^{fr} consisting of objects which are diffomeorphic to disjoint copies of D^n. This is because the space of rectilinear embeddings is homotopy equivalent to the space of framed embeddings.

So given an algebra A, the question is whether we can extend the functor A to the whole of Mfld:

$$\begin{array}{ccc} \mathsf{Mfld}_n^{fr} & \overset{?}{\dashrightarrow} & \mathsf{Chain}_k \\ \uparrow & \nearrow_{\mathsf{A}} & \\ \mathsf{Disk}_n^{fr} & & \end{array}$$

There is in fact a general way of doing this since Chain_k itself is a well-behaved category:

Definition 7 (*Factorization Homology*) Let $\int : \mathsf{Mfld}_n^{fr} \to \mathsf{Chain}_k$ be the *left Kan extension* of the functor A along the inclusion $\mathsf{Disk}_n^{fr} \to \mathsf{Mfld}_n^{fr}$. We call the resulting functor *factorization homology*, and write

$$X \mapsto \int_X \mathsf{A}$$

for any n-manifold X. We say that $\int_X \mathsf{A}$ is *the factorization homology of X with coefficients in* A.

Remark 6 (**Left Kan extensions**) Intuitively, any n-manifold is understood by seeing how it is glued together from many copies of $\mathbb{R}^n$. So one can express a manifold as a gigantic diagram of embedded copies of $\mathbb{R}^n$, together with gluing maps. To define the left Kan extension, one simply writes down the same diagram in the category of chain complexes, and glues along the corresponding maps given by the functor A. (i.e., one takes the colimit of the corresponding diagram.) Also, the left Kan extension we take is not a naive left Kan extension, but the ∞-categorical Kan extension. Equivalently, one takes the homotopy left Kan extension.

Remark 7 If one begins with an E_n-algebra in a general symmetric monoidal, topologically enriched category $\mathcal{C}$, one can still define factorization homology as left Kan extension so long as $\mathcal{C}$ admits enough colimits.

2.4 The Main Theorem

We first record some properties of factorization homology:

Theorem 1 *Factorization homology satisfies the following properties:*

1. *It sends disjoint unions of manifolds to tensor products of chain complexes.*
2. *It sends (higher) isotopies of embeddings to (higher) homotopies of chain maps.*
3. *It satisfies excision. That is, if a manifold X can be written as a union*

$$X = X_0 \cup_{Y \times D^1} X_1$$

where $X_0 \cap X_1 \cong Y \times D^1$ as framed manifolds, then the factorization homology of X is given by the bar construction

$$\int_X \mathsf{A} \cong \int_{X_0} \mathsf{A} \otimes_{\int_{Y \times D^1} \mathsf{A}} \int_{X_1} \mathsf{A}.$$

Remark 8 (Excision) The difficulty of the theorem is not in (1) and (2), but in the excision property. As discussed before, any E_1 algebra is an A_∞ algebra. And we see that the natural inclusion

$$\{id_Y\} \times \mathsf{Emb}((D^1)^{\sqcup i}, D^1) \subset \mathsf{Emb}((Y \times D^1)^{\sqcup i}, Y \times D^1)$$

gives the structure of an E_1 algebra to $\int_{Y \times D^1} \mathsf{A}$.

Moreover, we have a family of natural embeddings

$$(Y \times D^1) \sqcup X_0 \to X_0, \qquad (Y \times D^1) \sqcup X_1 \to X_1$$

which, by the monoidal property of factorization homology, give rise to maps

$$(\int_{X_0} \mathsf{A}) \otimes (\int_{Y \times D^1} \mathsf{A}) \to \int_{X_0} \mathsf{A}, \qquad (\int_{Y \times D^1} \mathsf{A}) \otimes (\int_{X_1} \mathsf{A}) \to \int_{X_1} \mathsf{A}.$$

In other words, the decomposition gives $\int_{X_0} A$ and $\int_{X_1} A$ the structure of a right- and left-modules over $\int_{Y \times D^1} A$, respectively. Thus the bar construction makes sense in (3).

Conversely, let $\mathcal{H}$ be the category of all functors $H : \mathsf{Mfld}_n^{fr} \to \mathsf{Chain}_k$ satisfying the properties (1)–(3) in the theorem above. There is a clear map

$$ev_{\mathbb{R}^n} : \mathcal{H} \to E_n\text{-alg}$$

given by evaluating $\mathcal{H}$ at the manifold $\mathbb{R}^n$. The following recognition principle was proven by John Francis in [7]:

Theorem 2 (Francis) *$ev_{\mathbb{R}^n}$ is an equivalence of categories. An inverse functor is given by factorization homology.*

My joint work with David Ayala and John Francis [2] replaces Mfld_n^{fr} by a category SMfld_n of *stratified* n-manifolds.[2] The point is that, even for a stratified n-manifold X, we know what the local structure of X looks like. For instance, a graph locally looks like an interval or an i-valent vertex for some i. And an embedded submanifold $A \subset B$ looks locally like an open neighborhood of B, or like the tubular neighborhood of an open patch in A. Such local pieces form a category Loc—this is in analogy with the case of smooth manifolds, where the local pieces form the category Disk_n^{fr}. Roughly speaking, Loc is a category whose objects are disjoint unions of local pieces, and whose morphisms are embeddings between them.

Then a functor $\mathsf{A} : \mathsf{Loc} \to \mathsf{Chain}_k$ is called a Loc-algebra, and these often give structures that look like modules over an E_n-algebra, where n is the top dimension of pieces in Loc. (I will give examples in the next subsection.)

Once more one can define factorization homology, for stratified manifolds, by taking the left Kan extension

$$\begin{array}{ccc} \mathsf{SMfld}_n & \overset{\int}{\dashrightarrow} & \mathsf{Chain}_k \, . \\ \uparrow & \nearrow_{\mathsf{A}} & \\ \mathsf{Loc} & & \end{array}$$

[2] The notation is somewhat misleading, since there is not a unique category of stratified manifolds—one can choose to include or exclude certain kinds of stratifications, but this is irrelevant to the philosophy of this talk.

This functor still satisfies excision, and the main result of our joint work is the following generalization of the previous theorem:

Theorem 3 (Ayala-Francis-T) *Let $\mathcal{H}_S$ be the category of functors $H : \mathsf{SMfld}_n \to \mathsf{Chain}_k$ which satisfy excision, are monoidal, and send (higher) isotopies of embeddings to (higher) chain homotopies. Then the restriction map to Loc induces an equivalence of categories*

$$\mathcal{H}_S \cong \mathsf{Loc}\text{-}alg.$$

The proof of this theorem appears in [2].

2.5 Examples

Example 4 (*The circle and Hochschild homology)* Recall from Chap. 13 that the basic example is when $n = 1$ and $X = S^1$. Then the excision axiom tells us that

$$\int_{S^1} \mathsf{A} \cong A \otimes_{A \otimes A^{op}} A.$$

The right-handside of this equivalence is a well-known object—it is the *Hochschild homology of A with coefficients in A*, or in short, the Hochschild homology of A. It is the derived 'abelianization' of A, in that H^0 of the right-hand-side recovers the group

$$A/[A, A].$$

Geometrically, one can see this as the ability to collide two points from the left, or from the right, on a circle.

Example 5 (*Hochschild homology with coefficients)* Now let us consider the category SMfld whose objects are smooth 1-manifolds with marked points. Then Loc is a category generated by two objects: The open interval, and the open interval with a single marked point. (Any 1-manifold with marked points can be constructed by gluing disjoint unions of these 'local pieces' together.) Let us suppose we have a Loc-algebra $\mathsf{A} : \mathsf{Loc} \to \mathsf{Chain}_k$. Then to the interval with the marked point, we associate a chain complex M, and to an open interval with no marked point, we associate a chain complex A. A is an A_∞-algebra as before.

Given an interval $(-1, 1)$ with a marked point at 0, one can include a copy of the interval $(0, 1)$ on either side of the marked point. These two inclusions induce maps

$$A \otimes M \to M, \qquad M \otimes A \to M$$

and one can check that a Loc-algebra in this case is precisely the data of an A_∞-algebra A, and a *pointed* bimodule M. Pointed simply means that there is a map $k \to M$ compatible with the A action. (This corresponds to the inclusion of the empty set into the interval with a marked point.)

Now let $X = (S^1, t_0)$ be a circle with a marked point t_0. Then by excision, we see that

$$\int_X \mathsf{A} \cong M \otimes_{A \otimes A^{op}} A.$$

The right-hand-side is the chain complex giving rise to *Hochschild homology of A with coefficients in M*. In general, a collection of k marked points on the circle will have factorization homology equal to Hochschild homology of A with coefficients in $M^{\otimes k}$.

Example 6 (*Hochschild homology with more coefficients)* More generally, if we let SMfld contain one-manifolds with *colored* marked points, each color will correspond to a different bimodule M_i over A, and the circle with various colored, marked points will yield Hochschild homology of A with coefficients in the appropriate tensor powers of the M_i.

Example 7 (*Link invariants)* Now let Loc be the category generated by two objects: $\mathbb{R}^3$, and a copy of $\mathbb{R}^1$ linearly embedded into $\mathbb{R}^3$. We will refer to the latter by the pair $(\mathbb{R}^3, \mathbb{R}^1)$. The stratified manifolds which can be built out of such pieces are precisely 3-manifolds with embedded links. Moreover, one can describe the structure that a Loc-algebra A has. Let us denote $A := \mathsf{A}(\mathbb{R}^3)$ and $M := \mathsf{A}((\mathbb{R}^3, \mathbb{R}^1))$.

Clearly, the stratified embeddings of copies of $(\mathbb{R}^3, \mathbb{R}^1)$ into itself give the structure of an E_1 algebra to M, and A as usual has the structure of an E_3 algebra. Moreover, the inclusions

$$\mathbb{R}^3 \sqcup (\mathbb{R}^3, \mathbb{R}^1) \to (\mathbb{R}^3, \mathbb{R}^1)$$

yield maps

$$A \otimes M \to M$$

which are compatible with all multiplication maps. Hence, M is an E_1 algebra *receiving a compatible action* from the E_3-algebra A.

Remark 9 Though we have not talked about the notion of push-forward, one can take a generic map from $L \subset \mathbb{R}^3$ to $\mathbb{R}^2$ to obtain a link diagram in $\mathbb{R}^2$. This is a stratified manifold, and its factorization homology is the same as that of the link itself. One verifies easily that the Reidemeister relations hold for this invariant.

Remark 10 As Witten explained in his seminal paper [19], it was his and Atiyah's aim to give a definition of a link invariant which is manifestly three-dimensional; that is, one that does not crucially rely on the Reidemeister relations, and is closer in philosophy to an embedding invariant. One can view this formulation, in terms of factorization homology, as a continuation of this arc.

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Deligne-Beilinson Cohomology in U(1) Chern-Simons Theories

Frank Thuillier

1 Introduction

In the early years of the 19th Century, after having studied the trajectories of some celestial objects as Asteroid Ceres or Comet Biela, Carl Freidrich Gauss set forth his famous formula for the linking number, thus providing one of the first mathematical tools allowing a characterization of celestial orbits configurations [1, 2]. One hundred years later, in 1931, Heinz Hopf indirectly highlighted a relation between Gauss linking number and homotopy classification of maps $\psi : S^3 \to S^2$, thanks to the Hopf fibration and the Hopf invariant $H(\psi)$ [3]. In 1947 John Henry Whitehead gave an integral formula for the Hopf invariant $H(\psi)$ [4]. In 1958, while dealing with astrophysical applications of Hydromagnetics, Lodewijk Woltjer exhibited an integral defining a conserved quantity [6] that Keith Moffat called helicity [5] and which turned out to be closely related to Whitehead's integral. Last but not least, helicity on its turn appears related with the abelian version of the Chern-Simons characteristic forms introduced by Shiing-Shen Chern and James Simons in 1971 [7]. All these historical references may suggest the existence of an intrinsic relation between Gauss linking number and the $U(1)$ Chern-Simons theory.

On the other hand, Deligne-Beilinson cohomology goes back to an article of Pierre Deligne published in 1972 [8] and independently to one by Alexander A. Beilinson published in 1985 [9]. Along with this, Jeff Cheeger and James Simons, after a series of lectures at Stanford in 1973, wrote an article where they introduced the notion of Differential Characters [10], thereby extending Chern and Simons original work. More recently, Reese Harvey, Blaine Lawson and John Zweck proposed an alternative description of Cheeger-Simons Differential Characters, based on de Rham-Federer currents, that they called Sparks [11]. The work of Michael Jerome Hopkins and Isadore Manuel Singer on Differential Cohomology [12], which is also

F. Thuillier (✉)
LAPTh, Université de Savoie, CNRS, 9, Chemin de Bellevue, BP 110,
74941 Annecy-le-Vieux Cedex, France
e-mail: frank.thuillier@lapth.cnrs.fr

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related to Differential Characters, has also to be mentioned. Recently, James Simons and Dennis Sullivan have clarified the relation between of all these notions, showing their equivalence [13]. In this review, Deligne-Beilinson point of view will be preferred.

Although it takes its roots in algebraic geometry and L-functions [14–16], Deligne-Beilinson cohomology proved extremely effective in the study of flat vector bundles with connections [17, 18], K-theory [19], or in the classification of abelian Gerbes with connections [20, 21]. Recently Deligne-Beilinson cohomology has extended its scope to that of Theoretical Physics [22–31]. An emblematic example is provided by the Ehrenberg-Siday-Aharonov-Bohm effect [32, 33] which can be revisited through Deligne-Beilinson cohomology. And as this effect plays an important role in Geometric Quantization [34], Deligne-Beilinson cohomology should enter in the landscape of this geometrical formulation of Quantum Mechanics and even suggest possible generalizations.

It is well-known that in a non-abelian Chern-Simons theory expectation values of Wilson loops yield link and knot invariants [35–45], at least perturbatively in Quantum Field Theory. In the $U(1)$ Chern-Simons theory, the use of Deligne-Beilinson cohomology yields the relation between linking numbers and expectation values of Wilson loops [28–31]. The corresponding relation is much hazier in the non-abelian case, even though the non-abelian action is a Deligne-Beilinson cohomology class just as in the abelian case. In this review we would like to exhibit some benefits in the use of Deligne-Beilinson cohomology in the context of the $U(1)$ Chern-Simons Quantum Field Theory on a 3-dimensional closed manifold M. As it will appear, all computations are performed on M itself, without having to resort to Dehn surgery. Among the results thus obtained, let us point out the quantization of the coupling constant and of charges of the colored knots, the explicit determination of link invariants from the expectation value of Wilson loops, the triviality of the link invariants for homologically non trivial links, Reshetikhin-Turaev invariants of 3-manifolds etc. Note that most—if not all—of the results presented in this review can be generalised to $(4l + 3)$-dimensional smooth closed manifolds [30]. We will mention some of these generalizations.

All manifolds considered in this review will be *closed smooth oriented manifolds endowed with a good cover*. Let us recall that a cover of a manifold M is good if any non-empty intersection of its open sets is contractible, thus allowing to apply Poincaré lemma in such an intersection. Furthermore, as they play a key role in this review, the reader will find a brief reminder of Čech-de Rham technics in the Appendix.

2 A Short Review of Deligne-Beilinson Cohomology

In this section we will present Deligne-Beilinson cohomology with the use of the Čech-de Rham bi-complex. This approach allows for instance to establish an isomorphism between de Rham and Čech real cohomologies as reminded in the Appendix. Once a truncation in the de Rham complex is introduced this approach gives birth to the sought cohomology. The determination of the cohomology groups so generated then yields two exact sequences into which these groups are embedded. In a second subsection the most important properties of Deligne-Beilinson cohomology will be exhibited as well as their relation with Pontryagin duality which naturally arises from the construction.

The notation $\stackrel{\mathbb{Z}}{=}$ will mean equality modulo integers.

2.1 From the Cech-de Rham Representatives to the Canonical Exact Sequences

We would like to introduce Deligne-Beilinson cohomology so that, like Monsieur Jourdain, theoretical physicists realize they often used it without even knowing it. This cohomology naturally occurs in the mathematical context of $U(1)$ principal bundles with connections and is physically realized through the Ehrenberg-Siday-Aharonov-Bohm effect. Indeed, the shift in the interference pattern which reflects this effect is semi-classically characterized by the quantity $\exp\{i \frac{e}{\hbar} \oint_\gamma A\}$. This suggests that up to some normalization factor the quantity $\oint_\gamma A$ is defined modulo integers. Deligne-Beilinson cohomology gives a mathematical substance to this idea. This reminds the Stern and Gerlach experiment which exhibited half-valued angular momentum for the electron thus giving rise to the notion of spin, mathematics then showing how spin is related to the eigenvalues of the generators of $SU(2)$, itself naturally generalizing the classical group of rotation in space.

Lemma 1 *The Deligne-Beilinson cohomology group $H^p_D(M, \mathbb{Z})$ can be defined by various exact sequences, including the following two*:

$$0 \to \frac{\Omega^p(M)}{\Omega^p_{\mathbb{Z}}(M)} \xrightarrow{j_1} H^p_D(M, \mathbb{Z}) \xrightarrow{\epsilon} H^{p+1}(M, \mathbb{Z}) \to 0$$

$$0 \to H^p(M, \mathbb{R}/\mathbb{Z}) \xrightarrow{j_2} H^p_D(M, \mathbb{Z}) \xrightarrow{cv} \Omega^{p+1}_{\mathbb{Z}}(M) \to 0.$$

There is a Čech-de Rham realization of the Deligne-Beilinson cohomology groups.

2.1.1 The Čech-de Rham Construction

Let M be a m-dimensional manifold, $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ an open cover of M and $P(M, U(1))$ a $U(1)$ principal bundle over M. The transition functions $g_{\alpha\beta}$: $U_\alpha \cap U_\beta \to U(1)$ of P can be written

$$g_{\alpha\beta} = e^{2i\pi \Lambda_{\alpha\beta}}, \tag{1}$$

for some collection of smooth function $\Lambda_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}$. On these local functions the cocycle condition $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$ that the transition functions have to satisfy in the intersection $U_\alpha \cap U_\beta \cap U_\gamma$ reads as:

$$\Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha} = n_{\alpha\beta\gamma} \in \mathbb{Z}, \tag{2}$$

The functions $\Lambda_{\alpha\beta}$ are not uniquely determined by Eq. (1), since the collection:

$$\widetilde{\Lambda}_{\alpha\beta} = \Lambda_{\alpha\beta} + m_{\alpha\beta}, \tag{3}$$

yields the same transition functions for any $m_{\alpha\beta} \in \mathbb{Z}$. In fact the transition functions are themselves ambiguous since $\widetilde{g}_{\alpha\beta} = h_\alpha^{-1} g_{\alpha\beta} h_\beta$ generate an equivalent principal bundle over M. The collection of $U(1)$-valued functions h_α can on their turn be written as:

$$h_\alpha = e^{2i\pi\xi_\alpha}, \tag{4}$$

thus inducing the change:

$$\Lambda_{\alpha\beta} \to \Lambda_{\alpha\beta} + \xi_\beta - \xi_\alpha. \tag{5}$$

Finally, the collection of integers $n_{\alpha\beta\gamma}$ defined in the intersections $U_\alpha \cap U_\beta \cap U_\gamma$ by Eq. (2) tautologically satisfies:

$$n_{\beta\gamma\rho} - n_{\alpha\gamma\rho} + n_{\alpha\beta\rho} - n_{\alpha\beta\gamma} = 0, \tag{6}$$

in the intersections $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\rho$. Equation (6) means that the collection $(n_{\alpha\beta\gamma})$ is a Čech 2-cocycle for the good cover $\mathcal{U}$ of M (see Appendix).
A $U(1)$ connection of M can be defined as a collection of local 1-forms A_α defined in each U_α and such that:

$$A_\beta - A_\alpha = d\Lambda_{\alpha\beta}, \tag{7}$$

in the intersections $U_\alpha \cap U_\beta$. The collection of local 1-forms A_α can be obtained by pulling back a 1-form $\mathcal{A}$ on $P(M, U(1))$ with local sections s_α according to:

$$A_\alpha = s_\alpha^* \mathcal{A}, \tag{8}$$

in each U_α. The collections of local fields and integers thus generated are then collated into the following triplet:

$$\mathbf{A} = (A_\alpha, \Lambda_{\alpha\beta}, n_{\alpha\beta\gamma}), \tag{9}$$

whose *components* satisfy:

$$\begin{cases} (\delta_0 A)_{\alpha\beta} := A_\beta - A_\alpha = d_0 \Lambda_{\alpha\beta} \\ (\delta_1 \Lambda)_{\alpha\beta\gamma} := \Lambda_{\beta\gamma} - \Lambda_{\alpha\gamma} + \Lambda_{\alpha\beta} = d_{-1} n_{\alpha\beta\gamma} \\ (\delta_2 n)_{\alpha\beta\gamma\rho} := n_{\beta\gamma\rho} - n_{\alpha\gamma\rho} + n_{\alpha\beta\rho} - n_{\alpha\beta\gamma} = 0 \end{cases}, \tag{10}$$

in the appropriate intersections. We have specified the degree of the various Čech and de Rham differential operators appearing in these equations, and we have denoted by d_{-1} the extension of the de Rham differential which is nothing but the injection of numbers into functions (i.e. 0-forms).

As already noticed, a collection like (9) suffers from the following ambiguities:

$$\begin{cases} A_\alpha \to A_\alpha + d_0 \xi_\alpha \\ \Lambda_{\alpha\beta} \to \Lambda_{\alpha\beta} + \xi_\beta - \xi_\alpha - d_{-1} m_{\alpha\beta} = \Lambda_{\alpha\beta} + (\delta_0 \xi)_{\alpha\beta} - d_{-1} m_{\alpha\beta} \\ n_{\alpha\beta\gamma} \to n_{\alpha\beta\gamma} - m_{\beta\gamma} + m_{\alpha\gamma} - m_{\alpha\beta} = n_{\alpha\beta\gamma} - (\delta_1 m)_{\alpha\beta\gamma} \end{cases} . \tag{11}$$

From a physicist point of view these ambiguities are nothing but (enlarged) gauge transformations. Remarkably, Eqs. (10) and (11) can be written in a more compact way as:

$$\begin{cases} D_{(1,1)} \mathbf{A} = 0 \\ \mathbf{A} \to \mathbf{A} + D_{(0,1)} \boldsymbol{\Xi} \end{cases} , \tag{12}$$

with:

$$\begin{cases} D_{(1,1)} := (\delta_0 + 0) - (\delta_1 + d_0) + (\delta_2 + d_{-1}) \\ D_{(0,1)} := (\delta_0 + d_0) - (\delta_1 + d_{-1}) \end{cases} . \tag{13}$$

The alternating signs ensure that:

$$D_{(1,1)} \circ D_{(0,1)} = 0. \tag{14}$$

This provides a cohomological interpretation of the construction. More precisely $\mathbf{A}$ will be refereed as a Deligne-Beilinson (DB) cocycle and $D_{(0,1)} \boldsymbol{\Xi}$ as a DB coboundary. To understand the double indexation of the Deligne-Beilinson operators appearing in Eq. (13) one can look closer at the expression of $D_{(1,1)}$ where the de Rham operator d_1 which should appear in the first parenthesis has been replaced by the tautological zero operator. However, this “truncation” has not been performed on $D_{(0,1)}$. In other words, the second index in a DB operator specifies where the truncation is done in the de Rham complex (here on 1-forms) whereas the first index refers to the degree of the objects on which this DB operator acts. Accordingly, the DB degree of $\mathbf{A}$ is made of the Čech and de Rham degrees of the components forming $\mathbf{A}$, and one should rather write $\mathbf{A}^{(1,1)}$ and $\boldsymbol{\Xi}^{(0,1)}$.

Finally, taking the quotient of the set of DB (1, 1)-cocycles with the set of DB (0, 1)-coboundaries one obtains the space of (1, 1)-DB cohomology classes:

$$H_D^{(1,1)}(M, \mathbb{Z}), \tag{15}$$

just like the quotient of the space of closed p-forms with the space of exact p-forms generates the pth de Rham cohomology group of M.

In fact, as in the more usual case of the Čech-de Rham formalism, what has been defined is a cohomology group subordinated to the good cover $\mathcal{U}$. In order to get the corresponding DB cohomology group for M one has to take the inductive limit over refinements of $\mathcal{U}$ [46]. We will either assume that this limit as been taken or that we deal with DB cohomology spaces subordinate to a given good cover $\mathcal{U}$, knowing that if one respects the rules of the DB cohomology theory all the procedures and constructions are eventually independent of $\mathcal{U}$ [27] .

It is quite obvious how to generalize the previous construction to obtain DB cohomology spaces $H_D^{(p,q)}(M,\mathbb{Z})$. Nonetheless, all the DB cohomology spaces we encounter in the sequel will be such that $p = q$. Hence, to simplify notations we replace (p, p) by p in all the DB terminology since no confusion will be possible. Hence, a DB p-cocycle is a $(p + 2)$-tuple:

$$\boldsymbol{\omega} = (\omega^{(0,p)}_{\alpha_0}, \omega^{(1,p-1)}_{\alpha_0\alpha_1}, \dots, \omega^{(p,0)}_{\alpha_0,\dots,\alpha_p}, n^{(p+1,-1)}_{\alpha_0,\dots,\alpha_{p+1}}), \tag{16}$$

where $\omega^{(k,p-k)}$ is a collection of $(p-k)$-forms defined in the intersections of degree k of elements of $\mathcal{U}$ and $n^{(p+1,-1)}$ is an integral Čech $(p + 1)$-cocycle, which is annihilated by the DB differential operator:

$$D_p := (\delta_0 + 0) - (\delta_1 + d_{p-1}) + \cdots + (-1)^{p+1}(\delta_{p+1} + d_{-1}). \tag{17}$$

Coboundaries are defined with respect to the standard Čech-de Rham operator $\sum_{k=0}^{p}(-1)^k(\delta_k + d_{p-1-k})$. The DB class of a DB cocycle $\boldsymbol{\omega}$ will be denoted $\bar{\boldsymbol{\omega}}$ and the corresponding DB group $H_D^p(M,\mathbb{Z})$.

We made an unconventional choice in the degree of the DB classes with respect to the original literature where the DB cohomology groups are increased by one degree. In fact, the de Rham complex defining the DB cohomology was originally truncated at $\Omega^{p-1}(M)$ in order to define the pth DB cohomology group, and not at $\Omega^p(M)$ as we have chosen here. Hence what we call here a DB class of degree p is described as a DB class of degree $p+1$ in this original formalism. In particular, with our convention the DB degree 1 coincides with the form degree of the local representative of a connection, whereas this connection would be an object of DB degree 2 in the original convention, which is the form degree of the curvature of the connection.

The next step is to try to determine more precisely the DB cohomology groups.

2.1.2 Canonical Exact Sequences, Affine Structure and Torsion Origins

Let us solve Eqs. (10) and (11) (or equivalently (12)) and then extrapolate to the general case. Let $\mathbf{A}$ be a DB 1-cocycle written as in Eq. (9). Our aim is to identify the DB 1-cocycles which are not equivalent to $\mathbf{A}$. A first way to do this is to notice that the integral Čech 2-cocycle n which defines the last components of $\mathbf{A}$, determines an integral Čech cohomology class $[n]$ once equivalence (11) is taken into account. Thus the DB class $\bar{\mathbf{A}}$ determines a Čech cohomology class $[n]$, yielding the surjective mapping:

$$H^1_D(M,\mathbb{Z}) \xrightarrow{\epsilon} H^2(M,\mathbb{Z}), \tag{18}$$

where $H^2(M,\mathbb{Z})$ denotes the second Čech cohomology group of M.

Let us assume that the cohomology class of n has been fixed, and let $\mathbf{A} = (A_\alpha, \Lambda_{\alpha\beta}, n_{\alpha\beta\gamma})$ and $\widetilde{\mathbf{A}} = (\widetilde{A}_\alpha, \widetilde{\Lambda}_{\alpha\beta}, n_{\alpha\beta\gamma})$ be two inequivalent $U(1)$ connections on M. Then:

$$(\delta_1(\widetilde{\Lambda} - \Lambda))_{\alpha\beta\gamma} = d_{-1}n_{\alpha\beta\gamma} - d_{-1}n_{\alpha\beta\gamma} = 0, \tag{19}$$

which implies that $\widetilde{\Lambda}_{\alpha\beta} - \Lambda_{\alpha\beta} = (\delta_0\rho)_{\alpha\beta}$. This yields:

$$(\delta_0(\widetilde{A} - A))_{\alpha\beta} = d_0(\widetilde{\Lambda} - \Lambda)_{\alpha\beta} = d(\delta_0\rho)_{\alpha\beta} = (\delta_0 d\rho)_{\alpha\beta}, \tag{20}$$

which means that

$$\widetilde{A}_\alpha - A_\alpha = d\rho_\alpha + (\delta_{-1}\omega)_\alpha \tag{21}$$

where $(\delta_{-1}\omega)_\alpha := \omega|_\alpha$ denotes the restriction to U_α of a 1-form ω on M. On the other hand, for any 1-form ω, the collection $((\delta_{-1}\omega)_\alpha, 0, 0)$ generates a DB cocycle since $\delta_0(\delta_{-1}\omega)_{\alpha\beta} = \omega|_\beta - \omega|_\alpha = 0 = d0$, $\delta_1 0 = d_{-1}0 = 0$ and $\delta_2 0 = 0$. This defines the mapping:

$$j_1(\omega) := (\omega|_\alpha, 0, 0), \tag{22}$$

which trivially goes to the DB classes yielding:

$$\Omega^1(M) \xrightarrow{j_1} H^1_D(M,\mathbb{Z}), \tag{23}$$

where $\Omega^1(M)$ denotes the space of 1-forms on M. This mapping reflects the well known fact that for a fixed principal bundle one moves among $U(1)$ connections with 1-forms.

When ω is a closed 1-form $\omega|_\alpha = d_0\zeta_\alpha$ in each U_α. The DB cocycle $(\omega|_\alpha = d_0\zeta_\alpha, 0, 0)$ is equivalent to the DB cocycle $(0, (\delta_0\zeta)_{\alpha\beta}, 0)$, and in the intersections $U_\alpha \cap U_\beta$ one has $(\delta_0\zeta)_{\alpha\beta} = \zeta_\beta - \zeta_\alpha = d_{-1}r_{\alpha\beta}$ since $d_0(\delta_0\zeta)_{\alpha\beta} = (\delta_0 d_0\zeta)_{\alpha\beta} = (\delta_0\delta_{-1}\omega)_{\alpha\beta} = 0$. Furthermore $(\delta_1 r)_{\alpha\beta\gamma} = (\delta_1\delta_0\Lambda)_{\alpha\beta\gamma} = 0$. If the real Čech 1-cocycle $r_{\alpha\beta}$ is not homologous to an integral cocycle, we cannot go any further and we end with the non trivial DB cocycle $(0, d_{-1}r_{\alpha\beta}, 0)$. However, when ω is closed and has integral periods it is always possible to find a descent for which $r_{\alpha\beta} = m_{\alpha\beta}$ where the $m_{\alpha\beta}$'s are integers. According to Eq. (11) the cocycle $(0, d_{-1}r_{\alpha\beta} = d_{-1}m_{\alpha\beta}, 0)$

is equivalent to the zero cocycle $(0, 0, -(\delta_1 m)_{\alpha\beta\gamma} = -(\delta_1 r)_{\alpha\beta\gamma}) = (0, 0, 0)$ which trivially defines the zero-connection on $M \times U(1)$. This shows that $\mathbf{A} + \boldsymbol{\omega} = (A_\alpha + (\delta_{-1}\omega)_\alpha, \Lambda_{\alpha\beta}, n_{\alpha\beta\gamma})$ is a DB cocycle equivalent to $\mathbf{A}$ whenever ω belongs to $\Omega^1_{\mathbb{Z}}(M)$, the space of closed 1-form with integral periods on M, while in any other case $\mathbf{A} + \boldsymbol{\omega}$ and $\mathbf{A}$ are inequivalent. Hence, we have to factorize $\Omega^1_{\mathbb{Z}}(M)$ out of $\Omega^1(M)$ in order to identify inequivalent DB classes.

To sum up the situation, once the Čech cohomology class $[n]$ has been fixed, a change of DB cohomology class is ensured by an element of $\Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$. This yields the first exact sequence:

$$0 \to \frac{\Omega^1(M)}{\Omega^1_{\mathbb{Z}}(M)} \xrightarrow{j_1} H^1_D(M, \mathbb{Z}) \xrightarrow{\epsilon} H^2(M, \mathbb{Z}) \to 0, \tag{24}$$

into which $H^1_D(M, \mathbb{Z})$ is canonically embedded. This sequence plays a crucial role in the $U(1)$ Chern-Simons Quantum Field Theory.

One can obtain a second exact sequence into which $H^1_D(M, \mathbb{Z})$ is canonically embedded by considering the de Rham derivative of the top component of $\mathbf{A}$, that is to say:

$$F_\alpha := dA_\alpha. \tag{25}$$

Due to the first equation of (10) the local two forms F_α satisfy:

$$F_\beta - F_\alpha := d(A_\beta - A_\alpha) = d^2\Lambda_{\alpha\beta} = 0. \tag{26}$$

in the intersections $U_\alpha \cap U_\beta$. Hence, they glue together to form a closed 2-form $\mathbf{F}$ on M: the curvature 2-form of the $U(1)$ connection $\mathbf{A}$. One can check that two $U(1)$ connections on M which do not have the same curvature are necessarily inequivalent DB cocycles. Remembering that $U(1)$ curvatures can be normalized in such a way that they have integral periods,[1] we conclude that there is a canonical surjective mapping:

$$H^1_D(M, \mathbb{Z}) \xrightarrow{cv} \Omega^2_{\mathbb{Z}}(M), \tag{27}$$

where $\Omega^2_{\mathbb{Z}}(M)$ denotes the space of closed 2-form with integral periods on M.

Let us assume that the curvature 2-form is fixed. Two $U(1)$ connections $\mathbf{A} = (A_\alpha, \Lambda_{\alpha\beta}, n_{\alpha\beta\gamma})$ and $\widetilde{\mathbf{A}} = (\widetilde{A}_\alpha, \widetilde{\Lambda}_{\alpha\beta}, \widetilde{n}_{\alpha\beta\gamma})$ with the same curvature satisfy:

$$\widetilde{A}_\alpha - A_\alpha = d\mu_\alpha, \tag{28}$$

[1] The $U(1)$ gauge fields used by physicists are related to the top component of the corresponding DB 1-cocycles according to $\mathcal{A}_\alpha = 2\pi(\hbar c/e)A_\alpha$, where e is the charge of the electron, $\hbar$ is the Planck constant and c is the speed of light; the field strength tensor is then $\mathcal{F} = 2\pi(\hbar c/e)\mathbf{F}$.

in each U_α. Hence:

$$(\delta_0(\widetilde{A} - A))_{\alpha\beta} = d(\widetilde{\Lambda} - \Lambda)_{\alpha\beta} = (\delta_0 d\mu)_{\alpha\beta} = d(\delta_0\mu)_{\alpha\beta}, \tag{29}$$

and therefore:

$$\widetilde{\Lambda}_{\alpha\beta} - \Lambda_{\alpha\beta} = (\delta_0\mu)_{\alpha\beta} + d_{-1}r_{\alpha\beta}, \tag{30}$$

in the intersections $U_\alpha \cap U_\beta$. This finally implies that:

$$(\delta_1 r)_{\alpha\beta\gamma} = \widetilde{n}_{\alpha\beta\gamma} - n_{\alpha\beta\gamma}, \tag{31}$$

which means that the Čech cochain $r_{\alpha\beta}$ is a cocycle in $\mathbb{R}/\mathbb{Z}$ which defines an element of $H^1(M, \mathbb{R}/\mathbb{Z})$, the $\mathbb{R}/\mathbb{Z}$-valued Čech cohomology group. This leads to the second exact sequence:

$$0 \to H^1(M, \mathbb{R}/\mathbb{Z}) \xrightarrow{j_2} H^1_D(M, \mathbb{Z}) \xrightarrow{cv} \Omega^2_{\mathbb{Z}}(M) \to 0. \tag{32}$$

Note that the injection $H^1(M, \mathbb{R}/\mathbb{Z}) \xrightarrow{j_2} H^1_D(M, \mathbb{Z})$ is defined, at the level of representatives, by:

$$j_2(r_{\alpha\beta}) := (0, d_{-1}r_{\alpha\beta}, 0), \tag{33}$$

for any $r_{\alpha\beta}$ defining a cohomology class in $H^1(M, \mathbb{R}/\mathbb{Z})$. We leave to the reader the task to check in detail that the two sequences (24) and (32) are truly exact.

The generalization to $H^p_D(M, \mathbb{Z})$ leads to the exact sequences:

$$\begin{aligned} &0 \to \frac{\Omega^p(M)}{\Omega^p_{\mathbb{Z}}(M)} \xrightarrow{j_1} H^p_D(M, \mathbb{Z}) \xrightarrow{\epsilon} H^{p+1}(M, \mathbb{Z}) \to 0 \\ &0 \to H^p(M, \mathbb{R}/\mathbb{Z}) \xrightarrow{j_2} H^p_D(M, \mathbb{Z}) \xrightarrow{cv} \Omega^{p+1}_{\mathbb{Z}}(M) \to 0. \end{aligned} \tag{34}$$

Elements of $H^p_D(M, \mathbb{Z})$ will be refereed as p-connections of M, but 1-connections will be simply called connections, as usual.

One can note that:

$$H^m_D(M, \mathbb{Z}) \cong \frac{\Omega^m(M)}{\Omega^m_{\mathbb{Z}}(M)} \cong H^m(M, \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z} \simeq S^1. \tag{35}$$

There is a very useful interpretation of the exact sequences (34): with respect to the first exact sequence $H^p_D(M, \mathbb{Z})$ is a fiber space over $H^{p+1}(M, \mathbb{Z})$ whose (affine) fibers have $\Omega^p(M)/\Omega^p_{\mathbb{Z}}(M)$ as translation group, while with respect to the second exact sequence $H^p_D(M, \mathbb{Z})$ is a fiber space over $\Omega^{p+1}_{\mathbb{Z}}(M)$ whose fibers have

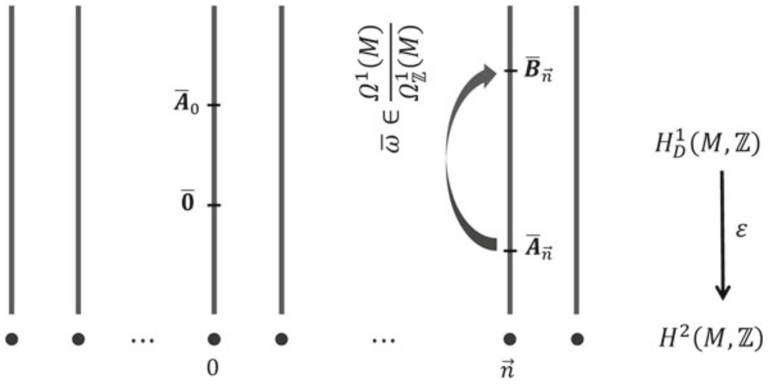

Fig. 1 Representation of the Deligne-Beilinson group $H^1_D(M,\mathbb{Z})$ as a fiber space over $H^2(M,\mathbb{Z})$

$H^p(M,\mathbb{R}/\mathbb{Z})$ as translation group. The space $\Omega^p_{\mathbb{Z}}(M)$ corresponds to what physicists call the (enlarged or full) gauge group, while $\Omega^{p+1}_{\mathbb{Z}}(M)$ defines the space of generalized field strength tensors (or $(p+1)$-curvature) (Fig. 1).

For later convenience, let us have a closer look at the case $m = 3$ and $p = 1$. First, the Universal Coefficient theorem together with Poincaré duality [46] allow to replace the base space $H^2(M) := H^2(M,\mathbb{Z})$ of $H^1_D(M,\mathbb{Z})$ with the homology space $H_1(M) := H_1(M,\mathbb{Z})$ since these two spaces are isomorphic in that case. As a finite abelian group, this later group can be decomposed as:

$$H_1(M) = F_1(M) \oplus T_1(M) \tag{36}$$

where $F_1(M)$ is the free part and $T_1(M)$ the torsion (or cyclic) part of $H_1(M)$. Finally, one can apply to $H_1(M)$ the universal decomposition theorem of finitely generated abelian groups [47], thus getting:

$$H_1(M) = \mathbb{Z}^q \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_N}, \tag{37}$$

where the p_i's are integers such that $p_i > 1$ and p_i divides p_{i+1}, and $\mathbb{Z}^q$ means q-times the direct sum of $\mathbb{Z}$ with itself. Let us recall that a cycle z on M is a torsion cycle if there exist an integer $p > 1$ such that $p.z$ is a boundary. A typical example of 3-dimensional manifold with torsion is $\mathbb{R}P^3 \simeq SO(3) \simeq SU(2)/\mathbb{Z}_2$. A larger set of 3-dimensional manifolds with torsion is provided by lens spaces $L(p;r)$ [35, 43, 48].

At this stage it is important to remark that there is no particular origin on the fibers of $H^1_D(M,\mathbb{Z})$, except for the fiber over $\mathbf{0} \in H^2(M)$ on which the DB class $\bar{\mathbf{0}}$—a representative of which is the zero connection defined by the trivial DB cocycle (0,0,0)—plays the role of a *canonical origin*. Accordingly, the translation group $\Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$ can be canonically identified with the *trivial fiber* of $H^1_D(M,\mathbb{Z})$, i.e. the fiber over $\mathbf{0} \in H^2(M)$. This is the reason why the usual Quantum Field Theory approach only deals with this fiber, missing all the others.

Actually, there are also particular origins on the fibers over the torsion elements of $H^2(M) \simeq H_1(M)$. Such a fiber will be refereed as a *torsion fiber*. Let us assume decomposition (37) for $H_1(M)$. Let κ_i be a Čech cocycle which generates the component $\mathbb{Z}_{p_i}$ of $T^2(M) \simeq T_1(M)$. This means that there exists a Čech 1-cochain ζ_i such that $p_i.\kappa_i = \delta\zeta_i$ and for any $0 \leq m_i \leq p_i - 1$ the cocycle $m_i.\kappa_i$ generates a non trivial class $m_i.\boldsymbol{\kappa}_i \in \mathbb{Z}_{p_i}$. Then, by seeing κ_i as a real Čech cochain, one can write $\kappa_i = \delta(\zeta_i/p_i)$. The collection $\mathbf{A}^0_{p_i} = (0, d_{-1}(\zeta_i/p), \kappa_i)$ is a DB cocycle since it fulfills (10). As the first component of $\mathbf{A}^0_{p_i}$ is zero, the corresponding 2-curvature is zero and hence $\mathbf{A}^0_{p_i}$ is a *flat* $U(1)$ connection of M. By construction the DB class $\bar{\mathbf{A}}^0_{p_i}$ belongs to the fiber over $\boldsymbol{\kappa}_i \in T^2(M)$ and hence can be chosen as origin on this torsion fiber. We call such an origin a *torsion origin*. One can wonder whether a torsion origin is unique on its torsion fiber. All the connections on the fiber of $\bar{\mathbf{A}}^0_{p_i}$ are of the form $\bar{\mathbf{A}}^0_{p_i} + \bar{\boldsymbol{\omega}}$ with $\bar{\boldsymbol{\omega}} \in \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$. On the other hand any other torsion origin on this fiber has to correspond to a flat connection. Hence the DB class $\bar{\mathbf{A}}^0_{p_i} + \bar{\boldsymbol{\omega}}$ is a torsion origin if and only if $\omega \in \Omega^1_0(M)$, the space of closed 1-forms of M. Consequently, the translation subgroup which allows to move among torsion origins on a given torsion fiber is the q-dimensional torus $\Omega^1_0(M)/\Omega^1_{\mathbb{Z}}(M)$ and the torsion origins are unique on their torsion fiber if and only if $F^2(M) \simeq F_1(M) = 0$. The torsion origin $\bar{\mathbf{A}}^0_{p_i} + \bar{\boldsymbol{\omega}}$ can be represented by $(\delta_{-1}\omega, d_{-1}(\zeta/p), \kappa)$ with $\omega \in \Omega^1_0(M)$, and as ω fulfills a Čech-de Rham descent (see Appendix) which generates a real Čech cocycle r, this torsion origin can equivalently be represented by $(0, d_{-1}((\zeta/p) + r), \kappa)$. When r is an integral cocycle, which happens if an only if ω has integral periods, then the latter DB cocycle is a representative of the original class $\bar{\mathbf{A}}^0_{p_i}$. Last but not least, $m_i.\bar{\mathbf{A}}^0_{p_i}$ is a torsion origin on the torsion fiber over $m_i.\boldsymbol{\kappa}_i$ for $0 \leq m_i \leq p_i - 1$, and $i = 1, \ldots, N$.

On the *free fibers* of $H^1_D(M, \mathbb{Z})$, i.e. the fibers over $F^2(M)$, not only there is no canonical origin, but there are also no particular ones unlike the torsion case. Nevertheless, there are free flat connections which are nothing but connections defined by closed 1-forms of M. The corresponding DB classes necessarily belong to the trivial fiber. One then deduces that the torus $\Omega^1_0(M)/\Omega^1_{\mathbb{Z}}(M)$ canonically identifies with the space of these free flat elements of $H^1_D(M, \mathbb{Z})$. Such a class can always be represented by $(\delta_{-1}\omega, 0, 0)$ or equivalently—after a Čech-de Rham descent—by $(0, d_{-1}r, 0)$, where r is a real non integral Čech cocycle.

In the sequel generic origins of $H^1_D(M, \mathbb{Z})$ will be denoted by $\bar{\mathbf{A}}^0_{\mathbf{n}}$, origins on free fibers by $\bar{\mathbf{A}}^0_{\mathbf{a}}$ and origins on torsion fibers by $\bar{\mathbf{A}}^0_{\boldsymbol{\kappa}}$. We keep the notation $\bar{\mathbf{A}}^0_{p_i}$ for a torsion origin over the torsion class generating $\mathbb{Z}_{p_i}$ in the decomposition (37) of $H_1(M) \simeq H^2(M)$, origins on the other classes being chosen as $m_i.\bar{\mathbf{A}}^0_{p_i}$.

2.2 Fundamental Properties and Pontryagin Duality

The DB cohomology spaces have a great set of properties that we would like to exhibit now. these properties play a fundamental role in the use of DB cohomology within the Quantum Field Theory framework.

Lemma 2 1) *There is a canonical graded pairing between DB cohomology spaces*:

$$\star : H_D^p(M,\mathbb{Z}) \times H_D^q(M,\mathbb{Z}) \to H_D^{p+q+1}(M,\mathbb{Z}), \tag{38}$$

called the DB product.

2) *There is a natural pairing between p-cycles of M and $H_D^p(M,\mathbb{Z})$*:

$$\int : Z_p(M) \times H_D^p(M,\mathbb{Z}) \to \mathbb{R}/\mathbb{Z},$$

which defines integration of DB classes over cycles.

3) *The Pontryagin dual of $H_D^p(M,\mathbb{Z})$ is a fiber space over $H^{m-p}(M,\mathbb{Z})$ that contains $Z_p(M)$ and into which $H_D^{m-p-1}(M,\mathbb{Z})$ is canonically injected ($m = dim M$).*

4) *Any cycle z in M can be represented by a unique distributional DB class $\bar{\eta}_z$ which satisfies*:

$$\int_z \bar{\boldsymbol{\omega}} = \int_M \bar{\boldsymbol{\omega}} \star \bar{\boldsymbol{\eta}}_z,$$

for any $\bar{\boldsymbol{\omega}} \in H_D^p(M,\mathbb{Z})$.

2.2.1 Deligne-Beilinson Product

Instead of expressing this product in the general case, let us return to the more specific one of $U(1)$ connections on a 3-dimensional manifold M which is of particular interest in this review. In that case:

$$\star : H_D^1(M,\mathbb{Z}) \times H_D^1(M,\mathbb{Z}) \to H_D^3(M,\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}, \tag{39}$$

where the DB class $\bar{\mathbf{A}} \star \bar{\mathbf{B}}$ is defined at the level of representatives by:

$$(A_\alpha \wedge dB_\alpha, \Lambda_{\alpha\beta} \wedge dB_\beta, n_{\alpha\beta\gamma} \wedge B_\gamma, n_{\alpha\beta\gamma} \wedge \Pi_{\gamma\rho}, n_{\alpha\beta\gamma} \wedge m_{\gamma\rho\sigma}), \tag{40}$$

with $\mathbf{A} = (A_\alpha, \Lambda_{\alpha\beta}, n_{\alpha\beta\gamma})$ and $\mathbf{B} = (B_\alpha, \Pi_{\alpha\beta}, m_{\alpha\beta\gamma})$. The curvature of this product is the exterior product of the curvatures of $\mathbf{A}$ and $\mathbf{B}$, whereas its last component is the cup product[2] of n and m. In the even more particular case where $\bar{\mathbf{B}} = \bar{\mathbf{A}}$, the

[2] which in Čech cohomology is the equivalent of the exterior product.

top component of the DB square $\bar{\mathbf{A}} \star \bar{\mathbf{A}}$ is (up to some normalization) nothing but the local expression used by theoretical physicists to define the $U(1)$ Chern-Simons lagrangian. This seems an excellent argument to consider on a general manifold M the $U(1)$ Chern-Simons lagrangian to be given by $\bar{\mathbf{A}} \star \bar{\mathbf{A}}$ instead of $A \wedge dA$.

The DB product naturally reduces to $\Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$, and when $\omega_0 \in \Omega^1_0(M)$ it is clear that Eq. (40) implies that:

$$\bar{\boldsymbol{\omega}} \star \bar{\boldsymbol{\omega}}_0 = 0 = \bar{\boldsymbol{\omega}}_0 \star \bar{\boldsymbol{\omega}}, \tag{41}$$

for any $\bar{\boldsymbol{\omega}} \in \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$. Property (41) will lead to the zero-modes property of the Chern-Simons functional measure, the space of zero-modes being $\Omega^1_0(M)/\Omega^1_{\mathbb{Z}}(M)$.

The generalization to all DB cohomology classes is straightforward. We also leave to the reader the care to check that:

$$\boldsymbol{\omega} \star \boldsymbol{\eta} = (-1)^{(p+1)(q+1)} \boldsymbol{\eta} \star \boldsymbol{\omega}, \tag{42}$$

for $\boldsymbol{\omega} \in H^p_D(M, \mathbb{Z})$ and $\boldsymbol{\eta} \in H^q_D(M, \mathbb{Z})$. Nonetheless, let us point out that $\boldsymbol{\omega} \star \boldsymbol{\omega} = 0$ as soon as $\boldsymbol{\omega}$ is a connection of even degree. This will imply that the higher dimensional $U(1)$ Chern-Simons action is non trivial if and only if it deals with $(2l+1)$-connections.

2.2.2 Integration Over Cycles

The second important property fulfilled by DB classes allows to exhibit the relation between Deligne-Beilinson cohomology and Cheeger-Simons Differential Characters. It concerns integration of DB classes over cycles of M. As before we are going to treat the simple case of $U(1)$ connections and extend straightforwardly the result to $H^p_D(M, \mathbb{Z})$.

Let $\mathbf{A} = (A_\alpha, \Lambda_{\alpha\beta}, n_{\alpha\beta\gamma})$ be a DB 1-cocycle and let z be a 1-cycle in M. We are going to assume that this cycle can be endowed with a *polyhedral decomposition* subordinate to the good cover $\mathcal{U}$ (see Fig. 2). This means that we can decompose z into 1-chains z_α each compactly supported in U_α and such that $z = \sum_\alpha z_\alpha$. Furthermore, each boundary bz_α satisfies $bz_\alpha = \sum_\beta (x_{\beta\alpha} - x_{\alpha\beta})$ where each 0-chain, i.e. point) $x_{\alpha\beta}$ is contained in $U_{\alpha\beta}$. Note that for a given good cover, not every 1-cycle can be decomposed accordingly. However for any 1-cycle z of M there exists a refinement of this good cover with respect to which z admits a subordinate polyhedral decomposition.

Then we define the integral of $\bar{\mathbf{A}}$ over z as the following $\mathbb{R}/\mathbb{Z}$-valued integral:

$$\int_z \bar{\mathbf{A}} \stackrel{\mathbb{Z}}{=} \sum_\alpha \int_{z_\alpha} A_\alpha - \sum_{\alpha,\beta} \int_{x_{\alpha\beta}} \Lambda_{\alpha\beta}, \tag{43}$$

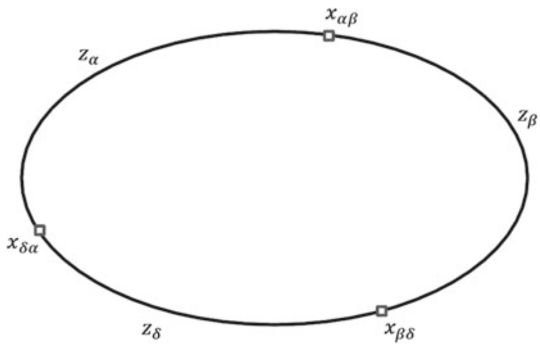

Fig. 2 Polyhedral decomposition of a cycle

where integration over 0-chains (i.e. points) means (linear) evaluation. One can easily verify that a change of representative of $\bar{\mathbf{A}}$ or of the polyhedral decomposition of z only produce integral contributions thus ensuring the $\mathbb{R}/\mathbb{Z}$-valuedness of the expression.

A straightforward generalization for a DB p-class $\bar{\boldsymbol{\omega}}$ and a p-cycle z is:

$$\int_z \bar{\boldsymbol{\omega}} \stackrel{\mathbb{Z}}{=} \sum_{\alpha_0} \int_{z^p_{\alpha_0}} \omega^{(0,p)}_{\alpha_0} - \sum_{\alpha_0,\alpha_1} \int_{z^{p-1}_{\alpha_0\alpha_1}} \omega^{(1,p-1)}_{\alpha_0\alpha_1} + \cdots + (-1)^p \sum_{\alpha_0,\ldots,\alpha_p} \int_{z^0_{\alpha_0,\ldots,\alpha_p}} \omega^{(1,p-1)}_{\alpha_0,\ldots,\alpha_p}, \tag{44}$$

for a polyhedral decomposition $(z^p_{\alpha_0}, z^{p-1}_{\alpha_0\alpha_1}, \ldots, z^0_{\alpha_0,\ldots,\alpha_p})$ of z subordinate to the good cover $\mathcal{U}$ of M.

In particular, on a 3-dimensional manifold M and for a $U(1)$ connection $\mathbf{A}$ we obtain:

$$\begin{aligned} \int_M \bar{\mathbf{A}} \star \bar{\mathbf{A}} \stackrel{\mathbb{Z}}{=} & \sum_\alpha \int_{M_\alpha} A_\alpha \wedge dA_\alpha - \sum_{\alpha,\beta} \int_{S_{\alpha\beta}} \Lambda_{\alpha\beta} \wedge dA_\beta \\ & + \sum_{\alpha,\beta,\gamma} \int_{L_{\alpha\beta\gamma}} n_{\alpha\beta\gamma} \wedge A_\gamma - \sum_{\alpha,\beta,\gamma,\rho} \int_{X_{\alpha\beta\gamma\rho}} n_{\alpha\beta\gamma} \wedge \Lambda_{\gamma\rho}, \end{aligned} \tag{45}$$

where a polyhedral decomposition $(M_\alpha, S_{\alpha\beta}, L_{\alpha\beta\gamma}, X_{\alpha\beta\gamma\rho})$ of M itself has been used.

Although one recognizes in the first term of expression (45) the usual $U(1)$ Chern-Simon action, one also sees that the other terms of this expression are required for $\int_M \bar{\mathbf{A}} \star \bar{\mathbf{A}}$ to be defined in $\mathbb{R}/\mathbb{Z}$, what is necessary if one wants the "quantum weight" $e^{2i\pi \int_M \bar{\mathbf{A}} \star \bar{\mathbf{A}}}$ to be well-defined. In the simple case of S^3, since $H^2(S^3) \simeq H_1(S^3) = 0$ the fiber space $H^1_D(S^3, \mathbb{Z})$ is made of only one fiber on which one can chose the DB class of the zero connection as origin. This natural choice of origin implies that any $U(1)$ connection $\mathbf{A}$ on S^3 admits a representative of the form $(\omega|_\alpha, 0, 0)$, where $\omega|_\alpha$ denotes the restriction to U_α of the 1-form ω of M. In other words one can identify $H^1_D(S^3, \mathbb{Z})$ with $\Omega^1(S^3)/d\Omega^0(S^3)$ and hence write $\bar{\mathbf{A}} = \bar{\mathbf{0}} + \bar{\boldsymbol{\omega}} = \bar{\boldsymbol{\omega}}$, with $\omega \in \Omega^1(S^3)$. Up to some normalization factor, expression (40) then reduces to the standard expression of the $U(1)$ Chern-Simon action on S^3 $\omega \wedge d\omega$. Nevertheless, even if one uses such a simple representative, the action is still defined modulo integers. This is because in the action we have omitted the contributions of $\bar{\mathbf{0}} \star \bar{\mathbf{0}}$ and $\bar{\mathbf{0}} \star \bar{\boldsymbol{\omega}}$ whose integrals are zero in $\mathbb{R}/\mathbb{Z}$, and not in $\mathbb{R}$. Other origins than the zero connection can be chosen, thus providing other expressions of the action. Definition (45) ensures that all these expressions differ only by integers.

To simplify notations we now use $=$ instead of $\stackrel{\mathbb{Z}}{=}$ when dealing with integration of DB classes.

2.2.3 Pontryagin Duality, Dual Exact Sequences and Cycle Map

Pontryagin duality is equivalent to Poincaré duality except that it is taken with respect to $\mathbb{R}/\mathbb{Z}$ instead of $\mathbb{R}$. Under the light of the previous subsections, it seems much more natural to consider Pontryagin duality than Poincaré ones. Accordingly, one introduces:

$$H^p_D(M, \mathbb{Z})^* := Hom(H^p_D(M, \mathbb{Z}), \mathbb{R}/\mathbb{Z}). \tag{46}$$

The spaces $H^p_D(M, \mathbb{Z})^*$ are embedded into exact sequences obtained by taking the Pontryagin dual of the sequences (34), giving:

$$\begin{aligned}
&0 \to \Omega^{p+1}_{\mathbb{Z}}(M)^* \stackrel{\bar{j}_1}{\to} H^p_D(M, \mathbb{Z})^* \stackrel{\bar{\epsilon}}{\to} H^{m-p}(M, \mathbb{Z}) \to 0 \\
&0 \to H^{m-p-1}(M, \mathbb{R}/\mathbb{Z}) \stackrel{\bar{j}_2}{\to} H^p_D(M, \mathbb{Z})^* \stackrel{\overline{cv}}{\to} \left(\frac{\Omega^p(M)}{\Omega^p_{\mathbb{Z}}(M)}\right)^* \to 0,
\end{aligned} \tag{47}$$

where $\Omega^{p+1}_{\mathbb{Z}}(M)^* := Hom(\Omega^{p+1}_{\mathbb{Z}}(M), \mathbb{R}/\mathbb{Z})$ and $(\Omega^p(M)/\Omega^p_{\mathbb{Z}}(M))^* := Hom$ $(\frac{\Omega^p(M)}{\Omega^p_{\mathbb{Z}}(M)}, \mathbb{R}/\mathbb{Z})$. One had to use the fact that $Hom(H^p(M, \mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z}) \cong H^{m-p}$ $(M, \mathbb{Z})$ and similarly that $Hom(H^{p+1}(M, \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \cong H^{m-p-1}_D(M, \mathbb{Z})$.

There are some noticeable facts concerning these two exact sequences. First, they exhibit a fiber space structure of $H^p_D(M, \mathbb{Z})^*$, with the same base space than $H^p_D(M, \mathbb{Z})$ for the first sequence and the same translation space for the second one. Furthermore with respect to integration modulo integers, one has the canonical "inclusions":

$$\begin{aligned} \frac{\Omega^{m-p-1}(M)}{\Omega_{\mathbb{Z}}^{m-p-1}(M)} &\hookrightarrow \Omega_{\mathbb{Z}}^{p+1}(M)^* \\ \Omega_{\mathbb{Z}}^{m-p}(M) &\hookrightarrow \left(\frac{\Omega^p(M)}{\Omega_{\mathbb{Z}}^p(M)}\right)^* . \end{aligned} \tag{48}$$

This yields the canonical "inclusion":

$$H_D^{m-p-1}(M,\mathbb{Z}) \hookrightarrow H_D^p(M,\mathbb{Z})^*. \tag{49}$$

Elements of $H_D^p(M,\mathbb{Z})^*$ are refereed as *generalized* p-connections, so that injection (49) means that smooth $(m-p-1)$-connections are *regular* elements of $H_D^1(M,\mathbb{Z})^*$, in the same way as $(m-p)$-forms are regular p-currents. In the case where $m = 4l+3$ and $p = 2l+1$ this injection takes the noticeable form $H_D^{2l+1}(M,\mathbb{Z}) \hookrightarrow H_D^{2l+1}(M,\mathbb{Z})^*$, allowing to identify regular $(2l+1)$-connections within generalized $(2l+1)$-connections. In particular these two fiber spaces have the same base $H^{2l+2}(M) \simeq H_{2l+1}(M)$ which fulfills the universal decomposition (37).

Similarly, integration along p-cycles yields the canonical inclusion:

$$Z_p(M) \subset H_D^p(M,\mathbb{Z})^*, \tag{50}$$

where $Z_p(M)$ is the space of p-cycles on M. This inclusion means that p-cycles of M are generalized p-connections. This provide some sort of geometrical interpretation of generalized connections. Again in the particular case $m = 4l+3$ and $p = 2l+1$, since $H^{2l+2}(M,\mathbb{Z}) \simeq H_{2l+1}(M,\mathbb{Z})$, inclusion (50) implies that a $(2l+1)$-cycle z of M belongs to the fiber of $H_D^p(M,\mathbb{Z})^*$ which stands over the homology class of z, whereas injection (49) implies that this fiber contains regular connections. One can sum up this by writing: $Z_{2l+1}(M) \oplus H_D^{2l+1}(M,\mathbb{Z}) \subset H_D^{2l+1}(M,\mathbb{Z})^*$.

One can combine the two previous results in order to construct a cycle map. It is a well known result that any p-chain on a manifold M defines a de Rham current on M in such a way that, formally one can write:

$$\int_{z^p} \omega^p := \int_M \omega^p \wedge j_{z^p}^{m-p}, \tag{51}$$

for any smooth p-form ω of M. Remarkably, there is an equivalent construction in DB cohomology. More precisely, one can canonically associate to any p-cycle z of M a DB $(m-p-1)$-class $\bar{\boldsymbol{\eta}}_z$ such that:

$$\int_z \bar{\boldsymbol{\omega}} = \int_M \bar{\boldsymbol{\omega}} \star \bar{\boldsymbol{\eta}}_z, \tag{52}$$

for any $\bar{\omega} \in H_D^p(M, \mathbb{Z})$. This is the *cycle map* for the DB cohomology. Like the de Rham current canonically associated with a p-chain is by essence distributional, the components of the DB class $\bar{\eta}_z$ of the p-cycle z are currents, except for the last component which is an integral cocycle, which represent the Čech Poincaré dual of the cycle z. The cycle map will play an important role in the $U(1)$ Chern-Simons theory. Note that the generalized curvature of the DB class of z is nothing but the de Rham current of z. In particular, if z is homologically trivial, that is to say if there is a chain C such that $z = bC$, then a representative of $\bar{\eta}_z$ is given by $(\delta_{-1} j_C^{m-p}, 0, 0)$, where j_C^{m-p} is the de Rham current of C.

3 Revisiting the U(1) Chern-Simons Theory on 3-Manifolds

This section will be fully devoted to the applications of the Deligne-Beilinson cohomology within the $U(1)$ Chern-Simons theory over a smooth closed manifold M.

Lemma 3 1) *The $U(1)$ Chern-Simons action is:*

$$S[\bar{\mathbf{A}}] = 2\pi k \int_M \bar{\mathbf{A}} \star \bar{\mathbf{A}},$$

where the coupling constant k has to be an integer.

2) *The $U(1)$ Chern-Simons action defines on $H_D^1(M, \mathbb{Z})$ the quadratic functional measure:*

$$d\mu(\bar{\mathbf{A}}) = d\bar{\mathbf{A}} \times e^{iS[\bar{\mathbf{A}}]},$$

which has zero-mode property.

3) *The fundamental observables of the theory identify with Wilson lines which are also $U(1)$ holonomies:*

$$W(\bar{\mathbf{A}}, L) = \exp\left\{2i\pi \int_L \bar{\mathbf{A}}\right\} = \exp\left\{2i\pi \int_M \bar{\mathbf{A}} \star \bar{\eta}_L\right\},$$

for any link L. The charges of the knot components of L have to be quantized.

3.1 The U(1) Chern-Simons Actions, the Coupling Constant Quantization and the Zero-Mode Property

As already explained, a good candidate for the $U(1)$ Chern-Simons lagrangian on a generic closed smooth manifold M is provided by $\bar{\mathbf{A}} \star \bar{\mathbf{A}}$ instead of $A \wedge dA$ which is not properly defined in general, when A is a $U(1)$ gauge field on M. Some important results directly stem from this choice.[3]

It is standard to introduce a coupling constant k in the theory so that the $U(1)$ Chern-Simons action reads:

$$S[\bar{\mathbf{A}}] = 2\pi k \int_M \bar{\mathbf{A}} \star \bar{\mathbf{A}}. \tag{53}$$

Since $\bar{\mathbf{A}} \star \bar{\mathbf{A}} \in H_D^m(M, \mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$, one has to impose that:

$$k \in \mathbb{Z}, \tag{54}$$

for the quantity $\exp(i S_{CS}[\bar{\mathbf{A}}])$ to be well defined. In other words, the DB cohomology framework leads to the **quantization of the coupling constant**. Note that even in the trivial case where $M = S^3$, the coupling constant is quantized if one uses the DB approach. In the non-abelian (3-dimensional) Chern-Simons theory the quantization of k is usually granted because non-abelian gauge transformations produce integers via the famous Wess-Zumino term. However, the non-abelian Chern-Simons lagrangian can also be interpreted as an element of $H_D^3(M, \mathbb{Z})$, what also straightforwardly leads to the quantization of k. In fact the usual normalization for the $U(1)$ Chern-Simons theory comes from the non-abelian one. Indeed, the $SU(n)$ Chern-Simons lagrangian is $\frac{1}{2} Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$ and hence the convention $\frac{1}{2} A \wedge dA$ is chosen. The 2π factor appearing in (53) is then correct [4] if one wants $dA \wedge dA$ to be the second Chern form of the curvature defined by A, just as the $\frac{1}{2}$ factor in the non-abelian case is correct for $\frac{1}{2} Tr(F_A \wedge F_A)$ to be the second Chern form of the non-abelian curvature F_A. From the Quantum Field theoretical point of view, which is actually standing in $\mathbb{R}^3$ and not even S^3, this is an irrelevant convention since k is not quantized: the action being $\int_{S^3} \omega \wedge d\omega$, gauge transformations produce zero contributions so that there is no reason to quantize the coupling constant k and then k or $k/2$ are allowed coupling constant. This becomes untrue in the DB approach.

In order to define a Quantum Field Theory within the Feynman path integration framework, we consider the $U(1)$ Chern-Simons functional quadratic measure:

$$d\mu(\bar{\mathbf{A}}) = d\bar{\mathbf{A}} \times e^{iS[\bar{\mathbf{A}}]}, \tag{55}$$

[3] The Chern-Simons action can be generalized to $(4l + 3)$-dimensional manifolds as it is the only dimension where the DB square $\bar{\mathbf{A}} \star \bar{\mathbf{A}}$ is not zero, $\bar{\mathbf{A}}$ being a $(2l + 1)$-connection.

[4] Strictly speaking there is also a factor $(e/2\pi\hbar c)^2$ in front of the abelian and non-abelian lagrangians for the reason explained in the first footnote.

where $d\bar{\mathbf{A}}$ is the purely formal Lebesgue measure on some infinite dimensional space of fields, as usual in Quantum Field Theory. We do not want to discuss here the existence of $d\bar{\mathbf{A}}$ or the precise meaning of $d\mu(\bar{\mathbf{A}})$. We will only assume that there exists on $H^1_D(M,\mathbb{Z})$, or on its Pontryagin dual $H^1_D(M,\mathbb{Z})^*$, a functional measure $d\mu(\bar{\mathbf{A}})$ such that it satisfies the Cameron-Martin like property:

$$d\mu(\bar{\mathbf{A}}+\bar{\boldsymbol{\omega}}) = \exp\left(2i\pi k \int\limits_M (2\bar{\mathbf{A}}\star\bar{\boldsymbol{\omega}}+\bar{\boldsymbol{\omega}}\star\bar{\boldsymbol{\omega}})\right)\times d\mu(\bar{\mathbf{A}}), \tag{56}$$

for any $\bar{\boldsymbol{\omega}} \in \Omega^1(M)/\Omega^{+1}_{\mathbb{Z}}(M)$. The Cameron-Martin like property will allow to use many of the standard technics of quadratic measures.

If one decided to work with “fields” in $H^1_D(M,\mathbb{Z})^*$ one will have to deal with regularization of DB products since in this case the DB classes are mainely distributional, and in particular $\bar{\boldsymbol{\omega}} \in \Omega^2_{\mathbb{Z}}(M)^*$. This is a standard problem occurring in Quantum Field Theory.

Note that there is no metric in this game, and no need to gauge fix the action since everything is done at the level of the gauge fixed objects, that is to say the DB cohomology classes.

Before showing how the DB formalism allows to give a more precise meaning to the Chern-Simons measure (55), let us exhibit one of its most important properties: the *zero-mode* property. Let us consider a closed surface Σ in a 3-manifold M which is not homologically trivial, that is to say which doesn't bound a volume in M. Note that such a surface does not exist in S^3, but does as soon as $F_1(M) \neq 0$. The de Rham current of Σ, j_Σ, defines a distributional DB class $\bar{\mathbf{j}}_\Sigma$ with representative $(\delta_{-1}j_\Sigma, 0, 0)$. Although $\bar{\mathbf{j}}_\Sigma = 0 \in \Omega^2_{\mathbb{Z}}(M)^*$ since $\int_M \varpi \in \mathbb{Z}$ for any $\varpi \in \Omega^2_{\mathbb{Z}}$, the DB class $\overline{\mathbf{j}_\Sigma/2k}$ associated with $(\delta_{-1}j_\Sigma/2k, 0, 0)$ is not trivial.[5] Applying, formally, Eq. (56) one obtains, for any integer m:

$$d\mu(\bar{\mathbf{A}}+m(\overline{\frac{\mathbf{j}_\Sigma}{2k}})) = \exp\left(2i\pi \int\limits_M 2mk(\bar{\mathbf{A}}\star\overline{\frac{\mathbf{j}_\Sigma}{2k}}) + m^2k(\overline{\frac{\mathbf{j}_\Sigma}{2k}}\star\overline{\frac{\mathbf{j}_\Sigma}{2k}})\right)\times d\mu(\bar{\mathbf{A}}). \tag{57}$$

The first integral in the exponential is zero because $2k\overline{(\mathbf{j}_\Sigma/2k)} = \bar{\mathbf{j}}_\Sigma = \bar{\mathbf{0}}$ as previously noticed, and by definition of the integral of DB classes:

$$k\int\limits_M \overline{\frac{\mathbf{j}_\Sigma}{2k}}\star\overline{\frac{\mathbf{j}_\Sigma}{2k}} = \frac{1}{4k}\int\limits_M j_\Sigma \wedge dj_\Sigma = 0, \tag{58}$$

since $dj_\Sigma = 0$. As an example, let us consider $S^1 \times S^2$ represented as two nested spheres S^2 whose points which face each other are identified, as depicted in Fig. 3. On this figure γ_0 is a 1-cycle generating $H_1(S^1\times S^2) \cong \mathbb{Z}$ and Σ_0 is a closed

[5] The first component of this DB cocycle defines a closed 1-current which doesn't have integral periods, these periods being defined as intersections with $\Sigma/2k$.

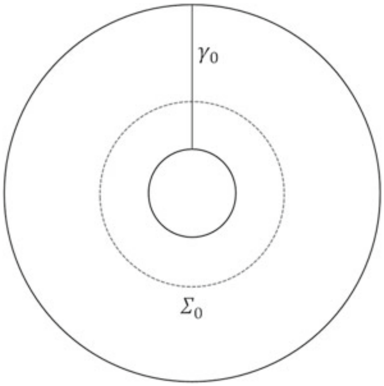

Fig. 3 The closed surface Σ_0 generates $H_2(S^1 \times S^2) \cong \mathbb{Z}$ and thus defines a zero-mode of the $U(1)$ Chern-Simons functional measure on $S^1 \times S^2$. As for the cycle γ_0, it generates $H_1(S^1 \times S^2) \cong \mathbb{Z}$

surface which generates $H_2(S^1 \times S^2) \cong \mathbb{Z}$ as well as the zero modes of the $U(1)$ Chern-Simons functional measure of $S^1 \times S^2$.

One deduces that for any integer m:

$$d\mu(\bar{\mathbf{A}} + m\frac{\overline{\mathbf{j}_\Sigma}}{2k}) = d\mu(\bar{\mathbf{A}}). \tag{59}$$

This is the so-called *zero-mode property* of the Chern-Simons measure on M [28]. It means that the Chern-Simons measure on M has a residual gauge invariance based on the free homology of M, even if the usual gauge invariance represented by $\Omega^1_{\mathbb{Z}}(M)$ has been washed out when we decided to work with DB classes, and not DB cocycles as theoretical physicists usually do.

Note that we have ignored the problem which may arise when $\bar{\mathbf{A}}$ is itself distributional. This is related to the regularisation issue already mentioned when dealing with $H^1_D(M, \mathbb{Z})^*$ instead of $H^1_D(M, \mathbb{Z})$. We leave this question aside for the moment.

The generalization to $(4l + 3)$-dimensional manifolds is straightforward [30].

In the non-abelian case—let say for $SU(n)$—the Chern-Simons action on a 3-dimensional manifold M takes, up to a $2i\pi$ factor, the form $\frac{k}{8\pi^2} Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$ where A is a $SU(n)$ connection on M. The action has such a "simple" expression because any $SU(n)$ principal bundle over a 3-dimensional manifold M is isomorphic to $M \times SU(n)$. Hence one can pick a global section up in order to pullback on M the $SU(n)$ connection $\mathcal{A}$ initially defined on $M \times SU(n)$, thus obtaining a globally defined 1-form A on M. A gauge transformation on A simply reflects a change in the global section used to perform the pull-back. The corresponding Chern-Simons action can also be seen as a 3-form on M whose integral over M is well-defined. This action changes by the Wess-Zumino 3-form $\frac{k}{24\pi^2} Tr[(g^{-1} \wedge dg]^3$ under a gauge transformation $g : M \to SU(n)$. Moreover this 3-form is tautologically closed and has integral periods. Therefore the quantum measure defined by the $SU(n)$ Chern-Simons action is gauge invariant. This also means that the $SU(n)$ Chern-Simons lagrangian can be interpreted as an element of $\Omega^3(M)/\Omega^3_{\mathbb{Z}}(M) \simeq H^3_D(M,\mathbb{Z})$. Hence, the coupling constant k in the non-abelian Chern-Simons theory is quantized for exactly the same reason as in the abelian case: the lagrangian is a DB class.

3.2 The U(1) Chern-Simons Observables and the Charges Quantization

In the Chern-Simons theory (whether it is abelian or non-abelian), observables are chosen to be Wilson lines, that is to say, in the abelian case discussed in this review, $U(1)$ holonomies of M. From our previous discussion concerning the fundamental properties of the DB classes, we know that the holonomy of a connection $\mathbf{A}$ defines the holonomy of the DB class $\bar{\mathbf{A}}$, according to:

$$W(\bar{\mathbf{A}}, z) := \exp\left\{2i\pi \int_z \bar{\mathbf{A}}\right\}, \tag{60}$$

where the integration over 1-cycles has been defined in Eq. (43). Note that no path ordering is required unlike the non-abelian case. Using the cycle map property expressed by Eq. (50), one can write this Wilson line as:

$$W(\bar{\mathbf{A}}, z) := \exp\left\{2i\pi \int_M \bar{\mathbf{A}} \star \bar{\eta}_z\right\}, \tag{61}$$

where $\bar{\eta}_z$ denotes the canonical DB class associated with z. Since we only deal with integral cycles, if z can be decomposed has $z = q.z_0$, by linearity of the integral one deduces that:

$$W(\bar{\mathbf{A}}, q.z_0) := \exp\left\{2i\pi q \int_{z_0} \bar{\mathbf{A}}\right\}, \tag{62}$$

and that q has to be an integer for this expression to be well-defined. A 1-cycle which generates one of the component of the decomposition (37) of $H_1(M)$ will be called a *fundamental cycle*. A *link* L on M is then a formal (i.e. homological) combination:

$$L = \sum_{i=1}^{N} q^i z_i, \tag{63}$$

where each cycle z_i is a fundamental cycle on M. By construction:

$$q^i \in \mathbb{Z}. \tag{64}$$

In other words, the **charges are quantized**. From a physicists point of view the fundamental charge would be the one of the electron, and since this charge has been absorbed into the definition of a connection, quantization of charges reduces to integrality of charges. Note that charges are quantized for exactly the same reason that the coupling constant is.

3.3 Fine Structure of the Functional Measure

Before looking how to use what we have just done let us see how the fiber structure of the Deligne-Beilinson cohomology spaces allows to give a more precise meaning to the $U(1)$ the Chern-Simons functional measure (55). Since the base space of $H^1_D(M,\mathbb{Z})$ is discrete the measure $d\mu(\bar{\mathbf{A}})$ actually reads

$$d\mu(\bar{\mathbf{A}}) = \sum_{\mathbf{n}\in H^2(M)} d\mu(\bar{\mathbf{A}}_{\mathbf{n}}), \tag{65}$$

where $\bar{\mathbf{A}}_{\mathbf{n}}$ denotes a generic element of the fiber over $\mathbf{n} \in H^2(M)$, and $d\mu(\bar{\mathbf{A}}_{\mathbf{n}})$ is the Chern-Simons functional measure on this fiber. The fibers being affine, one can pick an origin up on each of them, $\bar{\mathbf{A}}^0_{\mathbf{n}}$, and hence write:

$$d\mu(\bar{\mathbf{A}}_{\mathbf{n}}) = d\mu(\bar{\mathbf{A}}^0_{\mathbf{n}} + \bar{\boldsymbol{\omega}}), \tag{66}$$

with $\bar{\boldsymbol{\omega}}$ a generic element of $\Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$. Choosing an origin on each fiber of $H^1_D(M,\mathbb{Z})$ is nothing but a way to define a (discrete) section on this fiber space. Note that there is no canonical origin on the fibers of $H^1_D(M,\mathbb{Z})$ except for the fiber over $\mathbf{0} \in H^2(M)$ which canonically contains (the DB class of) the zero connection as already mentioned.

The Cameron-Martin type property (56) then allows to write:

$$d\mu(\bar{\mathbf{A}}_{\mathbf{n}}) = \exp\left(2i\pi k \int_M (2\bar{\mathbf{A}}^0_{\mathbf{n}} \star \bar{\boldsymbol{\omega}} + \bar{\boldsymbol{\omega}} \star \bar{\boldsymbol{\omega}})\right) \times D\bar{\boldsymbol{\omega}}, \tag{67}$$

with $D\bar{\boldsymbol{\omega}}$ the formal Lebesgue measure on $\Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$. Putting all this together one obtains the following finer expression for the $U(1)$ Chern-Simons functional measure:

$$d\mu(\bar{\mathbf{A}}) = \sum_{\mathbf{n}\in H^2(M)} \exp\left(2i\pi k \int_M (2\bar{\mathbf{A}}^0_{\mathbf{n}} \star \bar{\boldsymbol{\omega}} + \bar{\boldsymbol{\omega}} \star \bar{\boldsymbol{\omega}})\right) \times D\bar{\boldsymbol{\omega}}. \tag{68}$$

At this stage one can use the universal decomposition (36), thus writing $\bar{\mathbf{A}}^0_{\mathbf{n}} = \bar{\mathbf{A}}^0_{\mathbf{a}} + \bar{\mathbf{A}}^0_{\boldsymbol{\kappa}}$ and then getting a new expression for the functional measure:

$$d\mu(\bar{\mathbf{A}}) = \sum_{\mathbf{a}\in F^2(M)} \sum_{\boldsymbol{\kappa}\in T^2(M)} e^{2i\pi k \int_M 2(\bar{\mathbf{A}}^0_{\mathbf{a}} + \bar{\mathbf{A}}^0_{\boldsymbol{\kappa}})\star\bar{\boldsymbol{\omega}} + \bar{\boldsymbol{\omega}}\star\bar{\boldsymbol{\omega}}} \times D\bar{\boldsymbol{\omega}}, \tag{69}$$

where $F^2(M)$ and $T^2(M)$ can themselves be decomposed according to (37) if necessary.

Thanks to property (41) it seems natural to consider the following decomposition:

$$\frac{\Omega^1(M)}{\Omega^1_{\mathbb{Z}}(M)} = \left(\frac{\Omega^1(M)}{\Omega^1_0(M)}\right) \times \left(\frac{\Omega^1_0(M)}{\Omega^1_{\mathbb{Z}}(M)}\right), \tag{70}$$

where the space of zero-modes $\Omega^1_0(M)/\Omega^1_{\mathbb{Z}}(M)$ is a isomorphic to the torus $(\mathbb{R}/\mathbb{Z})^q$, $q = dim(F_1(M))$. This decomposition is a priori not canonical. Accordingly, the measure $D\bar{\boldsymbol{\omega}}$ decomposes as:

$$D\bar{\boldsymbol{\omega}} = D\bar{\boldsymbol{\omega}}_\perp \times d^q\boldsymbol{\theta}, \tag{71}$$

where $D\bar{\boldsymbol{\omega}}_\perp$ is the formal Lebesgue measure on $\Omega^1(M)/\Omega^1_0(M)$, and $d^q\boldsymbol{\theta}$ the canonical measure on $(\mathbb{R}/\mathbb{Z})^q$. This latter measure thus fulfills

$$\int_{(\mathbb{R}/\mathbb{Z})^q} d^q\boldsymbol{\theta} = 1. \tag{72}$$

Finally, choosing the canonical origin $\bar{\mathbf{0}}$ on the trivial fiber and using property (41) one concludes that on this fiber the Chern-Simons measure can be written:

$$d\mu(\bar{\mathbf{0}}) = e^{iS[\bar{\boldsymbol{\omega}}]}\, D\bar{\boldsymbol{\omega}} = \left(e^{iS[\bar{\boldsymbol{\omega}}_\perp]}\, D\bar{\boldsymbol{\omega}}_0\right) \times d^q\boldsymbol{\theta}. \tag{73}$$

This holds true on the trivial fiber of $H^1_D(M,\mathbb{Z})$ because this fiber is canonically identified with $\Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$ through the use of the canonical origin. On non-trivial fibers, one cannot write $d\mu(\bar{\mathbf{A}}_{\mathbf{n}})$ in a such simple way. This will be discussed in the next section.

Despite the regularization issues, it is interesting to consider the Chern-Simons functional measure on the Pontryagin dual $H^1_D(M,\mathbb{Z})^*$ because its contains 1-cycles of M (cf. inclusion (50)). Then, one can use 1-cycles as origin of the fibers of $H^1_D(M,\mathbb{Z})^*$. The fiber on which a cycle z has been chosen as origin is necessarily the fiber over the homology class of z. But, there is no canonical or even particular 1-cycle on M, except the zero one. Nevertheless, if one considers a fundamental cycle z which generates one of the free or torsion component of $H_1(M)$ then $m.z$ can be chosen as origin on the fiber over the class of $m\,.\,z$. This can simplify the computations. On torsion fibers however one always prefers to use torsion origins.

4 Abelian Links Invariants, Manifold Invariants and Surgery Discussion

This section will be devoted to most of the computations that we can made using the Deligne-Beilinson formalism. We will first show how to compute the partition function of a manifold M and then how to compute expectation values of Wilson lines thus yielding link invariants for M. As we will see everything is performed in M itself and not with the help of a Dehn surgery of M in S^3 as it is usual to proceed.

Lemma 4 1) *The only required regularization in the $U(1)$ Chern-Simons theory is the zero-regularization, a.k.a. regularization by framing:*

$$\bar{\boldsymbol{\eta}}_i \star \bar{\boldsymbol{\eta}}_j = 0,$$

where $\bar{\boldsymbol{\eta}}_i$ is the DB class of some cycle in M.

2) *The partition function of the $U(1)$ Chern-Simons takes the form:*

$$Z_k(M) = \sum_{\kappa^1=0}^{p_1-1} \cdots \sum_{\kappa^N=0}^{p_N-1} e^{-2i\pi k \sum_{i,j} \kappa^i \mathbf{Q}_{ij} \kappa^j},$$

using a universal decomposition of $H^2(M)$ for which $\mathbf{Q}$ is the linking form of M.

3) *The manifold invariant defined by the partition function can be written in the Reshetikhin-Turaev form:*

$$Z_k(M) = (k)^{-m/2} \sum_{q^1=0}^{2k-1} \cdots \sum_{q^{|\mathcal{L}|}=0}^{2k-1} \exp\left\{ -\frac{2i\pi}{4k} \sum_{i,j=1}^{|\mathcal{L}|} q^i \mathcal{L}_{ij}\, q^j \right\},$$

where $|\mathcal{L}|$ is the number of fundamental cycles of an algebraic link $\mathcal{L}_M$ in S^3 which is not necessarily a surgery link of M.

4) *The expectation value of a Wilson line on a smooth closed 3-dimensional manifold M can be computed without resorting to a Dehn surgery of M in S^3. In particular the expectation value of the Wilson line of a free and non homologically trivial link is zero. The expectation value of a link on M can be written in a Reshetikhin-Turaev form.*

4.1 Partition Functions, Zero-Regularization and Manifold Invariants

One considers the *normalized* $U(1)$ *Chern-Simons partition function* for the manifold M defined as:

$$Z_k(M) := \sum_{\mathbf{n}\in H_1(M)} \frac{\int D\bar{\boldsymbol{\omega}}\, e^{iS[\bar{\mathbf{A}}_{\mathbf{n}}+\bar{\boldsymbol{\omega}}]}}{\int D\bar{\boldsymbol{\omega}}\, e^{iS[\bar{\boldsymbol{\omega}}]}}, \tag{74}$$

Using decomposition (36) the partition function takes the more specific form:

$$Z_k(M) = \sum_{\mathbf{a}\in F_1(M)} \sum_{\kappa\in T_1(M)} \frac{\int D\bar{\boldsymbol{\omega}}\, e^{iS[\bar{\mathbf{A}}_{\mathbf{a}}+\bar{\mathbf{A}}_{\kappa}+\bar{\boldsymbol{\omega}}]}}{\int D\bar{\boldsymbol{\omega}}\, e^{iS[\bar{\boldsymbol{\omega}}]}}. \tag{75}$$

From now on we furthermore assume that $H^2(M) \simeq H_1(M)$ and hence $F_1(M)$ and $T_1(M)$ fulfill decomposition (37).

Firstly, when $H^2(M) = 0 \simeq H_1(M)$, the space $H_D^1(M,\mathbb{Z})$ reduces to the trivial fiber and the partition function simplifies to:

$$Z_k(M) = \frac{\int D\bar{\boldsymbol{\omega}}\, e^{iS[\bar{\boldsymbol{\omega}}]}}{\int D\bar{\boldsymbol{\omega}}\, e^{iS[\bar{\boldsymbol{\omega}}]}} = 1. \tag{76}$$

This is what happens for S^3 and more generally for any homology 3-sphere. Note that in this case one can straightforwardly replace $H_D^1(M,\mathbb{Z})$ by $H_D^1(M,\mathbb{Z})^*$.

Let us now assume that $H^2(M) \simeq H_1(M) = \mathbb{Z}^q$. One considers $H_D^1(M,\mathbb{Z})^*$ as the quantum configuration space instead of $H_D^1(M,\mathbb{Z})$. These two spaces only have free fibers. As already mentioned, some 1-cycles of M can be chosen as origins of the fiber: if z_i is a cycle generating the ith component $\mathbb{Z}$ of $H_1(M) = F_1(M)$, and if one denotes $\bar{\boldsymbol{\eta}}_i$ the corresponding DB class, then one chooses the DB class $m.\bar{\boldsymbol{\eta}}_i$ as origin on the fiber over the class of $m.z_i$. Thus, $Z_k(M)$ takes the form:

$$Z_k(M) = \sum_{a^1\in\mathbb{Z}} \cdots \sum_{a^q\in\mathbb{Z}} \frac{\int D\bar{\boldsymbol{\omega}}\, e^{2i\pi k \int_M \sum_{i,j} a^i a^j \bar{\boldsymbol{\eta}}_i \star \bar{\boldsymbol{\eta}}_j + 2\sum_i a^i \bar{\boldsymbol{\eta}}_i \star \bar{\boldsymbol{\omega}} + \bar{\boldsymbol{\omega}} \star \bar{\boldsymbol{\omega}}}}{\int D\bar{\boldsymbol{\omega}} e^{iS[\bar{\boldsymbol{\omega}}]}}. \tag{77}$$

The products $\bar{\boldsymbol{\eta}}_i \star \bar{\boldsymbol{\eta}}_i$ are ill-defined since they involve products of distributions. In order to give these DB products a meaning, one decides to use the *zero-regularization* procedure [28–31] defined by:

$$\bar{\eta}_i \star \bar{\eta}_j = 0, \tag{78}$$

for the DB class of any 1-cycle of M, whether it is free or torsion. The DB products $\bar{\boldsymbol{\eta}}_i \star \bar{\boldsymbol{\omega}}$ appearing in (77) do not require so much care as we will see.

Under zero-regularization, Eq. (77) simplifies to:

$$Z_k(M) = \sum_{a^1 \in \mathbb{Z}} \cdots \sum_{a^q \in \mathbb{Z}} \frac{\int D\bar{\boldsymbol{\omega}}\, e^{2i\pi k \int_M 2\sum_i a^i \bar{\boldsymbol{\eta}}_i \star \bar{\boldsymbol{\omega}} + \bar{\boldsymbol{\omega}} \star \bar{\boldsymbol{\omega}}}}{\int D\bar{\boldsymbol{\omega}}\, e^{iS[\bar{\boldsymbol{\omega}}]}}. \tag{79}$$

By injecting decomposition (73) into the denominator of this expression and then integrating over $\boldsymbol{\theta}$, the denominator of (79) reduces to the integral:

$$\int D\bar{\boldsymbol{\omega}}_\perp\, e^{iS[\bar{\boldsymbol{\omega}}_\perp]}, \tag{80}$$

performed on $\Omega^1(M)/\Omega^1_0(M)$ or its distributional equivalent.

The numerator on its turn can be rewritten according to:

$$\int D\bar{\boldsymbol{\omega}} \prod_{i=1}^{q} \left(\sum_{a_i \in \mathbb{Z}} e^{2i\pi(2ka^i) \int\limits_M \bar{\boldsymbol{\eta}}_i \star \bar{\boldsymbol{\omega}}} \right) e^{2i\pi k \int\limits_M \bar{\boldsymbol{\omega}} \star \bar{\boldsymbol{\omega}}}. \tag{81}$$

According to (70) one can decompose $\bar{\boldsymbol{\omega}}$ as:

$$\bar{\boldsymbol{\omega}} = \bar{\boldsymbol{\omega}}_\perp + \sum_i \overline{\theta^i \boldsymbol{\rho}_i}, \tag{82}$$

with $\theta^i \in \mathbb{R}/\mathbb{Z}$ and $\rho_i \in \Omega^1_{\mathbb{Z}}(M)$ such that:

$$\int\limits_{z_i} \rho_j = \delta_{ij}. \tag{83}$$

Such normalised 1-forms always exists since $F^2(M) \simeq F_1(M)$. Using decomposition (82) into (81) leads to

$$\left(\int D\bar{\boldsymbol{\omega}}_\perp\, e^{2i\pi k \int\limits_M (\bar{\boldsymbol{\omega}}_\perp \star \bar{\boldsymbol{\omega}}_\perp + 2\sum_i a^i \bar{\boldsymbol{\eta}}_i \star \bar{\boldsymbol{\omega}}_\perp)} \right) \times \left(\prod_{i=1}^{q} \int\limits_{\mathbb{R}/\mathbb{Z}} d\theta^i\, e^{2i\pi k(2a^i\theta^j) \int\limits_{z_i} \rho_j} \right), \tag{84}$$

where property (41) has been used. Due to constraint (83), each integral forming the second factor of this expression vanishes except when $a^i = 0$. When $a^i = 0$ for all i this factor reduces to 1, and hence expression (84) reduces to:

$$\int D\bar{\boldsymbol{\omega}}_\perp\, e^{2i\pi k \int\limits_M (\bar{\boldsymbol{\omega}}_\perp \star \bar{\boldsymbol{\omega}}_\perp + 2\sum_i a^i \bar{\boldsymbol{\eta}}_i \star \bar{\boldsymbol{\omega}}_\perp)} \delta_{\mathbf{a},\mathbf{0}}, \tag{85}$$

with $\mathbf{a} = (a^1, \ldots, a^q) \in \mathbb{Z}^q$. Taking into account the infinite sums over a^i, one obtains:

$$\begin{aligned} Z_k(M) &= \sum_{a^1\in\mathbb{Z}} \cdots \sum_{a^q\in\mathbb{Z}} \frac{\int D\bar{\boldsymbol{\omega}}_0\, e^{2i\pi k \int_M (\bar{\omega}_0\star\bar{\omega}_0 + 2\sum_i a^i \bar{\boldsymbol{\eta}}_i \star \bar{\omega}_0)} \delta_{\mathbf{a},\mathbf{0}}}{\int D\bar{\boldsymbol{\omega}}_0\, e^{iS[\bar{\omega}_0]}} \\ &= \frac{\int D\bar{\boldsymbol{\omega}}_0\, e^{2i\pi k \int_M \bar{\omega}_0\star\bar{\omega}_0}}{\int D\bar{\boldsymbol{\omega}}_0\, e^{iS[\bar{\omega}_0]}} \\ &= 1. \end{aligned} \tag{86}$$

In the case where $H^2(M) \simeq H_1(M) = \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_N}$, instead of using cycles as origins on the fibers of $H^1_D(M)$ one can use torsion origins introduced at the end of Sect. 2.1.2 and whose representatives are:

$$\mathbf{A}^0_{p_i} = (0, d_{-1}\frac{\zeta^{(i)}_{\alpha\beta}}{p_i}, \kappa^{(i)}_{\alpha\beta\gamma}), \tag{87}$$

with $p_i.\kappa^{(i)} = \delta\zeta^{(i)}$, $\kappa^{(i)}$ generating $\mathbb{Z}_{p_i}$. Since here $F_1(M) = 0$, the torsion origins defined by (87) are unique on torsion fibers. Using Eq. (45), one deduces on the one hand that:

$$\int_M \mathbf{A}^0_{p_i} \star \bar{\boldsymbol{\omega}} = 0. \tag{88}$$

for any $\bar{\boldsymbol{\omega}} \in \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$, and on the second hand that:

$$\int_M \mathbf{A}^0_{p_i} \star \mathbf{A}^0_{p_j} = -\sum_{\alpha,\beta,\gamma,\rho} \int_{X_{\alpha\beta\gamma\rho}} d_{-1}(\kappa^{(i)}_{\alpha\beta\gamma}.\frac{\zeta^{(j)}_{\gamma\rho}}{p_j}) = -\frac{< \kappa^{(i)} \smile \zeta^{(j)}, M >}{p_j}. \tag{89}$$

Up to the minus sign, the right hand side of this expression is the cohomological version of the intersection product which itself defines the linking form $\mathbf{Q} : T_1(M) \times T_1(M) \to \mathbb{Q}/\mathbb{Z}$ of M [48]. In other words

$$\mathbf{Q}_{ij} := \mathbf{Q}(\kappa^{(i)}, \kappa^{(j)}) = -\int_M \mathbf{A}^0_{p_i} \star \mathbf{A}^0_{p_j}. \tag{90}$$

Let us point out that this equality as a meaning because integration in $H^3_D(M)$ is $\mathbb{R}/\mathbb{Z}$-valued. After using property (88) and (90), the partition function reduces to:

$$Z_k(M) = \sum_{\kappa^1=0}^{p_1-1} \cdots \sum_{\kappa^N=0}^{p_N-1} e^{-2i\pi k \sum_{i,j} \kappa^i \mathbf{Q}_{ij} \kappa^j}. \tag{91}$$

This expression of $Z_k(M)$ also holds when $H_1(M) = \mathbb{Z}^q \oplus \mathbb{Z}_{p_1} \cdots \oplus \mathbb{Z}_{p_N}$ and defines an invariant for M.

One can use Dehn surgery technics to determine $\mathbf{Q}_{ij}$, or more traditional homological technics like in the case of the Tietze-de Rham representation of the lens space $L(p;r)$ [31]. We will discuss in Sect. 4.3 the possibility to push the computation of $Z_k(M)$ a little farther in order to obtain an expression based on a computation made in S^3 rather than in M directly. Such an idea is natural when using Dehn surgery representation of M, since the surgery is performed into S^3. However we will see that the idea of expressing $Z_k(M)$ as the expectation value of some link in S^3 can be understood without any reference to Dehn surgery.

4.2 Expectation Values of Wilson Lines, 2k Nilpotency and Link Invariants

Let $\mathcal{X}$ be a physical quantity which defines a *quantum observable* $X : H^1_D(M,\mathbb{Z}) \to U(1)$ associated with X. Once a full set of origins has been chosen on the fibers of $H^1_D(M,\mathbb{Z})$, one defines:

$$\langle X(\bar{\mathbf{A}})\rangle\Big|_M := \frac{\int d\mu(\bar{\mathbf{A}})X(\bar{\mathbf{A}})}{\int d\mu(\bar{\mathbf{A}})} = \frac{\sum_{\mathbf{n}\in H_1(M)}\int D\bar{\boldsymbol{\omega}}\, e^{iS[\bar{\mathbf{A}}_\mathbf{n}+\bar{\boldsymbol{\omega}}]}\, X[\bar{\mathbf{A}}_\mathbf{n}+\bar{\boldsymbol{\omega}}]}{\sum_{\mathbf{n}\in H_1(M)}\int D\bar{\boldsymbol{\omega}}\, e^{iS[\bar{\mathbf{A}}_\mathbf{n}+\bar{\boldsymbol{\omega}}]}}\,, \quad (92)$$

which is the *expectation value of the quantum observable* $X(\bar{\mathbf{A}})$ with respect to the so-called "standard" normalization. The normalisation used here takes into account the whole DB space $H^1_D(M,\mathbb{Z})$ whereas the normalization chosen to define the partition function $Z_k(M)$ deals only with the fiber over $\mathbf{0} \in H^2(M)$. The quantum observables we will deal with are the $U(1)$ Wilson lines introduced in Sect. 3.1. Note that they are defined from integrals of DB classes which are $\mathbb{R}/\mathbb{Z}$-valued, and therefore not classical quantities:

$$X(\bar{\mathbf{A}}) = W(\bar{\mathbf{A}},L) = \exp\left\{2i\pi \int_L \bar{\mathbf{A}}\right\} = \exp\left\{2i\pi \int_M \bar{\mathbf{A}} \star \bar{\eta}_L\right\}, \quad (93)$$

where L is a link in M and η_L its DB class.

As for the partition function, the quantum configuration space is taken to be $H^1_D(M,\mathbb{Z})^*$ instead of $H^1_D(M,\mathbb{Z})$, deferring regularization considerations. In this way, one is free to use 1-cycles as origins on the fibers of the configuration space.

Let us exhibit the so-called $2k$ *nilpotency property* (or *periodicity*) of the $U(1)$ Chern-Simons theory. For a link L, the link $2k.L$ corresponds to multiply by $2k$ all the charges of the fundamental cycles forming L. Then:

$$\langle W(\bar{\mathbf{A}}, 2kL)\rangle\Big|_M = \frac{\int d\mu(\bar{\mathbf{A}}) e^{2i\pi \int_M \bar{\mathbf{A}}\star\bar{\eta}_{2kL}}}{\int d\mu(\bar{\mathbf{A}})} = \frac{\int d\mu(\bar{\mathbf{A}}) e^{2i\pi(2k)\int_M \bar{\mathbf{A}}\star\bar{\eta}_L}}{\int d\mu(\bar{\mathbf{A}})}. \quad (94)$$

One performs the change of variable $\bar{\mathbf{A}} \to \bar{\mathbf{A}} + \bar{\boldsymbol{\eta}}_L$ and then use the Cameron-Martin property, thus yielding:

$$\langle W(\bar{\mathbf{A}}, 2kL)\rangle\Big|_M = \frac{\int d\mu(\bar{\mathbf{A}})}{\int d\mu(\bar{\mathbf{A}})} = 1. \quad (95)$$

This means that taking $2k$ times a link L turns the expectation value of the Wilson line along L into the expectation value of the zero knot. The degeneracy property can be obviously apply to the components of L. Consequently one has to consider charges as taking values in $\mathbb{Z}_{2k}$ instead of $\mathbb{Z}$. We will assume this from now on. This is some sort of *algebraic torsion* of the $U(1)$ Chern-Simons theory of M, which has nothing to do with the torsion of M.

One can redo most of what has been done for the partition function, like decomposing the numerator in definition (92) according to (36). This yields:

$$\langle W(\bar{\mathbf{A}}, L)\rangle\Big|_M = \mathcal{Z}^{-1} \sum_{\mathbf{a}\in F_1(M)} \sum_{\kappa\in T_1(M)} \int D\bar{\boldsymbol{\omega}}\, e^{iS[\bar{\mathbf{A}}_\mathbf{a}+\bar{\mathbf{A}}_\kappa+\bar{\boldsymbol{\omega}}]}\, W(\bar{\mathbf{A}}_\mathbf{a} + \bar{\mathbf{A}}_\kappa + \bar{\boldsymbol{\omega}}, L), \quad (96)$$

where the denominator of Eq. (92) has been denoted $\mathcal{Z}$.

Let us first consider the case of a homologically trivial 1-cycle z_0. There is necessarily a 2-chain Σ_0 which is bounded by z_0. The restrictions of the de Rham current of Σ_0 to the open sets of the good cover of M define a representative of the DB class[6] $\bar{\eta}_0$ associated with z_0. The expectation value (96) takes the form:

$$\langle W(\bar{\mathbf{A}}, z_0)\rangle\Big|_M = \mathcal{Z}^{-1} \int d\mu(\bar{\mathbf{A}})\, e^{2i\pi \int\limits_M \bar{\mathbf{A}}\star\bar{\eta}_0}. \quad (97)$$

One then performs the shift:

$$\bar{\mathbf{A}} \to \bar{\mathbf{A}} + \frac{\bar{\eta}_0}{2k}, \quad (98)$$

and uses Cameron-Martin property of $d\mu(\bar{\mathbf{A}})$, thus obtaining:

$$\begin{aligned}\langle W(\bar{\mathbf{A}}, z_0)\rangle\Big|_M &= \mathcal{Z}^{-1} \left(\int d\mu(\bar{\mathbf{A}})\right) \times e^{-2i\pi k \int\limits_M (\bar{\eta}_0/2k)\star(\bar{\eta}_0/2k)} \\ &= e^{-2i\pi k \int\limits_M (\bar{\eta}_0/2k)\star(\bar{\eta}_0/2k)},\end{aligned} \quad (99)$$

[6] This class rather belongs to the translation group $\Omega^2_{\mathbb{Z}}(M)^*$.

since $2k(\bar{\mathbf{A}} \star (\bar{\boldsymbol{\eta}}_0/2k)) = \bar{\mathbf{A}} \star \bar{\boldsymbol{\eta}}_0$. Finally, using Eq. (45) as well as the fact that $\bar{\boldsymbol{\eta}}_0$ is defined by the de Rham current of Σ_0, one concludes that:

$$\langle W(\bar{\mathbf{A}}, z_0)\rangle\Big|_M = \exp\left\{-\frac{2i\pi}{4k}\int_M j_0 \wedge dj_0\right\} = \exp\left\{-\frac{2i\pi}{4k}L(z_0, z_0)\right\} . \quad (100)$$

One has to define the self-linking number $L(z_0, z_0)$. The usual choice made is to attach a framing z_0^f to z_0 and then set $L(z_0, z_0) := L(z_0^f, z_0)$. This is equivalent to give a meaning to $\int_M j_0 \wedge dj_0$ which is ill-defined since there is a product of distributions appearing in this integral. Note that this is in perfect agreement with the zero-regularization choice since $\int_M \bar{\boldsymbol{\eta}}_0 \star \bar{\boldsymbol{\eta}}_0 \overset{\mathbb{Z}}{=} \int_M j_0 \wedge dj_0 \overset{\mathbb{Z}}{=} 0$ when z_0 is homologically trivial. However, the zero-regularization appears rougher than the framing procedure because in the case of homologically trivial cycles linking numbers are well-defined integers. For non trivial free cycles the linking number is not a well-fixed integer.

For a homologically trivial link L fulfilling decomposition (63):

$$L = \sum_{i=1}^{N} q^i z_i , \quad (101)$$

one will get:

$$\langle W(\bar{\mathbf{A}}, L)\rangle\Big|_M = \exp\left\{-\frac{2i\pi}{4k}q^i L(z_i, z_j)q^j\right\} = \exp\left\{-\frac{2i\pi}{4k}q^i L_{ij} q^j\right\} . \quad (102)$$

The *linking matrix* (L_{ij}) is well-defined outside its diagonal which is made of linking numbers. The diagonal of (L_{ij}) is defined using framed cycles z_i^f. Note that the corresponding integrals made of de Rham currents are also well-defined for $i \neq j$. Equation (102) coincide with the result obtained within the usual Quantum Field theoretical framework [30]. It is time to discuss the left aside question concerning the true nature of the Quantum Fields: are they belonging to $H_D^1(M, \mathbb{Z})$ or to its Pontryagin dual $H_D^1(M, \mathbb{Z})^*$, this last space containing distributional connections, as well as smooth ones (in a way already explained) and 1-cycles. Since the DB class of z_0 is necessarily in $H_D^1(M, \mathbb{Z})^*$ one immediately sees that the shift (98) has to be performed into $H_D^1(M, \mathbb{Z})^*$. This seems to imply that the natural space of Quantum Fields is $H_D^1(M, \mathbb{Z})^*$ (or a suitable subset of it). Nonetheless, instead of working with z_0 and its de Rham current, one can consider a Poincaré dual representative of z_0 given by a smooth closed 2-form ϖ_0 whose compact support is in a neighbourhood of z_0. The de Rham current of the surface Σ_0 bounded by z_0 is then replaced by a smooth 1-form ω_0 such that $\varpi_0 = d\omega_0$. The corresponding DB class $\bar{\boldsymbol{\omega}}_0$ is smooth and then belongs to $H_D^1(M, \mathbb{Z})$, and hence the shift (98) can be made within $H_D^1(M, \mathbb{Z})$. One will obtain (102) once a limit procedure $\omega_0 \to j_0$ is specified. When working with $H_D^1(M, \mathbb{Z})^*$ this limit procedure is not required and one directly deals with geometrical objects like cycles and links. In particular, the linking numbers are well defined at the level of currents, as long as the involved cycles are not intersecting. However in both cases one has to define the self-linking terms and the choice we made is zero-regularization (78).

Let us now consider a homologically non-trivial free cycle z on M. Such a cycle exists if and only if $F_1(M) \neq 0$. Then $F_2(M) \neq 0$ since $F_2(M) \simeq F^2(M) \simeq F_1(M)$. Furthermore $F_2(M) = \mathbb{Z}^q$, as in decomposition (37). Let Σ_i be a surface generating the ith $\mathbb{Z}$ component of $F_2(M)$, and let j_{Σ_i} be its de Rham current. Then one can use the zero-modes property (59) with j_{Σ_i} as well as the Cameron-Martin property, thus getting:

$$\langle W(\bar{\mathbf{A}} + m\frac{j_{\Sigma_i}}{2k}, z)\rangle\Big|_M = \langle W(\bar{\mathbf{A}}, L)\rangle\Big|_M \exp\left\{2i\pi m \int\limits_L \frac{\bar{\mathbf{j}}_{\Sigma_i}}{2k}\right\} \tag{103}$$

for any integer m. This equation can only hold true if $\exp\left\{2i\pi m \int_L \frac{\bar{\mathbf{j}}_{\Sigma_i}}{2k}\right\} = 1$ for any integer m, which implies that z and Σ_i must have trivial intersection. Since the surface Σ_i generates $F_2(M)$ this means that z is trivial, what contradicts our hypothesis. Accordingly this implies that:

$$\langle W(\bar{\mathbf{A}}, L)\rangle\Big|_M = 0 \tag{104}$$

for any homologically non trivial free link L of M. This very important result is naturally derived from the Deligne-Beilinson cohomology approach. The traditional Quantum Field Theory is performed on $\mathbb{R}^3$ where there are no non-trivial free cycles. In that case one first argues that the expectation values in $\mathbb{R}^3$ are the same as in S^3 and then use Dehn surgery theory to establish (104). Here all is done on M. Surprisingly, we never used the detailed expression for the functional measure in order to obtain (104). This result can also be obtained by using (73) and (82), in a computation similar to the one which led us to (86). The $2k$ nilpotency manifests itself in an interesting way in this computation whereas it is not involved in the partition function. Note that $2k$ nilpotency also implies that the expectation value of a link with charges multiple of $2k$ is one and not zero, even if the link is not homologically trivial. This also comes from Eq. (103) where the triviality of the link as to be consider modulo $2k$.

The third case to consider is the one of a (non trivial) torsion 1-cycle. Let κ_i be a torsion cycle on M generating the component $\mathbb{Z}_{p_i}$ of $T_1(M)$. Let $\boldsymbol{\kappa}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ denotes the class of κ_i in $\mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_N}$, the collection $(\boldsymbol{\kappa}_i)$ thus defining a basis of $T_1(M)$. Equation (96) now reads:

$$\langle W(\bar{\mathbf{A}}, \kappa_i)\rangle\Big|_M = \mathcal{Z}^{-1} \sum_{\mathbf{a}\in F_1(M)} \sum_{\kappa\in T_1(M)} \Bigg(\int D\bar{\boldsymbol{\omega}} e^{iS[\bar{\mathbf{A}}_\mathbf{a}+\bar{\mathbf{A}}_\kappa+\bar{\boldsymbol{\omega}}]} \times e^{2i\pi \int_M (\bar{\mathbf{A}}_\mathbf{a}+\bar{\mathbf{A}}_\kappa+\bar{\boldsymbol{\omega}})\star(\bar{\mathbf{A}}_{\tau_i}+\overline{\frac{\mathbf{j}_i}{p_i}})} \Bigg), \tag{105}$$

where j_i is the de Rham current of a surface Σ_i which is bounded by the trivial cycle $p_i.\kappa_i$. Recall that $\bar{\mathbf{A}}_{\kappa_i} + \overline{\frac{\mathbf{j}_i}{p_i}}$ is then the DB class of κ_i. After some algebraic juggles one finally obtains

$$\langle W(\bar{\mathbf{A}}, \kappa_i)\rangle\Big|_M = \sum_{\kappa\in T_1(M)} \frac{e^{-2i\pi k\mathbf{Q}(\kappa-\frac{\kappa_i}{2k},\kappa-\frac{\kappa_i}{2k})}}{Z_k(M)}. \tag{106}$$

One can combine all the different results to compute the expectation value of a general link in M. In particular, one will find that for the expectation value to be non-vanishing the link must not have a free component. A non vanishing expectation value will then be made of two factors, one similar to (106) with $(\boldsymbol{\tau}_i)$ replaced by a generic vector in $\mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_N}$ representing the purely torsion contribution to the link, the other factor being similar to (102) and representing the homologically trivial contribution of the link [51]. More precisely if the link decomposed according to:

$$L = \sum_{i=1}^{N} \sum_{\alpha_i} \zeta^i_{\alpha_i} \kappa_i^{\alpha_i} + \sum_{a=1}^{F} q^a z_a \tag{107}$$

where each cycle $\kappa_i^{\alpha_i}$ is a torsion cycle generating $\mathbb{Z}_{p_i}$, and each z_a is a homologically trivial cycle. The integers $\zeta^i_{\alpha_i}$ and q^a are respectively the torsion and trivial charges of L. The homology class of L is given by $\kappa_L \in \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_N}$ whose components are $\kappa_i = \sum_{\alpha_i} \kappa_i^{\alpha_i}$. One also introduces $\mathbf{q}_L \in \mathbb{Z}^F$ whose components are the trivial charges q_a of L.

$$\langle W(\bar{\mathbf{A}}, L)\rangle\Big|_M = \left(\sum_{\kappa\in T_1(M)} \frac{e^{-2i\pi k\mathbf{Q}(\kappa-\frac{\kappa_L}{2k},\kappa-\frac{\kappa_L}{2k})}}{Z_k(M)} \right) \times e^{-\frac{2i\pi}{4k}\mathbf{L}(\mathbf{q}_L,\mathbf{q}_L)}, \tag{108}$$

where $\mathbf{Q}$ is the linking form of M and $\mathbf{L}$ is the quadratic form associated with the linking matrix defined by the free cycles z_a, but also by the mixed linking of the z_a with the torsion cycles $\kappa_i^{\alpha_i}$ and of the linking matrix of the torsion cycles $\kappa_i^{\alpha_i}$ [51].

4.3 Some Remarks on Dehn Surgery Results Without Surgery

Let us consider in this last section the case of a 3-dimensional manifold M such that $H^2(M) \simeq H_1(M) = T_1(M) = \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_N}$. On the one hand we have seen that the $U(1)$ Chern-Simons partition function of M reads:

$$Z_k(M) = \sum_{\kappa^1=0}^{p_1-1} \cdots \sum_{\kappa^N=0}^{p_N-1} e^{-2i\pi k \int_M \kappa^i \mathbf{Q}_{ij} \kappa^j} \tag{109}$$

There is another way to compute $Z_k(M)$ based on Dehn surgery of M in S^3. Let $\mathcal{L}_M$ be a surgery link of M in S^3. Each fundamental cycle $\mathcal{Z}_i$ of $\mathcal{L}_M$ has a self-linking number r_i named the surgery coefficient of $\mathcal{Z}_i$, and the linking matrix $\mathcal{L}_{ij}$ of $\mathcal{L}_M$ is thus perfectly well-defined. One then defines the partition function of M [43] as:

$$Z_k(M) = (k)^{-m/2} \sum_{q^1=0}^{2k-1} \cdots \sum_{q^{|\mathcal{L}|}=0}^{2k-1} \exp\left\{ -\frac{2i\pi}{4k} \sum_{i,j=1}^{|\mathcal{L}|} q^i \mathcal{L}_{ij}\, q^j \right\}, \tag{110}$$

where $|\mathcal{L}|$ is the number of fundamental cycles of $\mathcal{L}_M$.

One can wonder about the relationship between (109) and (110). A result from F. Deloup and V. Turaev [49] allows to go from (110) to (109) when $F_1(M) = 0$, mainly because the linking matrix of $\mathcal{L}_M$ is then invertible so that $H_1(M) = T_1(M) = Coker\mathcal{L}$.

But expression (109) can also be considered as such, without any reference to a Dehn surgery of M. In this case, by using a theorem of C. T. C. Wall about quadratic forms on cyclic groups [50], one can recover (up to the normalization factor $(k)^{-m/2}$) (110) but with a link $\mathcal{L}_Q$ in S^3 which is not necessarily a surgery link of M. Such an *algebraic link* $\mathcal{L}_Q$ yields the matrix of the linking form $\mathbf{Q}$ of M as the $\mathbb{Q}/\mathbb{Z}$-valued inverse of the linking matrix of this algebraic link. In general, the linking form $\mathbf{Q}$ of M does not completely specify the 3-dimensional manifold M. Hence, the algebraic link $\mathcal{L}_Q$ contains less information than a Dehn surgery link $\mathcal{L}_M$ that completely characterises M. On the other hand, let us recall that for a given 3-dimensional manifold M there are many different Dehn surgery links of M in S^3. Wall algebraic link is even more degenerate since different 3-dimensional manifolds may have the same linking form.

This discussion can be extended to expectation values of links in M. Wall theorem allows to show that an expectation value obtained from a computation done in M itself and based on Deligne-Beilinson cohomology can be rewritten as the expectation value of an algebraic link in S^3. This new algebraic link is the sum of the previous algebraic link $\mathcal{L}_Q$ that defined the partition function of M and represents the linking form of M, together with a second link which represents in S^3 the original link L in M [51]. Of course, a Dehn surgery description in S^3 of $L \subset M$ provides such an algebraic link.

The advantage of Dehn surgery is that it gives a powerful representation of 3-manifolds: drawing M is simple and the computation of properties like the homology groups, the linking form or the homotopy groups of M are also simple. The drawback is that Dehn surgery is quite specific to 3-dimensional manifolds, whereas we already noticed that $U(1)$ Chern-Simons theories exist for $(4l+3)$-dimensional manifolds. In theses case, the algebraic approach based on Wall theorem might be particularly useful.

5 Conclusion

The use of Deligne-Beilinson cohomology in Quantum Field Theory is not yet very popular even if some authors have done some noticeable effort to shed light on this mathematical structure within the theoretical physics context [26, 27]. However, the simple case of $U(1)$ Chern-Simons Quantum field theories show the whole benefit this approach might bring. In particular, the usual Quantum Field theoretical approach—performing a gauge fixing of the action by the mean of a metric, finding a propagator, inverting it in order to obtain expectation values—cannot have access to cohomological property of the theory whereas the DB approach does.

There remains to see if Deligne-Beilinson cohomology is also involved in the non-abelian Chern-Simons theory apart in the action itself. This is far from clear up to now even if some indices like the non-abelian Stokes theorem [52], the Localization procedure [53] or the form of the Reshetikhin-Turaev non-abelian Partition Function [43] suggest not to give up too quickly the Deligne-Beilinson track.

Appendix: Few Reminders About the Čech-de Rham "Descent"

Let M be a m-manifold endowed with a good open cover $\mathcal{U} = (U_\alpha)_{\alpha \in I}$. Let $\omega^{(-1,p)}$ be a closed p-form on M ($p \leq m$): $d_p \omega^{(-1,p)} = 0$. The notation may look strange at this stage, but its relevance will appear in the sequel. The restriction $(\delta_{-1}\omega^{(-1,p)})_\alpha := \omega^{(-1,p)}|_\alpha$ of $\omega^{(-1,p)}$ to U_α is also closed and since U_α is contractible Poincaré lemma can be applied to $(\delta_{-1}\omega^{(-1,p)})_\alpha$, which gives:

$$(\delta_{-1}\omega^{(-1,p)})_\alpha = d_{p-1}\omega_\alpha^{(0,p-1)}, \tag{111}$$

for some $(p-1)$-form ξ_α defined in U_α. This can be iterated for all the open sets U_α of $\mathcal{U}$, thus generating a collection $(\delta_{-1}\omega^{(-1,p)})$ of local p-forms (the restrictions of $\omega^{(-1,p)}$) and a collection $\omega^{(0,p-1)}$ of local $(p-1)$-forms, which are not in general the restriction of a $(p-1)$-form on M, in such a way that:

$$(\delta_{-1}\omega^{(-1,p)}) = d_{p-1}\omega^{(0,p-1)}. \tag{112}$$

This is the first step of the Čech-de Rham procedure.

The Čech derivative associated to the intersections $U_\alpha \cap U_\beta$ is $(\delta_0\omega^{(0,p-1)})_{\alpha\beta} := \omega_\beta^{(0,p-1)} - \omega_\alpha^{(0,p-1)}$ and it clearly satisfies: $\delta_0\delta_{-1} = 0$. Applying this to the previous equation one obtains:

$$d_{p-1}(\delta_0\omega^{(0,p-1)}) := \delta_0 d_{p-1}\omega^{(0,p-1)} = \delta_0\delta_{-1}\omega^{(-1,p)} = 0, \tag{113}$$

since $\omega^{(0,p-1)}$ is globally defined. The commutativity of the Čech derivative with the de Rham one is trivial since the former is made of linear combinations and the latter is a linear operation. Since the intersections $U_\alpha \cap U_\beta$ are contractible (the cover $\mathcal{U}$ is good), Poincaré lemma can be again applied, thus leading to:

$$(\delta_0\omega^{(0,p-1)}) = d_{p-2}\omega^{(1,p-2)}, \tag{114}$$

in each $U_\alpha \cap U_\beta$. This generates another collection $\omega^{(1,p-2)}$, now made of local $(p-2)$-forms defined in these intersections. The next step is to introduce the Čech derivative for the intersections $U_\alpha \cap U_\beta \cap U_\gamma$: $(\delta_1\omega^{(1,p-2)})_{\alpha\beta\gamma} := \omega_{\beta\gamma}^{(0,p-1)} - \omega_{\alpha\gamma}^{(0,p-1)} + \omega_{\alpha\beta}^{(0,p-1)}$ which clearly satisfies: $\delta_1\delta_0 = 0$. Applying this derivative to the previous equation leads to:

$$(\delta_1\omega^{(1,p-2)}) = d_{p-3}\omega^{(2,p-3)}, \tag{115}$$

in each $U_\alpha \cap U_\beta \cap U_\gamma$. On then introduce the generic Čech derivative δ_k associated with the intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_{k+1}}$ defined by:

$$(\delta_k\omega^{(k,p-k-1)})_{\alpha_0,\ldots,\alpha_{k+1}} := \sum_{j=0}^{k+1}(-1)^j\,\omega^{(k,p-k-1)}_{\alpha_0,\ldots,\check{\alpha}_j,\ldots,\alpha_{k+1}}, \tag{116}$$

where $\omega^{(k,p-k-1)}_{\alpha_0,\ldots,\check{\alpha}_j,\ldots,\alpha_{k+1}}$ means that the index α_j has been omitted. These derivatives satisfy:

$$\delta_k\delta_{k-1} = 0, \tag{117}$$

and they allows to continue the procedure initiated previously, leading to:

$$(\delta_k\omega^{(k,p-k-1)}) = d_{p-k-2}\omega^{(k+1,p-k-2)}. \tag{118}$$

The before last step is reached when $k = p-2$ and gives:

$$(\delta_{p-2}\omega^{(p-2,1)}) = d_0\omega^{(p-1,0)}. \tag{119}$$

One now applies δ_{p-1} to this equation, and then Poincaré lemma in the corresponding contractible intersections, thus obtaining:

$$d_0(\delta_{p-1}\omega^{(p-1,0)}) = 0. \tag{120}$$

The collection $\delta_{p-1}\omega^{(p-1,0)}$ is therefore made of real numbers. One finally introduces the following extension of the de Rham derivatives:

$$d_{-1} : \mathbb{R} \to \Omega^0(M) \tag{121}$$

which associates to a real number the corresponding constant function, which allows to write:

$$\delta_{p-1}\omega^{(p-1,0)} = d_{-1}\omega^{(p,-1)}, \tag{122}$$

where the collection $\omega^{(p,-1)}$ is made of real numbers in the intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$. Such an object is a "pure" Čech p-cochain of $\mathcal{U}$. Furthermore, since $\delta_p\delta_{p-1} = 0$, one conclude that $\omega^{(p,-1)}$ is a Čech p-cocycle and that the procedure we have juste presented defined a relation between closed p-forms on M and Čech p-cocycle of the good cover $\mathcal{U}$ of M. One can check that this procedure goes to cohomology class because if the closed form is actually exact then the corresponding Čech cocycle is a coboundary. Hence we have an injection:

$$H^p_{dR}(M) \to \check{H}^p(\mathcal{U}, \mathbb{R}), \tag{123}$$

where $H^p_{dR}(M)$ is the pth de Rham cohomology group of M and $\check{H}^p(M, \mathbb{R})$ is the pth Čech cohomology group of $\mathcal{U}$. Using a partition of unity subordinate to $\mathcal{U}$, one can associate to a Čech p-cocycle of $\mathcal{U}$ a closed p-form of M in such a way that this goes to cohomologies [46], thus yielding the inverse injection, which allows to conclude that:

$$H^p_{dR}(M) \simeq \check{H}^p(\mathcal{U}, \mathbb{R}). \tag{124}$$

Furthermore, if the closed p-form $\omega^{(-1,p)}$ has integral periods, one can show that the associated Čech cocycle $\omega^{(p,-1)}$ is cohomologous to an integral Čech cocycle, that is to say: $\omega^{(p,-1)} = n^{(p,-1)} + \delta_{p-1}\rho^{(p-1,-1)}$. This provide an injection:

$$H^p_{dR,\mathbb{Z}}(M) \to \check{H}^p(\mathcal{U}, \mathbb{Z}), \tag{125}$$

of the space of integral de Rham classes into the integral Čech cohomology of $\mathcal{U}$. However it is not possible to reverse this injection in general because integral Čech cohomology groups, as abelian finite groups, are made of a free part and a torsion (i.e. cyclic) part, and the torsion part is not accessible to de Rham cohomology. This can be seen as follows. Let us assume that z is a torsion p-cycle on M. This means that z is not homologically trivial although there exist an integer q such that $q.z = bC$. Let ω be a closed p-form on M and let us consider:

$$\oint_z \omega. \tag{126}$$

Then, by linearity of the integral and Stokes theorem:

$$q \oint_z \omega = \oint_{q.z} \omega = \oint_{bC} \omega = \oint_C d_p \omega = 0. \tag{127}$$

This automatically implies that:

$$\oint_z \omega = 0, \tag{128}$$

which means that torsion cycles are "transparent" to forms, that is to say, torsion cycles appear as homologically trivial cycles to forms. It is also well-known that by taking the tensor product with real numbers torsion vanishes. This is what we actually did when we dealt with real Čech cocycles instead of integral ones in the beginning, and this is why we "obtained" the isomorphism (124).

Last but not least, all we have done here was related to given good cover $\mathcal{U}$ of M. One has to show that by taking the inductive limit over refinements of $\mathcal{U}$ one obtain the so-called Čech cohomology of M and that this cohomology is actually isomorphic to the one of $\mathcal{U}$, as long as this cover is good. This is purely technical and far beyond the scope of this appendix. all such details can be found, for instance, in [46].

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Part III
(Semi-)Classical Field Theories

Classical field theories appear to be of an independent interest, after the work of Alexandrov-Kontsevich-Schwarz-Zaboronski (see also D.S. Freed). Moreover, understanding them is also important in order to identify, when trying to provide rigorous constructions of QFTs, what are the difficulties specific to the quantum side.

Part III begins with "Semiclassical Quantization of Classical Field Theories" (written by Alberto Cattaneo, Pavel Mnev and Nicolai Reshetikhin) introducing some very recent work on the treatment of constraints and boundary conditions in classical field theories, with emphasis on the BV and BFV formalisms.

It continues with "Local BRST Cohomology for AKSZ Field Theories: A Global Approach" (written by Giuseppe Bonavolontà and Alexei Kotov) discussing the BV formalism in the context of AKSZ sigma models, using Vinogradov's secondary calculus as a tool.

"Symplectic and Poisson Geometry of the Moduli Spaces of Flat Connections Over Quilted Surfaces" (written by David Li-Bland and Pavol Ševera) explores the (quasi-)hamiltonian and Poisson geometry of various moduli spaces that are relevant to $2d$ and $3d$ classical TFTs.

Part III ends with "Groupoids, Frobenius Algebras and Poisson Sigma Models" (written by Ivan Contreras), that is about understanding the construction of the symplectic groupoid by means of a two-dimensional topological theory from the axiomatics of Frobenius algebras.

Semiclassical Quantization of Classical Field Theories

Alberto S. Cattaneo, Pavel Mnev and Nicolai Reshetikhin

Abstract These lectures are an introduction to formal semiclassical quantization of classical field theory. First we develop the Hamiltonian formalism for classical field theories on space time with boundary. It does not have to be a cylinder as in the usual Hamiltonian framework. Then we outline formal semiclassical quantization in the finite dimensional case. Towards the end we give an example of such a quantization in the case of Abelian Chern-Simons theory.

A.S. Cattaneo (✉) · P. Mnev
Institut Für Mathematik, Universität Zürich, Winterthurerstrasse 190,
8057 Zürich, Switzerland
e-mail: alberto.cattaneo@math.uzh.ch

P. Mnev
St. Petersburg Department of V.A. Steklov Institute of Mathematics
of the Russian Academy of Sciences, Fontanka 27,
191023 St. Petersburg, Russia
e-mail: pmnev@pdmi.ras.ru

P. Mnev
Chebyshev Laboratory, St. Petersburg State University, 14th Line, 29b,
199178 St. Petersburg, Russia

N. Reshetikhin
Department of Mathematics, University of California,
94305 Berkeley, CA, USA
e-mail: reshetik@math.berkeley.edu

N. Reshetikhin
KdV Institute for Mathematics, University of Amsterdam,
Science Park 904, 1098 XH Amsterdam, The Netherlands

N. Reshetikhin
ITMO University,
Kronverkskii ave. 49, 197101 Saint Petersburg, Russia

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1 Introduction

The goal of these lectures is an introduction to the formal semiclassical quantization of classical gauge theories.

In high energy physics space time is traditionally treated as a flat Minkowski manifold without boundary. This is consistent with the fact the characteristic scale in high energy is so much smaller then any characteristic scale of the Universe.

As one of the main paradigms in quantum field theory, quantum fields are usually assigned to elementary particles. The corresponding classical field theories are described by relativistically invariant local action functionals. The locality of interactions between elementary particles is one of the key assumptions of a local quantum field theories and of the Standard Model itself.

The path integral formulation of quantum field theory makes it mathematically very similar to statistical mechanics. It also suggests that in order to understand the mathematical nature of local quantum field theory it is natural to extend this notion from Minkowski space time to a space time with boundary. It is definitely natural to do it for the corresponding classical field theories.

The concept of topological and conformal field theories on space time manifolds with boundary was advocated in [3, 28]. The renormalizability of local quantum field theory on a space time with boundary was studied earlier in [30]. Here we develop the gauge fixing approach for space time manifolds with boundary by adjusting the Faddeev-Popov (FP) framework to this setting. This gauge fixing approach is a particular case of the more general Batalin-Vilkovisky (BV) formalism for quantization of gauge theories. The classical Hamiltonian part of the BV quantization on space time manifolds with boundary, the BV-BFV formalism, is developed in [13]. In a subsequent publication we will extend it to the quantum level.

The goal of these notes is an overview of the FP framework in the context of space time manifolds with boundary. As a first step we present the Hamiltonian structure for such theories. We focus on the Hamiltonian formalism for first order theories. Other theories can be treated similarly, see for example [14] and references therein. In a subsequent publication we will connect this approach with the BV-BFV program.

In Sect. 2 we recall the concept of local quantum field theory as a functor from the category of space time cobordisms to the category of vector spaces. The Sect. 3 contains examples: the scalar field theory, Yang-Mills theory, Chern-Simons and BF theories. The concept of semiclassical quantization of first order quantum field theories is explained in Sect. 4 where we present a finite dimensional model for the gauge fixing for space time manifolds with or without boundary. In Sect. 5 we briefly discuss the example of Abelian Chern-Simons theory. The nonabelian case and the details of the gluing of partition functions for semiclassical Chern-Simons theories will be given elsewhere.

2 First Order Classical Field Theories

2.1 Space Time Categories

In order to define a classical field theory one has to specify a space time category, a space of fields for each space time and the action functional on the space of fields.

Two space time categories which are most important for these lectures are the category of smooth n-dimensional cobordisms and the category of smooth n-dimensional Riemannian manifolds.

The d-dimensional smooth category. Objects are smooth, compact, oriented $(d-1)$-dimensional manifolds with smooth d-dimensional collars. A morphism between Σ_1 and Σ_2 is a smooth d-dimensional compact oriented manifolds with $\partial M = \Sigma_1 \sqcup \overline{\Sigma_2}$ and the smooth structure on M agrees with smooth structure on collars near the boundary. The orientation on M should agree with the orientations of Σ_1 and be opposite to the one on Σ_2 in a natural way.

The composition consists of gluing two morphisms along the common boundary in such a way that collars with smooth structure on them fit.

In this and the subsequent examples of space time categories identity morphisms have to be adjoined formally. Note also that we are not taking the quotient of cobordisms by diffeomorphisms.

The d-dimensional Riemannian category. Objects are $(d-1)$-dimensional Riemannian manifolds with d-dimensional collars. Morphisms between two oriented $(d-1)$-dimensional Riemannian manifolds N_1 and N_2 are oriented d-dimensional Riemannian manifolds M with collars near the boundary, such that $\partial M = N_1 \sqcup \overline{N_2}$. The orientation on all three manifolds should naturally agree, and the metric on M agrees with the metric on N_1 and N_2 and on collar near the boundary. The composition is the gluing of such Riemannian cobordisms. For the details see [29].

This category is important for many reasons. One of them is that it is the underlying structure for statistical quantum field theories.

The d-dimensional pseudo-Riemannian category. The difference between this category and the Riemannian category is that morphisms are pseudo-Riemannian with the signature $(d-1, 1)$ while objects remain $(d-1)$-dimensional Riemannian. This is the most interesting category for particle physics.

Both objects and morphisms may have an extra structure such as a fiber bundle (or a sheaf) over it. In this case such structures for objects should agree with the structures for morphisms.

2.2 General Structure of First Order Theories

2.2.1 First Order Classical Field Theories

A first order classical field theory[1] is defined by the following data:

- A choice of space time category.
- A choice of the space of fields F_M for each space time manifold M. This comes together with the definition of the space of fields $F_{\partial M}$ for the boundary of the space time and the restriction mapping $\pi : F_M \to F_{\partial M}$.
- A choice of the action functional on the space F_M which is local and first order in derivatives of fields, i.e.

$$S_M(\phi) = \int_M L(d\phi, \phi)$$

  Here $L(d\phi, \phi)$ is linear in $d\phi$.

These data define:

- The space EL_M of solutions of the Euler-Lagrange equations.
- The 1-form $\alpha_{\partial M}$ on the space of boundary fields arising as the boundary term of the variation of the action [14].
- The Cauchy data subspace $C_{\partial M}$ of boundary values (at $\{0\} \times \partial M$) of solutions of the Euler-Lagrange equations in $[0, \epsilon) \times \partial M$.
- The subspace $L_M \subset C_{\partial M}$ of boundary values of solutions of the Euler-Lagrange equations in M, $L_M = \pi(EL_M)$.

When $C_{\partial M} \neq F_{\partial M}$ the Cauchy problem is overdetermined and therefore the action is degenerate. Typically it is degenerate because of the gauge symmetry.

A natural boundary condition for such system is given by a Lagrangian fibration[2] on the space of boundary fields such that the form $\alpha_{\partial M}$ vanishes at the fibers. The last conditions guarantees that solutions of Euler-Lagrange equations which are constrained to a leaf of such fibration are critical points of the action functional, i.e. not only the bulk term vanishes but also the boundary terms.

[1] It is not essential that we consider here only first order theories. Higher order theories where $L(d\phi, \phi)$ is not necessary a linear function in $d\phi$ can also be treated in a similar way, see for example [14] and references therein. In first order theories the space of boundary fields is the pull-back of fields in the bulk.

[2] In our examples, fibrations are actually fiber bundles. By abuse of terminology, terms "fibration" and "foliation" will be used interchangeably.

2.2.2 First Order Classical Field Theory as a Functor

First order classical field theory can be regarded as a functor from the category of space times to the category which we will call Euler-Lagrange category and will denote $\underline{EL}$. Here is an outline of this category:

An *object* of $\underline{EL}$ is a symplectic manifold F with a prequantum line bundle, i.e. a line bundle with a connection α_F, such that the symplectic form is the curvature of this connection. It should also have a Lagrangian foliation which is α_F-exact, i.e. the pull-back of α_F to each fiber vanishes.[3]

A *morphism* between F_1 and F_2 is a manifold F together with two surjective submersions $\pi_1 : F \to F_1$ and $\pi_2 : F \to F_2$, with a function S_F on F and with the subspace $EL \subset F$ such that $dS_F|_{EL}$ is the pull-back of $-\alpha_{F_1} + \alpha_{F_2}$ on $F_1 \times F_2$. The image of EL in $(F_1, -\omega_1) \times (F_2, \omega_2)$ is automatically an isotropic submanifold. Here $\omega_i = d\alpha_{F_i}$. We will focus on theories where these subspaces are Lagrangian.

The composition of morphisms (F, S_F) and $(F', S_{F'})$ is the fiber product of the morphism spaces F and F' over the intermediate object and $S_{F' \circ F} = S_F + S_{F'}$. This category is the $gh = 0$ part of the BV-BFV category from [13].

A first order classical field theory defines a functor from the space time category to the Euler-Lagrange category. An object N of the space time category is mapped to the space of fields F_N, a morphism M is mapped to (F_M, S_M), etc. Composition of morphisms is mapped to the fiber product of spaces of fields[4] and because of the assumption of locality of the action functional, it is additive with respect to the gluing.

This is just an outline of the Euler-Lagrange category and of the functor. For our purpose of constructing formal semiclassical quantization we will not need the precise details of this construction. But it is important to have this more general picture in mind.

2.3 Symmetries in First Order Classical Field Theories

The theory is relativistically invariant if the action is invariant with respect to geometric automorphisms of the space time. These are diffeomorphisms for the smooth category, isometries for the Riemannian category etc. In such theory the action is constructed using geometric operations such as de Rham differential and exterior multiplication of forms for smooth category. In Riemannian category in addition to these two operations we have Hodge star (or the metric).

If the space time category has an additional structure such as fiber bundle, the automorphisms of this additional structure give additional symmetries of the theory. In Yang-Mills, Chern-Simons and BF theories, gauge symmetry, or automorphisms

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[3] Here we are assuming for simplicity of the exposition that the prequantum line bundle is trivial and thus we can identify the connection with its 1-form on F.

[4] We are not precise at this point. Rather, the value of the functor on a composition is *homotopic* (in the appropriate sense) to the fiber product.

of the corresponding principal G-bundle, are such a symmetry. A theory with such space time with the gauge invariant action is called gauge invariant. The Yang-Mills theory is gauge invariant, the Chern-Simons and the BF theories are gauge invariant only up to boundary terms.

There are more complicated symmetries when a distribution, not necessary integrable, is given on the space of fields and the action is annihilated by corresponding vector fields. Nonlinear Poisson σ-model is an example of such field theory [18].

3 Examples

3.1 First Order Lagrangian Mechanics

3.1.1 The Action and Boundary Conditions

In Lagrangian mechanics the main component which determines the dynamics is the Lagrangian function. This is a function on the tangent bundle to the configuration space $L(\xi, x)$ where $\xi \in T_x N$. In Newtonian mechanics the Lagrangian function is quadratic in velocity and the quadratic term is positive definite which turns N into a Riemannian manifold.

The most general form of first order Lagrangian is $L(\xi, x) =< \alpha(x), \xi > -H(x)$ where α is a 1-form on N and H is a function on N. The action of a first order Lagrangian mechanics is the following functional on parameterized paths $F_{[t_1,t_2]} = C^\infty([t_1, t_2], N)$

$$S_{[t_2,t_1]}[\gamma] = \int_{t_1}^{t_2} (\langle\alpha(\gamma(t)), \dot{\gamma}(t)\rangle - H(\gamma(t)))\, dt, \tag{1}$$

where γ is a parametrized path.

The Euler-Lagrange equations for this action are:

$$\omega(\dot{\gamma}(t)) - dH(\gamma(t)) = 0,$$

where $\omega = d\alpha$. Naturally, the first order Lagrangian system is called *non-degenerate*, if the form ω is non-degenerate. We will focus on non-degenerate theories here. Denote the space of solutions to Euler-Lagrange equations by $EL_{[t_1,t_2]}$.

Thus, a non-degenerate first order Lagrangian system defines an exact symplectic structure $\omega = d\alpha$ on a manifold N. The Euler-Lagrange equations for such system are equations for flow lines of the Hamiltonian on the symplectic manifold (N, ω) generated by the Hamiltonian H. It is clear that the action of a non-degenerate first order system is exactly the action for this Hamiltonian system.

The variation of the action on solutions of the Euler-Lagrange equations is given by the boundary terms:

$$\delta S_{[t_2,t_1]}[\gamma] = \langle \alpha(\gamma(t)), \delta\gamma(t)\rangle \,|_{t_1}^{t_2} .$$

If $\gamma(t_1)$ and $\gamma(t_2)$ are constrained to Lagrangian submanifolds in $L_{1,2} \subset N$ with $TL_{1,2} \subset \ker(\alpha)$, these terms vanish.

The restriction to boundary points gives the projection $\pi : F_{[t_1,t_2]} \to N \times N$. The image of the space of solutions of the Euler-Lagrange equations $L_{[t_1,t_2]} \subset N \times N$ for small $[t_1, t_2]$ is a Lagrangian submanifold with respect to the symplectic form $(d\alpha)_1 - (d\alpha)_2$ on $N \times N$.

Note that solutions of the Euler-Lagrange equation with boundary conditions in $L_1 \times L_2$ correspond to the intersections points $(L_1 \times L_2) \cap L_{[t_1,t_2]}$ which is generically a discrete set.

3.1.2 More on Boundary Conditions

The evolution of the system from time t_1 to t_2 and then to t_3 can be regarded as gluing of space times $[t_1, t_2] \times [t_2, t_3] \to [t_1, t_2] \cup [t_2, t_3] = [t_1, t_3]$. If we impose boundary conditions L_1, L_2, L_3 at times t_1, t_2, t_3 respectively there may be no continuous solutions of equations of motion for intervals $[t_1, t_2]$ and $[t_2, t_3]$ which would compose into a continuous solution for the interval $[t_1, t_3]$. This is why boundary conditions should come in families of Lagrangian submanifolds, so that by varying the boundary condition at t_2 we could choose L_2 in such a way that solutions for $[t_1, t_2]$ and $[t_2, t_3]$ would compose to a continuous solution.

This is why we will say that a boundary condition for a first order theory is a Lagrangian fibration on the space of boundary values of classical fields. In case of first order classical mechanics this is a Lagrangian fibration on N, boundary condition is a Lagrangian fibration of $(N, \omega) \times (N, -\omega)$. It is natural to choose boundary conditions independently for each connected component of the boundary of the space time. In case of classical mechanics this means a choice of Lagrangian fibration $p : N \to B$ for each endpoint of $[t_1, t_2]$. The form α should vanish on fibers of this fibration.

Remark 1 For semiclassical quantization we will need only classical solutions and infinitesimal neighborhood of classical solutions. This means that we need in this case a Lagrangian fibration on the space of boundary fields defined only locally, not necessary globally.

Let N be a configuration space (such as $\mathbb{R}^n$) and $T^*(N)$ be the corresponding phase space. Let γ be a parameterized path in $T^*(N)$ such that, writing $\gamma(t) = (p(t), q(t))$ (where p is momenta and q is position), we have $q(t_i) = q_i$ for two fixed points q_1, q_2. If γ_{cl} is a solution to the Euler-Lagrange equations, then

$$dS^{\gamma_{\text{cl}}}_{t_1,t_2}(q_1, q_2) = \pi^*(p_1\, dq_1 - p_2\, dq_2) \tag{2}$$

where $p_1 = p(t_1)$, $p_2 = p(t_2)$ are determined by t_1, t_2, q_1, q_2. The function $S^{\gamma_{\text{cl}}}_{t_1,t_2}$ is the Hamilton-Jacobi function.

3.2 Scalar Field Theory in an n-dimensional Space Time

The space time in this case is a smooth oriented compact Riemannian manifold M with $\dim M = n$. The space of fields is

$$F_M = \Omega^0(M) \oplus \Omega^{n-1}(M). \tag{3}$$

where we write φ for an element of $\Omega^0(M)$ and p for an element of $\Omega^{n-1}(M)$. The action functional is

$$S_M(p, \varphi) = \int_M p \wedge d\varphi - \frac{1}{2}\int_M p \wedge *p - \int_M V(\varphi)\, dx. \tag{4}$$

with $V \in C^\infty(\mathbb{R})$ a fixed potential; dx stands for the metric volume form.

The first term is topological and analogous to $\int_\gamma \alpha$ in (1). The second and third terms use the metric and together yield an analog of the integral of the Hamiltonian in (1).

The variation of the action is

$$\int_M \delta p \wedge (d\varphi - *p) - (-1)^{n-1} \int_M dp \wedge \delta\varphi + (-1)^{n-1} \int_{\partial M} p\, \delta\varphi - \int_M V'(\varphi)\, \delta\varphi\, dx. \tag{5}$$

The Euler-Lagrange equations are therefore

$$d\varphi - *p = 0, (-1)^{n-1} dp + V'(\varphi)\, dx = 0. \tag{6}$$

The first equation gives $p = (-1)^{n-1} * d\varphi$, and substituting this into the second equation gives

$$\Delta\varphi + V'(\varphi) = 0. \tag{7}$$

where $\Delta = *d * d$ is the Laplacian acting of functions.

Thus the space of all solutions of Euler-Lagrange equations is

$$EL_M = \{(p, \varphi) | p = (-1)^{n-1} * d\varphi, \quad \Delta\varphi + V'(\varphi) = 0\}$$

Remark 2 To recover the second-order Lagrangian compute the action at the critical point in p, i.e. substitute $p = (-1)^{n-1} * d\varphi$ into the action functional:

$$S_M((-1)^{n-1} * d\varphi, \varphi) = \int_M (-1)^{n-1} * d\varphi \wedge d\varphi - \frac{1}{2}\int * d\varphi \wedge * * d\varphi$$
$$- \int_M V(\varphi)\, dx = \frac{1}{2}\int_M d\varphi \wedge * d\varphi - \int_M V(\varphi)\, dx$$
$$= \int_M \left(\frac{1}{2}(d\varphi, d\varphi) - V(\varphi)\right) dx.$$

The boundary term in the variation gives the 1-form on boundary fields

$$\alpha_{\partial M} = \int_{\partial M} p\, \delta\varphi \in \Omega^1(F_{\partial M}). \tag{8}$$

Here δ is the de Rham differential on $\Omega^\bullet(F_{\partial M})$. The differential of this 1-form gives the symplectic form $\omega_{\partial M} = \delta\alpha_{\partial M}$ on $F_{\partial M}$.

Note that we can think of the space $F_{\partial M}$ of boundary fields as $T^*(\Omega^0(\partial M))$ in the following manner: if $\delta\varphi \in T_\varphi(\Omega^0(\partial M)) \cong \Omega^0(\partial M)$ is a tangent vector, then the value of the cotangent vector $A \in \Omega^{n-1}(\partial M)$ is

$$A(\delta\varphi) = \int_{\partial M} A \wedge \delta\varphi. \tag{9}$$

The symplectic form $\omega_{\partial M}$ is the natural symplectic form on $T^*\Omega^0(\partial M)$.

The image of the space EL_M of all solutions to the Euler-Lagrange equations with respect to the restriction map $\pi : F_M \to F_{\partial M}$ gives a subspace $L_M = \pi(EL_M) \subset F_{\partial M}$.

Proposition 1 *Suppose there is a unique solution*[5] *to* $\Delta\varphi + V'(\varphi) = 0$ *for any Dirichlet boundary condition* $\varphi|_{\partial M} = \eta$. *Then* $\pi(EL_M)$ *is a Lagrangian submanifold of* $F_{\partial M}$.

Indeed, in this case L_M is the graph of a map $\Omega^0(\partial M) \to F_{\partial M}$ given by $\eta \mapsto (p_\partial = \pi((-1)^{n-1} * d\varphi), \eta)$ where φ is the unique solution to the Dirichlet problem with boundary conditions η.

The space of boundary fields has a natural Lagrangian fibration $\pi_\partial : T^*(\Omega^0(\partial M)) \to \Omega^0(\partial M)$. This fibration corresponds to Dirichlet boundary conditions: we fix the value $\varphi|_{\partial M} = \eta$ and impose no conditions on $p|_{\partial M}$, i.e. we impose boundary condition $(p, \varphi)|_{\partial M} \in \pi_\partial^{-1}(\eta)$.

Another natural family of boundary conditions, Neumann boundary conditions, correspond to the Larganian fibration of $T^*(\Omega^0(\partial M)) \simeq \Omega^{n-1}(\partial M) \oplus \Omega^0(\partial M)$ where the base is $\Omega^{n-1}(\partial M)$. In the case we fix $*_\partial i^*(p) = \eta \in \Omega^0(\partial M)$. The

[5] It is unique if $-V(\varphi)$ is convex.

intersection of L_M and the fiber over η is the set of pairs $(*_\partial \eta, \xi) \in \Omega^{n-1}(\partial M) \oplus \Omega^0(\partial M)$ where $\xi = i^*(\phi)$ and ϕ is a solution to the Neumann problem

$$\Delta\phi + V'(\phi) = 0, \quad \partial_n \phi|_{\partial M} = \eta$$

where ∂_n is the normal derivative of ϕ at the boundary.

3.3 Classical Yang-Mills Theory

Space time is again a smooth compact oriented Riemannian manifold M. Let G be a compact semisimple, connected, simply-connected Lie group with Lie algebra $\mathfrak{g}$. We assume that it is a matrix group, i.e. we fix an embedding of G into $\mathrm{Aut}(V)$, and hence an embedding of $\mathfrak{g}$ into $\mathrm{End}(V)$ such that the Killing form on $\mathfrak{g}$ is $< a, b >= tr(ab)$. The space of fields in the first order Yang-Mills theory is

$$F_M = \Omega^1(M, \mathfrak{g}) \oplus \Omega^{n-2}(M, \mathfrak{g}) \tag{10}$$

where we think of $\Omega^1(M, \mathfrak{g})$ as the space of connections on a trivial G-bundle over M. If we use a nontrivial G-bundle over M then the first term should be replaced by the corresponding space of connections. We denote an element of F_M by an ordered pair (A, B), $A \in \Omega^1(M, \mathfrak{g})$ and $B \in \Omega^{n-2}(M, \mathfrak{g})$. The action functional is

$$S_M(A, B) = \int_M \mathrm{tr}(B \wedge F(A)) - \frac{1}{2}\int_M \mathrm{tr}(B \wedge *B) \tag{11}$$

where $F(A) = dA + A \wedge A$ is the curvature of A as a connection.[6]

After integrating by part we can write the variation of the action as the sum of bulk and boundary parts:

$$\delta S_M(A, B) = \int_M \mathrm{tr}(\delta B \wedge (F(A) - *B) + \delta A \wedge d_A B) - \int_{\partial M} \mathrm{tr}(\delta A \wedge B) \tag{12}$$

The space EL_M of all solution to Euler-Lagrange equations is the space of pairs (A, B) which satisfy

$$B = *F(A), \quad d_A B = 0$$

[6] We will use notations $A \wedge B = \sum_{\{i\}\{j\}} A_{\{i\}} B_{\{j\}} dx^{\{i\}} \wedge dx^{\{j\}}$ for matrix-valued forms A and B. Here $\{i\}$ is a multiindex $\{i_1, \dots, i_k\}$ and x^i are local coordinates on M. We will also write $[A \wedge B]$ for $\sum_{\{i\}\{j\}}[A_{\{i\}}, B_{\{j\}}] dx^{\{i\}} \wedge dx^{\{j\}}$.

3.3.1 Boundary structure

The boundary term of the variation defines the one-form on the space boundary fields $F_{\partial M} = \Omega^1(\partial M, \mathfrak{g}) \oplus \Omega^{n-2}(\partial M, \mathfrak{g})$.

$$\alpha_{\partial M} = -\mathrm{tr} \int\limits_{\partial M} \delta A \wedge B \in \Omega^1(F_{\partial M}). \tag{13}$$

Its differential defines the symplectic form $\omega_{\partial M} = \int\limits_{\partial M} \mathrm{tr}(\delta A \wedge \delta B)$.

Note that, similarly to the scalar field theory, boundary fields can be regarded as $T^*\Omega^1(\partial M, \mathfrak{g})$ where we identify cotangent spaces with $\Omega^{n-2}(\partial M, \mathfrak{g})$, tangent spaces with $\Omega^1(\partial M, \mathfrak{g})$ with the natural pairing

$$\beta(\alpha) = \mathrm{tr} \int\limits_{\partial M} \alpha \wedge \beta$$

The projection map $\pi : F_M \to F_{\partial M}$ which is the restriction (pull-back) of forms to the boundary defines the subspace $L_M = \pi(EL_M)$ of the space of boundary values of solutions to the Euler-Lagrange equations on M.

3.3.2 On Lagrange property of L_M

Let us show that this subspace is Lagrangian for Maxwell's electrodynamics, i.e. for the Abelian Yang-Mills with $G = \mathbb{R}$. In this case Euler-Lagrange equations are

$$B = *dA, \quad d * dA = 0$$

Fix Dirichlet boundary condition $i^*(A) = a$. Let A_0 be a solution to this equation satisfying Laurenz gauge condition $d^*A_0 = 0$. Such solution is a harmonic 1-form, $(dd^* + d^*d)A_0 = 0$ with boundary condition $i^*(A_0) = a$. If A_0' is another such form, then $A_0 - A_0'$ is a harmonic 1-form with boundary condition $i^*(A_0 - A_0') = 0$. The space of such forms is naturally isomorphic to $H^1(M, \partial M)$. Each of these solutions gives the same value for $B = *dA = *dA_0$ and therefore its boundary value $b = i^*(B)$ is uniquely determined by a. Therefore the projection of EL_M to the boundary is a graph of the map $a \to b$ and thus L_M is a Lagrangian submanifold.

The Dirichlet and Neumann boundary value problems for Yang-Mills theory were studied in [25].

Conjecture 1 *The submanifold L_M is Lagrangian for non-Abelian Yang-Mills theory.*

It is clear that this is true for small connections, when we can rely on perturbation theory starting from an Abelian connection. It is also easy to prove that L_M is isotropic.

3.3.3 The Cauchy subspace

Define the *Cauchy subspace*

$$C_{\partial M} = \pi_\epsilon(EL_{\partial M_\epsilon}) \tag{14}$$

where $\partial M_\epsilon = [0, \epsilon) \times \partial M$ and $\pi_\epsilon : F_{\partial M_\epsilon} \to F_{\partial M}$ is the restriction of fields to $\{0\} \times \partial M$. In other words $C_{\partial M}$ is the space of boundary values of solution to Euler-Lagrange equations in $\partial M_\epsilon = [0, \epsilon) \times \partial M$. It is easy to see that[7]

$$C_{\partial M} = \{(A, B)|d_A B = 0\}$$

We have natural inclusions

$$L_M \subset C_{\partial M} \subset F_{\partial M}$$

3.3.4 Gauge transformations

The automorphism group of the trivial principal G-bundle over M can be naturally identified with $C^\infty(M, G)$. Bundle automorphisms act on the space of Yang-Mills fields. Thinking of a connection A as an element $A \in \Omega^1(M, \mathfrak{g})$ we have the following formulae for the action of the bundle automorphism (gauge transformation) g on fields:

$$g : A \mapsto A^g = g^{-1}Ag + g^{-1}dg, \quad B \mapsto B^g = g^{-1}Bg. \tag{15}$$

Note that the curvature $F(A)$ is a 2-form and it transforms as $F(A^g) = g^{-1}F(A)g$. Also, if we have two connections A_1 and A_2, their difference is a 1-form and $A_1^g - A_2^g = g^{-1}(A_1 - A_2)g$.

The Yang-Mills functional is invariant under this symmetry:

$$S_M(A^g, B^g) = S_M(A, B) \tag{16}$$

which is just the consequence of the cyclic property of the trace.

The restriction to the boundary gives the projection map of gauge groups $\tilde{\pi} : G_M \to G_{\partial M}$ which is a group homomorphism. This map is surjective, so we obtain an exact sequence

$$0 \to \mathrm{Ker}(\tilde{\pi}) \to G_M \to G_{\partial M} \to 0 \tag{17}$$

---

[7] The subspace $C_{\partial M}$ also makes sense also in scalar field theory, where explicitly it consists of pairs $(p, \varphi) \in \Omega^{n-1}(\partial M) \oplus \Omega^0(\partial M)$ where p is the pullback of $p_0 = *d\varphi_0$ and φ is the boundary value of φ_0 which solves the Euler-Lagrange equation $\Delta\varphi_0 - V'(\varphi_0) = 0$. Since Cauchy problem has unique solution in a small neighborhood of the boundary, $C_{\partial M} = F_{\partial M}$ for the scalar field.

where $\mathrm{Ker}(\tilde{\pi})$ is the group of gauge transformations acting trivially at the boundary.

It is easy to check that boundary gauge transformations $G_{\partial M}$ preserve the symplectic form $\omega_{\partial M}$. The action of G_M induces an infinitesimal action of the Lie algebra $\mathfrak{g}_M = C^\infty(M, \mathfrak{g})$ of G_M by vector fields on F_M. For $\lambda \in \mathfrak{g}_M$ we denote by $(\delta_\lambda A, \delta_\lambda B)$ the tangent vector to F_M at the point (A, B) corresponding to the action of λ:

$$\delta_\lambda A = -[\lambda, A] + d\lambda = d_A \lambda, \delta_\lambda B = -[\lambda, B] \tag{18}$$

where the bracket is the pointwise commutator (we assume that $\mathfrak{g}$ is a matrix Lie algebra). Recall that the action of a Lie group on a symplectic manifold is Hamiltonian if vector fields describing the action of the Lie algebra $Lie(G)$ are Hamiltonian.

We have the following

Theorem 1 *The action of $G_{\partial M}$ on $F_{\partial M}$ is Hamiltonian.*

Indeed, let f be a function on $F_{\partial M}$ and let $\lambda \in \mathfrak{g}_{\partial M}$. Let $\delta_\lambda f$ denote the Lie derivative of the corresponding infinitesimal gauge transformation. Then

$$\delta_\lambda f(A, B) = \int_{\partial M} \mathrm{tr}\left(\frac{\delta f}{\delta A} \wedge d_A \lambda + \frac{\delta f}{\delta B} \wedge [\lambda, B]\right). \tag{19}$$

Let us show that this is the Poisson bracket $\{H_\lambda, f\}$ where

$$H_\lambda = \int_{\partial M} \mathrm{tr}(\lambda d_A B). \tag{20}$$

The Poisson bracket on functions on $F_{\partial M}$ is given by

$$\{f, g\} = \int_{\partial M} \mathrm{tr}\left(\frac{\delta f}{\delta A} \wedge \frac{\delta g}{\delta B} - \frac{\delta g}{\delta A} \wedge \frac{\delta f}{\delta B}\right). \tag{21}$$

We have

$$\frac{\delta H_\lambda}{\delta A} = \frac{\delta}{\delta A}\left(\int_{\partial M} \mathrm{tr}(\lambda\, dB + \lambda[A \wedge b])\right) = [\lambda, B] \tag{22}$$

and, using integration by parts:

$$\frac{\delta H_\lambda}{\delta B} = d_A B = dB + [A \wedge B]. \tag{23}$$

This proves the statement.

An important corollary of this fact is that the Hamiltonian action of G_M induces a moment map $\mu : F_{\partial M} \to \mathfrak{g}^*_{\partial M}$, and it is clear that

$$C_{\partial M} = \mu^{-1}(0)$$

This implies that $C_{\partial M} \subset F_{\partial M}$ *is a coisotropic submanifold.*

Remark 3 Let us show directly that $C_{\partial M} \subset F_{\partial M}$ is a coisotropic subspace of the symplectic space $F_{\partial M}$ when $\mathfrak{g} = \mathbb{R}$. We need to show that $C^{\perp}_{\partial M} \subset C_{\partial M}$ where $C^{\perp}$ is the symplectic orthogonal to C.

The subspace $C^{\perp}_{\partial M}$ consists of all $(\alpha, \beta) \in \Omega^1(\partial M) \oplus \Omega^{n-2}(\partial M)$ such that

$$\int_{\partial M} a \wedge \beta + \int_{\partial M} \alpha \wedge b = 0 \tag{24}$$

for all $(a, b) \in C_{\partial M} \subset \Omega^1(\partial M) \oplus \Omega^{n-2}(\partial M)$. This condition for all a gives that $\beta = 0$ and requiring this condition for all b gives that α is exact, so we have $C^{\perp}_{\partial M} = \Omega^1_{\text{ex}}(\partial M) \subset C_{\partial M}$ as desired.

3.3.5 Reduction by gauge symmetry

The differential δS_M of the action functional is the sum of the bulk term defining the Euler-Lagrange equations and of the boundary term defining the 1-form $\alpha_{\partial M}$ on the space of boundary fields. The bulk term vanishes on solutions of the Euler-Lagrange equations, so we have

$$\delta S_M|_{EL_M} = \pi^*(\alpha_{\partial M}|_{L_M}) \tag{25}$$

where $\pi : F_M \to F_{\partial M}$ is the restriction to the boundary and $L_M = \pi(EL_M)$. This is analogous to the property of the Hamilton-Jacobi action in classical mechanics.

Because S_M is gauge invariant, it defines the functional on gauge classes of fields and thus, on gauge classes of solutions to Euler-Lagrange equations. Passing to gauge classes we now replace the chain of inclusions of gauge invariant subspaces $L_M \subset C_{\partial M} \subset F_{\partial M}$ with the chain of inclusions of corresponding gauge classes

$$L_M/G_{\partial M} \subset C_{\partial M}/G_{\partial M} \subset F_{\partial M}/G_{\partial M}. \tag{26}$$

The rightmost space is a Poisson manifold since the action of $G_{\partial M}$ is Hamiltonian. The middle space is the Hamiltonian reduction of $C_{\partial M}$ and is a symplectic leaf in the rightmost space. The leftmost space is still Lagrangian by the standard arguments from symplectic geometry.

3.3.6 Gauge invariant Lagrangian fibrations on the boundary

A natural Lagrangian fibration $p_\partial : \Omega^{n-2}(\partial M, \mathfrak{g}) \oplus \Omega^1(\partial M, \mathfrak{g}) \to \Omega^1(\partial M, \mathfrak{g})$ corresponds to the Dirichlet boundary conditions when we fix the pull-back of A to the boundary: $a = i^*(A)$. Such boundary conditions are compatible with the gauge action. Another example of the family of gauge invariant boundary conditions corresponds to Neumann boundary conditions and is given by the Lagrangian fibration $p_\partial : \Omega^{n-2}(\partial M, \mathfrak{g}) \oplus \Omega^1(\partial M, \mathfrak{g}) \to \Omega^{n-2}(\partial M, \mathfrak{g})$.

3.4 Classical Chern–Simons Theory

3.4.1 Classical theory with boundary

Spacetimes for classical Chern-Simons field theory are smooth, compact, oriented 3-manifolds. Let M be such manifold fields F_M on M are connections on the trivial G-bundle over M with G being compact, semisimple, connected, simply connected Lie group. We will identify the space of connections with the space of 1-forms $\Omega^1(M, \mathfrak{g})$. The action functional is

$$S(A) = \int_M \mathrm{tr}\left(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A\right) \tag{27}$$

where A is a connection.

The variation is

$$\delta S_M(A) = \int_M \mathrm{tr}(F(A) \wedge \delta A) + \frac{1}{2} \int_{\partial M} \mathrm{tr}(A \wedge \delta A) \tag{28}$$

so the space of solutions EL_M to the Euler-Lagrange equations is the space of flat connections:

$$EL_M = \{A | F(A) = 0\}$$

The boundary term defines the 1-form on boundary fields (connections on the trivial G-bundle over the boundary which we will identify with $\Omega^1(\partial M)$):

$$\alpha_{\partial M} = -\frac{1}{2} \int_{\partial M} \mathrm{tr}(A \wedge \delta A). \tag{29}$$

This 1-form on boundary fields defines the symplectic structure on the space of boundary fields:

$$\omega_{\partial M} = \delta\alpha_{\partial M} = -\frac{1}{2}\mathrm{tr}\int_{\partial M} \delta A \wedge \delta A \tag{30}$$

3.4.2 Gauge symmetry and the boundary cocycle

The gauge group G_M is the group of bundle automorphisms of of the trivial principal G-bundle over M. It can be naturally be identified with the space of smooth maps $M \to G$ which transform connections as in (15) and we have:

$$S_M(A^g) = S_M(A) + \frac{1}{2}\mathrm{tr}\int_{\partial M} (g^{-1}Ag \wedge g^{-1}\,dg) - \frac{1}{6}\mathrm{tr}\int_M g^{-1}\,dg \wedge g^{-1}\,dg \wedge g^{-1}\,dg. \tag{31}$$

Assume the integrality of the Maurer-Cartan form on G:

$$\theta = -\frac{1}{6}\mathrm{tr}(dg\,g^{-1} \wedge dg\,g^{-1} \wedge dg\,g^{-1})$$

i.e. we assume that the normalization of the Killing form is chosen in such a way that $[\theta] \in H^3(M, \mathbb{Z})$. Then for a closed manifold M the expression

$$W_M(g) = -\frac{1}{6}\mathrm{tr}\int_M dg\,g^{-1} \wedge dg\,g^{-1} \wedge dg\,g^{-1}$$

is an integer and therefore $S_M \mod \mathbb{Z}$ is gauge invariant (for details see for example [20]).

Proposition 2 *When the manifold M has a boundary, the functional $W_M(g) \mod \mathbb{Z}$ depends only on the restriction of g to ∂M.*

Indeed, let M' be another manifold with the boundary $\partial M'$ which differs from ∂M only by reversing the orientation, so that the result of the gluing $M \cup M'$ along the common boundary is smooth. Then

$$W_M(g) - W_{M'}(g') = -\frac{1}{6}\mathrm{tr}\int_{M\cup M'}\int_M d\tilde{g}\tilde{g}^{-1} \wedge d\tilde{g}\tilde{g}^{-1} \wedge d\tilde{g}\tilde{g}^{-1} \in \mathbb{Z}$$

Here $\tilde{g}$ is the result of gluing maps g and g' into a map $M \cup M' \to G$. Therefore, modulo integers, it does not depend on g and g'.

For a a connection on the trivial principal G-bundle over a 2-dimensional manifold Σ and for $g \in C^\infty(\Sigma, G)$ define

$$c_\Sigma(a, g) = \exp\left(2\pi i\left(\frac{1}{2}\int_{\partial M} \mathrm{tr}(g^{-1}ag \wedge g^{-1}dg) + W_\Sigma(g)\right)\right)$$

Here we wrote $W_\Sigma(g)$ because $W_M(g) \mod \mathbb{Z}$ depends only on the value of g on ∂M.

The transformation property (31) of the Chern-Simons action implies that the functional

$$\exp(2\pi i S_M(A))$$

transforms as

$$\exp(2\pi i S_M(A^g)) = \exp(2\pi i S_M(A)) c_{\partial M}(i^*(A), i^*(g))$$

where i^* is the restriction to the boundary (pull-back). For further details on gauge aspects of Chern-Simons theory see [20, 21].

Now we can define the gauge invariant version of the Chern-Simons action. Consider the trivial circle bundle $\mathcal{L}_M = S^1 \times F_M$ with the natural projection $\mathcal{L}_M \to F_M$. Define the action of G_M on $\mathcal{L}_M$ as

$$g : (\lambda, A) \mapsto (\lambda c_{\partial M}(i^*(A), i^*(g)), A^g)$$

The functional $\exp(2\pi i S_M(A))$ is a G_M-invariant section of this bundle. The restriction of $\mathcal{L}_M$ to the boundary gives the trivial S^1-bundle over $F_{\partial M}$ with the $G_{\partial M}$-action

$$g : (\lambda, A) \mapsto (\lambda c_{\partial M}(A, g), A^g)$$

The 1-form $\alpha_{\partial M}$ is a $G_{\partial M}$-invariant connection of $\mathcal{L}_{\partial M}$. The curvature of this connection is the $G_{\partial M}$-invariant symplectic form ω_∂.

By definition of $\alpha_{\partial M}$ we have the Hamilton-Jacobi property of the action:

$$\delta S_M|_{EL_M} = \pi^*(\alpha_{\partial M}|_{L_M}). \tag{32}$$

3.4.3 Reduction

Now, when the gauge symmetry of the Chern-Simons theory is clarified, let us pass to gauge classes. The action of boundary gauge transformations on $F_{\partial M}$ is Hamiltonian with respect to the symplectic form (30). It is easy to check (and it is well known) that the vector field on $F_{\partial M}$ generating infinitesimal gauge transformation $A \to A + d_A\lambda$ is Hamiltonian with the generating function

$$H_\lambda(A) = \int_{\partial M} \operatorname{tr}(F(A)\lambda). \tag{33}$$

This induces the moment map $\mu : F_{\partial M} \to \mathfrak{g}^*_{\partial M}$ given by $\mu(A)(\lambda) = H_\lambda(A)$.

Let $C_{\partial M}$ be the space of Cauchy data, i.e. boundary values of connections which are flat in a small neighborhood of the boundary. It can be naturally identified with the space of flat G-connections on ∂M and thus, $C_{\partial M} = \mu^{-1}(0)$. Hence $C_{\partial M}$ is a coisotropic submanifold of $F_{\partial M}$. We have a chain of inclusions

$$L_M = \pi(EL_M) \subset C_{\partial M} \subset F_{\partial M} \tag{34}$$

where L_M is the space of flat connections on ∂M which extend to flat connections on M. Using Poincaré-Lefschetz duality for de Rham cohomology with coefficients in a local system, one can easily show that L_M is Lagrangian.

We have following inclusions of the spaces of gauge classes

$$L_M/G_{\partial M} \subset C_{\partial M}/G_{\partial M} \subset F_{\partial M}/G_{\partial M} \tag{35}$$

where the middle term is the Hamiltonian reduction $\mu^{-1}(0)/G_{\partial M} \cong \underline{C}_{\partial M}$, which is symplectic. The left term is Lagrangian, and the right term is Poisson. Note that the middle term is a finite dimensional symplectic leaf of the infinite dimensional Poisson manifold $F_{\partial M}/G_{\partial M}$.

The middle term $C_{\partial M}/G_{\partial M}$ is the moduli space $\mathcal{M}^G_{\partial M}$ of flat G-connections on ∂M. It is naturally isomorphic to the representation variety:

$$\mathcal{M}^G_{\partial M} \cong \mathrm{Hom}(\pi_1(\partial M), G)/G$$

where G acts on $\mathrm{Hom}(\pi_1(M), G)$ by conjugation. We will denote the symplectic structure on this space by $\underline{\omega}_{\partial M}$.

Similarly, we have $EL_M/G_M = \mathcal{M}^G_M \cong \mathrm{Hom}(\pi_1(M), G)/G$, which is the moduli space of flat G-connections on M. Unlike in Yang-Mills case, these spaces are finite-dimensional.

The image of the natural projection $\pi : \mathcal{M}^G_M \to \mathcal{M}^G_{\partial M}$ is the reduction of L_M which we will denote by $\underline{L}_M = L_M/G_M$.

Reduction of $\mathcal{L}_M$ and of $\mathcal{L}_{\partial M}$ gives line bundles $\underline{\mathcal{L}}_M = \mathcal{L}_M/G_M$ and $\underline{\mathcal{L}}_{\partial M} = \mathcal{L}_{\partial M}/G_{\partial M}$ over $\mathcal{M}^G_M$ and $\mathcal{M}^G_{\partial M}$ respectively. The 1-form $\alpha_{\partial M}$ which is also a $G_{\partial M}$-invariant connection on $\mathcal{L}_{\partial M}$ becomes a connection on $\underline{\mathcal{L}}_{\partial M}$ with the curvature $\underline{\omega}_{\partial M}$.

The Chern-Simons action yields a section cs of the pull-back of the line bundle $\mathcal{L}_{\partial M}$ over $\mathcal{M}^G_{\partial M}$. Because $\underline{L}_M$ is a Lagrangian submanifold, the symplectic form $\underline{\omega}_{\partial M}$ vanishes on it and the restriction of the connection $\underline{\alpha_{\partial M}}$ to $\underline{L}_M$ results in a flat connection over $\mathcal{L}_{\partial M}|_{\underline{L}_M}$. The section cs is horizontal with respect to the pull-back of the connection $\underline{\alpha_{\partial M}}$. It can be written as

$$(d - \pi^*(\underline{\alpha_{\partial M}}|_{L_M}))cs = 0. \tag{36}$$

This collection of data is the *reduced Hamiltonian structure* of the Chern-Simons theory.

3.4.4 Complex polarization

There are no natural non-singular Lagrangian fibrations on the space of connections on the boundary which are compatible with the gauge action. However, for formal semiclassical quantization we need such fibration only to exist locally near a preferred point in the space of connections. Now we will describe another structure on the space of boundary fields for the Chern-Simons theory which is used in geometric quantization [4].

Instead of looking for a real Lagrangian fibration, let us choose a complex polarization of $\Omega^1(M, \mathfrak{g})_{\mathbb{C}}$. Fixing a complex structure on the boundary, gives us the natural decomposition

$$\Omega^1(\partial M, \mathfrak{g})_{\mathbb{C}} = \Omega^{1,0}(\partial M, \mathfrak{g})_{\mathbb{C}} \oplus \Omega^{0,1}(\partial M, \mathfrak{g})_{\mathbb{C}}$$

and we can define boundary fibration as the natural projection to $\Omega^{1,0}(\partial M, \mathfrak{g})_{\mathbb{C}}$. Here elements of $\Omega^{1,0}(\partial M, \mathfrak{g})_{\mathbb{C}}$ are $\mathfrak{g}_{\mathbb{C}}$-valued forms which locally can be written as $a(z, \overline{z})\, dz$ and elements of $\Omega^{0,1}(\partial M, \mathfrak{g})_{\mathbb{C}}$ can be written as $b(z, \overline{z})\, d\overline{z}$. The decomposition above locally works as follows:

$$A = \mathcal{A} + \overline{\mathcal{A}}$$

where $\mathcal{A} = a(z, \overline{z})\, dz$.

In terms of this decomposition the symplectic form is

$$\omega = \int_{\partial M} \mathrm{tr}\, \delta\mathcal{A} \wedge \delta\overline{\mathcal{A}}$$

It is clear that subspaces $\mathcal{A} + \Omega^{0,1}(\partial M)$ are Lagrangian in the complexification of $\Omega(M, \mathfrak{g})$. Thus, we have a Lagrangian fibration $\Omega(M, \mathfrak{g})_{\mathbb{C}} \to \Omega^{0,1}(M, \mathfrak{g})_{\mathbb{C}}$. The action of the gauge group preserves the fibers.

However, the form $\alpha_{\partial M}$ does not vanish of these fibers. To make it vanish we should modify the action as

$$\tilde{S}_M = S_M + \frac{1}{2} \int_{\partial M} \mathrm{tr}\, (\mathcal{A} \wedge \overline{\mathcal{A}})$$

After this modification, the boundary term in the variation of the action gives the form

$$\tilde{\alpha}_{\partial M} = -\int_{\partial M} \mathrm{tr}\, (\overline{\mathcal{A}} \wedge \delta\mathcal{A})$$

This form vanishes on fibers. It is not gauge invariant as well as the modified action. The modified action transforms under gauge transformations as

$$\tilde{S}_M(A^g) = \tilde{S}_M(A) + \frac{1}{2}\mathrm{tr}\int\limits_{\partial M} (g^{-1}\mathcal{A}g \wedge g^{-1}\overline{\partial}g) + W_M(g)$$

This gives the following cocycle on the boundary gauge group

$$\tilde{c}_\Sigma(A, g) = \exp(2\pi i(\frac{1}{2}\int\limits_\Sigma \mathrm{tr}(g^{-1}\mathcal{A}g \wedge g^{-1}\overline{\partial}g) + W_\Sigma(g)))$$

This modification of the action and this complex polarization of the space of boundary fields is important for geometric quantization in Chern-Simons theory [4] and is important for understanding the relation between the Chern-Simons theory and the WZW theory, see for example [1, 16]. We will not expand this direction here, since we are interested in formal semiclassical quantization where real polarizations are needed.

3.5 BF-Theory

Space time M is smooth, oriented[8] and compact and is equipped with a trivial G-bundle where G is connected, simple or abelian compact Lie group. Fields are

$$F_M = \Omega^1(M, \mathfrak{g}) \oplus \Omega^{n-2}(M, \mathfrak{g}) \tag{37}$$

where $\Omega^1(M, \mathfrak{g})$ describes connections on the trivial G-bundle.

The action functional of the BF theory is the topological term of Yang-Mills action:

$$S_M(A, B) = \int\limits_M \mathrm{tr}(B \wedge F(A)). \tag{38}$$

For the variation of S_M we have:

$$\delta S_M = \mathrm{tr}\int\limits_M \delta B \wedge F(A) + (-1)^{n-1}\mathrm{tr}\int\limits_M d_A B \wedge \delta A + (-1)^{n-1}\mathrm{tr}\int\limits_{\partial M} B \wedge \delta A. \tag{39}$$

The bulk term gives Euler-Lagrange equations:

$$EL_M = \{(A, B) : F(A) = 0, \quad d_A B = 0\}. \tag{40}$$

The boundary term gives a 1-form on the space of boundary fields $F_{\partial M} = \Omega^1(\partial M, \mathfrak{g}) \oplus \Omega^{n-2}(\partial M, \mathfrak{g})$:

[8] The orientability assumption can be dropped, see [15].

$$\alpha_{\partial M} = \int_{\partial M} \mathrm{tr}(B \wedge \delta A). \tag{41}$$

The corresponding exact symplectic form is

$$\omega_{\partial M} = \delta\alpha_{\partial M} = \int_{\partial M} \mathrm{tr}(\delta B \wedge \delta A). \tag{42}$$

The space of Cauchy data is

$$C_{\partial M} = \{(A, B)|F_A = 0, \quad d_A B = 0\}$$

Boundary values of solutions of the Euler-Lagrange equations on M define the submanifold $L_M = \pi(EL_M) \subset F_{\partial M}$. This submanifold is Lagrangian. Thus we have the embedding:

$$L_M \subset C_{\partial M} \subset F_{\partial M}$$

where $F_{\partial M}$ is exact symplectic, $C_{\partial M}$ is co-isotropic, and L_M is Lagrangian.

3.5.1 Gauge symmetry and reduction

The space of bundle automorphisms G_M is the space of smooth maps $M \to G$. They act on $A \in \Omega^1(M, \mathfrak{g})$ by $A \mapsto g^{-1}Ag + g^{-1}dg$ and on $B \in \Omega^{n-2}(M, \mathfrak{g})$ by $B \mapsto g^{-1}Bg$. As in Yang-Mills theory the action is invariant with respect to these transformations.

In addition, it is almost invariant with respect to transformations $A \mapsto A$, $B \mapsto B + d_A\beta$ where $\beta \in \Omega^{n-3}(M, \mathfrak{g})$:

$$S_M(A, B + d_A\beta) = S_M(A, B) + \int_M \mathrm{tr}(d_A\beta \wedge F(A)). \tag{43}$$

After integration by parts in the second term we write it as

$$\int_M \mathrm{tr}(\beta \wedge d_A F(A)) + \int_{\partial M} \mathrm{tr}(\beta \wedge F(A)). \tag{44}$$

The bulk term here vanishes because of the Bianchi identity and the only additional contribution is a boundary term, thus:

$$S_M(A, B + d_A\beta) = S_M(A, B) + \operatorname{tr}\int_{\partial M} (\beta \wedge F(A))$$

The additional gauge symmetry $B \mapsto B + d_A\beta$ gives us a larger gauge group

$$G_M^{BF} = G_M \times \Omega_M^{n-3}. \tag{45}$$

Its restriction to the boundary gives the boundary gauge group

$$G_{\partial M}^{BF} = G_{\partial M} \times \Omega_{\partial M}^{n-3}. \tag{46}$$

The action is invariant up to a boundary term. This means that the 1-form $\alpha_{\partial M}$ is not gauge invariant. Indeed, it is invariant with respect to G_M-transformations, but when $(A, B) \mapsto (A, B + d_A\beta)$ the forms $\alpha_{\partial M}$ transforms as

$$\alpha_{\partial M} \mapsto \alpha_{\partial M} + \int_{\partial M} \operatorname{tr} d_A\beta \wedge \delta A$$

However, it is clear that the symplectic form $\omega_{\partial M} = \delta\alpha_{\partial M}$ is gauge invariant. Moreover, we have the following.

Theorem 2 *The action of $G_{\partial M}^{BF}$ is Hamiltonian.*

Indeed, if $\alpha \in \Omega^0(\partial M, \mathfrak{g})$ is an element of the Lie algebra of boundary gauge transformations and $\beta \in \Omega^{n-3}(\partial M, \mathfrak{g})$, then we can take

$$H_\alpha(A, B) = \int_{\partial M} \operatorname{tr}(B \wedge d_A\alpha) \tag{47}$$

$$H_\beta(A, B) = \int_{\partial M} \operatorname{tr}(A \wedge d_A\beta) \tag{48}$$

as Hamiltonians generating the action of corresponding infinite dimensional Lie algebra.

This defines a moment map $\mu : F_{\partial M} \to \Omega^0(\partial M, \mathfrak{g}) \oplus \Omega^{n-3}(\partial M, \mathfrak{g})$. It is clear that Cauchy submanifold is also $C_{\partial M} = \mu^{-1}(0)$. This proves that it is a co-isotropic submanifold.

Note also, that the restriction of $\alpha_{\partial M}$ to $C_{\partial M}$ is $G_{\partial M}^{BF}$-invariant. Indeed $\operatorname{tr}\int_{\partial M} d_A\beta \wedge \delta A = -\operatorname{tr}\int_{\partial M} \beta \wedge d_A\delta A$, and this expression vanishes when the form is pulled-back to the space of flat connections where $d_A\delta A = 0$. Therefore the Hamiltonian reduction of $F_{\partial M}$ which is $\underline{F}_{\partial M} = C_{\partial M}/G_{\partial M}^{BF}$ is an exact symplectic manifold.

It is easy to see that the reduced space of fields on the boundary $\underline{F}_{\partial M}$ can be naturally identified, as a symplectic manifold, with $T^*\mathcal{M}_{\partial M}^G$, the cotangent bundle to the moduli space of flat connections $\mathcal{M}^G_{\partial M} = \mathrm{Hom}(\pi_1(\partial M), G)/G$. The canonical 1-form on this cotangent bundle corresponds to the form $\alpha_{\partial M}$ restricted to $C_{\partial M}$. The Lagrangian subspace $L_M \subset F_{\partial M}$ is gauge invariant. It defines the Lagrangian submanifold

$$L_M/G^{BF}_{\partial M} \subset T^*\mathcal{M}^G_{\partial M}$$

The restriction of the action functional to EL_M is gauge invariant and defines the the function $\underline{S}_M$ on $EL_M/G^{BF}_{\partial M}$. The formula for the variation of the action gives the analog of the Hamilton-Jacobi formula:

$$d\underline{S}_M = \pi^*(\theta|_{L_M}) \tag{49}$$

where θ is the canonical 1-form on the cotangent bundle $T^*\mathcal{M}^G_{\partial M}$ restricted to $L_M/G^{BF}_{\partial M}$.

3.5.2 A gauge invariant Lagrangian fibration

One of the natural choices of boundary conditions is the Dirichlet boundary conditions. This is the Lagrangian fibration $\Omega^1(M, \mathfrak{g}) \oplus \Omega^{n-2}(M, \mathfrak{g}) \to \Omega^1(M, \mathfrak{g})$. This fibration is gauge invariant. After the reduction it gives the standard Lagrangian fibration $T^*\mathcal{M}^G_{\partial M} \to \mathcal{M}^G_{\partial M}$.

4 Semiclassical Quantization of First Order Field Theories

In this section, after reminding briefly the general framework of local quantum field theory, we will concentrate on a finite-dimensional toy model for the path integral. In this model partition functions satisfy the gluing axiom by general properties of measure theoretic integrals (the Fubini theorem). One can also model the gauge symmetry in this setting, treated by a version of the Faddeev-Popov trick. We will see that the arising integrals can be evaluated, in the asymptotics $h \to 0$, by the stationary phase formula. The result of such an evaluation we call a "formal integral" (alluding to integration over a formal neighborhood of a critical point, as well as the fact that we forget the measure-theoretic definition of the integral we started with). We will obtain the gluing formula for such formal integrals which is satisfied a priori, since the construction comes from measure integrals. In the usual setting of local quantum field theory, partition functions are the path integrals where a measure theoretic definition is not accessible, while the "formal integral" can be defined as a formal power series in h where coefficients are the Feynman diagrams. In this setting the gluing formulae are not automatic and have to be proven, cf. e.g. [22].

4.1 The Framework of Local Quantum Field Theory

We will follow the framework of local quantum field theory which was outlined by Atiyah and Segal for topological and conformal field theories. In a nutshell it is a functor from an appropriate category of cobordisms to the category of vector spaces (or, more generally, to some category).

In this sense, a quantum field theory is the assignment of a vector space to the boundary $N = \partial M$ of a space time manifold M and a vector in this vector space to the manifold M:

$$N \mapsto H(N), \quad M \mapsto Z_M \in H(\partial M).$$

The identification of such assignments with linear maps is natural assuming that the vector space assigned to the boundary is the tensor product of vector spaces assigned to connected components of the boundary and that changing the orientation replaces the corresponding vector space by its dual.

The vector space assigned to the boundary is the space of boundary states. It may depend on the extra structure at the boundary. In this case it is a vector bundle over the space of admissible geometric data and Z_M is a section of this vector bundle. The vector Z_M is called the partition function or the amplitude.

These data should satisfy natural axioms, which can by summarized as follows:

1. The locality properties of boundary states:

$$H(\varnothing) = \mathbb{C}, \quad H(N_1 \sqcup N_2) = H(N_1) \otimes H(N_2),$$

2. The locality property of the partition function

$$Z_{M_1 \sqcup M_2} = Z_{M_1} \otimes Z_{M_2} \in H(\partial M_1) \otimes H(\partial M_2).$$

3. For each space N (an object of the space time category) there is a non-degenerate pairing

$$\langle ., . \rangle_N : H(\overline{N}) \otimes H(N) \to \mathbb{C}$$

such that $\langle ., . \rangle_{N_1 \sqcup N_2} = \langle ., . \rangle_{N_1} \otimes \langle ., . \rangle_{N_2}$.

4. The canonical orientation reversing isomorphism $\sigma : N \to \overline{N}$ induces a $\mathbb{C}$-antilinear mapping $\widehat{\sigma}_N : H(N) \to H(\overline{N})$ which agrees with locality of N and $\widehat{\sigma}_{\overline{N}} \widehat{\sigma}_N = id_N$. Together with the pairing $\langle ., . \rangle_N$ the orientation reversing mapping induces the Hilbert space structure on $H(N)$.
5. An orientation preserving isomorphism[9] $f : N_1 \to N_2$ induces a linear isomorphism

[9] By an isomorphism here we mean a mapping preserving the corresponding geometric structure.

$$T_f : H(N_1) \to H(N_2).$$

which is compatible with the pairing and $T_{f \sqcup g} = T_f \otimes T_g, \quad T_{f \circ g} = T_f T_g$ (possibly twisted by a cocycle of the group of automorphisms of the boundary).

6. The gluing axiom. This pairing should agree with partition functions in the following sense. Let $\partial M = N \sqcup \overline{N} \sqcup N'$, then

$$(\langle ., . \rangle \otimes id) Z_M = Z_{\tilde{M}} \in H(N') \quad (50)$$

where $\tilde{M}$ is the result of gluing of N with $\overline{N}$. The operation is known as the gluing axiom. For more details see [8].

7. The quantum field theory is (projectively) invariant with respect to transformations of the space time (diffeomorphisms, gauge transformations etc.) if for such transformation $f : M_1 \to M_2$,

$$T_{f_\partial} Z_{M_1} = c_{M_1}(f) Z_{M_2}$$

Here $c_M(f)$ is a cocycle $c_M(fg) = c_{gM}(f) c_M(g)$. When the theory is invariant, not only projectively invariant, $c_M(f) = 1$.

Remark 4 The gluing axiom in particular implies the functoriality of Z:

$$Z_{M_1 \circ M_2} = Z_{M_1} \circ Z_{M_2} .$$

Here $M_1 \circ M_2$ is the composition of cobordisms in the category of space time manifolds. In case of cylinders this is the semigroup property of propagators in the operator formulation of QFT.

Remark 5 This framework is very natural in models of statistical mechanics on cell complexes with open boundary conditions, also known as lattice models.

Remark 6 The main physical concept behind this framework is the locality of the interaction. Indeed, we can cut our space time manifold in small pieces and the resulting partition function Z_M in such framework is expected to be the composition of partition functions of small pieces. Thus, the theory is determined by its structure on 'small' space time manifolds, or at 'short distances'. This is the concept of *locality*. To fully implement this concept one should consider the field theory on manifolds with corners where we can glue along parts of the boundary. In the case of topological theories, a particular realization of the concept of locality is the formalism of *extended/fully extended* topological quantum field theories of Baez-Dolan [7] and Lurie [23].

4.2 Path Integral and Its Finite Dimensional Model

4.2.1 Quantum Field Theory via Path Integrals

Given a first order classical field theory with boundary conditions given by Lagrangian fibrations, one can try to construct a quantum field theory by the path integral quantization. In this framework the space of boundary states $H(\partial M)$ is taken as the space of functionals on the base $B_{\partial M}$ of the Lagrangian fibration on boundary fields $F_{\partial M}$. The vector Z_M is the Feynman integral over the fields on the bulk with given boundary conditions

$$Z_M(b) = \int_{f \in \pi^{-1} p_\partial^{-1}(b)} e^{\frac{i}{\hbar} S_M(f)} Df \tag{51}$$

where Df is the integration measure, $\pi : F_M \to F_{\partial M}$ is the restriction map and $p_\partial : F_{\partial M} \to B_{\partial M}$ is the boundary fibration.

The integral above is difficult to define when the space of fields is infinite dimensional. To clarify the functorial structure of this construction and to define the formal semiclassical path integral let us start with a model case when the space of fields is finite dimensional, when the integrals are defined and absolutely convergent. A "lattice approximation" of a continuous theory is a good example of such a finite dimensional model.

4.2.2 Finite Dimensional Classical Model

A finite dimensional model of a first order classical field theory on a space time manifold with boundary consists of the following data. Three finite dimensional manifolds F, F_∂, B_∂ should be complemented by the following structures.

- The manifold F_∂ is endowed with an exact symplectic form $\omega_\partial = d\alpha_\partial$.
- A surjective submersion $\pi : F \to F_\partial$.
- A function S on F, such that the submanifold $EL \subset F$, on which the form $dS - \pi^*(\alpha_\partial)$ vanishes, projects to a Lagrangian submanifold in F_∂.
- A Lagrangian fibration of F_∂ given by $p_\partial : F_\partial \to B_\partial$ such that α_∂ vanishes on fibers. We also assume that fibers are transversal to $L = \pi(EL) \subset F_\partial$.

We will say that this is a finite dimensional model of a *non-degenerate theory* if S has finitely many simple critical points on each fiber $\pi^{-1} p_\partial^{-1}(b)$.

The model is gauge invariant with the bulk gauge group G and the boundary gauge group G_∂ if the following holds.

- The group G acts on F, and G_∂ acts on F_∂.
- There is a group homomorphism $\tilde{\pi} : G \to G_\partial$ such that the restriction map satisfies $\pi(gx) = \tilde{\pi}(g)\pi(x)$.

- The function S is invariant under the G-action up to boundary terms:

$$S(gx) = S(x) + c_\partial(\pi(x), \tilde{\pi}(g))$$

where $c_\partial(x, g)$ is a cocycle for G_∂ acting on F_∂:

$$c_\partial(x, gh) = c_\partial(hx, g) + c_\partial(x, h)$$

- The action of G_∂ is compatible with the fibration p_∂, i.e. it maps fibers to fibers. Assuming that the stabilizer subgroups $\mathrm{Stab}_b \subset G_\partial$ coincide for different fibers $\pi_\partial^{-1}(b)$, one can introduce a quotient group $\Gamma_\partial = G_\partial/\mathrm{Stab}_b$ acting on B_∂. One has then the quotient homomorphism $\tilde{p}_\partial : G_\partial \to \Gamma_\partial$. We require that the cocycle $c(g, x)$ is constant on fibers of p_∂, i.e. is a pullback of a cocycle $\tilde{c}$ of Γ_∂ acting on B_∂: $c(x, g) = \tilde{c}(p_\partial(x), \tilde{p}_\partial(g))$.

We will say that the theory with gauge invariance is non-degenerate if critical points of S form finitely many G-orbits and if the corresponding points on $F(b)/G$ are simple (i.e. isolated) on each fiber $F(b)$ of $p_\partial\pi$.

4.2.3 Finite Dimensional Quantum Model

To define quantum theory assume that F and B_∂ are defined together with measures dx and db respectively. Assume also that there is a measure $\frac{dx}{db}$ on each fiber $F(b) = \pi^{-1} p_\partial^{-1}(b)$ such that $dx = \frac{dx}{db} db$.

Define the vector space H_∂ together with the Hilbert space structure on it as follows:

$$H_\partial = L^2(B_\partial)$$

When the function S is only projectively invariant with respect to the gauge group, the space of boundary states is the space of L^2-sections of the corresponding line bundle.

Remark 7 It is better to consider the space of half-forms on B_∂ which are square integrable but we will not do it here. For details see for example [9].

The partition function Z_F is defined as an element of H_∂ given by the integral over the fiber $F(b)$:

$$Z_F(b) = \int_{F(b)} \exp\left(\frac{i}{h} S(x)\right) \frac{dx}{db} \tag{52}$$

When there is a gauge group the partition function transforms as

$$Z_F(\gamma b) = Z_F(b)\exp(\frac{i}{h}c_\partial(b,\gamma))$$

In such a finite dimensional model the gluing property follows from Fubini's theorem allowing to change the order of integration. Suppose we have two spaces F_1 and F_2 fibered over B_∂ and two functions S_1 and S_2 defined on F_1 and F_2 respectively such that integrals $Z_{F_1}(b)$ and $Z_{F_2}(b)$ converge absolutely for generic b. For example, we can assume that all spaces F, F_∂ and B_∂ are compact. Then changing the order of integration we have

$$\int_{B_\partial} Z_{F_1}(b)Z_{F_2}(b)\,db = Z_{F_1\times_{B_\partial}F_2} \tag{53}$$

where

$$Z_{F_1\times_{B_\partial}F_2} = \int_{F_1\times_{B_\partial}F_2} \exp(\frac{i}{h}(S_1(x_1)+S_2(x_2)))\frac{dx_1}{db}\frac{dx_2}{db}\,db$$

Here $F_1 \times_{B_\partial} F_2 = \{(x,x') \in F_1 \times F_2 | \pi_1(x) = \pi_2(x')\}$ is the fiber product of F_1 and F_2 over B_∂. The measure $\frac{dx}{db}\frac{dx'}{db}db$ is induced by measures on $F_1(b)$, $F_2(b)$ and on B_∂.

Remark 8 The quantization is not functorial. We need to make a choice of measure of integration.

Remark 9 We will not discuss here quantum statistical mechanics where instead of oscillatory integrals we have integrals of probabilistic type representing Boltzmann measure. Wiener integral is among the examples of such integrals.

Remark 10 When the gauge group is non-trivial, the important subgroup in the total gauge group is the bulk gauge group, i.e. the symmetry of the integrand in the formula for $Z_F(b)$. If Γ_∂ is the gauge group acting on the base of the boundary Lagrangian fibration, then the bulk gauge group G^B is the kernel in the exact sequence of groups $1 \to G^B \to G \to \Gamma_\partial \to 1$.

An example of such construction is the discrete time quantum mechanics which is described in Appendix A.

4.2.4 The Semiclassical Limit, Non-degenerate Case

The asymptotical expansion of the integral (52) can be computed by the method of stationary phase (see for example [19, 26] and references therein).

Here we assume that the function S has finitely many simple critical points on the fiber $F(b)$ for each $b \in B_\partial$. Denote the set of such critical points by $C(b)$.

Using the stationary phase approximation we obtain the following expression for the asymptotical expansion of the partition function as $h \to 0$:

$$Z(b) \simeq \sum_{c \in C(b)} Z_c \tag{54}$$

where Z_c is the contribution to the asymptotical expansion from the critical point c. To describe Z_c let us choose local coordinates x^i on $F(b)$ near c, then

$$Z_c = (2\pi h)^{\frac{N}{2}} \frac{1}{\sqrt{|\det(B_c)|}} e^{\frac{iS(c)}{h} + \frac{i\pi}{4}\operatorname{sign}(B_c)} (v(c) + \sum_{\Gamma} \frac{(ih)^{-\chi(\Gamma)} F_c(\Gamma)}{|\operatorname{Aut}(\Gamma)|}) \tag{55}$$

Here $N = \dim F(b)$ and $(B_c)_{ij} = \frac{\partial^2 S(c)}{\partial x^i \partial x^j}$, $v(x)$ is the volume density in local coordinates $\{x^i\}_{i=1}^N$ on $F(b)$, $\frac{dx}{db} = v(x)\, dx^1, \ldots, dx^N$, $\chi(\Gamma)$ is the Euler characteristic of the graph Γ, $|\operatorname{Aut}(\Gamma)|$ is the number of automorphisms of the graph and the summation is taken over finite graphs where each vertex has valency at least 3. The weight of a graph $F_c(\Gamma)$ is given by the "state sum" which is described in the Appendix B. Note that this formula by the construction is invariant with respect to changes of local coordinates. This is particularly clear at the level of determinants. Indeed, let J be the Jacobian of the coordinate change $x^i \mapsto f^i(x)$. Then $v \mapsto v|\det(J)|$ and $|\det(B_c)| \mapsto |\det(B_c)| \det(J)^2$ and the Jacobians cancel. For higher level contributions, see [22].

4.2.5 Gluing Formal Semiclassical Partition Functions in the Non-degenerate Case

The image $L = \pi(EL)$, according to our assumptions is transversal to generic fibers of $p_\partial : F_\partial \to B_\partial$. By varying the classical background c we can span the subspace $T_{\pi(c)} L \subset T_{\pi(c)} F_\partial$ which is, according to the assumption of transversality, isomorphic to $T_{p_\partial \pi(c)} B_\partial$.

We will call the partition function Z_c the *formal semiclassical partition function* on the classical background c. We will also say that it is given by the formal integral of $\exp(\frac{iS}{h})$ over the formal neighborhood of c:

$$Z_c = \int_{T_c F(b)}^{formal} \exp(\frac{iS}{h}) \frac{dx}{db}$$

with $b = p_\partial \pi(c)$. The formal integral on the right hand side here is defined to be the right hand side of (55).

Passing to the limit $h \to 0$ in (53) we obtain the gluing formula for formal semiclassical partition functions (under the assumption of non-degeneracy of critical points):

$$\int_{T_{b_0} B_\partial}^{formal} Z_{c_1(b)} Z_{c_2(b)} db = Z_c \tag{56}$$

Here c is a simple critical point of S on $F_1 \times_{B_\partial} F_2$, $b_0 = p_\partial \pi_1 \pi(c) = p_\partial \pi_2 \pi'(c)$ where $\pi : F_1 \times_{B_\partial} F_2 \to F_1$ and $\pi' : F_1 \times_{B_\partial} F_2 \to F_2$ are natural projections and $c_1(b)$ and $c_2(b)$ are critical points of S_1 and S_2 on fibers $F_1(b)$ and $F_2(b)$ respectively which are formal deformations of $c_1(b_0) = \pi_1(c)$ and of $c_2(b_0) = \pi_2(c)$. The left hand side of (56) stands for the stationary phase evaluation of the integral (note that the integrand has the appropriate asymptotics at $h \to 0$). In [22] this formula was used to prove that formal semiclassical propagator satisfies the composition property.

4.3 Gauge Fixing

4.3.1 Gauge Fixing in the Integral

Here we will outline a version of the Faddeev-Popov trick for gauge fixing in the finite dimensional model in the presence of boundary. We assume that the action function S, the choice of boundary conditions, and group action on F satisfy all properties described in Sect. 4.2.2.

The goal here is to calculate the asymptotics of the partition function

$$Z_F(b) = \int_{F(b)} e^{\frac{i}{h} S(x)} \frac{dx}{db} \tag{57}$$

when $h \to 0$. Here, as in the previous section $F(b) = \pi^{-1} p_\partial^{-1}(b)$ but now a Lie group G acts on F and the function S and the integration measure dx are G-invariant. As in Sect. 4.2.2 we assume that there is an exact sequence $1 \to G^B \to G \to \Gamma_\partial \to 1$, where Γ_∂ acts on B_∂ in such a way that db is Γ_∂-invariant and the subgroup G^B acts fiberwise so that the measure $\frac{dx}{db}$ is G^B-invariant. We will denote the Lie algebra of G^B by $\mathfrak{g}^B$.

Assume that the function S has finitely many isolated G^B-orbits of critical points on $F(b)$ and that the measure of integration is supported on a neighborhood of these points.[10] We denote by $v(x)$ the density of the measure in local coordinates, $\frac{dx}{db} = v(x)\, dx^1, \ldots, dx^N$ with $\{x^i\}$ the local coordinates on F_b.

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[10] In the asymptotics $h \to 0$, one can replace any invariant G^B-invariant measure by one with this property, since we are working with oscillatory integrals.

Let $O \subset F(b)$ be a critical G^B-orbit of the action S. Denote $U_O \subset F(b)$ the connected component[11] of the support of the density v containing O. The integral (57) is a sum of contributions of individual critical orbits:

$$\int_{F(b)} e^{\frac{i}{h}S(x)} \frac{dx}{db} = \sum_O \int_{U_O} e^{\frac{i}{h}S(x)} \frac{dx}{db}$$

For a fixed critical orbit O, let $\varphi : U_O \to \mathfrak{g}^B$ be some function with zero a regular value. Denote $\Lambda_\varphi = \varphi^{-1}(0) \subset U_O$ – the "gauge-fixing surface". Assume that Λ_φ intersects O transversally. Note that we do not assume that Λ_φ is a section of the G^B-action (i.e. of the projection $U_O \to U_O/G^B$).

Let c be one of the intersection points of the orbit O with Λ_φ. Denote $V_{O,c} \subset U_O$ the connected component of c in the intersection $U_O \cap \Lambda_\varphi$ and let $U_{O,c} \subset U_O$ be an open tubular neighborhood of $V_{O,c}$ in U_O (thin enough not to contain zeroes of φ lying outside $V_{O,c}$). Using Faddeev-Popov construction, the contribution of U_O to the integral (57) can be written as follows:

$$\int_{U_O} e^{\frac{i}{h}S(x)} \frac{dx}{db} = |G^B| \int_{U_{O,c}} e^{\frac{i}{h}S(x)} \det(L_\varphi(x))\delta(\varphi(x)) \frac{dx}{db} \qquad (58)$$

We have a natural isomorphism $U_O \simeq V_{O,c} \times G^B$ given by the action of G^B on points of $V_{O,c}$, hence $V_{O,c} \simeq U_O/G^B$ and therefore the integral on the right hand side of (58) can be thought of as an integral supported on the quotient U_O/G^B. To describe $L_\varphi(x)$ choose a basis e_a in the Lie algebra $\mathfrak{g}^B$. The action of e_a on F_b is given by the vector field $\sum_i l_a^i(x)\partial_i$. Matrix elements of $L_\varphi(x)$ are $\sum_i l_a^i(x)\partial_i\varphi^b(x)$. The factor $|G^B|$ in (58) stands for the volume of the group G^B (with respect to the Haar measure compatible with the basis $\{e_a\}$ in $\mathfrak{g}^B$).

It is convenient to write (58) as a Grassmann integral:

$$\frac{|G^B|}{(2\pi i)^{\dim G^B}} \int_{\mathcal{F}_c(b)} \exp \frac{i}{h}\left(S(x) + \sum_a \lambda_a \varphi^a(x) + \sum_a \bar{\mathsf{c}}_a L_\varphi(x)^a_b \mathsf{c}^b \right) \frac{dx}{db}\, d\lambda\, d\bar{\mathsf{c}}\, d\mathsf{c} \qquad (59)$$

where $\mathcal{F}_c(b) = U_{O,c} \oplus \mathfrak{g}^B_{odd} \oplus (\mathfrak{g}^B_{odd})^* \oplus (\mathfrak{g}^B_{even})^*$ and $\bar{\mathsf{c}}$ and c are odd variables. See for example [19] for details on Grassman integration. The asymptotical stationary phase expansion of (58) as $h \to 0$ can be understood[12] as a formal integral over the (formal) neighborhood of c in the supermanifold $\mathcal{F}_c(b)$. The functions $S(x)$, $\varphi^a(x)$, $L_\varphi(x)^a_b$ should be understood as the Taylor expansions in parameter $\frac{x-c}{\sqrt{h}}$, just as in the previous section. The result is the asymptotical expression given by Feynman diagrams where two types of edges correspond to the even and odd Gaussian terms in the

[11] In the case when the group G^B is disconnected, we define U_O to be the union of connected components of O_k in $\mathrm{supp}(v)$, where O_k are the connected components of O.

[12] The logic is that the formal integral is *defined* to be stationary phase asymptotics of (58).

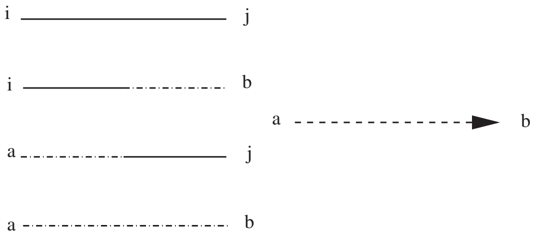

Fig. 1 Bosonic (*left*) and fermionic (*right*) edges for Feynman diagrams in (60) with states at their endpoints

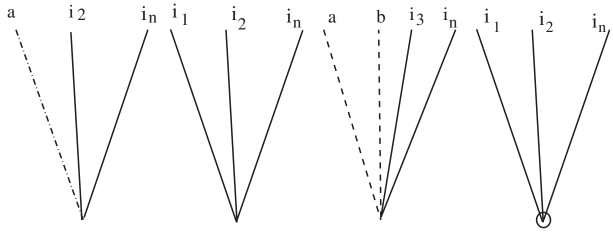

Fig. 2 Vertices for Feynman diagrams in (60) with states on their stars

integral :

$$Z_c = \int_{T_c\mathcal{F}(b)}^{formal} e^{\frac{i}{h}S(x)}\frac{dx}{db} = |G^B|(2\pi h)^{\frac{\dim F(b)-\dim G^B}{2}} \times \frac{1}{\sqrt{|\det(B(c))|}}\det(L_\varphi(c))\cdot\exp\left(\frac{i}{h}S(c)+\frac{i\pi}{4}\,\mathrm{sign}(B(c))\right) \times\left(v(c)+\sum_{\Gamma\neq\varnothing}\frac{(ih)^{-\chi(\Gamma)}(-1)^{c(D(\Gamma))}F_c(D(\Gamma))}{|\operatorname{Aut}(\Gamma)|}\right), \tag{60}$$

Here $D(\Gamma)$ is the planar projection of Γ, a Feynman diagram. Feynman diagrams in this formula have bosonic edges and fermionic oriented edges, $c(D(\Gamma))$ is the number of crossings of fermionic edges.[13] The structure of Feynman diagrams is the same as in (55). The propagators corresponding to Bose and Fermi edges are shown in Fig. 1. The weights of vertices are shown on Fig. 2.

The weight of the fermionic edge on Fig. 1 is $(L_\varphi(c)^{-1})_{ab}$. Weights of the bosonic edges from Fig. 1 correspond to matrix elements of $B(c)^{-1}$ where

[13] The sign rule is equivalent to the usual $(-1)^{\#\text{fermionic loops}}$ which is used in physics literature.

$$B(c) = \begin{pmatrix} \frac{\partial^2 S(c)}{\partial x^i \partial x^j} & \frac{\partial \varphi^a(c)}{\partial x^i} \\ \frac{\partial \varphi^b(c)}{\partial x^j} & 0 \end{pmatrix}$$

The weights of vertices with states on their stars from Fig. 2 are (from left to right):

$$\frac{\partial^{n-1}\varphi^a(c)}{\partial x^{i_2}, \ldots, \partial x^{i_n}}, \quad \frac{\partial^n S(c)}{\partial x^{i_1}, \ldots, \partial x^{i_n}}, \quad \frac{\partial^{n-2} L_\varphi(c)^a_b}{\partial x^{i_3}, \ldots, \partial x^{i_n}}, \quad \frac{\partial^n v(c)}{\partial x^{i_1}, \ldots, \partial x^{i_n}}$$

The last vertex should appear exactly once in each diagram.

This formula, by construction, does not depend on the choice of local coordinates. It is easy to see this explicitly at the level of determinants. Indeed, when we change local coordinates, we have

$$B(c) \mapsto \begin{pmatrix} J^T & 0 \\ 0 & 1 \end{pmatrix} B(c) \begin{pmatrix} J & 0 \\ 0 & 1 \end{pmatrix}, \quad v \mapsto |\det(J)| v$$

where J is the Jacbian of the coordinate transformation. It is clear that the ratio $v/|\det(B(c))|$ is invariant with respect to such transformations.

Note that because we defined the formal integral (60) as the contribution to the asymptotical expansion of the integral (58) from the critical orbit of S passing through c, the coefficients in (60) do not depend on the choice of gauge constraint φ and

$$Z_c = Z_{[c]}$$

where $[c] = O$ is the orbit of G^B passing through c.

4.3.2 Gluing Formal Integrals for Gauge Theories

Assume that as in Sect. 4.2.3 we have two spaces F_1 and F_2 fibered over B_∂ and two functions S_1 and S_2 defined on F_1 and F_2 respectively such that the integrals $Z_{F_1}(b)$ and $Z_{F_2}(b)$ converge absolutely for generic b. For example, we can assume that spaces F_1, F_2 and B_∂ are compact. Denote by F the fiber product $F_1 \times_{B_\partial} F_2$ and set $N_i = \dim F_i$, $N_\partial = \dim B_\partial$. Let Lie groups G_1, G_2 and Γ_∂ act as $G_i : F_i \to F_i$ and $\Gamma_\partial : B_\partial \to B_\partial$ and assume that functions S_i are G_i-invariant and Γ_∂ appears in exact sequences:

$$1 \to G_1^B \to G_1 \to \Gamma_\partial \to 1, \quad 1 \to G_2^B \to G_2 \to \Gamma_\partial \to 1$$

where kernels G_1^B and G_2^B are bulk gauge groups for F_1 and F_2.

Changing the order of integration we obtain (53). As $h \to 0$ the gluing identity (53) becomes the identity between formal integrals just as in the non-degenerate case

$$\int_{T_{b_0}B_\partial}^{formal} Z_{[c_1(b)]} Z_{[c_2(b)]} db = Z_{[c]}$$

which should be regarded as the contribution of the critical point c to Z_F written as an iterated integral.[14] After a gauge fixing in the integral over b we arrive to the following formula for the left side:

$$\begin{aligned} Z_{[c]} = |G_1^B||G_2^B||\Gamma_\partial|(2\pi h)^{\frac{N-n}{2}} & \frac{\det(L_{\varphi_1}(c_1))\ \det(L_{\varphi_2}(c_2))\ \det(L_{\varphi_\partial}(c_\partial))}{\sqrt{|\det(B_1(c_1))|\ |\det(B_2(c_2))|\ |\det(B_\partial(c_\partial))|}} \\ & \times \exp\left(\frac{i}{h}(S_1(c_1) + S_2(c_2)) + \frac{i\pi}{4}(\mathrm{sign}(B_1(c_1)) + \mathrm{sign}(B_2(c_2)) + \mathrm{sign}(B_\partial(c_\partial)))\right) \\ & \times \left(v_1(c_1) v_2(c_2) v_\partial(c_\partial) + \sum_{\Gamma \neq \varnothing} \text{composite Feynman diagrams} \right), \end{aligned} \tag{61}$$

Here $N = N_1 + N_2 - N_\partial = \dim F$ and $n = n_1 + n_2 - n_\partial$ were $n_i = \dim G_i$ and $n_\partial = \dim \Gamma_\partial$. Composite Feynman diagrams consist of Feynman diagrams for F_1, Feynman diagrams for F_2 and Feynman diagrams connecting them which come from formal integration over boundary fields in the formal neighborhood of b_0. Factors $v_1(c_1)$, $v_2(c_2)$, $v_\partial(c_\partial)$ are densities of corresponding measures in local coordinates which we used in (61).

Comparing this expression with (60) besides the obvious identity $S(c) = S(c_1) + S(c_2)$ we obtain identities

$$\begin{aligned} & \frac{\det(L_{\varphi_1}(c_1))\ \det(L_{\varphi_2}(c_2))\ \det(L_{\varphi_\partial}(c_\partial))}{\sqrt{|\det(B_1(c_1))|\ |\det(B_2(c_2))|\ |\det(B_\partial(c_\partial))|}} \\ & \quad \cdot \exp\left(\frac{i\pi}{4}(\mathrm{sign}(B_1(c_1)) + \mathrm{sign}(B_2(c_2)) + \mathrm{sign}(B_\partial(c_\partial)))\right) \\ & \quad = \frac{\det(L_\varphi(c))}{\sqrt{|\det(B(c))|}} \ \exp\left(\frac{i\pi}{4}\,\mathrm{sign}(B(c))\right) \end{aligned} \tag{62}$$

In addition to this, in each order h^m with $m > 0$ we will have the following identity: the sum of all composite Feynman diagrams of order m for F_1, F_2, B_∂ equals the sum of all Feynman diagrams of order m for F.

5 Abelian Chern-Simons Theory

In TQFT's there are no ultraviolet divergencies but there is a gauge symmetry to deal with. Perhaps the simplest non-trivial example of TQFT is the Abelian Chern-Simons theory with the Lie group $\mathbb{R}$. Fields in such theory are connections on the

[14] Recall that db is a Γ_∂-invariant measure on B_∂ such that $\frac{dx}{db}db$ is a G-invariant measure on F.

trivial $\mathbb{R}$-bundle over a compact, smooth, oriented 3-dimensional manifold M. We will identify fields with 1-forms on M. The action is

$$S(A) = \frac{1}{2} \int_M A \wedge dA$$

Solutions of the Euler-Lagrange equations are closed 1-forms on M. The variation of this action induces the exact symplectic form on $\Omega^1(\partial M)$ (see Sect. 3.4).

5.1 The Classical Action and Boundary Conditions

A choice of metric on M induces a metric on ∂M and the Hodge decomposition:

$$\Omega(\partial M) = d\Omega(\partial M) \oplus H(\partial M) \oplus d^*\Omega(\partial M)$$

The Lagrangian subspace of boundary values of solutions to Euler-Lagrange equations is

$$L_M = H^1_M(\partial M) \oplus d\Omega^0(\partial M)$$

where $H_M(\partial M)$ is the space of harmonic representatives of cohomology classes on the boundary coming from cohomology classes $H^1(M)$ of the bulk by pull-back with respect to inclusion of the boundary.

Choose a decomposition of $H(\partial M)$ into a direct sum of two Lagrangian subspaces:

$$H(\partial M) = H_+(\partial M) \oplus H_-(\partial M)$$

This induces a decomposition of forms $\Omega(\partial M) = \Omega_+(\partial M) \oplus \Omega_-(\partial M)$ where

$$\Omega_+(\partial M) = H_+(\partial M) \oplus d\Omega(\partial M), \quad \Omega_-(\partial M) = H_-(\partial M) \oplus d^*\Omega(\partial M)$$

Choose the boundary Lagrangian fibration as

$$p_\partial : \Omega(\partial M) \to B(\partial M) = \Omega_+(\partial M)$$

with fibers

$$p_\partial^{-1}(b) = b + \Omega_-(\partial M) \simeq H_-(\partial M) \oplus d^*\Omega(\partial M).$$

This fibration is not $\alpha_{\partial M}$-exact, i.e. the restriction of $\alpha_{\partial M}$ to fibers is zero. Let us modify the action, by adding a boundary term such that the form $\alpha_{\partial M}$ will vanish on fibers of p. Define the new action as

$$\widetilde{S}(A) = S(A) + \frac{1}{2}\int_{\partial M} A_+ \wedge A_-$$

where $A_\pm$ are $\Omega_\pm$-components of $i^*(A)$.

The new form on boundary connections is

$$\widetilde{\alpha}_{\partial M}(a) = \alpha_{\partial M}(a) + \frac{1}{2}\delta\int_{\partial M} a_+ \wedge a_- = -\int_{\partial M} a_- \wedge \delta a_+$$

and it vanishes on the fibers of p_∂ because on each fiber $\delta a_+ = 0$.

Note that the modified action is gauge invariant. Indeed, on components $A_\pm$ gauge transformations act as $A_+ \mapsto A_+ + d\theta$ and $A_- \mapsto A_-$, i.e. gauge transformations act trivially on fibers.

5.2 Formal Semiclassical Partition Function

5.2.1 More on Boundary Conditions

For this choice of Lagrangian fibration the bulk gauge group G^B is $\Omega^0(M, \partial M)$. The boundary gauge group acts trivially on fibers. Indeed, the boundary gauge group $\Omega^0(\partial M)$ acts naturally on the base $B(\partial M) = H^1(\partial M)_+ \oplus d\Omega^0(\partial M), \alpha \mapsto \alpha + d\lambda$. It acts on the base shifting the fibers: $p(\beta + d\lambda) = p(\beta) + d\lambda$.

According to the general scheme outlined in Sect. 4.3, in order to define the formal semiclassical partition function we have to fix a background flat connection a and "integrate" over the fluctuations $\sqrt{h}\alpha$ with boundary condition $i^*(\alpha)_+ = 0$. We have

$$\widetilde{S}(a + \alpha) = \tilde{S}(\alpha) + \frac{1}{2}\int_{\partial M} a_+ \wedge a_-$$

Note that $da = 0$ which means that a restricted to the boundary is a closed form which we can write as $i^*(a) = [a]_+ + [a]_- + d\theta$ where $[a]_\pm \in H_\pm(\partial M)$. Therefore, for the action we have:

$$\widetilde{S}(a + \alpha) = \widetilde{S}(\alpha) + \frac{1}{2} < [a]_+, [a]_- >_{\partial M}$$

where $\langle ., . \rangle$ is the symplectic pairing in $H(\partial M)$.

For semiclassical quantization we should choose the gauge fixing submanifold $\Lambda \subset \Omega(M)$, such that $(T_a F_M)_+ = T_a EL \oplus T_a \Lambda$. Here $(T_a F_M)_+$ is the space of 1-forms (α-fields) with boundary condition $i^*(\alpha)_+ = 0$. As it is shown in Appendix D the action functional restricted to fields with boundary values in an isotropic subspace $I \subset \Omega^1(\partial M)$ is non-degenerate on

$$T_a\Lambda_I = d^*\Omega^2_N(M, I^\perp) \cap \Omega^1_D(M, I)$$

For our choice of boundary conditions $I = \Omega^1_-(\partial M)$.

5.2.2 Closed Space Time

First, assume the space time has no boundary. Then the formal semiclassical partition function is defined as the product of determinants which arise from gauge fixing and from the Gaussian integration as in (60). In the case of Abelian Chern-Simons the gauge condition is $d^*A = 0$ and the action of the gauge Lie algebra $\Omega^0(M)$ on the space of fields $\Omega^1(M)$ is given by the map $d : \Omega^0(M) \to \Omega^1(M)$ (here we identified $\Omega^1(M)$ with its tangent space at any point). Thus, the FP action (59) in our case is

$$S(A, \overline{c}, c, \lambda) = \frac{1}{2}\int_M A \wedge dA + \int_M \overline{c}\ \ \Delta c\ \ d^3x + \int_M \lambda\ \ d^*A\ \ d^3x$$

where $\overline{c}$, c are ghost fermion fields, and λ is the Lagrange multiplier for the constraint $d^*A = 0$.

By definition the corresponding Gaussian integral is

$$Z_a = C\frac{|\det'(\Delta_0)|}{\sqrt{|\det'(\widehat{*d})|}}\exp(\frac{i\pi}{4}(2\mathrm{sign}(\Delta_0) + \mathrm{sign}(\widehat{*d})))$$

Here $\det'$ is a regularized determinant and $\mathrm{sign}(A)$ is the signature of the differential operator A. The constant depends of the choice of regularization. The usual choice is the ζ-regularization. The signature is up to a normalization the eta invariant [31]. The operator $\widehat{*d}$ acts on $\Omega^1(M) \oplus \Omega^0(M)$ as

$$\begin{pmatrix} *d & d \\ d^* & 0 \end{pmatrix} \tag{63}$$

Its square is the direct sum of Laplacians:

$$\widehat{*d}^2 = \begin{pmatrix} d^*d + dd^* & 0 \\ 0 & d^*d \end{pmatrix}$$

Thus the regularized determinant of $\widehat{*d}$ is the product of determinants acting on 1-forms and on 0-forms:

$$|\det'(\widehat{*d})|^2 = |\det'(\Delta_1)||\det'(\Delta_0)|$$

This gives the following formula for the determinant contribution to the partition function:

$$\frac{|\det'(\Delta_0)|}{\sqrt{|\det'(\widehat{*d})|}} = \frac{|\det'(\Delta_0)|^{\frac{3}{4}}}{|\det'(\Delta_1)|^{\frac{1}{4}}} \tag{64}$$

Taking into account that $*\Omega^i(M) = \Omega^{3-i}(M)$ we can write this as

$$T^{1/2} = |\det'(\Delta_1)|^{\frac{1}{4}}|\det'(\Delta_2)|^{\frac{2}{4}}|\det'(\Delta_3)|^{\frac{3}{4}}$$

where T is the Ray-Singer torsion. This gives well-known formula for the absolute value of the partition function of the Abelian Chern-Simons theory on a closed manifold.

$$|Z| = CT^{1/2} \tag{65}$$

We will not discuss here the η-invariant part.

Remark 11 The operator $\widehat{*d}$ is easy to identify with $L_- = *d + d*$, acting on $\Omega^1(M) \oplus \Omega^3(M)$ from [31]. Indeed, using Hodge star we can identify $\Omega^0(M)$ and $\Omega^3(M)$. After this the operators are related as

$$L_- = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \widehat{*d} \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}^{-1}$$

Remark 12 There is one more formula in the literature for gauge fixing. Assume that a Lie group G has an invariant inner product, the space of fields F is a Riemannian manifold and G acts by isometries on F. In this case there is a natural gauge fixing which leads to the following formula for an integral of a G-invariant function [27]:

$$\int_F h(x)dx = |G| \int_{F/G} h(x)(\det'(\tau_x^*\tau_x))^{\frac{1}{2}}[dx]$$

Here we assume that the G-action does not have stabilizers. The linear mapping $\tau_x : \mathfrak{g} \to T_xF$ is given by the G-action, the Hermitian conjugate is taken with respect to the metric structure on F and on G, dx is the Riemannian volume on F and $[dx]$ is the Riemannian volume on F/G with respect to the natural Riemannian structure on the quotient space.

For the Abelian Chern-Simons a choice of metric on the space time induces metrics on $G = \Omega^0(M)$ and on $F = \Omega^1(M)$. The gauge group G acts on F by isometries and $\tau_x = d$, the de Rham differential. This gives another expression for the absolute value of the partition function

$$|Z| = C\frac{|\det'(\Delta_0)|^{\frac{1}{2}}}{|\det'(*d)|^{\frac{1}{2}}} \tag{66}$$

Here $*d : \Lambda \to \Lambda$, and $\Lambda = d^*\Omega^2(M)$ is the submanifold on which the action functional is non-degenerate. It is clear that this formula coincides with (65).

5.2.3 Space Time with Boundary

Now let us consider the case when ∂M is non-empty. In this case the bulk gauge group G^B is $\Omega_D(M, \{0\})$ which we will denote just $\Omega_D(M)$. The space of fluctuations is $\Omega^1_D(M, \Omega_-(\partial M))$. The bilinear from in the Faddeev-Popov action is

$$\frac{1}{2}\int_M \alpha \wedge d\alpha + \int_M \lambda\, d^*\alpha\, d^3x - i\int_M \bar{c}\, \Delta c\, d^3x$$

The even part of this form is symmetric if we impose the boundary condition $i^*(\lambda) = 0$. Similarly to the case of closed space time we can define the partition function as

$$Z_{a,M} = C|\det{}'(\widehat{*d})|^{-1/2}|\det{}'(\Delta_0^{D,\{0\}})|\exp(\frac{i\pi}{4}(2\,\text{sign}(\Delta_0) + \text{sign}(\widehat{*d})))$$
$$\exp(\frac{i}{h} < [a]_+, [a]_- >_{\partial M}) \tag{67}$$

Here $\Delta_0^{D,\{0\}}$ is the Laplace operator action on $\Omega_D(M, \{0\})$ and $[a]_\pm$ are the $\pm$ components of the cohomology class of the boundary value $i^*(a)$ of a. The operator $\widehat{*d}$ acts on $\Omega^1_D(M, \Omega_-(\partial M)) \oplus \Omega^0_D(M, \{0\})$ and is given by (63). This ratio of determinants is expected to give a version of the Ray-Singer torsion for appropriate boundary conditions. The signature contributions are expected to be the η-invariant with the appropriate boundary conditions. For the usual choices of boundary conditions, such as tangent, absolute, or APS boundary conditions at least some of these relations are known, for more general boundary conditions it is a work in progress.

5.2.4 Gluing

According to the finite dimensional gluing formula we expect a similar gluing formula for the partition function. A consequence of this formula is the multiplicativity of the version of the Ray-Singer torsion with boundary conditions described above. To illustrate this, let us take a closer look at the exponential part of (67).

Recall that $L_M \subset \Omega^1(\partial M)$ is the space of closed 1-forms which are boundary values of closed 1-forms on M. To fix boundary conditions we fixed the decomposition $\Omega^1(\partial M) = \Omega^1(\partial M)_+ \oplus \Omega^1(\partial M)_-$ (see above).

Let β be a tangent vector to L_M at the point $i^*(a) \in L_M$. We have natural identifications

$$T_{i^*(a)}\Omega^1(\partial M)_- = H^1(\partial M)_- \oplus d^*\Omega^2(\partial M), \;\; T_{i^*(a)}\Omega^1(\partial M)_+ = H^1(\partial M)_+ \oplus d\Omega^0(\partial M)$$

Denote by $\beta_\pm$ the components of β in $T_{i^*(a)}L_\pm$ respectively. Since $d\beta = 0$ we have $\beta_+ = [\beta]_+ + d\theta$, and $\beta_- = [\beta]_-$, where $[\beta]_\pm$ are components of the cohomology $[\beta]_\pm$ in $H^1(\partial M)_\pm$. If the reduced tangent spaces $[T_{i^*(a)}L_M] = H^1_\pm(\partial M)$ and $[T_{i^*(a)}\Omega^1(\partial M)_\pm] = H^1_M(\partial M)$ are transversal, which is what we assume here, projections to $[T_{i^*(a)}\Omega^1(\partial M)_\pm]$ give linear isomorphisms $A^{(\pm)}_M : H^1_M(\partial M) \to H^1_\pm(\partial M)$. This defines the linear isomorphism

$$B_M = A^{(-)}_M (A^{(+)}_M)^{-1} : H^1(\partial M)_+ \to H^1(\partial M)_-$$

acting as $B_M([\beta]_+) = [\beta]_-$ for each $[\beta] \in H^1_M(\partial M)$. This is the analog of the Dirichlet-to-Neumann operator.

Now considering small variations around a have

$$\begin{aligned} Z_{[a+\sqrt{h}\beta]} = Z_{[a]} \exp(\frac{i}{\sqrt{h}}(&< [i^*(a)]_+, B_M([i^*(\beta)]_+) >_{\partial M} \\ &+ < [i^*(\beta)]_+, [i^*(a)]_- >_{\partial M}) + i < [i^*(\beta)]_+, B_M([i^*(\beta)]_+) >_{\partial M}) \end{aligned} \tag{68}$$

The gluing formula for this semiclassical partition function at the level of exponents gives the gluing formula for Hamilton-Jacobi actions. At the level of pre-exponents it also gives the gluing formula for torsions and for the η-invariant for appropriate boundary conditions. Changing boundary conditions results in a boundary contribution to the partition function and to the gluing identity. One should also expect the gluing formula for correlation functions. The details of these statements require longer discussion and substantial analysis and will be done elsewhere.

There are many papers on Abelian Chern-Simons theory. The appearance of torsions and η-invariants in the semiclassical asymptotics of the path integral for the Chern-Simons action was first pointed out in [31]. For a geometric approach to compact Abelian Chern-Simons theory and a discussion of gauge fixing and the appearance of torsions in the semiclassical analysis see [24]. For the geometric quantization approach to the Chern-Simons theory with compact Abelian Lie groups see [2].

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Appendix

A Discrete Time Quantum Mechanics

An example of a finite dimensional version of a classical field theory is a discrete time approximation to the Hamiltonian classical mechanics of a free particle on $\mathbb{R}$. We denote coordinates on this space (p, q) where p represents the momentum and q represents the coordinate of the system.

In this case the space time is an ordered collection of n points which represent the discrete time interval. If we enumerate these points $\{1, \ldots, n\}$ the points $1, n$ represent the boundary of the space time. The space of fields is $\mathbb{R}^{n-1} \times \mathbb{R}^n$ with coordinates p_i where $i = 1, \ldots, n-1$ represents the "time interval" between points i and $i+1$ and q_i where $i = 1, \ldots, n$. The coordinates p_1, p_{n-1}, q_1, q_n are boundary fields.[15] The action is

$$S = \sum_{i=1}^{n-1} p_i(q_{i+1} - q_i) - \sum_{i=1}^{n-1} \frac{p_i^2}{2}$$

We have

$$dS = \sum_{i=1}^{n-2}(q_{i+1} - q_i - p_i)\, dp_i + \sum_{i=2}^{n-1}(p_{i-1} - p_i)\, dq_i + p_{n-1}dq_n - p_1 dq_1$$

From here we derive the Euler-Lagrange equations

$$q_{i+1} - q_i = p_i, \quad i = 1, \ldots n-1,$$

$$p_{i-1} - p_i = 0, \quad i = 2, \ldots, n-1$$

and the boundary 1-form

$$\alpha = p_{n-1}dq_n - p_1 dq_1$$

This gives the symplectic structure on the space of boundary fields with

$$\omega_\partial = dp_{n-1} \wedge dq_n - dp_1 \wedge dq_1$$

The boundary values of solutions of the Euler-Lagrange equations define the subspace

$$L = \pi(EL) = \{(p_1, q_1, p_{n-1}, q_n) | p_1 = p_{n-1}, q_n = q_1 + (n-1)p_1\}$$

---

[15] In other words the space time is a 1-dimensional cell complex. Fields assign coordinate function q_i to the vertex i and p_i to the edge $[i, i+1]$.

It is clear that this a Lagrangian subspace.

B Feynman Diagrams

Let Γ be a graph with vertices of valency ≥ 3 with one special vertex which may also have valency 0, 1, 2. We define the weight $F_c(\Gamma)$ as follows.

A state on Γ is a map from the set of half-edges of Γ to the set $1, \ldots, n$, for an example see Fig. 3. The weight of Γ is defined as

$$F_c(\Gamma) = \sum_{states} \left(\frac{\partial^l v}{\partial x^{j_1}, \ldots, \partial x^{j_l}}(c) \prod_{vertices} \frac{\partial^k S}{\partial x^{i_1}, \ldots, \partial x^{i_k}}(c) \prod_{edges} (B_c^{-1})_{ij} \right)$$

Here the sum is taken over all states on Γ, and $i_1, \ldots, i_k$ are states on the half-edges incident to a vertex. The first factor is the weight of the special vertex where v is the density of the integration measure in local coordinates $\frac{dx}{db} = v(x)dx^1, \ldots, dx^N$. The pair (i, j) is the pair of states at the half-edges comprising an edge. Note that weights of vertices and the matrix B_c are symmetric. This makes the definition meaningful.

C Gauge Fixing in Maxwell's Electromagnetism

In the special case of electromagnetism ($G = \mathbb{R}, \mathfrak{g} = \mathbb{R}$), the space of fields is $F_M = \Omega^1(M) \oplus \Omega^{n-2}(M)$ and similarly for the boundary. If M has no boundary, the gauge group $G_M = \Omega^0(M)$ acts on fields as follows: $A \mapsto A + d\alpha$, $B \mapsto B$. We can construct a global section of the corresponding quotient using Hodge decomposition: we know that

$$\Omega^\bullet(M) \cong \Omega^\bullet_{\text{exact}}(M) \oplus H^\bullet(M) \oplus \Omega^\bullet_{\text{coexact}}(M) \tag{69}$$

where the middle term consists of harmonic forms. In particular,

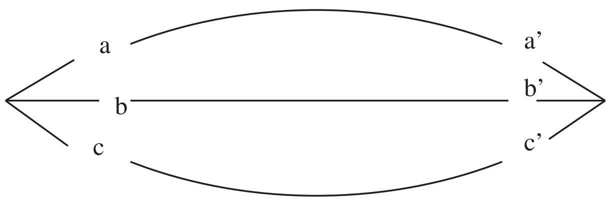

Fig. 3 The "theta" diagram

$$\Omega^1(M) = d\Omega^0(M) \oplus H^1(M) \oplus d^*\Omega^2(M) \tag{70}$$

where the last two terms give a global section. In physics, choosing a global section is called gauge fixing, and this particular choice of gauge is called the Lorentz gauge, where $d^*A = 0$.

D Hodge Decomposition for Riemannian Manifolds With Boundary

D.1 Hodge Decomposition With Dirichlet and Neumann Boundary Conditions

Let M be a smooth oriented Riemannian manifold with boundary ∂M. Recall some basic facts about the Hodge decomposition of differential forms on M. Choose local coordinates near the boundary in which the metric has the product structure with t being the coordinate in the normal direction. Near the boundary any smooth form can be written as

$$\omega = \omega_{tan} + \omega_{norm} \wedge dt$$

where ω_{tan} is the tangent component of ω near the boundary and ω_{norm} is the normal component.

We will denote by $\Omega_D(M)$ the space of forms satisfying the Dirichlet boundary conditions $\iota^*(\omega) = 0$ where ι^* is the pull-back of the form ω to the boundary. This condition can be also written as $\omega_{tan} = 0$.

We will denote by $\Omega_N(M)$ the space of forms satisfying the Neumann boundary conditions $\iota^*(*\omega) = 0$. Here $* : \Omega^i(M) \to \Omega^{n-i}(M)$ is the Hodge star operation, recall that $*^2 = (-1)^{i(n-i)}\mathrm{id}$ on $\Omega^i(M)$. Because $\omega_{norm} = *'\iota^*(*\omega)$ the Neumann boundary condition can be written as $\omega_{norm} = 0$.

Denote by $d^* = (-1)^i *^{-1} d*$ the formal adjoint of d, and by $\Delta = dd^* + d^*d$ the Laplacian on M. Denote by $\Omega_{cl}(M)$ closed forms on M, $\Omega_{ex}(M)$ exact forms on, $\Omega_{cocl}(M)$ the space of coclosed forms, i.e. closed with respect to d^* and by $\Omega_{coex}(M)$ the space of coexact forms.

Define subspaces:

$$\Omega_{cl,cocl}(M) = \Omega_{cl}(M) \cap \Omega_{cocl}(M), \quad \Omega_{cl,coex}(M) = \Omega_{cl}(M) \cap \Omega_{coex}(M)$$

and similarly $\Omega_{ex,cocl}(M)$, $\Omega_{cl,cocl,N}(M)$ and $\Omega_{cl,cocl,D}(M)$.

Theorem 3 (1) *The space of forms decomposes as*

$$\Omega(M) = d^*\Omega_N(M) \oplus \Omega_{cl,cocl}(M) \oplus d\Omega_D(M)$$

(2) *The space of closed, coclosed forms decomposes as*

$$\Omega_{cl,cocl}(M) = \Omega_{cl,cocl,N}(M) \oplus \Omega_{ex,cocl}(M)$$

$$\Omega_{cl,cocl}(M) = \Omega_{cl,cocl,D}(M) \oplus \Omega_{cl,coex}(M)$$

We will only outline the proof of this theorem. For more details and references on the Hodge decomposition for manifolds with boundary and Dirichlet and Neumann boundary conditions see [17]. Riemannian structure on M induces the scalar product on forms

$$(\omega, \omega') = \int_M \omega \wedge *\omega' \tag{71}$$

For two forms of the same degree we have $\omega(x) \wedge *\omega'(x) = \langle \omega(x), \omega'(x) \rangle \, dx$ where dx is the Riemannian volume form and $\langle ., . \rangle$ is the scalar product on $\wedge^k T_x^* M$ induced by the metric. This is why (71) is positive definite.

Lemma 1 *With respect to the scalar product* (71)

$$(d\Omega_D(M))^\perp = \Omega_{cocl}$$

Proof By the Stokes theorem for any form $\theta \in \Omega_D^{i-1}(M)$ we have

$$(\omega, d\theta) = \int_M \omega \wedge *d\theta = (-1)^{(i+1)(n-i)}\Big(\int_{\partial M} \iota^*(*\omega) \wedge \iota^*(\theta) + \int_M d * \omega \wedge \theta\Big)$$

The boundary integral is zero because $\theta \in \Omega_D(M)$. Thus $(\omega, d\theta) = 0$ for all θ if and only if $d * \omega = 0$ which is equivalent to $\omega \in \Omega_{cocl}(M)$.

Corollary 1 *Because* $d\Omega_D(M) \subset \Omega_{cl}(M)$, *we have* $\Omega_{cl}(M) = \Omega_{cl}(M) \cap (d\Omega_D(M))^\perp \oplus d\Omega_D(M)$. *i.e.*

$$\Omega_{cl}(M) = \Omega_{cl,cocl}(M) \oplus d\Omega_D(M)$$

Here we are sketchy on the analytical side of the story. If $U \subset V$ is a subspace in an inner product space, in the infinite dimensional setting more analysis might be required to prove that $V = U \oplus U^\perp$. Here and below we just assume that this does not create problems. Similarly to Lemma 1 we obtain

$$(d^*\Omega_N(M))^\perp = \Omega_{cl}(M)$$

This completes the sketch of the proof of the first part. The proof of the second part is similar.

Note that the spaces in the second part of the theorem are harmonic forms representing cohomology classes:

$$\Omega_{cl,cocl,N}(M) = H(M), \quad \Omega_{cl,cocl,D}(M) = H(M, \partial M)$$

D.2 More General Boundary Conditions

D.2.1 General setup

Assume that M is a smooth compact Riemannian manifold, possibly with non-empty boundary ∂M. Let $\pi : \Omega^i(M) \to \Omega^i(\partial M)$, $i = 0, \dots, n-1$ be the restriction map (the pull-back of a form to the boundary) and $\pi(\Omega^n(M)) = 0$.

The Riemannian structure on M induces the metric on ∂M. Denote by $*$ the Hodge star for M, and by $*_\partial$ the Hodge star for the boundary $*_\partial : \Omega^i(\partial M) \to \Omega^{n-1-i}(\partial M)$. Define the map $\tilde{\pi} : \Omega(M) \to \Omega(\partial M)$, $i = 1, \dots, n$ as the composition $\tilde{\pi}(\alpha) = *_\partial \pi(*\alpha)$. Note that $\tilde{\pi}(\Omega^0(M)) = 0$.

Denote by $\Omega_D(M, L)$ and $\Omega_N(M, L)$ the following subspaces:

$$\Omega_D(M, L) = \pi^{-1}(L), \quad \Omega_N(M, L) = \tilde{\pi}^{-1}(L)$$

where $L \subset \Omega(\partial M)$ is a subspace.

Denote by $L^\perp$ the orthogonal complement to L with respect to the Hodge inner product on the boundary. The following is clear:

Lemma 2

$$(*L^{(i)})^\perp = *(L^{(i)})^\perp, *(L^\perp) = L^{sort}$$

Here L^{sort} is the space which is symplectic orthogonal to L.

Proposition 3 $(d^*\Omega_N(M, L))^\perp = \Omega_D(M, L^\perp)_{cl}$

Proof Let ω be an i-form on M such that

$$\int_M \omega \wedge d * \alpha = 0$$

for any α. Applying Stocks theorem we obtain

$$\int_M \omega \wedge d * \alpha = (-1)^i \int_{\partial M} \pi(\omega) \wedge *_\partial \tilde{\pi}(\alpha) + (-1)^{i+1} \int_M d\omega \wedge *\alpha$$

The boundary integral is zero for any α if and only if $\pi(\omega) \in L^\perp$ and the bulk integral is zero for any α if and only if $d\omega = 0$.

As a corollary of this we have the orthogonal decomposition

$$\Omega(M) = \Omega_D(M, L^\perp)_{cl} \oplus d^*\Omega_N(M, L)$$

Similarly, for each subspace $L \subset \Omega(\partial M)$ we have the decomposition

$$\Omega(M) = \Omega_N(M, L^{\perp})_{cocl} \oplus d\Omega_D(M, L)$$

Now, assume that we have two subspaces $L, L_1 \subset \Omega(\partial M)$ such that

$$d_{\partial}(L_1^{\perp}) \subset L^{\perp}, \tag{72}$$

Note that this implies $d_{\partial}^* L \subset L_1$. Indeed, fix $\alpha \in L$, then (72) implies that for any $\beta \in L_1^{\perp}$ we have

$$\int_{\partial M} \alpha \wedge *d_{\partial}\beta = 0$$

This is possible if and only if

$$\int_{\partial M} *d_{\partial} * \alpha \wedge *\beta = 0$$

Thus, $d_{\partial}^*\alpha \in L_1$. Here we assumed that $(L_1^{\perp})^{\perp} = L_1$.

Because $\pi d = d_{\partial}\pi$ and $\tilde{\pi} d^* = d_{\partial}^* \tilde{\pi}$ we also have

$$d\Omega_D(M, L_1^{\perp}) \subset \Omega_D(M, L^{\perp})_{cl}, \quad d^*\Omega_N(M, L) \subset \Omega_N(M, L_1)_{cocl}$$

Theorem 4 *Under assumption* (72) *we have*

$$\Omega(M) = d^*\Omega_N(M, L) \oplus \Omega_D(M, L^{\perp})_{cl} \cap \Omega_N(M, L_1)_{cocl} \oplus d\Omega_D(M, L_1^{\perp}) \tag{73}$$

Indeed, if $V, W \subset \Omega$ are liner subspaces in the scalar product space Ω such that $W \subset V^{\perp}$ and $V \subset W^{\perp}$ then $\Omega = V \oplus V^{\perp} = W \oplus W^{\perp}$ and

$$\Omega = V \oplus W^{\perp} \cap V^{\perp} \oplus W$$

We will call the identity (73) the Hodge decomposition with boundary conditions. The following is clear:

Theorem 5 *The decomposition* (73) *agrees with the Hodge star operation if and only if*

$$*L_1^{\perp} = L$$

Remark 13 In the particular case $L = \{0\}$ and $L_1^{\perp} = \{0\}$ we obtain the decomposition from the previous section:

$$\Omega(M) = d^*\Omega_N(M) \oplus \Omega_{cl,cocl}(M) \oplus d\Omega_D(M)$$

Lemma 3 *If $L \subset \Omega(\partial M)$ is an isotropic subspace then $*L \subset \Omega(\partial M)$ is also an isotropic subspace.*

Indeed, if L is isotropic then for any $\alpha, \beta \in L$ we have $\int\limits_{\partial M} \alpha \wedge *\beta = 0$, but

$$\int\limits_{\partial M} *\alpha \wedge *^2\beta = \pm \int\limits_{\partial M} \alpha \wedge *\beta$$

therefore $*L$ is also isotropic.

Remark 14 We have

$$*\Omega_N(M) = \Omega_D(M), \;\; *H(M) = H(M, \partial M)$$

In the second formula $H(M)$ is the space of closed-coclosed forms with Neumann boundary conditions and $H(M, \partial M)$ is the space of closed-coclosed forms with Dirichlet boundary conditions. They are naturally isomorphic to corresponding cohomology spaces. Note that as a consequence of the first identity we have $*d^*\Omega_N(M) = d\Omega_D(M)$. We also have more general identity

$$*\Omega_N(M, L) = \Omega_D(M, *_\partial L)$$

and consequently $*\Omega_D(M, L) = \Omega_N(M, *_\partial L)$.

Let π and $\tilde{\pi}$ be maps defined at the beginning of this section. Because π commutes with de Rham differential and $\tilde{\pi}$ commutes with its Hodge dual, we have the following proposition

Proposition 4 *Let $H_M(\partial M)$ be the space of harmonic forms on ∂M extendable to closed forms on M, then*

$$\pi(\Omega_{cl}(M)) = H_M(\partial M) \oplus d\Omega(\partial M), \;\; \tilde{\pi}(\Omega_{cocl}(M)) = H_M(\partial M)^\perp \oplus d^*\Omega(\partial M)$$

Here is an outline of the proof. Indeed, let $\theta \in \Omega_{cl}(M)$ and $\sigma \in \Omega_{cocl}(M)$. Then

$$\int\limits_{\partial M} \pi(\theta) \wedge *_\partial\tilde{\pi}(\sigma) = \int\limits_{\partial M} \pi(\theta) \wedge \pi(*\sigma) = \int\limits_{M} d(\theta \wedge *\sigma)$$

The last expression is zero because by the assumption θ and $*\sigma$ are closed. The proposition follows now from the Hodge decomposition for forms on the boundary and from $\pi(\Omega_{cl}(M)) \subset \Omega_{cl}(\partial M)$, $\tilde{\pi}(\Omega_{cocl}(M)) \subset \Omega_{cocl}(\partial M)$.

D.2.2 dim $M = 3$

Let us look in details at the 3-dimensional case. In order to have the Hodge decomposition with boundary conditions we required

$$dL_1^\perp \subset L^\perp$$

If we want it to be *invariant with respect to the Hodge star* we should also have $*L_1^\perp = L$. Together these two conditions imply that L should satisfy $d * L \subset L^\perp$ or

$$\int_{\partial M} d * \alpha \wedge *\beta = 0$$

for any $\alpha, \beta \in L$. This condition is equivalent to

$$\int_{\partial M} d^* \alpha \wedge \beta = 0$$

for any $\alpha \in L^{(1)}$ and any $\beta \in L^{(2)}$.

Note that if $L^{(2)} = \{0\}$ we have no conditions on the subspace $L^{(1)}$. In this case for any choice of $L^{(0)}$ and $L^{(1)}$ the $*$-invariant Hodge decomposition is:

$$\Omega^0(M) = d^* \Omega^1_N(M, L^{(0)}) \oplus \Omega^0_D(M, L^{(0)\perp})_{cl}$$

$$\Omega^1(M) = d^* \Omega^2_N(M, L^{(1)}) \oplus \Omega^1_D(M, L^{(1)\perp})_{cl} \cap \Omega^1_N(M, L^{(0)})_{cocl} \oplus d\Omega^0_D(M)$$

Here we used $\Omega^i_N(M, L_1) = \Omega^i_N(M, L_1^{(i-1)}) = \Omega^i_N(M, (*L^{(3-i)})^\perp)$. The condition $L^{(2)} = \{0\}$ implies that $\Omega^1_N(M, (*L^{(2)})^\perp) = \Omega^1(M)$. We also used $\Omega^0_D(M, L_1^\perp) = \Omega^0(M, *L^{(2)}) = \Omega^0_D(M)$.

The decomposition of 2- and 3-forms is the result of application of Hodge star to these formulae.

D.2.3 The gauge-fixing subspace

Consider the bilinear form

$$B(\alpha, \beta) = \int_M \beta \wedge d\alpha \tag{74}$$

on the space $\Omega^\bullet(M)$.

Let $I \subset \Omega^\bullet(\partial M)$ be an isotropic subspace.

Proposition 5 *The form B is symmetric on the space $\Omega_D(M, I)$.*

Indeed

$$\int_M (\beta \wedge d\alpha) = (-1)^{|\beta|+1} \int_{\partial M} \pi(\beta) \wedge \pi(\alpha) + \int_M d\beta \wedge \alpha = (-1)^{(|\alpha|+1)(|\beta|+1)} B(\alpha, \beta)$$

The boundary term vanishes because boundary values of α and β are in an isotropic subspace I.

Proposition 6 *Let $I \subset \Omega(\partial M)$ be an isotropic subspace, then B is nondegenerate on $d^*\Omega_N(M, I^\perp) \cap \Omega_D(M, I)$.*

Proof If I is isotropic, $\beta \in \Omega_D(M, I)$ and $B(\beta, \alpha) = 0$ for any $\alpha \in \Omega_D(M, I)$, we have:

$$B(\beta, \alpha) = B(\alpha, \beta) = \int_M \alpha \wedge d\beta$$

and therefore $d\beta = 0$. Therefore $\Omega_D(M, I)_{cl}$ is the kernel of the form B on $\Omega_D(M, I)$. But we have the decomposition

$$\Omega(M) = \Omega_D(M, I)_{cl} \oplus d^*\Omega_N(M, I^\perp)$$

This implies

$$\Omega_D(M, I) = \Omega_D(M, I)_{cl} \oplus d^*\Omega_N(M, I^\perp) \cap \Omega_D(M, I)$$

This proves the statement.

In particular, the restriction of the bilinear form B is nondegenerate on $\Lambda_I = d^*\Omega^2_N(M, I^{(1)\perp}) \cap \Omega^1_D(M, I^{(1)})$. For the space of all 1-forms with boundary values in $I^{(1)}$ we have:

$$\Omega^1_D(M, I^{(1)}) = \Omega^1_D(M, I^{(1)})_{cl} \oplus d^*\Omega^2_N(M, I^{(1)\perp}) \cap \Omega^1_D(M, I^{(1)})$$

The first part is the space of solutions to the Euler-Lagrange equations with boundary values in $I^{(1)}$.

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Local BRST Cohomology for AKSZ Field Theories: A Global Approach

Giuseppe Bonavolontà and Alexei Kotov

Abstract We study the Lagrangian antifield BRST formalism, formulated in terms of exterior horizontal forms on the infinite order jet space of graded fields for topological field theories associated to Q-bundles. In the case of a trivial Q-bundle with a flat fiber and arbitrary base, we prove that the BRST cohomology are isomorphic to the cohomology of the target space differential "twisted" by the de Rham cohomology of the base manifold. This generalizes the local result of G. Barnich and M. Grigoriev, computed for a flat base manifold.

1 Introduction

Horizontal forms constitute a bicomplex with respect to the BRST operator $\mathbf{s}$ and the horizontal (or total) differential d_h. We are interested in the study of the iterated $\mathbf{s}$-cohomology $H^{*,*}(\mathbf{s}|\mathrm{d}_h)$ of the d_h-cohomology groups of this bicomplex. Otherwise stated we are interested in the term $\mathrm{E}_2^{*|*}$ of its spectral sequence. Particularly relevant for the applications are the terms $H^{*,n}(\mathbf{s}|\mathrm{d}_h)$ of top horizontal forms (n being the dimension of the base manifold) known as "local BRST cohomology", i.e. the cohomology groups of $\mathbf{s}$ in the space of local functionals. These groups control the deformation theory for gauge theories and encode classical observables, generalized symmetries and conservations laws (e.g. see [2–4]).

Here we will adapt the formalism of local BRST cohomology to the specific setting of (topological) gauge field theories associated to flat Q-bundles [1, 14, 15]. Recall that a Q-bundle is a fiber bundle in the category of Q-manifolds. In particular, a trivial Q-bundle over $T[1]X$ is a trivial bundle of graded manifolds

G. Bonavolontà (✉)
University of Luxembourg, 6, Rue R. Coudenhove-Kalergi,
1359 Luxembourg City, Luxembourg
e-mail: Giuseppe.Bonavolonta@uni.lu

A. Kotov
Department of Mathematics and Statistics, Faculty of Science and Technology,
University of Tromsø, 9037 Tromsø, Norway
e-mail: oleksii.kotov@uit.no

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$$\eta : T[1]X \times \mathcal{M} \to T[1]X,$$

where the cohomological vector field on the total space, $\mathcal{Q}$, is η-related to the de Rham operator of the base. The space of graded sections $\underline{\Gamma}(\eta)$ is identified with the space of graded maps $\underline{\mathrm{Hom}}(T[1]X, \mathcal{M})$ [11]. In this case the BRST differential $\mathbf{s}$ consists of the evolutionary vector field induced by $\mathcal{Q}$ on the space of infinite jets of $\underline{\Gamma}(\eta)$ (see [5]). The aforementioned BRST formalism has been studied in [5] in the case of coordinate neighborhoods for both (graded)manifolds, the base X and the target $\mathcal{M}$. In these hypotheses the iterated cohomologies are the following

(i) $H^{g,n}(\mathbf{s}|\mathrm{d}_h)$ is isomorphic to the cohomology $H^{g+n}(\mathbf{s} + \mathrm{d}_h)$ of the total BRST operator $\widetilde{\mathbf{s}} = \mathbf{s} + \mathrm{d}_h$ on horizontal forms of total degree $g + n$;
(ii) as a consequence of the hypothesis about the contractibility of the base space, the local BRST cohomology is isomorphic to the Q-cohomology of the target space functions.

These results are obtained by constructing (local) descent equations (in view of the contractibility assumption for the base X). The local BRST cohomology in the case of an arbitrary connected base manifold X and flat target space is given by Theorem 1.

Assume that the target $(\mathcal{M}, \mathcal{Q}_{\mathcal{M}}) := (L = \oplus_{i\in\mathbb{Z}} L^i, \mathcal{Q}_L)$ is a $\mathbb{Z}$-graded $\mathbb{R}$-vector space of finite type, i.e. with $\dim_{\mathbb{R}} L_i < \infty$ for all i. More precisely we will assume $\mathcal{M}$ to be a formal pointed Q-manifold. In this case the space of graded maps $\underline{\mathrm{Hom}}(T[1]X, L)$ is naturally identified with the module of differential forms on X twisted by L; this identification suggests the following generalization: we replace the de Rham operator of the base with Q_{DR}, a (linear) homological vector field given by the L-twisted de Rham operator. We prove that

Main result [Theorem 10.1] *The iterated* BRST *complex for* AKSZ *field theories with arbitrary connected base manifold X and target space* $(L = \oplus_{i\in\mathbb{Z}} L_i, \mathcal{Q}_L)$ has the following form:

$$H^{g|n}(\mathbf{s}|\mathrm{d}_h) \simeq \left(H^\bullet_{\mathrm{DR}}(X) \otimes H^\bullet_Q(L)\right)^{g+n}. \quad (1)$$

In other words, the local BRST cohomology are isomorphic to the Q-cohomology of the target space functions "twisted" by the de Rham cohomology of X.

An immediate interpretation for this result is the following: the $\mathbf{s}$-cohomology in the space of local functionals contains a very restrictive information. More general functionals are needed in order to incorporate TFT (and especially those of AKSZ-type) in the frame of variational calculus for Classical Field Theory [6, 7]. We shall continue investigating this subject in [10] by the use of different tools as "multivalued Lagrangians" and the theory of coverings for non linear PDE-s.

The paper has the following content. In Sect. 2 we recall basic notions about jet spaces as the Cartan distribution, evolutionary vector fields, $\mathcal{D}$-modules, variational bicomplex and the horizontal complex. Particularly relevant for the rest of the paper will be the choice of a specific subcomplex of the horizontal complex denoted with $\bar{\Omega}^\bullet_{\mathrm{poly}}(\pi)$ (see Proposition 6).

In Sect. 3 we construct the proof of Theorem 1 in two steps: first we prove that the local BRST cohomology $H^{g,n}(\mathbf{s}|\mathrm{d}_h)$ are still isomorphic to the total cohomology $H^{g+n}(\mathbf{s}+\mathrm{d}_h)$ and then we calculate the latter cohomology by the use of an argument based on the formal integrability for a compatibility complex (see [16, 17, 20]).

Here we introduce some of the notations employed in the paper. If $\mathtt{M}$ is a sheaf on a manifold X then $\mathtt{M}(U)$ is the space of its sections over an open set $U \subset X$; in the case of canonical sheaves, X will appear as a subscript e.g.: Ω_X is the sheaf of differential forms, $\mathcal{T}_X$—the sheaf of vector fields, $\mathcal{D}_X$—the sheaf of differential operators. With $\Omega(X)$ we mean the space of sections over X, that is, all forms; $\mathcal{T}(X)$—all vector fields, $\mathcal{D}(X)$—all differential operators. Analogously for the bundle of forms we write Λ_X.

2 Jet Bundles, $\mathcal{D}$-modules, and Local Functionals

In this section we review basic facts about jet spaces. Let $\pi\colon E \to X$ be a vector bundle over an n-dimensional smooth manifold. Let $\mathrm{J}^k(\pi)$ be the space of k-jets of its sections:

$$\mathrm{J}^k(\pi) = \{[s]^k_x \mid x \in X,\ s \in \Gamma(\pi)\}. \tag{2}$$

It is obvious that $\pi_k\colon \mathrm{J}^k(\pi) \to X$ inherits a vector bundle structure for all $k \geq 0$, where $\pi_k([s]^k_x) = x$. Furthermore, there exists a canonical surjective vector bundle morphism $\pi_{k,l}\colon \mathrm{J}^k(\pi) \to \mathrm{J}^l(\pi)$ for all $k \geq l$, so that $\pi_{k,l}([s]^k_x) = [s]^l_x$. The collection of vector bundles π_k together with projections $\pi_{k,l}$ constitutes an inverse system, which allows to define the projective limit $\pi_\infty\colon \mathrm{J}^\infty(\pi) \to X$, called the *infinite jet space*, along with projections $\pi_{\infty,k}\colon \mathrm{J}^\infty(\pi) \to \mathrm{J}^k(\pi)$, $k \geq 0$. The algebra of smooth functions on $\mathrm{J}^\infty(\pi)$, $\mathcal{F}(\pi)$, is defined to be the direct limit of $\mathcal{F}_k(\pi) = C^\infty(\mathrm{J}^k(\pi))$,

$$\mathcal{F}(\pi)\colon = \bigcup_k \mathcal{F}_k(\pi). \tag{3}$$

Each element of $\mathcal{F}_k(\pi)$ is regarded as a *nonlinear scalar differential operator* of order k acting on sections of π; this correspondence is established by the following formula:

$$s \mapsto f[s] = j_k(s)^*(f) \in C^\infty(X), \quad s \in \Gamma(\pi)\,,\, f \in \mathcal{F}_k(\pi), \tag{4}$$

where $j_k(s)$ is the k-jet of s, regarded as a section of π_k, so that $j_k(s)(x) = [s]^k_x$.

Let $\pi'\colon E' \to X$ be another bundle over the same manifold. We denote by $\mathcal{F}_k(\pi,\pi')$ and $\mathcal{F}(\pi,\pi')$ the space of smooth sections of the pull-back bundles $\pi_k^*(\pi')$ and $\pi_\infty^*(\pi')$, respectively. Similarly to scalar functions on the space of jets, $\mathcal{F}_k(\pi,\pi')$ is canonically identified with nonlinear PDEs of maximal order k acting from $\Gamma(\pi)$ to $\Gamma(\pi')$.

The tangent space to $j_{k-1}(s)$ at $x_{k-1} = [s]_x^{k-1}$ is uniquely determined by $x_k = [s]_x^k$; this allows to define a vector bundle $\tau_k \colon L^k \to \mathrm{J}^k(\pi)$, the fiber L_{x_k} of which is the tangent space to $j_{k-1}(s)$ at x_{k-1}.

Proposition 1 *It is easy to verify that following properties hold.*

1. $\mathrm{d}\pi_{k-1,k-2}(L_{x_k}) = L_{x_{k-1}}$ *for all* $k \geq 2$ *and* $\mathrm{d}\pi_{k-1}(L_{x_k}) = T_x X$ *for all* $k \geq 1$.[1]
2. *Therefore* $\tau_k \simeq \pi_k^*(\tau)$, *where* $\tau \colon TX \to X$ *is the tangent bundle.*
3. *Sections of* τ_k *can be viewed as derivations of* $\mathcal{F}_{k-1}(\pi)$ *with values in* $\mathcal{F}_k(\pi)$ *and sections of* $\tau_\infty = \pi_\infty^*(\tau)$*—as derivations of* $\mathcal{F}(\pi)$ *with values in* $\mathcal{F}(\pi)$, *respectively.*
4. *There exists a canonical bracket on* $\Gamma(\tau_k)$ *with values in* $\Gamma(\tau_{k+1})$, *which gives rise to a Lie bracket on* $\Gamma(\tau_\infty)$. *The latter coincides with the commutator of the corresponding derivations of* $\mathcal{F}(\pi)$, *hence* τ_∞ *determines an involutive distribution* $\mathcal{C}(\pi)$ *on* $\mathrm{J}_m^\infty(\pi)$, *called the infinite Cartan distribution.*
5. *Sections of* π_∞, *which are integral leaves of* $\mathcal{C}(\pi)$, *are of the form* $j_\infty(s)$ *for some* $s \in \Gamma(\pi)$.

Taking into account the above isomorphism $\mathcal{C}(\pi) \simeq \pi_\infty^*(\tau)$, we can canonically lift any vector field on X to a vector field on $\mathrm{J}^\infty(\pi)$, tangent to the Cartan distribution. Moreover, this lifting respects the Lie bracket, thus it can be viewed as a (non-linear) flat connection in π_∞. The canonical lift of a vector field v is called the *total derivative* along v. More concretely, let $U \subset X$ be a coordinate chart together with local coordinates $\{x^i\}$ and let $\{u^a\}$, $a = 1, \ldots, \mathrm{rk}(\pi)$, be the linear fiber coordinates corresponding to some trivialization of π_U, the restriction of π to U. Let $v = \sum_{i=1}^{n} h^i(x)\partial_{x^i}$ be a vector field in U. Then for any $f \in \mathcal{F}(\pi)$,

$$\bar{v}(f) = \sum_{i=1}^{n} h^i(x) D_{x^i} f, \ \text{ where } \ D_{x^i} = \partial_{x^i} + \sum_{a=1}^{\mathrm{rk}(\pi)} \sum_{(\sigma)} u^a_{(\sigma+1_i)} \partial_{u^a_{(\sigma)}}. \tag{5}$$

Here $(\sigma) = (\sigma_1, \ldots, \sigma_n)$ is a multi-index, $(\sigma + 1_i) = (\sigma_1, \ldots, \sigma_i + 1, \ldots, \sigma_n)$, and $\{u^a_{(\sigma)}\}$ are the fiber linear coordinates on the trivialization of $\mathrm{J}^\infty(\pi_U)$, such that the infinite jet of a section $u^a = u^a(x)$, $a = 1, \ldots, \mathrm{rk}(\pi)$ is represented by the formula $u^a_{(\sigma)}(x) = \partial_{(\sigma)} u^a(x)$. Henceforth we shall use the notation $\partial_{(\sigma)}$ for $\left(\partial_{x^1}\right)^{\sigma_1} \ldots (\partial_{x^n})^{\sigma_n}$ and $D_{(\sigma)}$ for $\left(D_{x^1}\right)^{\sigma_1} \ldots (D_{x^n})^{\sigma_n}$, respectively.

Proposition 2 *Given any* $v \in \mathcal{T}(X)$, $s \in \Gamma(\pi)$, *and* $f \in \mathcal{F}(\pi)$, *one has*

$$\bar{v}(f)[s] = v\left(f[s]\right). \tag{6}$$

The Cartan distribution on $\mathrm{J}^\infty(\pi)$ allows to define an $\mathcal{F}$-module of *horizontal* (tangent to the Cartan distribution) vector fields as well as an $\mathcal{F}$-module of $\mathcal{C}$*-differential*

[1] Hereafter one has $x_l = \pi_{k,l}(x_k)$ for all $k \geq l$ and $x = \pi_k(x_k)$ for all $k \geq 0$, unless the contrary is expressed.

operators $\mathcal{CD}(\pi)$, generated by $\mathcal{CT}(\pi)$. Apparently, $\mathcal{CT}(\pi)$, as an $\mathcal{F}$-module, admits a canonical (involutive) complement, consisting of π_∞-vertical vector fields.

Remark 1 The Lie subalgebra of horizontal vector fields lifted from X generates $\mathcal{CT}(\pi)$ over $\mathcal{F}$. Along with vector fields on X, we can canonically lift differential operators on X, $\mathcal{D}(X)$, to $\mathcal{C}$-differential operators. Furthermore, $\mathcal{CD}(\pi) = \mathcal{F} \otimes_{C^\infty(X)} \mathcal{D}(X)$.

Let us consider the Lie subalgebra vector fields on $J^\infty(\pi)$, which preserve $\mathcal{C}$, denoted by $\mathcal{T}_\mathcal{C}(\pi)$. Apparently, $\mathcal{CT}(\pi)$ is an ideal in $\mathcal{T}_\mathcal{C}(\pi)$. Let us define

$$\mathcal{T}_{\mathsf{sym}}(\pi) = \mathcal{T}_\mathcal{C}(\pi)/\mathcal{CT}(\pi). \tag{7}$$

Elements of $\mathcal{T}_{\mathsf{sym}}(\pi)$ are uniquely represented by π_∞-vertical vector fields which preserve $\mathcal{C}$, called *evolutionary vector fields*; they can be identified with sections of $\varkappa(\pi) = \pi^*_{\infty,0}(\pi)$ as follows:

$$\mathcal{T}_{\mathsf{sym}}(\pi) \ni v \mapsto \phi_v = v_{|\mathcal{F}_0}.$$

Remark 2 Taking into account that every total derivative is a π_∞-projectable vector field on $J^\infty(\pi)$, and thus it preserves the subspace of π_∞-vertical vector fields, we immediately conclude that evolutionary vector fields are those and only those which commute with all total derivatives. In other words, an evolutionary vector field is a derivation of $\mathcal{F}(\pi)$ over $\mathcal{D}(X)$. All sections of π_∞ which are integral leaves of the Cartan distribution, are in one-to-one correspondence with infinite jets of sections of π; therefore any infinitesimal bundle morphism of π_∞ preserving $\mathcal{C}(\pi)$, determines an infinitesimal flow on $\Gamma(\pi)$. Hence an evolutionary vector field is a "good candidate" for being a vector field on the space of sections. Indeed, evolutionary vector fields induce derivations of local functionals (see the later Remark 3). However, almost all evolutionary vector fields, except those which come from infinitesimal morphisms of π, will not generate a flow. What concerns bundle morphisms of π, they obviously act on $\Gamma(\pi)$, so that the corresponding infinitesimal generators, which are π-projectible vector fields on the total space of π, can be thought of as "honest" vector fields on $\Gamma(\pi)$. In other words, any π-projectible vector field v admits the unique lift $\tilde{v}$, which preserves the Cartan distribution, that is, $\tilde{v} \in \mathcal{T}_\mathcal{C}(\pi)$. In coordinates as in (5), if

$$v = \sum_{i=1}^{n} h^i(x)\partial_{x^i} + \sum_{a=1}^{\mathrm{rk}(\pi)} g^a(x,u)\partial_{u^a},$$

then

$$\tilde{v} = \sum_{i=1}^{n} h^i(x) D_{x^i} + \sum_{a=1}^{\mathrm{rk}(\pi)} \sum_{(\sigma)} D_{(\sigma)} \left(-\sum_{i=1}^{n} h^i u^a_i + g^a(x,u) \right) \partial_{u^a_{(\sigma)}}. \tag{8}$$

One can easily check that, in contrast to total derivatives, $\tilde{v}$ preserves $\mathcal{F}_k(\pi)$ for all k. The π_∞-vertical part of (8) is the evolutionary vector field corresponding to v.

Define the module $\Omega^i(\pi)$ of differential i-forms on $\mathrm{J}^\infty(\pi)$ by setting[2]

$$\Omega^i(\pi) := \bigcup_k \Omega^i(\pi_k),$$

where $\Omega^i(\pi_k)$ is the module of i-forms on $\mathrm{J}^k(\pi)$. Let us set $\Omega^*(\pi) = \oplus_{i=0}^{\infty}\Omega^i(\pi)$. The decomposition of vector fields on the infinite jets space into the sum of horizontal and vertical parts gives rise to a bicomplex structure on $\Omega^*(\pi)$, called the *variational bicomplex*:

$$\Omega^\bullet = \bigoplus_{p,q\geq 0} \Omega^{p,q}(\pi), \quad \mathrm{d} = \mathrm{d}_h + \mathrm{d}_v, \tag{9}$$

where

$$\begin{array}{ccc} & \Omega^{p,q}(\pi) & \\ {}^{\mathrm{d}_h}\swarrow & & \searrow^{\mathrm{d}_v} \\ \Omega^{p+1,q}(\pi) & & \Omega^{p,q+1}(\pi) \end{array}$$

such that $\Omega^{0,1}(\pi)$ is the annihilator of the Cartan distribution and $\Omega^{1,0}(\pi)$ is the space of π_∞-horizontal 1-forms. In local coordinates as in (5), one has

$$\mathrm{d}_h = \sum_{i=1}^{n} \mathrm{d}x^i D_{x^i}\,, \quad \mathrm{d}_v = \sum_{a=1}^{\mathrm{rk}(\pi)} \sum_{(\sigma)} \vartheta^a_{(\sigma)} \partial_{u^a_{(\sigma)}} \tag{10}$$

where $\vartheta^a_{(\sigma)}$ are the (local) Cartan 1-forms defined as follows:

$$\vartheta^a_{(\sigma)} = \mathrm{d}u^a_{(\sigma)} - \sum_{i=1}^{n} u^a_{(\sigma+1_i)} \mathrm{d}x^i. \tag{11}$$

Hereafter we use the notation $\bar{\Lambda}^p(\pi)$ for the bundle $\Lambda^{p,0}(\pi)$ of horizontal p-forms and $\left(\bar{\Omega}^\bullet(\pi), \mathrm{d}_h\right)$ for the horizontal part of the variational bicomplex (9), $\left(\Omega^{\bullet,0}(\pi), \mathrm{d}_h\right)$, respectively. Similarly to scalar functions, any p-form $\omega \in \bar{\Omega}^p(\pi)$ can be regarded as a nonlinear differential operator with values in p-forms on X, acting on sections of π by the following formula:

$$s \mapsto \omega[s] = j_k(s)^*(\omega) \in \Omega^p(X), \quad s \in \Gamma(\pi). \tag{12}$$

[2] Direct limit of differential forms and embeddings induced by the projections π and $\pi_{k+1,k}$.

The next property is immediate from (6) and (10):

$$(\mathrm{d}_h\omega)[s] = \mathrm{d}\,(\omega[s])\,. \tag{13}$$

By (12) we conclude that, if X is oriented, then any horizontal top-form $\omega \in \bar{\Omega}^n(\pi)$ determines a local (that is, a jet depending) functional on $\Gamma(\pi)$,

$$s \mapsto \int_X \omega[s], \tag{14}$$

so that, if X is a compact oriented manifold without boundary then the above functional is determined by the cohomology class of ω in $H^n(\bar{\Omega}^\bullet(\pi), \mathrm{d}_h)$. We denote the space of local functionals by $Loc(\pi)$ and summarize the above considerations as follows.

Proposition 3 *Let X be a compact oriented manifold without boundary, then*

$$Loc(\pi) \simeq H^n(\bar{\Omega}^\bullet(\pi), \mathrm{d}_h).$$

Remark 3 From the Remark 2 we conclude that any evolutionary vector field preserves the bicomplex structure (9). In particular, this implies that, if X is compact without boundary, then, by Proposition 3, evolutionary vector fields are acting in $Loc(\pi)$.

For a generic fiber bundle (E, π, X) we recall some standard results about horizontal cohomologies, see [7, 12, 21]. Note that all the aforementioned results about jet spaces (e.g. Cartan distribution, variational bicomplex, etc.) can be generalized to the case of an arbitrary smooth fiber bundle. The exterior algebra $\Omega^\bullet(\pi)$ provides the (infinite order) de Rham complex

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^0(\pi) \xrightarrow{\mathrm{d}} \Omega^1(\pi) \xrightarrow{\mathrm{d}} \cdots .$$

First we remind the following[3]

Proposition 4 *The cohomology $H^*(\Omega^\bullet(\pi))$ of the previous de Rham complex is equal to the de Rham cohomology $H^*(E)$ of the total space E.*

Recall that there is a canonical homomorphism between the de Rham cohomologies of the base and the total space

$$\pi^* : H^*(X) \to H^*(E);$$

if $s \in \Gamma(\pi)$ is a global section we denote with s^* the corresponding epimorphism $s^* : H^*(E) \to H^*(X)$. Whenever this epimorphism is defined, π^* becomes a mono

[3] It is based on the fact that jet bundles $\mathrm{J}^{k+1}(\pi) \to \mathrm{J}^k(\pi)$ are affine.

morphism. In this hypothesis we extend the monomorphism from the de Rham cohomology groups of the base X to those for the infinite jets space

$$\pi^* : H^*(X) \hookrightarrow H^*(\Omega^\bullet(\pi)).$$

In the previous paragraph we have already introduced the splitting of $\Omega^\bullet(\pi)$ into horizontal and vertical parts; we denote with

$$\pi^{\bullet,0} : \Omega^\bullet(\pi) \to \bar{\Omega}^\bullet(\pi) := \Omega^{\bullet,0}(\pi)$$

the horizontal projection. It is obvious that this projection is a chain map

$$\mathrm{d} \circ \pi^{\bullet,0} = \pi^{\bullet,0} \circ \mathrm{d}_h$$

and it defines a homomorphisms of groups

$$(\pi^{\bullet,0})^* : H^*(\Omega^\bullet(\pi)) \to H^*(\bar{\Omega}^\bullet(\pi)).$$

The composition of the previous two cohomology maps

$$(\pi^{\bullet,0})^* \circ \pi^* : H^*(X) \to H^*(\bar{\Omega}^\bullet(\pi)), \tag{15}$$

in the case (E, π, X) admits a global section, is still a monomorphism. It is again a well-known result (*loc.cit.*) the fact that $H^*(\bar{\Omega}^\bullet(\pi))$ for $* < n$ is equal to the de Rham cohomology of the total space $H^*(E)$.

We shall adapt these results to our specific setting, i.e. (E, π, X) is in particular a vector bundle. In this case the canonical choice for the aforementioned global section is the zero section and the cohomology $H^*(E)$ coincides with the de Rham cohomology of the base $H^*(X)$. Apparently, in our hypothesis, the horizontal cohomologies (of degree less than n) are provided by the image of the de Rham complex of the base, lifted by the pullback of the projection map. In the next paragraph we will restrict our attention to the subcomplex of horizontal forms which vanish on the infinite jet of the zero section; this subcomplex is complementary to the image of the forms from the base.

Among all functions on the space of k-jets of a (possibly graded super) vector bundle, there are two distinguished $\mathbb{Z}$-graded subalgebras: of fiber-wise polynomial functions, $\mathcal{S}_k^\bullet(\pi)$, and fiber-wise polynomial functions, vanishing on the zero section of π, $\mathcal{S}_k^+(\pi)$, which can be identified with sections of[4] $\mathrm{Sym}^\bullet(\pi_k^*)$ and $\mathrm{Sym}^+(\pi_k^*) = \oplus_{j>0}\mathrm{Sym}^j(\pi_k^*)$, respectively. In the case of a graded super vector bundle, the symmetric powers should be understood in the super sense. Given that π_∞^* is a direct limit of π_k^*, and thus $\mathrm{Sym}^p(\pi_\infty^*)$ is a direct limit of $\mathrm{Sym}^p(\pi_k^*)$, a section of $\mathrm{Sym}^p(\pi_\infty^*)$ is always a section of $\mathrm{Sym}^p(\pi_k^*)$ for some k. We denote by $\mathcal{S}^\bullet(\pi)$ and $\mathcal{S}^+(\pi)$ the direct limit of the corresponding algebras.

[4] Symmetric powers of the dual bundle.

Remark 4 According to (4), an element of $\mathcal{S}^p_k(\pi)$ can be viewed as a symmetric p-linear differential operator of maximal order k acting from sections of π to smooth functions on X: in order to verify this statement, we use the usual correspondence between polynomial and symmetric multi-linear maps.

From (5) one can see that the subspaces $\mathcal{S}^p(\pi)$ are preserved by total derivatives for all p, thus we obtain an action of $\mathcal{D}(X)$ on $\mathcal{S}^p(\pi)$, and finally on $\mathcal{S}^\bullet(\pi)$ and $\mathcal{S}^+(\pi)$. In order to determine the precise form of this action, we shall first give a very brief survey of the properties of modules over $\mathcal{D}_X$, the sheaf of differential operators on X, called *$\mathcal{D}$-modules*; nowadays it is a convenient language for talking about linear PDEs and their solutions. The structure sheaf of smooth functions on X will be denoted with $\mathcal{O}_X$ (its sections over U is just $C^\infty(U)$); the choice for this convention is so motivated: many properties stated hereafter can be generalized to the analytic and algebraic case.

Denote by $\mathsf{Mod}(X)$ and $\mathsf{Mod}(X)^{\mathsf{r}}$—the categories of *left and right $\mathcal{D}$-modules*, respectively. E.g. the structure sheaf $\mathcal{O}_X$ is a left $\mathcal{D}$-module, while Ω^n_X, the sheaf of top degree forms on X, is a right $\mathcal{D}$-module, where the right action on Ω^n_X is generated by

$$\omega v = -L_v(\omega) \quad \forall\, \omega \in \Omega^n_X\,,\, v \in \mathcal{T}_X\,.$$

Here $\mathcal{T}_X$ is the sheaf of vector fields on X. Recall that:

- if $\mathtt{M}$ and $\mathtt{N}$ belong to $\mathsf{Mod}(X)$, then so do $\mathrm{Hom}(\mathtt{M},\mathtt{N})$, $\mathtt{M}\otimes\mathtt{N}$, and $\mathrm{Sym}^p(\mathtt{M})$ for all p; the symmetrization is to respect the sign rule in the super case.[5]
- if $\mathtt{M} \in \mathsf{Mod}(X)$ and $\mathtt{N} \in \mathsf{Mod}(X)^{\mathsf{r}}$, then $\mathtt{N} \otimes \mathtt{M} \in \mathsf{Mod}(X)^{\mathsf{r}}$, where

$$(n \otimes m)v = nv \otimes m - n \otimes vm\,, \qquad \forall\, m \in \mathtt{M}\,, n \in \mathtt{N}\,, v \in \mathcal{T}_X\,.$$

- if $\mathtt{N}_1, \mathtt{N}_2 \in \mathsf{Mod}(X)^{\mathsf{r}}$ then $\mathrm{Hom}(\mathtt{N}_1, \mathtt{N}_2) \in \mathsf{Mod}(X)$, where

$$v\psi(n) = \psi(nv) - \psi(n)v\,, \quad \forall\, n \in \mathtt{N}_1\,, \psi \in \mathrm{Hom}(\mathtt{N}_1, \mathtt{N}_2)\,, v \in \mathcal{T}_X\,.$$

The tensor product $\otimes$ determines a symmetric monoidal structure in $\mathsf{Mod}(X)$ with $\mathcal{O}_X$ as unit.

Definition 1 A commutative $\mathcal{D}$-algebra is an algebra in the symmetric monoidal category ($\mathsf{Mod}(X)$, $\otimes$, $\mathcal{O}_X$), i.e. a commutative monoid in the category of $\mathcal{D}$-modules.

More explicitly, a commutative $\mathcal{D}$-algebra is a $\mathcal{D}$-module $\mathcal{A}$ together with two $\mathcal{D}_X$-linear maps, (product)

$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

and (unit)

$$i : \mathcal{O}_X \to \mathcal{A},$$

[5] The bifunctors Hom and $\otimes$ are defined over $\mathcal{O}_X$.

which respect the usual associativity, unitality and commutativity constraints. Note that the action of a vector field on M on a product $\mu(a \otimes a')$ verifies the Leibniz' rule for any $a, a' \in \mathcal{A}$.

Example 1 Given any vector bundle π, $\mathcal{F}(\pi)$ is a $\mathcal{D}$-algebra, where the $\mathcal{D}$-module structure is defined by total derivatives. Another example is the algebra of functions on an infinitely prolonged system of nonlinear partial differential equations, regarded as a "submanifold" in $\mathrm{J}^\infty(\pi)$.

Definition 2 An evolutionary vector field for a $\mathcal{D}$-algebra $\mathcal{A}$ is a derivation of $\mathcal{A}$ commuting with the action $\mathcal{D}_X$.

Denote by $\mathcal{D}(\alpha, \beta)$ the space of linear differential operators acting between sections of vector bundles α and β on X, and by $\mathbf{1}^k$ the trivial vector bundle of rank k. Then $\mathcal{D}(\alpha, \mathbf{1})$ is a left $\mathcal{D}$-module, which is isomorphic to $\mathcal{S}^1(\pi) = \Gamma(\pi^*_\infty)$ (see the Remark 4); here $\mathcal{D}(X)$ is acting from the left by composition. Likewise, $\mathcal{D}(\alpha, \Lambda^n_X)$ is right $\mathcal{D}$-module, where Λ^p_X is the bundle differential p-forms on X.

Definition 3 Let α be a vector bundle. Denote the conjugated vector bundle $\mathrm{Hom}(\alpha, \Lambda^n_X)$ by $\hat{\alpha}$.

Proposition 5 *There exists a canonical isomorphism of $\mathcal{O}_X$-bimodules $\mathcal{D}(\alpha, \beta) \simeq \mathcal{D}(\hat{\beta}, \hat{\alpha})$, determined by formal conjugation. In particular, $\mathcal{D}(\alpha, \Lambda^n_X) \simeq \mathcal{D}(\mathbf{1}, \hat{\alpha})$. The latter is also an isomorphism of right $\mathcal{D}$-modules.*

Consider the following complex of right $\mathcal{O}_X$-modules $\left(\mathrm{Sym}^l \mathcal{D}(\pi, \Lambda^\bullet_X), \mathrm{d}_{\mathrm{DR}}\right)$, where $\mathrm{Sym}^l \mathcal{D}(\alpha, \beta)$ is, by definition, the space the q-linear symmetric differential operators acting from sections of a vector bundle α to sections of another vector bundle β, and the differential d_{DR} is induced by the left composition with the de Rham operator. The statement from Remark 4 about polynomial functions on the space of jets can be easily extended to polynomial horizontal differential forms.

Proposition 6 *The following complexes are canonically isomorphic:*

$$\left(\bar{\Omega}^{\bullet,l}_{\mathrm{poly}}(\pi), \mathrm{d}_h\right) \simeq \left(\mathrm{Sym}^l \mathcal{D}(\pi, \Lambda^\bullet_X), \mathrm{d}_{\mathrm{DR}}\right) \tag{16}$$

where $\bar{\Omega}^{\bullet,l}_{\mathrm{poly}}(\pi)$ is a subcomplex of the horizontal complex $(\bar{\Omega}^\bullet(\pi), \mathrm{d}_h)$ consisting of horizontal differential forms which depend on jet variables as polynomials of the degree l.

3 BRST Cohomology in the Space of Local Functionals

Let η be a Q-bundle over $T[1]X$, that is, a bundle in the category of Q-manifolds (cf. [14], [15]), so that the Q-structure on the base is determined by the de Rham operator, regarded as a homological vector field. Apparently, not every section of η in the graded sense is a section in the category of Q-manifolds, that is, not necessarily a Q-morphism; sections of η, which are Q-morphisms at the same time,[6] are solutions to a certain system of PDEs. This system admits gauge symmetries (cf. [13]). The Q-stucture on the total space generates a homological vector field on the super space of sections $\underline{\Gamma}(\eta)$, denoted as Q_{BRST}; $\big(\underline{\Gamma}(\eta), Q_{\text{BRST}}\big)$ is the BV-BRST type model for the above system of PDEs. Q_{BRST} induces a nilpotent derivational of a (suitable) space of functionals $\mathscr{F}(\underline{\Gamma}(\eta))$; the problem is to compute the cohomology of the obtained complex.

In the case of a trivial bundle, the fiber of which is a PQ manifold, that is, a graded super symplectic manifold with a symplectic form of degree $\dim X - 1$, so that the corresponding Q-field is Hamiltonian, we come to the classical BV theory for AKSZ type topological sigma models [1, 8]. In usual differential geometry, sections of a trivial bundle are in one-to-one correspondence with maps from the base to the fiber. Likewise, in the super case

$$\underline{\Gamma}(\eta) \simeq \underline{\text{Hom}}(T[1]X, \mathcal{M}), \tag{17}$$

where $\mathcal{M}$ is the fiber and $\underline{\text{Hom}}$ is the super space of maps. In general, the construction of $\underline{\text{Hom}}$ in (17) is rather complicated (cf. [11] for the categorical approach; in [9], $\underline{\text{Hom}}$ is explicitly represented by an infinite-dimensional supermanifold), unless the target is flat.

The choice of an appropriate space of functionals $\mathcal{F}$ is not canonical. Furthermore, there is a tendency (even in non-super cases) to avoid possible troubles with an infinite-dimensional analysis by considering local ("jet depending") functionals in the sense of Sect. 2. It seems to be at least equally useful for those theories which involve super maps. However, in TFTs the space of local functionals contains a very restrictive information, and we shall explicitly show that in the particular case of $\mathcal{M}$ being a $\mathbb{Z}$-graded super vector space L of finite type, i.e.

$$L^{\bullet} = \bigoplus_{i \in \mathbb{Z}} L^i$$

with $\dim L^i < \infty$ for all i, endowed with a structure of a Lie_∞-algebra.

Definition 4 A Lie_∞-algebra is a formal Q-manifold with the homological vector field vanishing at the origin.

Example 2 In the particular case of a Lie algebra the corresponding Q-manifold is $L := \mathfrak{g}[1]$, where $\mathfrak{g}$ is the Lie agebra considered as a pure odd manifold, with Q-field given by the Chevalley–Eilenberg differential.

[6] Geometrically it means that those sections are tangent to the Q-stucture on the total space.

Remark 5 In general, formal pointed (i.e. vanishing at the origin) Q-structures on L are in one-to-one correspondence with nilpotent degree 1 coderivations of the coalgebra $\mathrm{Sym}_c^+(L)$, determined by an infinite sequence of maps $\mathrm{Sym}_c^i(L) \to L[1]$, $i \geq 1$. By use of the natural isomorphism $\mathrm{Sym}^i(\mathfrak{g}[1]) \simeq \Lambda^i(\mathfrak{g})[i]$, we obtain a sequence of super skew-symmetric operations

$$l_i \colon \Lambda^i(\mathfrak{g}) \to \mathfrak{g}[2-i]\,, \quad \forall i \geq 1, \tag{18}$$

where $\mathfrak{g} = L[-1]$; the latter was introduced under the name "homotopy Lie algebras" [19].

Denote with $\alpha^{j,i}$ the bundle of differential j-forms on X twisted by L^i, $\alpha^{j,i} = \Lambda_X^j \otimes L^i$.

Lemma 1

1. *The super space of maps is given by $\Gamma(\alpha^\bullet)$, where*

$$\alpha^\bullet = \bigoplus_{q\in\mathbb{Z}} \alpha^q\,, \quad \alpha^q = \bigoplus_{j=0}^{\dim X} \alpha^{j,q-j}$$

is regarded as a $\mathbb{Z}$-graded super vector bundle with the total $\mathbb{Z}$-grading induced by the degree of forms and the grading in L.

2. *$Q_{\mathrm{BRST}} = Q_{\mathrm{DR}} + Q_L$, where Q_{DR} is a (linear) homological vector field, given by the L-twisted de Rham operator, while Q_L is a pointed formal Q-field, determined by the super multi-linear over $\Omega^\bullet(X)$ extension of the coderivation of* $\mathrm{Sym}_c^+(L)$.

As it was previously mentioned, the choice of $\mathscr{F}$, the space of functionals, is not canonical. On the other hand, the super space of maps is now represented by sections of graded super vector bundle over an even ("bosonic") base X. One may address the naturally looking question of computing the cohomology in the space of local functionals, which are polynomials in jet variables, with respect to the differential $\mathbf{s}$, where $\mathbf{s}$ is the evolutionary vector field corresponding to Q_{BRST}. In other words, we are interested in $H^{\bullet,n}(\mathbf{s} \mid \mathrm{d}_h)$, where

$$H^{\bullet,n}(\mathbf{s} \mid \mathrm{d}_h) \colon = \bigoplus_{g\in\mathbb{Z}} H^g\left(H^n\left(\bar{\Omega}^{\bullet,\bullet}_{\mathrm{poly}}(\pi), \mathrm{d}_h\right), \mathbf{s}\right)$$

($\omega^{n,g} \in \Omega^{n,g}_{\mathrm{poly}}(\pi)$ is a n-horizontal form of $g \in \mathbb{Z}$ degree).

Proposition 7 *One has $H^{g,n}(\mathbf{s} \mid \mathrm{d}_h) \simeq H^{g+n}(\mathbf{s} + \mathrm{d}_h)$.*

Proof We apply the canonical isomorphism (16). Let us consider the corresponding bicomplex

$$\begin{array}{ccccccccc}
 & & \vdots & & & & \vdots & & \\
 & & \Big\uparrow{\scriptstyle \mathbf{s}} & & & & \Big\uparrow{\scriptstyle \mathbf{s}} & & \\
0 & \longrightarrow & \big[\mathrm{Sym}^{+}\mathcal{D}(\alpha^{\bullet}, \mathbf{1})\big]^{g+1} & \xrightarrow{\mathrm{d_{DR}}} & \cdots & \xrightarrow{\mathrm{d_{DR}}} & \big[\mathrm{Sym}^{+}\mathcal{D}(\alpha^{\bullet}, \Lambda^{n}_{X})\big]^{g+1} & \longrightarrow & 0 \\
 & & \Big\uparrow{\scriptstyle \mathbf{s}} & & & & \Big\uparrow{\scriptstyle \mathbf{s}} & & \\
0 & \longrightarrow & \big[\mathrm{Sym}^{+}\mathcal{D}(\alpha^{\bullet}, \mathbf{1})\big]^{g} & \xrightarrow{\mathrm{d_{DR}}} & \cdots & \xrightarrow{\mathrm{d_{DR}}} & \big[\mathrm{Sym}^{+}\mathcal{D}(\alpha^{\bullet}, \Lambda^{n}_{X})\big]^{g} & \longrightarrow & 0. \\
 & & \Big\uparrow{\scriptstyle \mathbf{s}} & & & & \Big\uparrow{\scriptstyle \mathbf{s}} & & \\
 & & \vdots & & & & \vdots & &
\end{array} \tag{19}$$

We examine the spectral sequence determined by (19), where the filtration is chosen such that the cohomology of the rows are to be taken at first. The E_1-term of the above spectral sequence can be computed by use of the following Lemma.

Lemma 2 *Let α be a vector bundle. Then one has for all $l > 0$*

$$H^{i}\left(\mathrm{Sym}^{l}\mathcal{D}(\alpha, \Lambda^{\bullet}_{X}), \mathrm{d_{DR}}\right) = \begin{cases} \mathrm{Sym}^{l-1}_{\mathrm{self}}\mathcal{D}(\alpha, \hat{\alpha}), & i = n \\ 0, & i < n \end{cases} \tag{20}$$

where $\mathrm{Sym}^{l-1}_{\mathrm{self}}\mathcal{D}(\alpha, \hat{\alpha})$ is the space the $(q-1)$-linear symmetric differential operators, (formally) self-adjoint with respect to each argument. In particular, for $q = 1$ one has

$$H^{i}\left(\mathcal{D}(\alpha, \Lambda^{\bullet}_{X}), \mathrm{d_{DR}}\right) = \begin{cases} \hat{\alpha}, & i = n \\ 0, & i < n. \end{cases}$$

The proof is rather standard; we notice that the differential in the above complex commutes with the right $\mathcal{O}_X$-action coming from the $\mathcal{O}_X$-module structure on $\Gamma(\alpha)$, thus one has a complex of locally trivial $\mathcal{O}_X$-modules or, equivalently, a complex of vector bundle morphisms. This implies that formula (20) can be derived in any local coordinates, using the symbolic filtration. A similar result, involving $\mathcal{C}$-differential operators instead of $\mathcal{D}_X$, is obtained in the case of the Vinogradov's $\mathcal{C}$-spectral sequence (cf. [7, 16]). Taking into account that the E_1-term is concentrated in degree n only, we immediately obtain that the above spectral sequence converges in the second term, thus the second term of the spectral sequence is isomorphic to the cohomology of the total complex with the differential $\mathbf{s} + \mathrm{d}_h$. Given that the second term of the spectral sequence is nothing but $H^{g,n}(\mathbf{s} \mid \mathrm{d}_h)$, we complete the proof of Proposition 7.

Theorem 1 *One has* $H^{g,n}(\mathbf{s} \mid \mathrm{d}_h) \simeq \left(H^{\bullet}_{\mathrm{DR}}(X) \otimes H^{\bullet}_{Q}(L)\right)^{g+n}$.

Proof The differential, given by the evolutionary vector field $\mathbf{s}$, splits into the two parts $\mathbf{s} = \mathcal{Q}_L + \delta_{\mathrm{DR}}$, which come from Q_L and Q_{DR}, respectively. In particular

$\delta_{\rm DR}$ is the derivation of $\mathrm{Sym}^+\mathcal{D}(\alpha^\bullet, \Lambda^p_X)$ induced by the right composition of $Q_{\rm DR}$ with differential operators $\mathcal{D}(\alpha^\bullet, \Lambda^p_X)$. The two independent gradings allow to define another bicomplex

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
\cdots \xrightarrow{\mathcal{Q}_L} & & \left[\mathrm{Sym}^+\mathcal{D}(\alpha^\bullet, \Lambda^p_X)\right]^{0,i} & \xrightarrow{\mathcal{Q}_L} & \left[\mathrm{Sym}^+\mathcal{D}(\alpha^\bullet, \Lambda^p_X)\right]^{0,i+1} & \xrightarrow{\mathcal{Q}_L} & \cdots \\
 & & \uparrow{\scriptstyle \delta_{\rm DR}} & & \uparrow{\scriptstyle \delta_{\rm DR}} & & \\
 & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle \delta_{\rm DR}} & & \uparrow{\scriptstyle \delta_{\rm DR}} & & \\
\cdots \xrightarrow{\mathcal{Q}_L} & & \left[\mathrm{Sym}^+\mathcal{D}(\alpha^\bullet, \Lambda^p_X)\right]^{-j+1,i} & \xrightarrow{\mathcal{Q}_L} & \left[\mathrm{Sym}^+\mathcal{D}(\alpha^\bullet, \Lambda^p_X)\right]^{-j+1,i+1} & \xrightarrow{\mathcal{Q}_L} & \cdots \\
 & & \uparrow{\scriptstyle \delta_{\rm DR}} & & \uparrow{\scriptstyle \delta_{\rm DR}} & & \\
\cdots \xrightarrow{\mathcal{Q}_L} & & \left[\mathrm{Sym}^+\mathcal{D}(\alpha^\bullet, \Lambda^p_X)\right]^{-j,i} & \xrightarrow{\mathcal{Q}_L} & \left[\mathrm{Sym}^+\mathcal{D}(\alpha^\bullet, \Lambda^p_X)\right]^{-j,i+1} & \xrightarrow{\mathcal{Q}_L} & \cdots . \\
 & & \uparrow{\scriptstyle \delta_{\rm DR}} & & \uparrow{\scriptstyle \delta_{\rm DR}} & & \\
 & & \vdots & & \vdots & &
\end{array} \tag{21}$$

Here $\mathrm{Sym}^+\mathcal{D}(\alpha^\bullet, \Lambda^p_X)$ is a canonically bi-graded vector space, such that, in particular, the first degree, corresponding to the one in α, is always non-positive. Furthermore, we are finally interested in the calculation of the total cohomology $\mathrm{d}_{\rm DR} + \mathbf{s}$ (due to Proposition 7) which is made up by three differentials $\mathrm{d}_{\rm DR}$, $\mathcal{Q}_L$, and $\delta_{\rm DR}$ with three independent gradings. We combine the fist two of them and construct a filtration for the bicomplex $(\mathrm{Sym}^+\mathcal{D}(\alpha^\bullet, \Lambda^\bullet_X), \mathcal{Q}_L + \mathrm{d}_{\rm DR}, \delta_{\rm DR})$, such that the cohomology with respect to $\delta_{\rm DR}$ are to be computed at first. Thus we need to calculate the cohomology of the columns in (21). We notice that the following complex is formally exact (in the sense of [17, 20])

$$\alpha^{0,i} \xrightarrow{Q_{\rm DR}} \alpha^{1,i} \xrightarrow{Q_{\rm DR}} \cdots \xrightarrow{Q_{\rm DR}} \alpha^{n,i} \longrightarrow 0.$$

In particular this means that applying to this complex the jet infinity functor, J^∞, we get an exact sequence of $\mathcal{O}_X$-modules

$$\mathrm{J}^\infty(\alpha^{0,i}) \longrightarrow \mathrm{J}^\infty(\alpha^{1,i}) \longrightarrow \cdots \longrightarrow \mathrm{J}^\infty(\alpha^{n,i}) \longrightarrow 0;$$

we extend it to the following exact sequence of $\mathcal{O}_X$-modules

$$\Lambda_X^{n-p} \otimes \mathrm{J}^\infty(\alpha^{0,i}) \longrightarrow \Lambda_X^{n-p} \otimes \mathrm{J}^\infty(\alpha^{1,i}) \longrightarrow \cdots \longrightarrow \Lambda_X^{n-p} \otimes \mathrm{J}^\infty(\alpha^{n,i}) \longrightarrow 0$$

(for $0 \leq p \leq n$), where the horizontal arrows are still induced by the operator Q_{DR}. Dualizing the previous sequence, by the use of the left-exact contravariant $\mathrm{Hom}(-, \Lambda_X^n)$ functor, we get a sequence of right $\mathcal{O}_X$-modules which is exact everywhere except at the zero spot (i.e. it is a resolution of a cokernel)

$$0 \to \mathcal{D}(\alpha^{n,i}, \Lambda_X^p) \xrightarrow{\delta_{\mathrm{DR}}} \mathcal{D}(\alpha^{n-1,i}, \Lambda_X^p) \xrightarrow{\delta_{\mathrm{DR}}} \ldots \mathcal{D}(\alpha^{0,i}, \Lambda_X^p) \xrightarrow{\delta_{\mathrm{DR}}} \mathrm{coker} \to 0. \tag{22}$$

More precisely, it means that

$$H^j\left(\mathcal{D}(\alpha^{\bullet,i}, \Lambda_X^p), \delta_{\mathrm{DR}}\right) = \begin{cases} \mathrm{Hom}(L^i, \Lambda_X^p), & j = 0 \\ 0, & j < 0. \end{cases}$$

Now we take the symmetric powers of (22) and we get

$$H^j\left(\left[\mathrm{Sym}^+\mathcal{D}(\alpha^\bullet, \Lambda_X^p)\right]^{\bullet,i}, \delta_{\mathrm{DR}}\right) = \begin{cases} \left[\mathrm{Sym}^+(L^*)\right]^i \otimes \Omega^p(X), & j = 0 \\ 0, & j < 0. \end{cases} \tag{23}$$

This completes the calculation of the term E_1. The second term of the above spectral sequence coincides with the total cohomology of the bicomplex $(\Omega^\bullet(X) \otimes \left[\mathrm{Sym}^+(L^*)\right]^\bullet, \mathrm{d}_{\mathrm{DR}}, \mathcal{Q}_L)$, which is simply the tensor product of $(\Omega^\bullet(X), \mathrm{d}_{\mathrm{DR}})$ and $(\left[\mathrm{Sym}^+(L^*)\right]^\bullet, \mathcal{Q}_L)$. Thus we have the following Künneth type formula (see [18])

$$H^p\left(\Omega^\bullet(X) \otimes \left[\mathrm{Sym}^+(L^*)\right]^\bullet, \mathrm{d}_{\mathrm{DR}} + \mathcal{Q}_L\right) = \bigoplus_{i+j=p} H^i_{\mathrm{DR}}(X) \otimes H^j_Q(L). \tag{24}$$

We observe that the bicomplex associated to the couple $(\mathrm{d}_{\mathrm{DR}} + \mathcal{Q}_L, \delta_{\mathrm{DR}})$ verifies the hypothesis of Remark 6 below, in view of Lemma 2 and Eq. (23). Therefore, the E_2-term of the associated spectral sequence coincides with the total cohomology with the differential $\mathrm{d}_{\mathrm{DR}} + \mathcal{Q}_L + \delta_{\mathrm{DR}}$ and thus, using (24), we accomplish the proof of Theorem 1.

The previous proof contains a result which can be stated in all generality in the following way.

Lemma 3 *Let β be a vector bundle and γ be another vector space endowed with a flat connection. Consider the following complex of left $\mathcal{O}_X$-modules $\left(\mathcal{D}(\Lambda_X^\bullet \otimes \gamma, \beta)\right.$, $\left.\delta_{\mathrm{DR}}\right)$, where the differential δ_{DR} is induced by the right composition with the de Rham operator twisted by the flat connection in γ. Then one has*

$$H^i\left(\mathrm{Sym}^+\mathcal{D}(\Lambda_X^\bullet \otimes \gamma, \beta), \delta_{\mathrm{DR}}\right) = \begin{cases} \Gamma\left(\mathrm{Sym}^+(\gamma^*) \otimes \beta\right), & i = 0 \\ 0, & i < 0. \end{cases}$$

Remark 6 Let $K^\bullet$ be the total complex of a bicomplex $K^{\bullet,\bullet}$ with linear maps

$$d^1\colon K^{p,q} \to K^{p+1,q}, \quad d_2\colon K^{p,q} \to K^{p,q+1},$$

such that $(d_1)^2 = 0,\ (d_2)^2 = 0$ and $d_2 d_1 + d_1 d_2 = 0$. There are two filtrations

$$K^i_p(1) = \bigoplus_{j+q=i,\ j\geq p} K^{j,p}, \quad K^i_q(2) = \bigoplus_{p+j=i,\ j\geq q} K^{p,j}.$$

These two filtrations yield two spectral sequences, denoted respectively by $E^{p,q}_r(1)$ and $E^{p,q}_r(2)$; in particular recall that $E^{p,q}_2(1) = H^p_1\left(H^q_2(K^{\bullet,\bullet})\right)$ and $E^{p,q}_2(2) = H^p_2\left(H^q_1(K^{\bullet,\bullet})\right)$. Now assume that both filtrations are regular. In this case both spectral sequences converge to the common limit $H^\bullet(K^\bullet)$.

Suppose that in the following diagram

$$\begin{array}{ccccccc}
0 & \longleftarrow & K^{2,0} & \xleftarrow[d_2]{} & K^{2,1} & \xleftarrow[d_2]{} & K^{2,2} & \longleftarrow \\
 & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1} & \\
0 & \longleftarrow & K^{1,0} & \xleftarrow[d_2]{} & K^{1,1} & \xleftarrow[d_2]{} & K^{2,2} & \longleftarrow \\
 & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1} & \\
0 & \longleftarrow & K^{0,0} & \xleftarrow[d_2]{} & K^{0,1} & \xleftarrow[d_2]{} & K^{0,2} & \longleftarrow \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & & 0 & & 0 &
\end{array}$$

all the sequences are exact except for the terms in the left column and bottom row. We have two complexes $Q^\bullet_1$ and $Q^\bullet_2$, where $Q^i_1 = H^0(K^{i,\bullet}, d_2)$ and $Q^i_2 = H^0(K^{\bullet,i}, d_1)$ and the differentials are induced by d_1 and d_2 respectively. It follows that $E^{p,q}_2(1) = E^{p,q}_3(1) = \ldots = E^{p,q}_\infty(1)$ is equal to $H^p(Q^\bullet_1)$ (if $q = 0$ and zero otherwise) and $E^{p,q}_2(2) = E^{p,q}_3(2) = \ldots = E^{p,q}_\infty(2)$ is equal to $H^q(Q^\bullet_2)$ (if $p = 0$ and zero otherwise). Since both spectral sequences converge to a common limit, we conclude that $H^i(Q^\bullet_1) = H^i(Q^\bullet_2)$.

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Symplectic and Poisson Geometry of the Moduli Spaces of Flat Connections Over Quilted Surfaces

David Li-Bland and Pavol Ševera

Abstract In this paper we study the symplectic and Poisson geometry of moduli spaces of flat connections over *quilted surfaces*. These are surfaces where the structure group varies from region to region in the surface, and where a reduction (or relation) of structure occurs along the boundaries of the regions. Our main theoretical tool is a new form moment-map reduction in the context of Dirac geometry. This reduction framework allows us to extend the results of [30, 40] to allow more general relations of structure groups, and to investigate both the symplectic and Poisson geometry of the resulting moduli spaces from a unified perspective. The moduli spaces we construct in this way include a number of important examples, including Poisson Lie groups and their Homogeneous spaces, moduli spaces for meromorphic connections over Riemann surfaces (following the work of Philip Boalch), and various symplectic groupoids. Realizing these examples as moduli spaces for *quilted surfaces* provides new insights into their geometry.

1 Introduction and Summary of Results

Suppose that G is a Lie group whose Lie algebra, $\mathfrak{g}$, is endowed with a G-invariant inner product, $\langle\cdot,\cdot\rangle$. Suppose that Σ is a closed oriented surface, and $P \to \Sigma$ is a principal G-bundle. Let $\mathcal{A}_{flat}(P \to \Sigma)$ denote the space of flat connections on P. Atiyah and Bott [6] showed that the moduli space

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D. Li-Bland(✉)
Department of Mathematics, University of California, Berkeley, USA
e-mail: libland@math.berkeley.edu

P. Ševera
Department of Mathematics, Université de Genève, Geneva, Switzerland
e-mail: pavol.severa@gmail.com

P. Ševera
on leave from Department of Theoretical Physics, FMFI UK, Bratislava, Slovakia

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$$\mathcal{M}(P \to \Sigma) := \mathcal{A}_{flat}(P \to \Sigma)/\mathrm{Aut}(P)$$

of flat connections on P carries a symplectic structure. Their construction involves infinite dimensional symplectic reduction. Somewhat later, Alekseev, Malkin, and Meinrenken introduced quasi-Hamiltonian geometry [4], equipping it with a toolkit of fusion and reduction operations, in order to provide a finite dimensional construction of this moduli space. Boalch [11] enlarged the quasi-Hamiltonian toolkit, introducing the fission operation, which enables a finite dimensional construction of the moduli space of flat connections with prescribed irregular singularities. Interestingly, this new fission operation also allowed Boalch to associate Poisson/sympletic/quasi-Hamiltonian spaces of connections to surfaces with different structure groups in different regions. Moreover, these techniques enabled Boalch to interpret additional Poisson spaces, including examples of Poisson Lie groups [8–12] and Lu-Weinstein double symplectic groupoids [10–12], as moduli spaces for connections.

In this paper we expand the quasi-Hamiltonian toolkit further. First we introduce a slight generalization of group-valued moment maps, so that the moduli space on a surface with several marked points on every boundary component is equipped with such a moment map.

Next, we subsume the quasi-Hamiltonian toolkit, consisting of reduction, fusion, and fission, into a single broad generalization of reduction. In particular, the moduli space for a triangulated surface is obtained via reduction from the moduli spaces for the triangles.

Consequently, we are able to construct symplectic structures on moduli spaces for:

- surfaces with boundary, where segments of the boundary are labelled by coisotropic subalgebras of $\mathfrak{g}$ (generalizing some results found in [40]),
- surfaces with domains labelled by distinct structure groups and domain walls labelled by coisotropic relations between the structure groups—also called *quilted surfaces* (generalizing some results found in [10–12]),
- branched surfaces, where the branch locus is labelled by a coisotropic interaction between the branches (generalizing some results found in [10–12]).

Even more generally, our techniques may be used to produce Poisson structures, and a natural generalization of quasi-Hamiltonian and quasi-Poisson structures.

As a result, we are able to construct of a number of well known spaces including: Lu's symplectic double groupoid integrating a Poisson Lie group [32], Boalch's Fission spaces [11, 12], Poisson Lie groups [17, 36], and Poisson homogeneous spaces [34], among others. Our approach builds upon the results and ideas of various authors including Fock and Rosly, Boalch, and the second author [7–12, 20, 38–40].

Some of these results appeared in [30], where the (quasi-) Poisson structures on moduli spaces are constructed in terms of an intersection pairing. Here we present the reduction theorems in full generality (unifying both the twists and reductions found in (quasi-) Poisson geometry) and with an emphasis on symplectic structures. We also formulate the results in more natural way, as morphisms of Manin pairs.

Among the morphisms of Manin pairs, we introduce the class of *exact morphisms*, corresponding to (quasi-) symplectic structures.

1.1 Notation and Terminology

At this point, we would like to introduce some notation. Suppose V_i is a family of vector spaces (or manifolds) indexed by a set I and $f : J \to I$ is a map. We use the notation

$$f^! : \prod_{i \in I} V_i \to \prod_{j \in J} V_{f(j)}$$
$$\{v_i\}_{i \in I} \mapsto \{v_{f(j)}\}_{j \in J}$$

for the induced pull-back map.

For any oriented graph Γ, we let E_Γ denote the set of edges, V_Γ the set of vertices and $\mathsf{in}, \mathsf{out} : E_\Gamma \to V_\Gamma$ the incidence maps. Γ is called a *permutation graph*[1] if both in and out are bijections.

A *quadratic Lie algebra* is a Lie algebra endowed with an invariant non-degenerate symmetric pairing.

To simplify our presentation, we will assume that the Lie group G is connected throughout this paper. The generalization to disconnected Lie groups is straightforward.

1.2 The Construction

1.2.1 Motivating Example: The Symplectic Form from a Triangulation

Let Σ be a closed oriented surface and let

$$\mathcal{M}_\Sigma(G) = \mathrm{Hom}(\pi_1(\Sigma), G)/\mathbf{Ad}(G)$$

be the moduli space of flat connections. Let us recall how to compute the Atiyah-Bott symplectic form ω on $\mathcal{M}_\Sigma(G)$ in terms of a triangulation of Σ.

Let $\mathcal{T}$ be a triangulation of Σ. Let $\mathcal{T}_0$ denote the set of its vertices, $\mathcal{T}_1$ the set of (unoriented) edges and $\mathcal{T}_2$ the set of triangles. We let $\tilde{\mathcal{T}}_1$ denote the set of oriented edges (we thus have a 2–1 map $\tilde{\mathcal{T}}_1 \to \mathcal{T}_1$) we let

$$(e \to \bar{e}) : \tilde{\mathcal{T}}_1 \to \tilde{\mathcal{T}}_1$$

denote the map which reverses the orientation of the edges.

[1] Such graphs are also called *directed cycle graphs* in the literature.

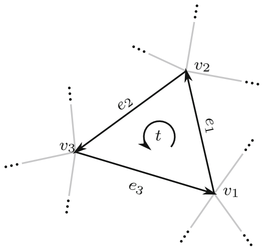

Fig. 1 In the figure, $t \in \mathcal{T}_2$ is a *triangle*, $e_1, e_2, e_3 \in \tilde{\mathcal{T}}_1$ are oriented edges, and $v_1, v_2, v_3 \in \mathcal{T}_0$ are vertices. We have $\partial t = \{e_1, e_2, e_3\}$, and $v_2 = \mathsf{in}\,(e_1)$ and $v_1 = \mathsf{out}\,(e_1)$

Let $\mathcal{A}_{\mathit{flat}}(\mathcal{T})$ be the space of "combinatorial flat connections" on Σ:

$$\mathcal{A}_{\mathit{flat}}(\mathcal{T}) = \{g \in G^{\tilde{\mathcal{T}}_1} \mid g_{\bar{e}} = g_e^{-1} \text{ for all } e \in \tilde{\mathcal{T}}_1, \text{ and } \prod_{e \in \partial t} g_e = 1 \text{ for all } t \in \mathcal{T}_2\},$$

here $\partial t \subset \tilde{\mathcal{T}}_1$ denotes the oriented boundary and the product is taken in the natural (cyclic) order (cf. Fig. 1). We have an action of $G^{\mathcal{T}^0}$ on $\mathcal{A}_{\mathit{flat}}(\mathcal{T})$ by "gauge transformations"

$$(g' \cdot g)_e = g'_{\mathsf{in}(e)} g_e (g'_{\mathsf{out}(e)})^{-1}, \quad g' \in G^{\mathcal{T}_0}, \quad g \in G^{\tilde{\mathcal{T}}_1} \tag{1}$$

and

$$\mathcal{M}_\Sigma(G) = \mathcal{A}_{\mathit{flat}}(\mathcal{T})/G^{\mathcal{T}^0}.$$

If t is an oriented triangle with edges e_1, e_2, e_3 (in their cyclic order), let

$$\mathcal{M}_t(G) = \{(g_{e_1}, g_{e_2}, g_{e_3}) \in G \times G \times G \mid g_{e_1} g_{e_2} g_{e_3} = 1\}. \tag{2}$$

We have an inclusion

$$i : \mathcal{A}_{\mathit{flat}}(\mathcal{T}) \subset \prod_{t \in \mathcal{T}_2} \mathcal{M}_t(G),$$

where the subset $\mathcal{A}_{\mathit{flat}}(\mathcal{T})$ is given by the condition $g_e = g_{\bar{e}}^{-1}$.

Let

$$\omega_t = \frac{1}{2} \langle g_{e_2}^{-1} \mathrm{d} g_{e_2}, \mathrm{d} g_{e_1}\, g_{e_1}^{-1} \rangle \in \Omega^2(\mathcal{M}_t(G)).$$

The 2-form ω_t is invariant under cyclic permutations of the edges.

The symplectic form ω on $\mathcal{M}_\Sigma(G)$ is given by

$$p^* \omega = i^* \sum_{t \in \mathcal{T}_2} \omega_t, \tag{3}$$

where $p : \mathcal{A}_{\mathit{flat}}(\mathcal{T}) \to \mathcal{M}_\Sigma(G)$ is the projection [45].

We shall interpret Eq. (3) in the following way: $\mathcal{M}_\Sigma(G)$ is obtained from $\prod_{t\in\mathcal{T}_2}\mathcal{M}_t(G)$ by a variant of Hamiltonian reduction. The subset $\mathcal{A}_{flat}(\mathcal{T}) \subset \prod_{t\in\mathcal{T}_2}\mathcal{M}_t(G)$ is given by a moment map condition, and then we need to take the quotient by the residual group $G^{\mathcal{T}^0}$ to get a symplectic manifold. To do it, we need to explain this (quasi-) Hamiltonian reduction and the (quasi-) Hamiltonian structure on $\mathcal{M}_t(G)$.

1.2.2 Quasi-Hamiltonian Reduction

Let $\mathfrak{d}$ be a quadratic Lie algebra and $\mathfrak{h} \subset \mathfrak{d}$ a Lagrangian subalgebra (i.e. $\mathfrak{h}^\perp = \mathfrak{h}$). In other words, $(\mathfrak{d}, \mathfrak{h})$ is a Manin pair.

Suppose that $\mathfrak{d}$ acts on a manifold N so that all the stabilizers are coisotropic Lie subalgebras of $\mathfrak{d}$. We shall recall below the following notions (introduced by Alekseev, Malkin and Meinrenken in [4] and by Alekseev, Kosmann-Schwarzbach and Meinrenken in [3], slightly generalized in this paper):

- A *quasi-Hamiltonian* $(\mathfrak{d}, \mathfrak{h}) \times N$*-manifold* (or quasi-Hamiltonian $\mathfrak{h}$-manifold, if $\mathfrak{d}$ and N are clear from the context) is a manifold M with an action of $\mathfrak{h}$, an $\mathfrak{h}$-equivariant map $\mu : M \to N$ (*moment map*), and a bivector field π on M, satisfying certain conditions.[2]
- Among the moment maps there are *exact moment maps*. In this case the bivector field π can be replaced by a 2-form (M is "quasi-symplectic").

One of our main results is the following reduction theorem:

Theorem 1 *Let M be a quasi-Hamiltonian $(\mathfrak{d}, \mathfrak{h}) \times N$-manifold, $\mathfrak{l} \subset \mathfrak{d}$ a Lagrangian Lie subalgebra, and $S \subset N$ an $\mathfrak{l}$-invariant submanifold.*

1. *There is a natural Poisson bracket on the algebra $C^\infty(M)^{\mathfrak{l}\cap\mathfrak{h}} \subset C^\infty(M)$ of $\mathfrak{l}\cap\mathfrak{h}$-invariant functions. In particular, if $M/(\mathfrak{l}\cap\mathfrak{h})$ is a manifold, it is a Poisson manifold.*
2. *The ideal $I \subset C^\infty(M)^{\mathfrak{l}\cap\mathfrak{h}}$ of functions vanishing on $\mu^{-1}(S)$ is a Poisson ideal. In particular, $\mu^{-1}(S)/(\mathfrak{l}\cap\mathfrak{h})$ is a Poisson manifold, provided it is a manifold.*
3. *If the moment map μ is exact and S is an $\mathfrak{l}$-orbit then the Poisson manifold $\mu^{-1}(S)/(\mathfrak{l}\cap\mathfrak{h})$ is symplectic.*

More generally, if in place of the Lagrangian subalgebra $\mathfrak{l}$ we use a coisotropic subalgebra, we have a similar result, where the reduced manifold is still quasi-Hamiltonian. This result is contained in Theorems 5 and 6, expressed in the more appropriate language of morphisms of Manin pairs.

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[2] Strictly speaking the bivector field π depends in an inessential way on a choice of a vector space complement $\mathfrak{k} \subset \mathfrak{d}$ to $\mathfrak{h}$, as in [2]. Similarly, in the exact case, the 2-form depends in an inessential way on some other choice. These choices can be made canonically in our cases of interest, and so we will ignore this subtlety until Sect. 2.3.

1.2.3 The Quasi-Hamiltonian Structure on Moduli Spaces

Let e be an (abstract) oriented edge, let $N_e = G$ and $\mathfrak{d}_e = \bar{\mathfrak{g}} \oplus \mathfrak{g}$, where $\bar{\mathfrak{g}}$ is $\mathfrak{g}$ with the inner product negated. The corresponding group $D_e = G \times G$ acts on $N_e = G$ via

$$(g_1, g_2) \cdot g = g_1\, g\, g_2^{-1}. \tag{4}$$

$N_e = G$ should be imagined as the space of possible holonomies along e, and the action of $D_e = G \times G$ as gauge transformations at the endpoints of e.

Let Σ be a compact oriented surface and $V \subset \partial\Sigma$ a finite subset such that every component of both Σ and $\partial\Sigma$ intersects V non-trivially. We shall call (Σ, V) a *marked surface*. The boundary circles of Σ are cut into a sequence of oriented edges with endpoints in V. Together these edges and vertices form a permutation graph Γ, the *boundary graph* of (Σ, V) (cf. Fig. 2). Let $\Pi_1(\Sigma, V)$ denote the fundamental groupoid of Σ with the base set V. Let

$$\mathcal{M}_{\Sigma,V}(G) = \mathrm{Hom}(\Pi_1(\Sigma, V), G)$$

be the moduli space of flat connections on G-bundles over Σ trivialized at V. We now describe the quasi-Hamiltonian on this moduli space.

We have an action of the group $H = G^V$ on $\mathcal{M}_{\Sigma,V}(G)$ by (residual) gauge transformations,

$$(h \cdot f)(e) = h_{\mathsf{in}(e)} f(e) h_{\mathsf{out}(e)}^{-1},$$

for $h \in G^V$, $f \in \mathcal{M}_{\Sigma,V}(G)$, and $e \in \Pi_1(\Sigma, V)$. We also have a map

$$\mu : \mathcal{M}_{\Sigma,V}(G) \to N := \prod_{e \in E_\Gamma} N_e.$$

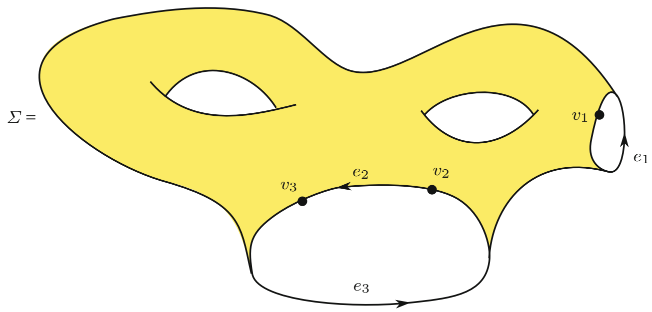


Fig. 2 The marked surface (Σ, V), with $V = \{v_1, v_2, v_3\}$. The boundary graph Γ has edges $E_\Gamma = \{e_1, e_2, e_3\}$ and vertices $V_\Gamma = V$

where the components of μ are given by

$$\mu(f)_e = f(e)$$

(in other words, μ is the list of holonomies along the boundary arcs). Notice that the map μ is H-equivariant, where $H = G^V$ embeds as a subgroup

$$G^V \subseteq D := \prod_{e \in E_\Gamma} D_e = \prod_{e \in E_\Gamma} (G \times G).$$

Here $g \in G^V$ is included as the element $\prod_{e \in E_\Gamma} (g_{\mathrm{in}(e)}, g_{\mathrm{out}(e)})$. Letting $\mathfrak{d}$ and $\mathfrak{h}$ denote the Lie algebras of D and H, we have:

Theorem 2 *There is a natural* $(\mathfrak{d}, \mathfrak{h}) \times N$*-quasi-Hamiltonian structure on* $\mathcal{M}_{\Sigma,V}(G)$ *with the moment map* μ*. The moment map is exact and the quasi-symplectic form* ω *on* $\mathcal{M}_{\Sigma,V}(G)$ *is given by the formula* (3)*, where* $\mathcal{T}$ *is any triangulation of* Σ *such that* $\mathcal{T}_0 \cap \partial\Sigma = V$.

We prove this theorem in Sect. 4.

Remark 1 In the case where every boundary component of Σ contains exactly one element of V, the theorem (except for the triangulation part) was proved by Alekseev, Malkin and Meinrenken in [4], and became the motivation for quasi-Hamiltonian structures.

1.2.4 Reduction Applied to Moduli Spaces

We can combine Theorems 1 and 2 to produce Poisson and symplectic manifolds: we choose a collection (Σ_i, V_i) of marked surfaces with boundary graphs Γ_i, and a collection G_i of Lie groups with quadratic Lie algebras. The manifold

$$\mathcal{M} := \prod_i \mathcal{M}_{\Sigma_i, V_i}(G_i)$$

is quasi-Hamiltonian, with the moment map $\mu : \mathcal{M} \to N = \prod_i N_i$. We choose a Lagrangian Lie subalgebra $\mathfrak{l} \subset \mathfrak{d}$ and a $\mathfrak{l}$-invariant submanifold $S \subset N$. Then by Theorem 1, if the transversality conditions are satisfied, the manifold

$$\mathcal{M}_{red} = \mu^{-1}(S)/(\mathfrak{l} \cap \mathfrak{h})$$

is symplectic or Poisson.

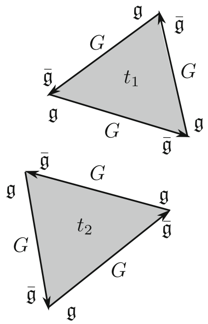
The reduced manifold $\mathcal{M}_{red}$ can be again seen as a moduli space of flat connections, with certain boundary (or sewing) conditions. Below we shall give various examples for simple choices of $\mathfrak{l}$ and S.

Example 1 As the first example, let Σ be a closed surface with a triangulation $\mathcal{T}$. Let (Σ', V') be the disjoint union of the triangles, with V' consisting of the vertices, and let

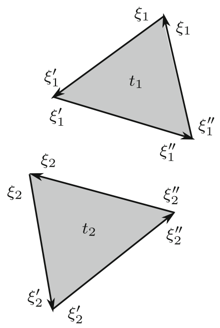
$$\mathcal{M} = \mathcal{M}_{\Sigma',V'}(G) = \prod_{t\in\mathcal{T}_2} M_t(G).$$

in the notation of Eq. (2).

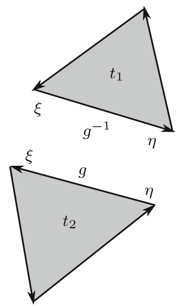
Let us now identify our data on a picture (showing just two triangles, with the parallel edges identified in Σ):



The Lie algebra $\mathfrak{d}$ is the direct sum of all the $\mathfrak{g}$'s and $\bar{\mathfrak{g}}$'s, situated at the half-edges of the triangles. N is the product of all G's. The Lie algebra $\mathfrak{h} \subset \mathfrak{d}$ is the direct sum of all the diagonal Lie subalgebras, $\mathfrak{g}_\Delta \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$, situated at the vertices of the triangles. Let the Lie algebra $\mathfrak{l} \subset \mathfrak{d}$ be the direct sum of all the diagonals $\mathfrak{g}_\Delta \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$ situated at the pairs of half-edges that are identified in Σ. Notice that $\mathfrak{h} \cap \mathfrak{l} = \mathfrak{g}^{\mathcal{T}_0}$.



Elements of $\mathfrak{h}$ lie in the direct sum of all the diagonals $\mathfrak{g}_\Delta \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$ at the vertices of the triangles.



Elements of $\mathfrak{l}$ lie in the direct sum of all the diagonals $\mathfrak{g}_\Delta \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$ at the pairs of half-edges that are identified in Σ. The $\mathfrak{l}$-orbit, S, consists of those elements $g \in \prod_{e\in\tilde{\mathcal{T}}_1}$ such that $g_e = g_{\bar{e}}^{-1}$.

For the $\mathfrak{l}$-orbit $S \subset N$ we take the subset given by the conditions $g_{\bar{e}} = g_e^{-1}$ for any pair of edges $e, \bar{e}$ that are identified in Σ. We have

$$\mathcal{M}_\Sigma(G) = \mathcal{M}_{red} := \mu^{-1}(S)/(\mathfrak{h} \cap \mathfrak{l}).$$

Thus we are able to obtain $\mathcal{M}_\Sigma(G)$ by quasi-Hamiltonian reduction from triangles.

So far we have not explicitly described the symplectic or Poisson structure on $\mathcal{M}_{red}$. In a special case it is very simple. Let

$$\mu_i : \mathcal{M}_{\Sigma_i, V_i}(G_i) \to N_i$$

denote the (exact) moment map, and let ω_i be the quasi-symplectic 2-form on $M_{\Sigma_i, V_i}(G_i)$ (given explicitly in Theorem 2). For every boundary arc e of Σ_i we have the involution of $\mathfrak{d}_e = \bar{\mathfrak{g}}_i \oplus \mathfrak{g}_i$ given by

$$(\xi, \eta) \mapsto (\eta, \xi).$$

If we apply the involution simultaneously at all the boundary arcs, we get an involution of

$$\mathfrak{d} = \bigoplus_i \bigoplus_{e \subset \partial \Sigma_i} \bar{\mathfrak{g}}_i \oplus \mathfrak{g}_i.$$

We shall say that a subalgebra $\mathfrak{l} \subset \mathfrak{d}$ is *symmetric* if it is invariant with respect to this involution.

Theorem 3 *If $\mathfrak{l} \subset \mathfrak{d}$ is a symmetric Lagrangian subalgebra and $S \subset N$ is the $\mathfrak{l}$-orbit through the identity element*

$$1 \in N = \prod_i G_i^{E_{\Gamma_i}},$$

then the symplectic form ω_{red} on

$$\mathcal{M}_{red} = \mu^{-1}(S)/\mathfrak{l} \cap \mathfrak{h}$$

is given by

$$p^* \omega_{red} = \sum_i \omega_i \Big|_{\mu^{-1}(S)}$$

where $p : \mu^{-1}(S) \to \mathcal{M}_{red}$ is the projection.

As explained in Remark 18, Theorem 3 will follow as a corollary to Proposition 2.

1.3 Colouring Edges

Suppose that $\mathfrak{c} \subseteq \mathfrak{g}$ is a coisotropic subalgebra (i.e. $\mathfrak{c}^{\perp} \subseteq \mathfrak{c}$). Then the subalgebra

$$\mathfrak{l}_{\mathfrak{c}} := \{(\xi, \eta) \in (\bar{\mathfrak{g}} \oplus \mathfrak{g}) \mid \xi, \eta \in \mathfrak{c} \text{ and } \xi - \eta \in \mathfrak{c}^{\perp}\}$$

is both Lagrangian and symmetric. The orbit of $\mathfrak{l}_{\mathfrak{c}}$ through the identity of G, with respect to the action Eq. (4), can be identified with the simply connected Lie group $C^{\perp}$ integrating the Lie algebra $\mathfrak{c}^{\perp}$.

Let (Σ, V) be a marked surface. For every boundary arc e (i.e. for every edge of the permutation graph $\Gamma_{\Sigma,V}$ with the vertex set V), let $\mathfrak{c}_e \in \mathfrak{g}$ be a coisotropic subalgebra, and consider the Lie subalgebra

$$\mathfrak{l} := \bigoplus_e \mathfrak{l}_{\mathfrak{c}_e} \subset \bigoplus_e \bar{\mathfrak{g}} \oplus \mathfrak{g} = \mathfrak{d}.$$

It is clear that $\mathfrak{l}$ is both Lagrangian and symmetric. Let $S \subset N = \prod_e G$ be the $\mathfrak{l}$-orbit passing through $1 \in \prod_e G$. Theorem 3 implies that if the quotient space

$$\mathcal{M}_{red} = \mu^{-1}(S)/\mathfrak{l} \cap \mathfrak{g}^V$$

is a manifold, it is symplectic.

Concretely,

$$\mathcal{M}_{red} = \{f : \Pi_1(\Sigma, V) \to G \mid f(e) \in C_e^{\perp} \text{ for every } e\}/\mathfrak{l} \cap \mathfrak{g}^V, \tag{5}$$

and $\mathfrak{l} \cap \mathfrak{g}^V \subset \mathfrak{g}^V$ is given by the conditions

$$\xi_v \in \mathfrak{c}_{e_1} \cap \mathfrak{c}_{e_2} \text{ where } v = \mathsf{in}(e_1) = \mathsf{out}(e_2) \tag{6a}$$

$$\xi_{\mathsf{in}(e)} - \xi_{\mathsf{out}(e)} \in \mathfrak{c}_e^{\perp}. \tag{6b}$$

Notice that if $\mathfrak{c}_e$'s are Lagrangian then the first condition implies the second one. If, moreover, $\mathfrak{c}_{e_1} \cap \mathfrak{c}_{e_2} = 0$ for any pair of consecutive boundary arcs then $\mathfrak{l} \cap \mathfrak{g}^V = 0$. Under these conditions the moduli space $\mathcal{M}_{red}$ was considered in [40].

Example 2 ([10–12, 39, 40]) Suppose that $\mathfrak{e}, \mathfrak{f} \subseteq \mathfrak{g}$ are transverse Lagrangian subalgebras, and let $E, F \subset G$ denote the corresponding connected Lie groups. We may colour alternate edges of a rectangle with $\mathfrak{e}$ and $\mathfrak{f}$, as in Fig. 3. From (Eq. 5) we see that

$$\mathcal{M}_{red} = \{(e_1, e_2, f_1, f_2) \in E^2 \times F^2 \mid e_1 f_1 e_2 f_2 = 1\}.$$

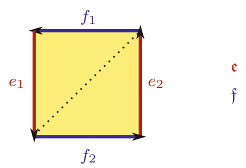


Fig. 3 The symplectic double groupoid integrating the Lie-Poisson structures on E and F

By Theorem 3 the moduli space $\mathcal{M}_{red}$ carries the symplectic form

$$\omega = \frac{1}{2}\langle e_1^{-1}\mathrm{d}e_1, \mathrm{d}f_1 f_1^{-1}\rangle + \frac{1}{2}\langle e_2^{-1}\mathrm{d}e_2, \mathrm{d}f_2 f_2^{-1}\rangle.$$

Here, the upper-left triangle in Fig. 3 contributed the term $\frac{1}{2}\langle e_1^{-1}\mathrm{d}e_1, \mathrm{d}f_1f_1^{-1}\rangle$ to this expression while the bottom-right triangle in Fig. 3 contributed the term $\frac{1}{2}\langle e_2^{-1}\mathrm{d}e_2, \mathrm{d}f_2f_2^{-1}\rangle$.

As explained in [39, 40], the symplectic manifold $(\mathcal{M}, \omega)$ is the Lu-Weinstein symplectic double groupoid integrating the Lie-Poisson structures on E and F [33].

Example 3 ([38, 40]) Let $\mathfrak{e}, \mathfrak{f} \subseteq \mathfrak{g}$ be as above. Divide each boundary component of the annulus into two segments and colour alternate edges with $\mathfrak{e}$ and $\mathfrak{f}$, as in Fig. 4. From Eq. (5) we see that

$$\mathcal{M}_{red} = \{(e_1, e_2, f_1, f_2, g) \in E^2 \times F^2 \times G \mid g e_1 f_1 g^{-1} f_2 e_2 = 1\}.$$

The moduli space $\mathcal{M}_{red}$ carries the symplectic form

$$\begin{aligned}\omega = &\frac{1}{2}\langle e_2^{-1}\mathrm{d}e_2, \mathrm{d}g\, g^{-1}\rangle + \frac{1}{2}\langle (e_2g)^{-1}\mathrm{d}(e_2g), \mathrm{d}e_1\, e_1^{-1}\rangle \\ &+ \frac{1}{2}\langle g f_1^{-1}\mathrm{d}(f_1g^{-1}), g f_2 f_2^{-1}\rangle - \frac{1}{2}\langle \mathrm{d}g\, g^{-1}, \mathrm{d}f_1 f_1^{-1}\rangle,\end{aligned}$$

which can be computed from the triangulation pictured in Fig. 4.

As explained in [38, 40], the symplectic manifold $(\mathcal{M}, \omega)$ is the symplectic double groupoid integrating the Lie-Poisson structure on G.

Example 4 Suppose that $\mathfrak{e}, \mathfrak{f} \subseteq \mathfrak{g}$ are transverse Lagrangian subalgebras and $\mathfrak{c} \subseteq \mathfrak{g}$ is a coisotropic subalgebra. Let $E, F, C, C^{\perp} \subset G$ denote the corresponding connected Lie subgroups, and suppose that $C \subset G$ is closed. Consider the annulus whose outer boundary is divided into two segments. Colour the outer boundary by the two Lie subalgebras $\mathfrak{e}$ and $\mathfrak{f}$ and the inner boundary by the inner boundary by the Lie subalgebra $\mathfrak{c}$, as in Fig. 5. We have

$$\mu^{-1}(S) = \{(e, f, g, c) \in E \times F \times G \times C^{\perp} \mid efgcg^{-1} = 1\}.$$

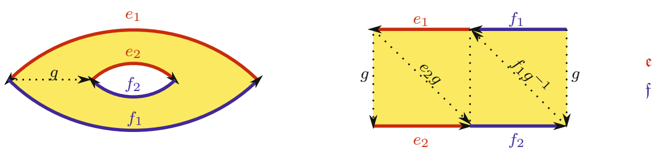


Fig. 4 The symplectic double groupoid integrating the Lie-Poisson structure on G

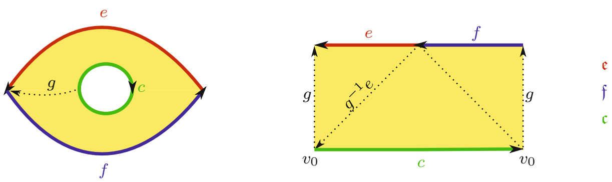


Fig. 5 The symplectic groupoid integrating the Lu-Yakimov Poisson on G/C

Meanwhile, Eq. (6a) yields

$$\mathfrak{l} \cap \mathfrak{g}^V = \{\xi \in \mathfrak{g}^V \mid \xi_{v_0} \in \mathfrak{c} \text{ and } \xi_v = 0 \text{ for all } v \neq v_0\},$$

where v_0 is the vertex labelled in Fig. 5. Thus the Lie group of residual gauge transformations is C, acting as

$$c' \cdot (e, f, g, c) = (e, f, gc'^{-1}, c'cc'^{-1}), \quad c' \in C, \quad (e, f, g, c) \in E \times F \times G \times C^{\perp}.$$

Since, by assumption this acts freely and properly on $\mu^{-1}(N)$, Theorem 3 implies that the moduli space

$$\mathcal{M}_{red} = \{(e, f, g, c) \in E \times F \times G \times C^{\perp} \mid efgcg^{-1} = 1\}/C$$

carries the symplectic form

$$\omega = -\frac{1}{2}\langle \mathrm{d}g\, g^{-1}, \mathrm{d}e\, e^{-1}\rangle + \frac{1}{2}\langle c^{-1}\mathrm{d}c, \mathrm{d}(g^{-1}e)\, e^{-1}g\rangle + \frac{1}{2}\langle f^{-1}\mathrm{d}f, \mathrm{d}g\, g^{-1}\rangle.$$

The symplectic manifold $(\mathcal{M}_{red}, \omega)$ is the symplectic groupoid integrating the Lu-Yakimov Poisson structure on the homogeneous space G/C [34]. The source and target maps are

$$\mathsf{s}(e, f, g, c) = g, \quad \mathsf{t}(e, f, g, c) = fg,$$

and the multiplication is

$$(e', f', g', c') \cdot (e, f, g, c) = (ee', f'f, g, c'c), \quad g' = fg.$$

1.4 Domain Walls and Branched Surfaces

Let (Σ_i, V_i) be a finite collection of marked surfaces with boundary graphs Γ_i, and G_i a collection of Lie groups with quadratic Lie algebras $\mathfrak{g}_i$. As we observed above, the space

$$\mathcal{M} = \prod_i \mathcal{M}_{\Sigma_i, V_i}(G_i)$$

is a $(\mathfrak{d}, \mathfrak{h}) \times N$-quasi-Hamiltonian for appropriate $(\mathfrak{d}, \mathfrak{h}, N)$, and if we choose a Lagrangian Lie subalgebra $\mathfrak{l} \subset \mathfrak{d}$ and a $\mathfrak{l}$-orbit $S \subset N$, then

$$\mathcal{M}_{red} = \mu^{-1}(S)/\mathfrak{l} \cap \mathfrak{h}$$

is symplectic. If the subalgebra $\mathfrak{l} \subset \mathfrak{d}$ is symmetric then Theorem 3 gives us a simple formula for the symplectic form on $\mathcal{M}_{red}$.

Let us now choose a symmetric $\mathfrak{l} \subset \mathfrak{d}$ in the following way. We first glue the boundary arcs of (Σ_i, V_i) in an arbitrary way. More precisely, let W be a finite collection of (disjoint) unit intervals called *domain walls*, let

$$\kappa : \sqcup_i E_{\Gamma_i} \to \mathsf{W}$$

be a surjective map assigning to every edge of every boundary graph Γ_i a domain wall, and let

$$\phi_e : e \to \kappa(e)$$

be a homeomorphism for every boundary edge e (not required to preserve the orientation). Let Σ be the topological space obtained from Σ_i's and the domain walls after we identify every boundary arc e with $\kappa(e)$ via the map ϕ_e.

For every boundary arc $e \in E_{\Gamma_i}$ let $i(e) = i$, and

$$\mathsf{sign}(e) = \begin{cases} +1 & \text{if } \phi_e \text{ is orientation-preserving} \\ -1 & \text{otherwise.} \end{cases}$$

For every domain wall $w \in \mathsf{W}$, let

$$\mathfrak{g}_w = \bigoplus_{\substack{e \in \kappa^{-1}(w) \\ \mathsf{sign}(e) = +1}} \mathfrak{g}_{i(e)} \oplus \bigoplus_{\substack{e \in \kappa^{-1}(w) \\ \mathsf{sign}(e) = -1}} \bar{\mathfrak{g}}_{i(e)}$$

and

$$\mathfrak{d}_w = \bar{\mathfrak{g}}_w \oplus \mathfrak{g}_w.$$

Notice that

$$\mathfrak{d} := \bigoplus_{w \in \mathsf{W}} \bar{\mathfrak{g}}_w \oplus \mathfrak{g}_w = \bigoplus_i (\bar{\mathfrak{g}}_i \oplus \mathfrak{g}_i)^{E_{\Gamma_i}}.$$

For every domain wall $w \in \mathsf{W}$ we now choose a coisotropic Lie subalgebra

$$\mathfrak{c}_w \subset \mathfrak{g}_w.$$

Using $\mathfrak{c}_w$ we construct the symmetric Lagrangian Lie subalgebra $\mathfrak{l}_w \subset \mathfrak{d}_w$,

$$\mathfrak{l}_w := \{(\xi, \eta) \in \bar{\mathfrak{g}}_w \oplus \mathfrak{g}_w \mid \xi, \eta \in \mathfrak{c}_w,\ \xi - \eta \in \mathfrak{c}_w^\perp\}.$$

Finally we set

$$\mathfrak{l} = \bigoplus_{w \in \mathsf{W}} \mathfrak{l}_w.$$

For every domain wall $w \in \mathsf{W}$, let $C_w^\perp \subset G_w$ denote the connected Lie subgroup with Lie algebra $\mathfrak{c}_w^\perp$, and

$$C^\perp = \prod_{w \in \mathsf{W}} C_w^\perp \subseteq \prod_i G_i^{E_{\Gamma_i}}.$$

Then $S := C^\perp \subset N$ is the $\mathfrak{l}$-orbit passing through $1 \in \prod_i G_i^{E_{\Gamma_i}} = N$. As before, we have

$$\mu^{-1}(S) = \Big\{\{f_i : \Pi_1(\Sigma_i, V_i) \to G_i\}_i \mid \prod_i \{f_i(e)^{\mathsf{sign}(e)}\}_{e \in E_{\Gamma_i}} \in C^\perp\Big\} \qquad (7)$$

and

$$\mathcal{M}_{red} = \mu^{-1}(S)/\mathfrak{l} \cap \bigoplus_i \mathfrak{g}_i^{V_i}.$$

Example 5 (Oriented surfaces with coloured boundaries) If we have just one domain and the gluing map κ is injective, then we are in the case described in Sect. 1.3.

1.4.1 Domain Walls

Suppose that the glued topological space Σ is still a (not necessarily oriented) surface. Equivalently, every domain wall $w \in \mathsf{W}$ borders either one or two domains (i.e. the preimage $\kappa^{-1}(w)$ has cardinality one or two). The resulting surface Σ was called a *quilted surface* in [30] (following [44]) (Fig. 6).

Remark 2 Quantizations of these moduli spaces have been studied in the physics community [21, 24, 25] for abelian structure groups and Lagrangian relations on the domain walls.

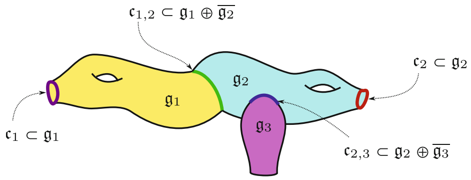


Fig. 6 Our surface is divided into domains with distinct structure groups, and the domain walls are coloured by coisotropic relations between the structure groups. As before, coisotropic boundary conditions are also chosen

Example 6 (Boalch [12]*)* Suppose that $\mathfrak{g} = \mathfrak{u}_+ \oplus \mathfrak{h} \oplus \mathfrak{u}_-$ as a vector space (but not as a Lie algebra), where $\mathfrak{p}_\pm := \mathfrak{h} \oplus \mathfrak{u}_\pm \subseteq \mathfrak{g}$ are coisotropic subalgebras satisfying $\mathfrak{p}_\pm^\perp = \mathfrak{u}_\pm$. Suppose further that the Lie subalgebras $\mathfrak{u}_\pm, \mathfrak{p}_\pm, \mathfrak{h}$ all integrate to closed subgroups $U_\pm, P_\pm, H \subseteq G$ such that $H = P_+ \cap P_-$. The metric on $\mathfrak{g}$ descends to a non-degenerate invariant metric on $\mathfrak{h} \subseteq \mathfrak{g}$, and

$$\mathfrak{c}_\pm := \{(\xi; \xi + \mu) \in \mathfrak{h} \oplus \mathfrak{g} \mid \xi \in \mathfrak{h} \text{ and } \mu \in \mathfrak{u}_\pm\} \tag{8}$$

is a coisotropic subalgebra (in fact, it is Lagrangian).

As in Fig. 7, let Σ denote the annulus, and let $\gamma \subset (\Sigma \setminus \partial\Sigma)$ be a simple closed curve representing the generator of the fundamental group. Cutting Σ along γ yields two annuli, $\Sigma_G, \Sigma_H \subset \Sigma$, which we label with the structure groups G and H, respectively. We divide γ into $2r$ segments with endpoints labelled $v_1, \ldots v_{2r}$, and colour alternating segments with the coisotropic Lie subalgebras $\mathfrak{c}_+$ and $\mathfrak{c}_-$. Finally, we mark the respective components of $\partial\Sigma$ with points x_G and x_H.

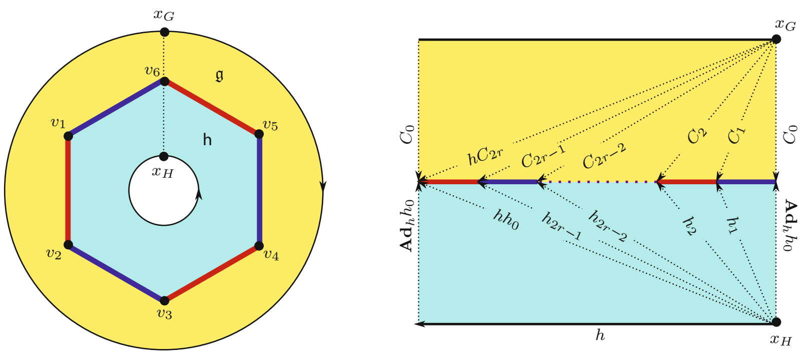


Fig. 7 On the surface pictured above, the structure group in the *yellow* domain is G while the structure group in the *blue* domain is H. Along the boundary of the two domains, *blue edges* are coloured with $\mathfrak{c}_+$ while the *red edges* are coloured with $\mathfrak{c}_-$. Cutting along the *dotted line* in the first picture yields yields the second picture. Acting by H at the vertices $v_1, \ldots, v_{2r}$ allows us to set the holonomies $h_0, \ldots, h_{2r-1}$ to the identity

The points $x_G, X_H, v_1, \ldots, v_{2r}$ form the vertices of a triangulation of Σ, as pictured in Fig. 7. Now the orbit of $\mathfrak{l}_\pm$ through the identity is $P_\pm$. Thus, from (7), we see that

$$\begin{aligned}\mu^{-1}S = \{&(h, h_0, \ldots, h_{2r-1}; C_0, C_1, \ldots, C_{2r}) \in H^{2r+1} \times G^{2r+1} \\ &\mid h_{2i+1}^{-1} C_{2i+1} C_{2i}^{-1} h_{2i} \in U_+ \text{ and } h_{2i}^{-1} C_{2i} C_{2i-1}^{-1} h_{2i-1} \in U_-, \},\end{aligned}$$

where the elements $h, h_0, \ldots, h_{2r-1} \in H$ and $C_0, C_1, \ldots, C_{2r} \in G$ denote the appropriately labelled holonomies in Fig. 7.

On the other hand,

$$\mathfrak{l} \cap \bigoplus_i \mathfrak{g}_i^{V_i} \cong \prod_{v_1,\ldots,v_{2r}} \mathfrak{h},$$

acting at the appropriate vertices. Thus, up to a gauge transformation, we may assume that $h_0 = h_1 = \cdots = h_{2r-1} = 1$. Setting $S_i = C_i C_{i-1}^{-1}$, we see that that the quotient space, $\mathsf{hol}^{-1}(\mathfrak{l} \cdot 1)/\big(\mathfrak{l} \cap \prod_{t \in \mathcal{T}_2} (\mathfrak{g}_t)_{\Gamma_{P_3}}\big)$, can be identified with

$${}_G\mathcal{A}_H^r := \{(h; S_{2r}, \ldots, S_1; C_0) \in H \times (U_- \times U_+)^r \times G\}.$$

We compute the two form to be

$$\begin{aligned}\omega = -\frac{1}{2}\Big(&\langle \mathrm{d}(hC_{2r})\,(hC_{2r})^{-1}, \mathrm{d}C_0\, C_0^{-1}\rangle + \langle (hC_{2r})^{-1}\mathrm{d}(hC_{2r}), C_{2r-1}^{-1}\mathrm{d}C_{2r-1}\rangle \\ &+ \sum_{i=1}^{2r-1} \langle C_i^{-1}\mathrm{d}C_i, C_{i-1}^{-1}\mathrm{d}C_{2i-1}\rangle\Big).\end{aligned}$$

Substituting $bC_0 = hC_{2r}$ in the first term and simplifying yields

$$\begin{aligned}\omega = \frac{1}{2}\Big(&\langle \mathrm{d}C_0\, C_0^{-1}, \mathbf{Ad}_b \mathrm{d}C_0\, C_0^{-1}\rangle + \langle \mathrm{d}C_0\, C_0^{-1}, \mathrm{d}b\, b^{-1}\rangle + \langle \mathrm{d}C_{2r}\, C_{2r}^{-1}, h^{-1}\mathrm{d}h\rangle \\ &+ \sum_{i=1}^{2r} \langle C_i^{-1}\mathrm{d}C_i, C_{i-1}^{-1}\mathrm{d}C_{2i-1}\rangle\Big),\end{aligned}$$

(here we have used the fact that $\langle \mathrm{d}S_{2r}\, S_{2r}^{-1}, h^{-1}\mathrm{d}h\rangle = 0$). Now, we haven't coloured the boundary of $\partial\Sigma$, so Theorem 3 does not imply that ω is symplectic. Nevertheless, as we shall see later, Theorem 6 implies that ω defines a quasi-Hamiltonian $G \times H$ structure on ${}_G\mathcal{A}_H^r$, where the moment map ${}_G\mathcal{A}_H^r \to G \times H$ is given by the holonomy along the (oriented) boundary components:

$$(h; S_{2r}, \ldots, S_1; C_0) \to (C_0^{-1} h S_{2r} \cdots S_1 C_0, h^{-1}),$$

and the G and H actions on ${}_G\mathcal{A}_H^r$ are precisely the residual gauge transformations. These act by G at x_G and by H at x_H:

$$(g, k) \cdot (h; S_{2r}, \ldots, S_1; C_0) = (khk^{-1}, kS_{2r}k^{-1}, \ldots, kS_1k^{-1}, kC_0g^{-1}),$$
$$g \in G, \quad k \in H.$$

Remark 3 This quasi-Hamiltonian $G \times H$-space was first discovered by Boalch [10–12], who used it to study meromorphic connections on Riemann surfaces.

1.4.2 Branched Surfaces

We can now consider examples where Σ is not a topological surface, i.e. where the domain walls may border more than two domains (Fig. 8).

Remark 4 Since our gauge fields (connections on Σ) are constrained to lie in $\mathfrak{c}_w \subseteq \bigoplus_{e \in \kappa^{-1}(w)} \mathfrak{g}_e$ along the domain wall $w \in W$, one may interpret $\mathfrak{c}_w$ as a "conservation law" for an interaction between the structure groups of the various domains glued to the domain wall w.

Example 7 ([10–12]) Let $V = \oplus_{i=1}^n V_i$ be a direct sum decomposition of a finite dimensional vector space, $G = \mathsf{Gl}(V)$, and $P_+ \subseteq G$ the stabilizer for the flag

$$F_1 \subset F_2 \subset \cdots \subset F_n = V,$$

where $F_k = \oplus_{i=1}^k V_i$. Similarly, let $P_- \subseteq G$ be the stabilizer for the flag

$$\tilde{F}_n \subset \cdots \subset \tilde{F}_2 \subset \tilde{F}_1 = V,$$

where $\tilde{F}_k = \oplus_{i=k}^n V_i$. Finally, let $H_i = \mathsf{Gl}(V_i)$ so that $\prod_{i=1}^n H_i = P_+ \cap P_-$. Let $H = \prod_{i=1}^n H_i$, let $U_\pm$ denote the unipotent radicals of $P_\pm$, and let $\mathfrak{g}, \mathfrak{p}_\pm, \mathfrak{u}_\pm, \mathfrak{h}, \mathfrak{h}_i$ denote the Lie algebras corresponding to the various Lie groups.

Now consider the moduli space

$${}_G\mathcal{A}_H^r := (\prod_{i=1}^n H_i) \times (U_- \times U_+)^r \times G$$

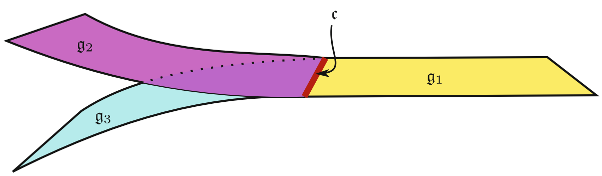


Fig. 8 Pictured above are three domains with structure Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}_3$. The three domains intersect at a branch locus, which we must colour by a coisotropic subalgebra $\mathfrak{c} \subseteq \oplus_{i=1}^3 \mathfrak{g}_i$

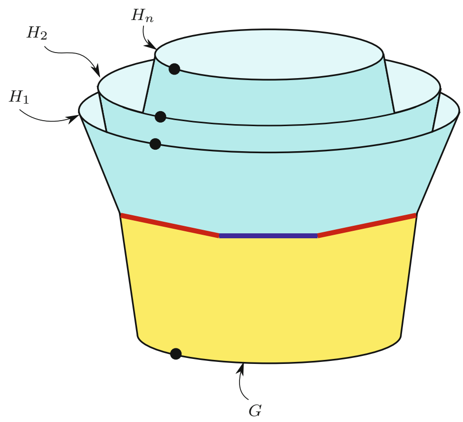


Fig. 9 A branched surface which arises in the work of Philip Boalch (see [11, p. 2675] and [12, Fig. 2])

described in Example 6. The coisotropic Lie algebra

$$\mathfrak{c}_{\pm} := \{(\sum_{i=1}^{n} \xi_i; \sum_{i=1}^{n} \xi_i + \mu) \mid \xi_i \in \mathfrak{h}_i \text{ and } \mu \in \mathfrak{u}_{\pm}\} \subset (\oplus_{i=1}^{n} \mathfrak{h}_i) \oplus \mathfrak{g}$$

defined in Eq. (8) can be used to colour the branch locus of $n+1$ domains with the structure groups $H_1, \ldots, H_n$ and G. Thus we may interpret ${}_G\mathcal{A}_H^r$ as the moduli space of flat connections for the branched surface pictured in Fig. 9.

Remark 5 The quasi-Hamiltonian space ${}_G\mathcal{A}_H^r$ first appeared in the work of Philip Boalch [10–12].

Example 8 (Quasi-triangular structures) Let $\mathfrak{g}$ be a quasi-triangular Lie quasi-bialgebra, i.e. $\mathfrak{g}$ is a Lie algebra with a chosen element $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$. Let $\mathfrak{d}$ be the Drinfel'd double of $\mathfrak{g}$. This means that $\mathfrak{d}$ is a quadratic Lie algebra, $\mathfrak{g} \subset \mathfrak{d}$ is a Lagrangian subalgebra, and $\mathfrak{p} \subset \mathfrak{d}$ is an ideal such that $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{p}$ as a vector space. Additionally, the restriction of the quadratic form on $\mathfrak{d}$ to $\mathfrak{p} \cong \mathfrak{g}^*$ is s.[3]

There is a natural groupoid structure on $\mathfrak{d}$, where $\mathfrak{g}$ is the space of objects and composition is defined by

$$(\xi + \alpha)(\xi + \beta) = \xi + \alpha + \beta \quad \forall \xi \in \mathfrak{g}, \alpha \in \mathfrak{p}, \beta \in \mathfrak{p}^{\perp},$$

and the source and target maps are

$$\mathsf{s}(\xi + \alpha) = \xi, \quad \mathsf{t}(\xi + \alpha) = \xi + s(\alpha, \cdot), \quad \xi \in \mathfrak{g}, \alpha \in \mathfrak{p}$$

[3] These properties uniquely define $\mathfrak{d}$. In particular, the Lie bracket is given by

$$[\xi + \alpha, \eta + \beta] = [\xi, \eta] + \mathbf{ad}^*_{\xi}\alpha - \mathbf{ad}^*_{\eta}\beta, \quad \xi, \eta \in \mathfrak{g}, \quad \alpha \in \mathfrak{p}, \beta \in \mathfrak{p}^{\perp}$$

where $\mathbf{ad}^*$ denotes the contragredient representation of $\mathfrak{g}$.

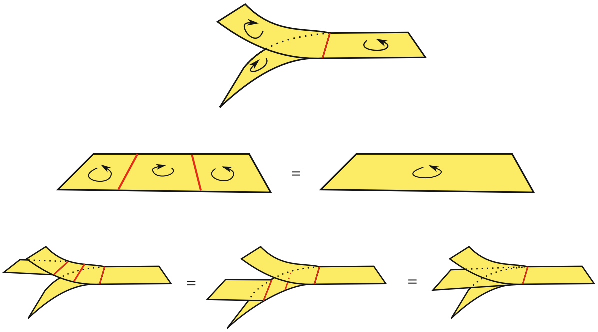


Fig. 10 Each of the edges above are coloured by $\mathfrak{c}_n$, where n is the number of domains branching off the given edge. As we cross a branch locus, the orientation of the domains reverses. As depicted, we may move branch loci past each other

The graph of multiplication,

$$\mathsf{gr}(\mathsf{Mult}) = \{(\xi\eta, \xi, \eta) \mid \xi, \eta \in \mathfrak{d} \text{ are composable}\} \subseteq \mathfrak{d} \oplus \bar{\mathfrak{d}} \oplus \bar{\mathfrak{d}}, \tag{9}$$

is a Lagrangian Lie subalgebra. See [18] and [28] for more details.

Similarly, the graph of iterated multiplication

$$\mathfrak{c}_n := \{(\xi_1, \dots, \xi_n) \mid \xi_1\xi_2 \cdots \xi_n \in \mathfrak{g}\} \subseteq \mathfrak{d}^n$$

is also a Lagrangian Lie subalgebra. As such it can be used to colour the branch locus of n domains each with structure group D (a connected Lie group with the Lie algebra $\mathfrak{d}$). Note that crossing such a branch locus reverses the orientation of the domain.

The associativity of multiplication on $\mathfrak{d}$ plays out as follows: paying attention to the orientations, if (as in Fig. 10) we

- move two branch loci past each other, or
- break a $\mathfrak{c}_{m+n-2}$-coloured branch locus into two separate $\mathfrak{c}_m$ and $\mathfrak{c}_n$ coloured branch loci (or vice-versa),

the resulting moduli spaces are canonically symplectomorphic.

In fact, there is a clear interpretation of the moduli spaces constructed by sewing domains together using the Lagrangian relations $\mathfrak{c}_n$. One may identify them with certain traditional moduli spaces in the following way: Suppose that $\cup\Sigma_d \to \Sigma$ is our $\mathfrak{c}_n$-coloured surface with domains Σ_d. First we form a two sheeted (branched cover) $\tilde{\Sigma}$ of Σ as follows: double each domain Σ_d to two sheets $\Sigma_d^+ \cup \Sigma_d^-$, where the sheet Σ_d^+ is canonically identified with Σ_d, while the sheet Σ_d^- is also canonically identified with Σ_d but with the opposite orientation. At each $\mathfrak{c}_n$-coloured domain wall with incident domains $\Sigma_{d_1}, \dots, \Sigma_{d_n}$, cyclically glue the sheets $\Sigma_{d_1}^{\pm}, \dots, \Sigma_{d_n}^{\pm}$

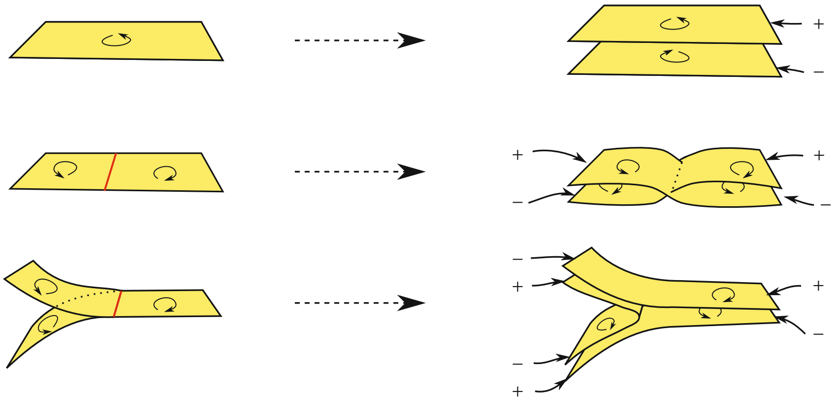


Fig. 11 We replace each domain in our branched surface by two copies, one with the same orientation (labelled $+$), and one with the opposite orientation (labelled $-$). Each $\mathfrak{c}_n$-coloured domain wall is replaced by cyclically gluing the incident sheets together, respecting the orientations. In this way we obtain an oriented surface from our $\mathfrak{c}_n$-coloured (branched) surface

together along their corresponding boundary segment, respecting the orientations, as in Fig. 11. In this way, one constructs the oriented surface $\tilde{\Sigma}$.

The groupoid inversion $\mathsf{Inv} : \mathfrak{d} \dashrightarrow \bar{\mathfrak{d}}$, being a morphism of Lie algebras, integrates to an involution of the Lie group D. This involution in turn lifts to an involution

$$\mathcal{A}_{\tilde{\Sigma}}(D) \to \mathcal{A}_{\tilde{\Sigma}}(D)$$

of the connections on $\tilde{\Sigma}$, mapping the fibre of the principal bundle over the $+$-sheet to the fibre over the $-$-sheet via $\mathsf{Inv} : D \to D$. The involution is a symplectomorphism which is compatible with the gauge transformations, and thus descends to a symplectomorphic involution on the moduli space, $\mathcal{M}_{\tilde{\Sigma}}(D)$. The fixed points of this involution are naturally identified with the moduli space $\mathcal{M}_{\Sigma}(D)$ of flat connections on the original $\mathfrak{c}_n$-coloured surface.

Example 9 ([10–12, 20]) Suppose that $\mathfrak{g}$ is a quasi-triangular Lie-bialgebra, where the $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$ is non-degenerate, i.e. $\mathfrak{g}$ is a quadratic Lie algebra. Equivalently, the double is $\mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$, and the Manin triple is $(\mathfrak{d}; \mathfrak{g}_\Delta, \mathfrak{h})$ where $\mathfrak{g}_\Delta \subset \mathfrak{g} \oplus \bar{\mathfrak{g}} = \mathfrak{d}$ is the diagonal, and $\mathfrak{h} \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$ is a complementary Lagrangian subalgebra. Notice that we may view $\mathfrak{h}$ as either a Lagrangian subalgebra of $\mathfrak{d}$, or a Lagrangian relation from $\mathfrak{g}$ to itself. We let G and $H \subset G \times G$ denote the simply connected (respectively connected) Lie groups corresponding to $\mathfrak{g}$ and $\mathfrak{h}$.

Of course G and H are Poisson Lie groups, and we may construct the Lu-Weinstein double symplectic groupoid integrating the Lie-Poisson structures on G and H as a moduli space $\mathcal{M}$, as in Example 2. Specifically, $\mathcal{M}$ is a moduli space of $\mathfrak{d}$-valued connections over a square, where alternating edges of the square are coloured with $\mathfrak{g}$ and $\mathfrak{h}$ as in the leftmost quilted surface pictured in Fig. 12.

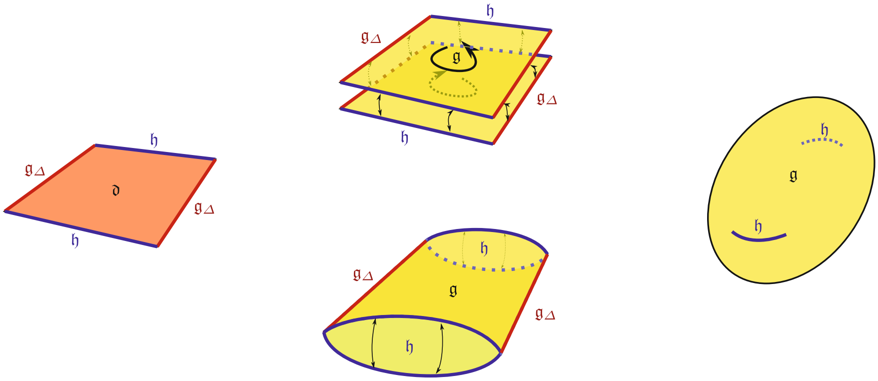


Fig. 12 The moduli spaces for the quilted surfaces pictured above may each be identified with the Lu-Weinstein double symplectic groupoid integrating the Lie-Poisson structures on G and H. In the *leftmost* quilted surface, a single domain carries the structure Lie algebra $\mathfrak{d}$, while each of the domains in the other quilted surfaces carry the structure Lie algebra $\mathfrak{g}$. The *double-ended arrows* between edges in the *middle* two quilted surfaces signify that those pairs of edges have been coloured by the corresponding Lagrangian relations. The *rightmost* quilted surface depicts a *sphere* containing two domain walls each coloured by $\mathfrak{h}$

However, since $\mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$, we may equally well view $\mathcal{M}$ as a moduli space of $\mathfrak{g}$ connections on two squares, where alternating edges of the first square are sewn to the corresponding edges of the second square using the Lagrangian relations $\mathfrak{g}_\Delta \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$ and $\mathfrak{h} \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$, respectively (see the middle two quilted surfaces in Fig. 12).

Now $\mathfrak{g}_\Delta \subseteq \mathfrak{g} \oplus \bar{\mathfrak{g}}$ is just the graph of the identity map, so a $\mathfrak{g}_\Delta$-coloured domain wall relates the (identical) structure groups in the incident domains by identifying them. Effectively, a $\mathfrak{g}_\Delta$-coloured domain wall can be erased. Thus $\mathcal{M}$ may be viewed as a moduli space of $\mathfrak{g}$-connections on the cylinder, where either boundary of the cylinder has been broken into two segments which are then sewn to each other using the Lagrangian relation $\mathfrak{h} \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$.

That is to say, $\mathcal{M}$ is a moduli space of $\mathfrak{g}$-connections over the sphere S^2, where two (contractible, non-intersecting) domain walls $\gamma_1, \gamma_2 \subset S^2$ have been coloured with $\mathfrak{h}$ (see the rightmost quilted surface in Fig. 12).

Thus, in this (quasi-triangular) case, the Lu-Weinstein double symplectic groupoid can be identified with a certain moduli space of connections on the sphere. This fact was first discovered by Fock and Rosly [20] (in terms of graph connections) and Boalch [10, 11] (in the case where $\mathfrak{g}$ is reductive and endowed with the standard quasi-triangular Lie bialgebra structure). Moreover, Boalch's perspective shows that placing these contractible domain walls on the sphere has the much deeper interpretation of prescribing certain irregular singularities for the connection.

Remark 6 In fact, Boalch [10–12] also provides an interpretation of these $\mathfrak{h}$-coloured domain wall in terms of quasi-Hamiltonian geometry (in the case where $\mathfrak{g}$ is reductive and endowed with the standard quasi-triangular Lie-bialgebra structure). Indeed, in this case, a neighborhood of each domain wall may be identified with the quilted surface described in Example 6 (for $r = 1$), for which the corresponding moduli space is Boalch's fission space.

1.5 Poisson Structures

In this section, we will describe some Poisson structures which may be constructed using our approach. Later, in Sect. 5 we will generalize these results to the case where $\mathfrak{g}$ is a quasi-triangular Lie quasi-bialgebra rather than a quadratic Lie algebra.

Let (Σ, V) be a marked surface with boundary graph Γ. First, recall from Theorem 2 that the moduli space $\mathcal{M}_{\Sigma,V}(G)$ for a marked surface (Σ, V) carries a $(\mathfrak{d}^{V_\Gamma}, \mathfrak{g}^{V_\Gamma}, G^{E_\Gamma})$-quasi-Hamiltonian structure, where $\mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$, the Lagrangian Lie subalgebra $\mathfrak{g}^{V_\Gamma} = \mathfrak{g}_\Delta^{V_\Gamma} \subset \mathfrak{d}^{V_\Gamma}$ is embedded as the diagonal, and $\mathfrak{d}^{V_\Gamma}$ acts on the $e \in E_\Gamma$th factor of G^{E_Γ} via the vector field

$$\xi^L_{\mathsf{out}(e)} - \eta^R_{\mathsf{in}(e)}, \quad (\xi, \eta) \in \mathfrak{g}^{V_\Gamma} \oplus \bar{\mathfrak{g}}^{V_\Gamma} = \mathfrak{d}^{V_\Gamma}. \tag{10}$$

Here the superscripts ^{L},^{R} denote left, right invariant vector fields. The bivector field on $\mathcal{M}_{\Sigma,V}(G)$ is computed in Sect. 5.2 and leads to the result of [30, Theorem 3], which we summarize briefly.[4]

If $a, b \in \Pi_1(\Sigma, V)$, let us represent them by transverse smooth paths α, β. For any point A in their intersection, let

$$\lambda(A) = \begin{cases} 1 & \text{if } A \in \partial\Sigma \\ 2 & \text{otherwise} \end{cases}$$

$$\mathsf{sign}(A) := \mathsf{sign}(\alpha, \beta; A) = \begin{cases} 1 & \text{if } (\beta'|_A, \alpha'|_A) \text{ is positively oriented} \\ -1 & \text{otherwise.} \end{cases}$$

as in Fig. 13.

Let α_A denote the portion of α parametrized from the beginning up to the point A. Finally, let

$$(a, b) := \sum_A \lambda(A)\mathsf{sign}(A)[\alpha_A^{-1}\beta_A] \in \mathbb{Z}\Pi_1(\Sigma, V).$$

[4] In fact, the computation in Sect. 5.2 results in minus the bivector field described in [30], due to us orienting $\partial\Sigma$ in the opposite way.

Strictly speaking, the bivector field on $\mathcal{M}_{\Sigma,V}(G)$ depends on the choice of a complement $\mathfrak{k} \subset \mathfrak{d}^{V_\Gamma}$ to $\mathfrak{h} = \mathfrak{g}^{V_\Gamma}$. In this case, $\mathfrak{k}$ can be chosen canonically as $\mathfrak{k} := \mathfrak{g}_{\bar{\Delta}}^{V_\Gamma}$, where

$$\mathfrak{g}_{\bar{\Delta}} := \{(\xi, -\xi) \in (\mathfrak{g} \oplus \bar{\mathfrak{g}})\}.$$

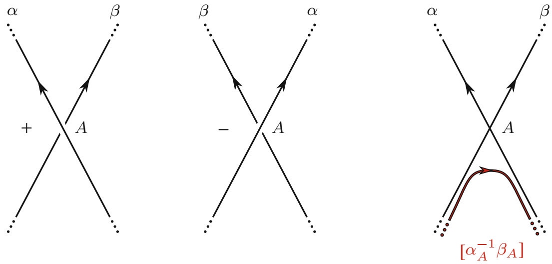


Fig. 13 $\mathsf{sign}(A) = \pm 1$ is determined by comparing the orientation of α and β with that of Σ. The path $[\alpha_A^{-1}\beta_A]$ is shown to the *right*

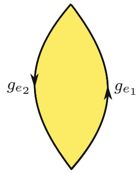


Fig. 14 The cyclic product of holonomies, $g_{e_2}g_{e_1}$, is trivial, so $g_{e_2} = g_{e_1}^{-1}$

(When V contains only one point, this is a skew symmetrized version of the intersection form described in [42]).

Then for any $a, b \in \Pi_1(\Sigma, V)$,

$$\pi\big(\mathsf{ev}_a^*(g^{-1}\mathrm{d}g), \mathsf{ev}_b^*(g^{-1}\mathrm{d}g)\big) = \frac{1}{2}(\mathbf{Ad}_{\mathsf{ev}_{(a,b)}} \otimes 1)\, s, \tag{11}$$

where $\mathsf{ev}_a, \mathsf{ev}_b : \mathcal{M}_{\Sigma,V}(G) = \mathrm{Hom}(\Pi_1(\Sigma, V), G) \to G$ denotes evaluation, $s \in \mathfrak{g} \otimes \mathfrak{g}$ is the inverse of the quadratic form, and $g^{-1}\mathrm{d}g$ denotes the left invariant Maurer-Cartan form on G.

Example 10 (The two sided polygon, P_2) Suppose $(\Sigma, V) = P_2$ is the disk with two marked points and $E_\Gamma = \{e_1, e_2\}$, as in Fig. 14. Then we may identify $\mathcal{M}_{P_2}(G)$ with G via g_{e_1} (since $g_{e_2} = g_{e_1}^{-1}$). Under this identification, the bivector field is trivial, $\pi_{P_2} = 0$ (cf. [30]).

1.5.1 Colouring Edges

Suppose now that we colour each marked point $v \in V_\Gamma$ with a Lagrangian subalgebra $\mathfrak{l}_v \subseteq \mathfrak{d}$, and each edge $e \in E_\Gamma$ with a submanifold $S_e \subseteq G$ in a compatible way: Specifically, we require that

$$S := \prod_{e \in E_\Gamma} S_e \subseteq G^{E_\Gamma}$$

be $\mathfrak{l}$-invariant, where

$$\mathfrak{l} := \bigoplus_{v \in V_\Gamma} \mathfrak{l}_v \subseteq \mathfrak{d}^{V_\Gamma},$$

and the action is given in (10). Then Theorem 1 implies that

$$\mathcal{M}_{red} := \mu^{-1}(S)/(\mathfrak{l} \cap \mathfrak{g}^{V_\Gamma})$$

is a Poisson manifold (provided it is a manifold).

It is not difficult to describe the bivector field on $\mu^{-1}(S)/(\mathfrak{l} \cap \mathfrak{g}^{V_\Gamma})$. First, let $\mathfrak{g}_\Delta \subseteq \mathfrak{d}$ denote the diagonal and $\mathfrak{g}_{\bar{\Delta}} \subseteq \mathfrak{d}$ the off-diagonal:

$$\mathfrak{g}_{\bar{\Delta}} := \{(\xi, -\xi) \in (\mathfrak{g} \oplus \overline{\mathfrak{g}})\}.$$

Each Lagrangian Lie subalgebra $\mathfrak{l}_v \subseteq \mathfrak{d}^{V_\Gamma}$ defines an element $\tau_v \in \wedge^2\big(\mathfrak{h}/(\mathfrak{l} \cap \mathfrak{h})\big)$ by the equation

$$\mathfrak{l}_v = \{(\alpha + \tau_v^\sharp \alpha) \mid \alpha \in \mathfrak{g}_{\bar{\Delta}}, \quad \langle \alpha, \mathfrak{l}_v \cap \mathfrak{g}_\Delta \rangle = 0\} + \mathfrak{l}_v \cap \mathfrak{g}_\Delta.$$

Here $\tau_v^\sharp \alpha \in \mathfrak{g}_\Delta/(\mathfrak{l}_v \cap \mathfrak{g}_\Delta)$ is defined by

$$\tau_v^\sharp \alpha := \frac{1}{2} \sum_{i,j} \tau_v^{ij} \langle \alpha, \xi_i \rangle \xi_j - \frac{1}{2} \sum_{i,j} \tau_v^{ij} \langle \alpha, \xi_j \rangle \xi_i$$

when we represent τ_v as $\tau_v = \frac{1}{2} \sum_{i,j} \tau_v^{ij} \xi_i \wedge \xi_j$.

We have the following theorem:

Theorem 4 *If the intersection, $\mu^{-1}(S)$, of $S \times \mathcal{M}_{\Sigma,V}(G)$ with the graph of μ is clean, and the $\mathfrak{l} \cap \mathfrak{g}^{V_\Gamma}$-orbits of $\mu^{-1}(S)$ form a regular foliation, then the bivector field*

$$\pi + \sum_{v \in V_\Gamma} \rho_v(\tau_v) \in \Gamma\big(\wedge^2 T(\mu^{-1}(S))/\rho(\mathfrak{l} \cap \mathfrak{g}^{V_\Gamma})\big)$$

is $\mathfrak{l} \cap \mathfrak{g}^{V_\Gamma}$ invariant and descends to define the Poisson structure on $\mu^{-1}(S)/(\mathfrak{l} \cap \mathfrak{g}^{V_\Gamma})$. Here $\rho : \mathfrak{g}^{V_\Gamma} \to \mathfrak{X}(\mathcal{M}_{\Sigma,V}(G))$ denotes the action by infinitesimal gauge transformations at the marked points, and ρ_v is the restriction of ρ to the $v \in V_\Gamma$th factor.

Moreover, for any $\mathfrak{l}$-orbit $O \subseteq S$, the image of $\mu^{-1}(O)$ in $\mu^{-1}(S)/(\mathfrak{l} \cap \mathfrak{g}^{V_\Gamma})$ will be a symplectic leaf.

Proof This will follow from Theorem 7, while the statement for the symplectic leaves will follow from Theorem 6.

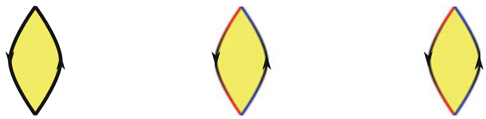


Fig. 15 The double Poisson Lie group. The holonomies $g_i \in G$ satisfy $g_1 g_2 = 1$

Remark 7 As in Sect. 1.4, one may also sew domains together to obtain Poisson structures on the moduli spaces of branched surfaces. The general reduction statement is Theorem 7.

Example 11 (Double Poisson Lie group [30]*)* Suppose that $\mathfrak{g} = \mathfrak{e} \oplus \mathfrak{f}$ as a vector space, where $\mathfrak{e}, \mathfrak{f} \subseteq \mathfrak{g}$ are Lagrangian Lie subalgebras, and that $\mathfrak{e}, \mathfrak{f} \subseteq \mathfrak{g}$ integrate to Lie subgroups $E, F \subseteq G$ such that $E \cap F = 1$. Let Σ be a disk with two marked points labelled as in Fig. 15. We colour the edges with the full group G,

$$S_{e_1} = G, \quad S_{e_2} = G,$$

and the vertices as

$$\mathfrak{l}_{v_1} = \mathfrak{e} \oplus \mathfrak{f}, \quad \mathfrak{l}_{v_2} = \mathfrak{f} \oplus \mathfrak{e}.$$

Therefore,

$$\mu^{-1}(S) = \mathcal{M}_{P_2} = \{(g_1, g_2) \in G \times G \mid g_1 g_2 = 1\}.$$

Meanwhile the residual gauge transformations,

$$\bigoplus_{v \in V_\Gamma} \mathfrak{l}_v \cap \mathfrak{g}_\Delta = 0$$

(since $\mathfrak{e} \cap \mathfrak{f} = 0$). Thus we may identify the moduli space $\mathcal{M}_{red} \cong G$, via the map $(g_1, g_2) \to g_1$.

Next, we compute the bivector field, $\pi = \pi_{P_2} + \rho_{v_1}(\tau_{v_1}) + \rho_{v_2}(\tau_{v_2})$. Now, as explained in Example 10, $\pi_{P_2} = 0$ so only the term $\sum_i \rho_{v_i}(\tau_{v_i})$ contributes. Now,

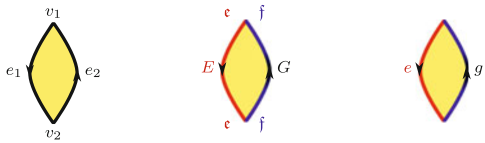
$$\tau_{v_1} = \frac{1}{2} \sum_i (\zeta^i, \zeta^i) \wedge (\eta_i, \eta_i), \quad \tau_{v_2} = \frac{1}{2} \sum_i (\eta_i, \eta_i) \wedge (\zeta_i, \zeta_i),$$

where $\{\eta_i\} \subset \mathfrak{e}$ and $\{\zeta^i\} \subset \mathfrak{f}$ are basis in duality. Therefore,

$$\pi = \frac{1}{2} \sum_i (\zeta^i)^L \wedge (\eta_i)^L + (\eta_i)^R \wedge (\zeta^i)^R.$$

In fact, π defines the Poisson Lie group structure on G corresponding the double Lie bialgebra structure on $\mathfrak{g}$ resulting from the Manin triple $(\mathfrak{g}, \mathfrak{e}, \mathfrak{f})$ [30, 32]. The

Fig. 16 The Poisson Lie group. The holonomies $g \in G$ and $e \in E$ satisfy $eg = 1$



symplectic leaves are computed as the restriction of the $\mathfrak{l}$-orbits, which in this case can be seen to correspond to the orbits of the dressing action on G.[5]

Remark 8 In the case where $\mathfrak{g}$ is a quasi-triangular Lie-bialgebra, the double Poisson Lie group was constructed as a moduli space of graph connections in the work of Fock and Rosly [20].

Example 12 (Poisson Lie group [30]*)* Suppose the Lie groups G, E, F and their Lie algebras are as in Example 11, and let Σ be a disk with two marked points, as in Fig. 16. We colour the vertices as in Example 11, but we colour the edges as

$$S_{e_1} = E, \quad S_{e_2} = G.$$

Therefore,

$$\mu^{-1}(S) = \{(e, g) \in E \times G \mid eg = 1\},$$

while the residual gauge transformations are trivial, as before. Thus we may identify the moduli space $\mathcal{M}_{red} \cong E$, via the map $(e, g) \to e$.

The bivector field, π, on E is computed to be the restriction of the bivector field

$$\frac{1}{2}\sum_i (\zeta^i)^L \wedge (\eta_i)^L + (\eta_i)^R \wedge (\zeta^i)^R.$$

on G to $E \subseteq G$. Thus, π defines the Poisson Lie group structure on E corresponding the Manin triple $(\mathfrak{g}, \mathfrak{e}, \mathfrak{f})$ [30, 32]. As before, the symplectic leaves are computed as the restriction of the $\mathfrak{l}$-orbits. Once again, they are precisely the orbits of the dressing action on E.

Example 13 (Poisson homogenous spaces) Suppose the Lie groups G, E, F and their Lie algebras are as in Example 11, and let Σ be a disk with two marked points, as in Fig. 17. Suppose further that $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lagrangian subalgebra such that $\mathfrak{k} := \mathfrak{h} \cap \mathfrak{e}$ integrates to a closed Lie subgroup $K \subseteq E$. We colour the edges as in Example 12, but we colour the vertices as

$$\mathfrak{l}_{v_1} = \mathfrak{e} \oplus \mathfrak{h}, \quad \mathfrak{l}_{v_2} = \mathfrak{f} \oplus \mathfrak{e}.$$

[5] In fact computing the symplectic leaves via Theorem 4 is precisely the computation found in [27].

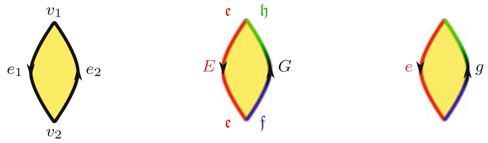


Fig. 17 The Poisson homogeneous space corresponding to the Lagrangian Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. The holonomies $g \in G$ and $e \in E$ satisfy $eg = 1$

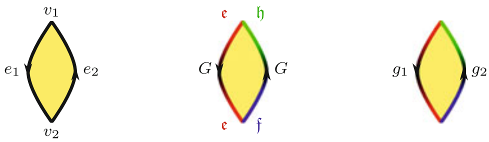


Fig. 18 The affine Poisson structure on G. The holonomies $g_i \in G$ satisfy $g_1 g_2 = 1$

Therefore,

$$\mu^{-1}(S) = \{(e, g) \in E \times G \mid eg = 1\},$$

while the residual gauge transformations are $G \times K$ acting as

$$(g, k) \cdot (e, g) = (ek^{-1},\ kg).$$

We may identify the moduli space $\mathcal{M}_{red} \cong E/K$, via the map $(e, g) \to [e]$. The Poisson structure on $\mathcal{M}_{red}$ is the Poisson homogenous structure corresponding to the Lagrangian Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ in Drinfel'd's classification [19]. We leave it to the reader to compute the bivector field and symplectic leaves on E/K via Theorem 4.

Example 14 (Affine Poisson structure on G [32]*)* Generalizing the setup found in Example 11, we suppose that $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lagrangian subalgebra which is also complementary (as a vector space) to $\mathfrak{e} \subseteq \mathfrak{g}$. As in Example 11, we colour the edges with the full group G, but the vertices as

$$\mathfrak{l}_{v_1} = \mathfrak{e} \oplus \mathfrak{h}, \quad \mathfrak{l}_{v_2} = \mathfrak{f} \oplus \mathfrak{e},$$

(cf. Fig. 18). As before, we have $\mu^{-1}(S) = \mathcal{M}_{P_2}$, and the residual gauge transformations are trivial. Thus, we may identify the moduli space $\mathcal{M}_{red} \cong G$, via the map $(g_1, g_2) \to g_1$.

Meanwhile the bivector field, π, is

$$\pi = \frac{1}{2} \sum_i (\zeta_{\mathfrak{f}}^i)^L \wedge (\eta_i)^L + (\eta_i)^R \wedge (\zeta_{\mathfrak{h}}^i)^R,$$

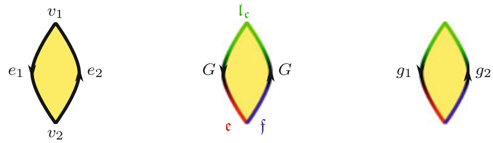


Fig. 19 Lu-Yakimov Poisson homogenous spaces. The holonomies $g_i \in G$ satisfy $g_1 g_2 = 1$

where the bases $\{\zeta^i_\mathfrak{f}\} \subseteq \mathfrak{f}$ and $\{\zeta^i_\mathfrak{h}\} \subseteq \mathfrak{h}$ are both dual to $\{\eta_i\} \subseteq \mathfrak{g}$. In fact, π defines the affine Poisson structure on G corresponding to the Manin triples $(\mathfrak{g}, \mathfrak{e}, \mathfrak{f})$ and $(\mathfrak{g}, \mathfrak{e}, \mathfrak{h})$ (as described by Lu [32]). The symplectic leaves are computed as the restriction of the $\mathfrak{l}$-orbits.

Example 15 (Lu Yakimov Poisson homogenous spaces [30]*)* Suppose the Lie groups G, E, F and their Lie algebras are as in Example 11, and that $C \subseteq G$ is a closed Lie subgroup whose Lie algebra $\mathfrak{c} \subseteq \mathfrak{g}$ is coisotropic. Let Σ be a disk with two marked points and edges and vertices labelled as in Fig. 19. We colour the edges with the full group G,

$$S_{e_1} = G, \quad S_{e_2} = G,$$

and the vertices as

$$\mathfrak{l}_{v_1} = \mathfrak{l}_\mathfrak{c} = \{(\xi, \xi') \in \mathfrak{c} \oplus \mathfrak{c} \mid \xi - \xi' \in \mathfrak{c}^\perp\}, \quad \mathfrak{l}_{v_2} = \mathfrak{f} \oplus \mathfrak{e}$$

(cf. Fig. 19). Therefore,

$$\mu^{-1}(S) = \mathcal{M}_{P_2} = \{(g_1, g_2) \in G \times G \mid g_1 g_2 = 1\}.$$

Meanwhile the residual gauge transformations,

$$\bigoplus_{v \in V_\Gamma} \mathfrak{l}_v \cap \mathfrak{g}_\Delta = \mathfrak{c}_{v_0},$$

where $\mathfrak{c}_{v_0} = \{(\xi, \xi) \in \mathfrak{c} \oplus \mathfrak{c}\} \subseteq \mathfrak{l}_{v_0}$. Thus, up to a gauge transformation, $(g_1, g_2) \sim (g_1 c^{-1}, c g_2)$ (for any $c \in C$), and we may identify the moduli space $\mathcal{M}_{red} \cong G/C$, via the map $(g_1, g_2) \to [g_1]$.

The bivector field, π, on G/C can be computed to be the projection of the bivector field

$$\frac{1}{2} \sum_i (\eta_i)^R \wedge (\zeta^i)^R.$$

on G to G/C. Thus, π defines the Lu-Yakimov Poisson structure on G/C corresponding the Manin triple $(\mathfrak{g}, \mathfrak{e}, \mathfrak{f})$ [30, 34].

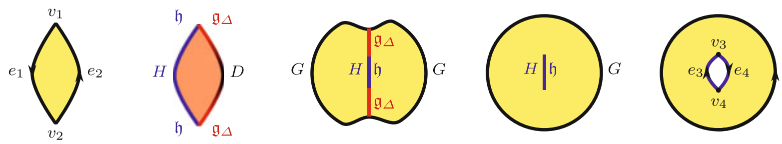


Fig. 20 The dual Poisson Lie group, H. The holonomies $h_1, d_2 \in D = G \times G$ along the edges e_1 and e_2 (respectively) satisfy $h_1 d_2 = 1$, with $h_1 \in H$; while the holonomies $g_3, g_4 \in G$ along the edges e_3 and e_4 satisfy $(g_3^{-1}, g_4) \in H$

Example 16 (Quasi-Triangular Poisson Lie groups [7–12]*)* As in Example 9, suppose that $(\mathfrak{d}; \mathfrak{g}_\Delta, \mathfrak{h})$ is the Manin triple corresponding to a (non-degenerate) quasi-triangular Lie-bialgebra where $\mathfrak{g}_\Delta \subset \mathfrak{g} \oplus \bar{\mathfrak{g}} = \mathfrak{d}$ is the diagonal, and $\mathfrak{h} \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$ is a complementary Lagrangian subalgebra. As before we may view $\mathfrak{h}$ as either a Lagrangian subalgebra of $\mathfrak{d}$, or a Lagrangian relation from $\mathfrak{g}$ to itself. We let G and $H \subset G \times G = D$ denote the simply connected (respectively connected) Lie groups corresponding to $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{d}$.

We may construct the Lie-Poisson structure on H by identifying it with a moduli space $H \cong \mathcal{M}$, as in Example 12. More specifically let Σ be a disk with two marked points, with edges and vertices labelled as in Fig. 20. As pictured in Fig. 20, we color the edges as

$$S_{e_1} = H, \quad S_{e_2} = D,$$

and the vertices as

$$\mathfrak{l}_{v_1} = \mathfrak{h} \oplus \mathfrak{g}_\Delta, \quad \mathfrak{l}_{v_2} = \mathfrak{g}_\Delta \oplus \mathfrak{h}.$$

Therefore,

$$\mu^{-1}(S) = \{(h_1, d_2) \in H \times D \mid h_1 d_2 = 1\}.$$

Meanwhile the residual gauge transformations are trivial (since $\mathfrak{h}$ and $\mathfrak{g}_\Delta$ are complements). Thus we may identify the moduli space $\mathcal{M} \cong H$, via the map $(h_1, d_2) \to h_1$.

As in Example 9, since $\mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$, we may also view $\mathcal{M}$ as a moduli space of $\mathfrak{g}$ connections on two stacked copies of Σ, where their left edges are sewn using the Lagrangian relation $\mathfrak{h} \subseteq \mathfrak{g} \oplus \bar{\mathfrak{g}}$, while their top and bottom right half-edges are sewn using the relation $\mathfrak{g}_\Delta \subseteq \mathfrak{g} \oplus \bar{\mathfrak{g}}$. In the same fashion as one opens a pita-pocket, we may imagine pulling the two copies of Σ apart starting from the middle of their right edges (leaving the left edges incident to each other), in which case the resulting quilted surface is pictured in the middle of Fig. 20.

Since $\mathfrak{g}_\Delta \subseteq \mathfrak{g} \oplus \bar{\mathfrak{g}}$ is just the graph of the identity map, a $\mathfrak{g}_\Delta$-coloured domain wall relates the (identical) structure groups in the incident domains by identifying them. Effectively, a $\mathfrak{g}_\Delta$-coloured domain wall can be erased. Thus $\mathcal{M}$ may be viewed as a moduli space of $\mathfrak{g}$-connections on the annulus, Σ', where the inner boundary has been broken into two segments which are then sewn together using the Lagrangian relation $\mathfrak{h} \subseteq \mathfrak{g} \oplus \bar{\mathfrak{g}}$.

More explicitly, the inner boundary of the annulus Σ' has two marked points, v_3 and v_4, dividing it into two segments e_3 and e_4, while the outer boundary has no marked points. As pictured in the right two quilted surfaces of Fig. 20, we colour the edges as

$$S = S_{e_3} \times S_{e_4} = \{(g_3, g_4) \in G \times G \mid (g_3^{-1}, g_4) \in H\} \cong H$$

and the vertices as

$$\mathfrak{l}_{v_3} = \mathfrak{h}^{-1} \subseteq \mathfrak{g} \oplus \bar{\mathfrak{g}}, \quad \mathfrak{l}_{v_4} = \mathfrak{h} \subseteq \mathfrak{g} \oplus \bar{\mathfrak{g}},$$

where $\mathfrak{h}^{-1} = \{(\xi, \eta) \in \mathfrak{g} \oplus \bar{\mathfrak{g}} \mid (\eta, \xi) \in \mathfrak{h}\}$ (the inverse refers to the pair-groupoid structure).

Since μ is a diffeomorphism, $\mu^{-1}(S) \cong S \cong H$. The residual gauge transformations are trivial, since $\mathfrak{h}$ and $\mathfrak{g}_\Delta$ are complements. As before, we may identify the moduli space $\mathcal{M} \cong H$.

That is to say, the (dual) Poisson Lie group $H \cong \mathcal{M}$ is naturally a moduli space of flat $\mathfrak{g}$-connections over the disk containing a (contractible) $\mathfrak{h}$-coloured domain wall. Thus, in this (quasi-triangular) case, the Poisson Lie group H can be identified with a certain moduli space of flat $\mathfrak{g}$-connections on the disk. This fact was first observed by Fock and Rosly [20] (in terms of graph connections) and Boalch [7–12] (in the case where $\mathfrak{g}$ is reductive and endowed with the standard quasi-triangular Lie-bialgebra structure). Moreover, Boalch's perspective shows that placing this contractible domain wall on the disk has the much deeper interpretation of prescribing a certain irregular singularity for the connection.

Remark 9 In fact, Boalch [10–12] also provides an interpretation of this $\mathfrak{h}$-coloured domain wall in terms of quasi-Hamiltonian geometry (in the case where $\mathfrak{g}$ is reductive and endowed with the standard quasi-triangular Lie-bialgebra structure). Indeed, in this case, a neighborhood of the domain wall may be identified with the quilted surface described in Example 6 (for $r = 1$), for which the corresponding moduli space is Boalch's fission space.

2 Background

2.1 Courant Algebroids

Courant algebroids and Dirac structures are the basic tools in the theory of generalized moment maps.

Definition 1 (*Liu* [31]) A *Courant algebroid* is a vector bundle $\mathbb{E} \to M$ endowed with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the fibres, a bundle map $\mathbf{a} : \mathbb{E} \to TM$ called the *anchor* and a bracket $[\![\cdot, \cdot]\!] : (\mathbb{E}) \times (\mathbb{E}) \to (\mathbb{E})$ called the *Courant bracket* satisfying the following axioms for sections $e_1, e_2, e_3 \in (\mathbb{E})$ and functions $f \in C^\infty(M)$:

(c1) $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$,
(c2) $\mathbf{a}(e_1)\langle e_2, e_3\rangle = \langle [\![e_1, e_2]\!], e_3\rangle + \langle e_2, [\![e_1, e_3]\!]\rangle$,
(c3) $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = \mathbf{a}^* \mathrm{d}\langle e_1, e_2\rangle$.

Here $\mathbf{a}^* : T^*M \to \mathbb{E}^* \cong \mathbb{E}$ is the map dual to the anchor.

A subbundle $E \subseteq \mathbb{E}|_S$ along a submanifold $S \subseteq M$ is called a *Dirac structure with support on S* if

$$e_1|_S, e_2|_S \in (E) \Rightarrow [\![e_1, e_2]\!]|_S \in (E),$$

(it is *involutive*) and $E^\perp = E$ (it is *Lagrangian*). If $S = M$, then E is simply called a *Dirac structure*.

Remark 10 As shown in [35, 43], one may also derive the following useful identities from the Courant axioms:

(c4) $[\![e_1, fe_2]\!] = f[\![e_1, e_2]\!] + (\mathbf{a}(e_1)f)e_2$
(c5) $[\![fe_1, e_2]\!] = f[\![e_1, e_2]\!] - (\mathbf{a}(e_2)f)e_1 + \langle e_1, e_2\rangle \mathbf{a}^*\mathrm{d}f$
(c6) $\mathbf{a}[\![e_1, e_2]\!] = [\mathbf{a}(e_1), \mathbf{a}(e_2)]$

For any Courant algebroid $\mathbb{E}$, we denote by $\overline{\mathbb{E}}$ the Courant algebroid with the same bracket and anchor, but with the metric negated.[6]

Example 17 A Courant algebroid over a point is a quadratic Lie algebra. Dirac structures are Lagrangian Lie subalgebras.

Example 18 *(Standard Courant algebroid* [15, 16]*)* The vector bundle $\mathbb{T}M := TM \oplus T^*M$ is a Courant algebroid with metric

$$\langle v_1 + \mu_1, v_2 + \mu_2\rangle = \mu_1(v_2) + \mu_2(v_1), \quad v_1, v_2 \in TM, \quad \mu_1, \mu_2 \in T^*M$$

and bracket

$$[\![X + \alpha, Y + \beta]\!] = [X, Y] + \mathcal{L}_X\beta - \iota_Y d\alpha, \quad X, Y \in \mathfrak{X}(M), \alpha, \beta \in \Omega^1(M).$$

The standard Courant algebroid is an example of an important class of Courant algebroids called *exact Courant algebroids*.

Definition 2 (*Exact Courant algebroids* [37, 41]) A Courant algebroid $\mathbb{E} \to M$ is called *exact* if the sequence

[6] Note that this also negates the map $\mathbf{a}^* : T^*M \to \mathbb{E}^* \cong \mathbb{E}$, so axiom c3) still holds.

$$0 \to T^*M \xrightarrow{\mathbf{a}^*} \mathbb{E} \xrightarrow{\mathbf{a}} TM \to 0 \tag{12}$$

is exact.

If a Dirac structure $E \subseteq \mathbb{E}$ is supported on $S \subseteq M$, then $\mathbf{a}(E) \subseteq TS$, and $\mathbf{a}^*\big(\mathsf{ann}(TS)\big) \subseteq E$.

Definition 3 (*Exact Dirac structures*) Suppose $\mathbb{E} \to M$ is a Courant algebroid (not necessarily exact), and $E \subseteq \mathbb{E}$ is a Dirac structure with support on $S \subseteq M$. We say that E is an *exact Dirac structure* if the sequence

$$0 \to \mathsf{ann}(TS) \xrightarrow{\mathbf{a}^*} E \xrightarrow{\mathbf{a}} TS \to 0 \tag{13}$$

is exact.

Lemma 1 *Suppose $\mathbb{E} \to M$ is a Courant algebroid, and $E \subseteq \mathbb{E}$ is a Dirac structure with support on $S \subseteq M$. Then the following two statements are equivalent:*

1. *E is an exact Dirac structure.*
2. *The Courant algebroid $\mathbb{E}$ is exact along S (that is, the sequence (12) is exact at every $x \in S$), and $\mathbf{a} : E \to TS$ is surjective.*

Proof Let $x \in S$, and consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{ann}(T_xS) & \xrightarrow{\mathbf{a}^*} & E_x & \xrightarrow{\mathbf{a}} & T_xS & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_x^*M & \xrightarrow{\mathbf{a}^*} & \mathbb{E}_x & \xrightarrow{\mathbf{a}} & T_xM & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_x^*M/\mathsf{ann}(T_xS) & \xrightarrow{\mathbf{a}^*} & E_x^* & \xrightarrow{\mathbf{a}} & T_xM/T_xS & \longrightarrow & 0
\end{array}$$

Note that the vertical sequences are exact.

Suppose that E is an exact Dirac structure. Then the top horizontal sequence is exact, by assumption. The lower horizontal sequence is dual to the top sequence, and hence also exact. The five lemma then implies that all terms in the long exact sequence vanish. In particular, the central horizontal sequence is exact.

Conversely, suppose that the Courant algebroid $\mathbb{E}$ is exact along S and $\mathbf{a} : E_x \to T_xS$ is surjective (and hence $T_x^*M/\mathsf{ann}(T_xS) \to E_x^*$ is injective). Once again, the five lemma implies that all terms in the long exact sequence vanish. We conclude that E is an exact Dirac structure.

Example 19 *(Action Courant algebroids* [27]*)* Suppose $\mathfrak{d}$ is a Lie algebra equipped with an invariant metric. Given a Lie algebra action $\rho\colon \mathfrak{d} \to \mathfrak{X}(M)$ on a manifold M, let $\mathbb{E} = \mathfrak{d} \times M$ with anchor map $\mathbf{a}(\xi, m) = \rho(\xi)_m$, and with the bundle metric coming from the metric on $\mathfrak{d}$. As shown in [27], the Lie bracket on constant sections $\mathfrak{d} \subseteq C^\infty(M, \mathfrak{d}) = (\mathbb{E})$ extends to a Courant bracket *if and only if* the stabilizers

$\mathfrak{d}_m \subseteq \mathfrak{d}$ are coisotropic, i.e. $\mathfrak{d}_m \supseteq \mathfrak{d}_m^{\perp}$. Explicitly, for $\xi_1, \xi_2 \in (\mathbb{E}) = C^{\infty}(M, \mathfrak{d})$ the Courant bracket reads (see [27, ß4])

$$[\![\xi_1, \xi_2]\!] = [\xi_1, \xi_2] + \mathcal{L}_{\rho(\xi_1)}\xi_2 - \mathcal{L}_{\rho(\xi_2)}\xi_1 + \rho^*\langle \mathrm{d}\xi_1, \xi_2\rangle. \tag{14}$$

Here $\rho^*\colon T^*M \to \mathfrak{d} \times M$ is the dual map to the action map $\rho\colon \mathfrak{d} \times M \to TM$, using the metric to identify $\mathfrak{d}^* \cong \mathfrak{d}$. We refer to $\mathfrak{d} \times M$ with bracket (14) as an *action Courant algebroid.*

Example 20 (Cartan Courant algebroid [41]*)* Suppose $\mathfrak{g}$ is a Lie group endowed with an invariant metric, $\langle \cdot, \cdot \rangle$. We let $\overline{\mathfrak{g}}$ denote the Lie algebra $\mathfrak{g}$ with the metric negated, $-\langle \cdot, \cdot \rangle$. Suppose G is a Lie group with Lie algebra $\mathfrak{g}$ and which preserves the metric. The Lie algebra $\overline{\mathfrak{g}} \oplus \mathfrak{g}$ acts on G by $\rho : (\overline{\mathfrak{g}} \oplus \mathfrak{g}) \times G \to TG$,

$$\rho(\xi, \eta) = -\xi^R + \eta^L, \quad \xi, \eta \in \mathfrak{g},$$

where $\xi^L, \xi^R \in \mathfrak{X}(G)$ denotes the left/right-invariant vector field on G which is equal to $\xi \in \mathfrak{g}$ at the identity element.

The stabilizer at the identity element is the diagonal subalgebra, $\mathfrak{g}_\Delta \subseteq \overline{\mathfrak{g}} \oplus \mathfrak{g}$ which is Lagrangian. Now ρ is equivariant with respect to the G-action on $(\overline{\mathfrak{g}} \oplus \mathfrak{g}) \times G$ given by

$$g' : (\xi, \eta; g) \to (\mathbf{Ad}_{g'}\xi, \eta, g' \cdot g), \quad g' \in G, (\xi, \eta; g) \in (\overline{\mathfrak{g}} \oplus \mathfrak{g}) \times G,$$

and the left action of G on TG. Since this action is transitive on the base of the vector bundles and G preserves the metric on $\mathfrak{g}$, it follows that all stabilizers are Lagrangian. Thus $(\overline{\mathfrak{g}} \oplus \mathfrak{g}) \times G$ is an action Courant algebroid, called the *Cartan Courant algebroid*.

The diagonal subalgebra $\mathfrak{g}_\Delta \subseteq (\overline{\mathfrak{g}} \oplus \mathfrak{g})$ defines a Dirac structure

$$\mathfrak{g}_\Delta \times G \subseteq (\overline{\mathfrak{g}} \oplus \mathfrak{g}) \times G$$

called the *Cartan-Dirac structure*.

Remark 11 The Cartan Courant algebroid was first introduced in [41], and later simplified to the above description in [1]. The Cartan-Dirac structure was discovered independently by Alekseev, Ševera and Strobl [3, 26, 41]. The description given above was found in [1]

The Dirac structure of central focus in this paper is the following generalization of the Cartan-Dirac structure.

Example 21 (Γ-twisted Cartan-Dirac structure) Suppose Γ is a permutation graph, with edge set E_Γ, vertex set V_Γ, and (bijective) incidence maps

$$\mathsf{in}, \mathsf{out} : E_\Gamma \to V_\Gamma.$$

The diagonal subalgebra $\mathfrak{g}_{\Delta}^{V_\Gamma} \subseteq (\bar{\mathfrak{g}} \oplus \mathfrak{g})^{V_\Gamma}$ is Lagrangian, and hence so is its image

$$\mathfrak{g}_\Gamma := (\mathsf{in} \oplus \mathsf{out})^!\big(\mathfrak{g}_\Delta^{V_\Gamma}\big) \subseteq (\bar{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}$$

under the isomorphism $(\mathsf{in} \oplus \mathsf{out})^! : (\bar{\mathfrak{g}} \oplus \mathfrak{g})^{V_\Gamma} \to (\bar{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}$. Thus

$$\mathfrak{g}_\Gamma \times G^{E_\Gamma} \subseteq \big((\bar{\mathfrak{g}} \oplus \mathfrak{g}) \times G\big)^{E_\Gamma}$$

is a Dirac structure, called the Γ-twisted Cartan-Dirac structure.

The following picture can be helpful. We associate a copy of the Courant algebroid $(\bar{\mathfrak{g}} \oplus \mathfrak{g}) \times G$ to each edge, as in Fig. 21a. The Dirac structure $\mathfrak{g}_\Gamma$ acts diagonally at each vertex, as pictured in Fig. 21b.

The Γ-twisted Cartan Courant algebroid. Here $\mathsf{in}(e) = v = \mathsf{out}(v')$, where $v \in V_\Gamma$ is the vertex and $e, e' \in E_\Gamma$ are edges.

The element $\xi^{(\cdot)} \in \mathfrak{g}_\Delta^{V_\Gamma} \cong \mathfrak{g}_\Gamma$ of the Γ-twisted Cartan Dirac structure acts diagonally by ξ^v at the vertex v.

Remark 12 Γ-twisted Cartan-Dirac structures were discovered independently by Alejandro Cabrera, who provides their construction in terms of Dirac reduction of the Lie-Poisson structure on the dual of the loop Lie algebra, $\Omega^1(S^1, \mathfrak{g})$.

Remark 13 The following was explained to the authors by Eckhard Meinrenken. Suppose $\sigma_\Gamma : V_\Gamma \to V_\Gamma$ is the permutation induced by Γ (by the discrete flow along the edges of Γ), and consider the group $G_{\mathsf{big}} := G^{V_\Gamma} \rtimes \mathbb{Z}$,

$$(g, i) \cdot (g', i') = (g(\sigma_\Gamma^i)^!(g'), i + i'),$$

where $\big((\sigma_\Gamma^i)^!(g')\big)_v = (g')_{\sigma_\Gamma^i(v)}$ for any $v \in V_\Gamma$. Consider the embedding of manifolds

$$\big((\mathsf{in}^{-1})^!, -1\big) : G^{E_\Gamma} \to G^{V_\Gamma} \rtimes \mathbb{Z} = G_{\mathsf{big}}.$$

The Lie algebra of G_{big} is $\mathfrak{g}^{V_\Gamma}$, and for every $\xi \in \mathfrak{g}^{V_\Gamma}$, the left and right invariant vector fields $\xi^L, \xi^R \in \mathfrak{X}(G_{\mathsf{big}})$ restrict to the $e \in E_\Gamma$'th factor of $\big((\mathsf{in}^{-1})^!, -1\big)(G^{E_\Gamma})$ as $\xi^R_{\mathsf{in}(e)}$ and $\xi^L_{\mathsf{out}(e)}$, (respectively). Thus, the Γ-twisted Cartan-Dirac structure on G^{E_Γ} may be canonically identified with the restriction of the Cartan-Dirac structure on G_{big} to $\big((\mathsf{in}^{-1})^!, -1\big)(G^{E_\Gamma})$.

2.2 Courant Relations and Morphisms of Manin Pairs

2.2.1 Relations

A smooth relation $S\colon M_1 \dashrightarrow M_2$ between manifolds is an immersed submanifold $S \subseteq M_2 \times M_1$. We will write $m_1 \sim_S m_2$ if $(m_2, m_1) \in S$. Given smooth relations $S\colon M_1 \dashrightarrow M_2$ and $S'\colon M_2 \dashrightarrow M_3$, the set-theoretic composition $S' \circ S$ is the image of

$$S' \diamond S = (S' \times S) \cap (M_3 \times (M_2)_\Delta \times M_1) \tag{15}$$

under projection to $M_3 \times M_1$, where $(M_2)_\Delta \subseteq M_2 \times M_2$ denotes the diagonal.

We say that the two relations *compose cleanly* if (15) is a clean intersection in the sense of Bott (i.e. it is smooth, and the intersection of the tangent bundles is the tangent bundle of the intersection), and the map from $S' \diamond S$ to $M_2 \times M_1$ has constant rank. In this case, the composition $S' \circ S\colon M_1 \dashrightarrow M_3$ is a well-defined smooth relation. See [28, AppendixA] for more information on the composition of smooth relations. For background on clean intersections of manifolds, see e.g. [22, p. 490].

For any relation $S : M_1 \dashrightarrow M_2$, we let $S^\top : M_2 \dashrightarrow M_1$ denote the *transpose* relation,

$$S^\top = \{(m_1, m_2) \in M_1 \times M_2 \mid (m_2, m_1) \in S\}.$$

2.2.2 Courant Relations

As popularized by the second author [37, 39], Dirac structures can be interpreted as the ‘canonical relations’ between Courant algebroids:

Definition 4 (*Courant relations and morphisms* [5, 14, 39]) Suppose $\mathbb{E}_1 \to M_1$ and $\mathbb{E}_2 \to M_2$ are two Courant algebroids. A relation

$$R : \mathbb{E}_1 \dashrightarrow \mathbb{E}_2$$

is called a *Courant relation* if $R \subseteq \mathbb{E}_2 \times \overline{\mathbb{E}_1}$ is a Dirac structure supported on a submanifold $S \subseteq M_2 \times M_1$. A Courant relation is called *exact* if the underlying Dirac structure is exact.

When $S = \mathsf{gr}(\mu)$ is the graph of a smooth map $\mu : M_1 \to M_2$, R is called a *Courant morphism*.

We define the range $\mathsf{ran}(R) \subseteq \mathbb{E}_2|_S$ and the kernel $\mathsf{ker}(R) \subseteq \mathbb{E}_2|_S$ of R by

$$\mathsf{ran}(R) := \{e \in \mathbb{E}_2|_S \mid e' \sim_R e \text{ for some } e' \in \mathbb{E}_1\}$$
$$\mathsf{ker}(R) := \{e \in \mathbb{E}_1|_S \mid e \sim_R 0\}.$$

As an example, any Dirac structure $E \subseteq \mathbb{E}$ defines a Courant morphism

$$E : \mathbb{E} \dashrightarrow *$$

to the trivial Courant algebroid (or a Courant relation from the trivial Courant algebroid). Similarly, the diagonal $\mathbb{E}_\Delta \subseteq \mathbb{E} \times \overline{\mathbb{E}}$ defines the Courant morphism

$$\mathbb{E}_\Delta : \mathbb{E} \dashrightarrow \mathbb{E}$$

corresponding to the identity map.

The key property of Courant relations is the ability to compose them:

Proposition 1 ([28, Proposition 1.4]) *Suppose $R : \mathbb{E}_1 \dashrightarrow \mathbb{E}_2$ and $R' : \mathbb{E}_2 \dashrightarrow \mathbb{E}_3$ are two Courant relations which compose cleanly, then their composition,*

$$R' \circ R : \mathbb{E}_1 \dashrightarrow \mathbb{E}_3,$$

is a Courant relation.

Example 22 (Standard lift) Suppose $S : M_1 \to M_2$ is a relation, then the *standard lift* of S,

$$R_S := TS \oplus \mathsf{ann}(TS) \subseteq \mathbb{T}M_2 \times \overline{\mathbb{T}M_1},$$

defines a Courant relation

$$R_S : \mathbb{T}M_1 \dashrightarrow \mathbb{T}M_2.$$

Example 23 (Coisotropic subalgebras) Suppose $\mathfrak{d}$ is a Lie algebra equipped with an invariant metric. A subalgebra $\mathfrak{c} \subseteq \mathfrak{d}$ is said to be *coisotropic* if $\mathfrak{c}^\perp \subseteq \mathfrak{c}$. In this case, $\mathfrak{c}^\perp \subseteq \mathfrak{c}$ is an ideal, and the metric on $\mathfrak{d}$ descends to define a metric on

$$\mathfrak{d}_\mathfrak{c} := \mathfrak{c}/\mathfrak{c}^\perp.$$

The natural relation

$$R_\mathfrak{c} : \mathfrak{d} \dashrightarrow \mathfrak{d}_\mathfrak{c}, \quad \xi \sim_{R_\mathfrak{c}} \xi + \mathfrak{c}^\perp \quad \text{for } \xi \in \mathfrak{c},$$

is a Courant relation, where $(\xi + \mathfrak{c}^\perp) \in \mathfrak{c}/\mathfrak{c}^\perp$ denotes the equivalence class of $\xi \in \mathfrak{c}$.

For any Lagrangian subalgebra $\mathfrak{h} \subseteq \mathfrak{d}$, Proposition 1 implies that

$$\mathfrak{h}_\mathfrak{c} := R_\mathfrak{c} \circ \mathfrak{h} = (\mathfrak{h} \cap \mathfrak{c})/(\mathfrak{h} \cap \mathfrak{c}^\perp)$$

is a Lagrangian subalgebra of $\mathfrak{d}_\mathfrak{c}$.

2.2.3 Morphisms of Manin Pairs

A pair $(\mathbb{E}, E)$ consisting of a Courant algebroid, $\mathbb{E}$, together with a Dirac structure $E \subseteq \mathbb{E}$ is known as a *Manin pair* [2, 14].

Definition 5 (*Bursztyn et al.* [14]) Suppose $\mathbb{E}_1 \to M_1$ and $\mathbb{E}_2 \to M_2$ are two Courant algebroids. A Courant morphism

$$R : \mathbb{E}_1 \dashrightarrow \mathbb{E}_2,$$

supported on the graph of a map $\mu : M_1 \to M_2$, defines a *morphism of Manin pairs*,

$$R : (\mathbb{E}_1, E_1) \dashrightarrow (\mathbb{E}_2, E_2) \tag{16}$$

if

(m1) $R \circ E_1 \subseteq E_2$, and
(m2) $\mathsf{ker}(R) \cap E_1 = 0$

Here $\mathsf{ker}(R) := (0 \times \overline{\mathbb{E}_1}) \cap R$.

The morphism of Manin pairs, (16), is said to be *exact* if the underlying Dirac structure is exact.

Suppose

$$R' : (\mathbb{E}_2, E_2) \dashrightarrow (\mathbb{E}_3, E_3) \tag{17}$$

is a second morphism of Manin pairs. Conditions (m1) and (m2) imply that the composition of relations $R' \circ R$ is clean. Moreover, the composition defines a morphism of Manin pairs

$$R' \circ R : (\mathbb{E}_1, E_1) \dashrightarrow (\mathbb{E}_3, E_3), \tag{18}$$

(cf. [14]).

Remark 14 In [29], a morphism of Manin pairs, (16) was said to be *full* if the map

$$\mathbf{a}|_R : R \to T\mathsf{gr}(\mu)$$

was a surjection. The concept of exact morphisms of Manin pairs is a stronger, but more natural, condition.

If Eq. (16) is a morphism of Manin pairs, then there exists map $\rho_R : \mu^* E_2 \to E_1$ uniquely determined by the condition

$$\rho_R(e) \sim_R e, \quad e \in E_2. \tag{19}$$

The induced map of section $\rho_R : (E_2) \to (E_1)$ is a morphism of Lie algebras. Thus Eq. (16) defines an action of E_2 on M_1 which factors through the action of E_1.

2.3 Quasi-Hamiltonian Manifolds

If $(\mathbb{E}, E)$ is a Manin pair, a *quasi-Hamiltonian* $(\mathbb{E}, E)$*-manifold* in the sense of [14] is a manifold M together with a morphism of Manin pairs

$$(\mathbb{T}M, TM) \dashrightarrow (\mathbb{E}, E).$$

We shall say that the quasi-Hamiltonian space is *exact* (or *quasi-symplectic*) if the morphism of Manin pairs is exact. To simplify notation (when M is a complicated expression), we will often denote the Manin pair on the left as $(\mathbb{T}M, TM) = (\mathbb{T}, T)M$.

Example 24 (Poisson and symplectic structures) Let 0 denote the trivial Courant algebroid over a point. Consider a morphism of Manin pairs

$$R : (\mathbb{T}, T)M \dashrightarrow (0, 0). \tag{20}$$

In this case, $R \subseteq \mathbb{T}M$ is just a Dirac structure with support on all of M. Condition (m1) is vacuous, while condition (m2) is equivalent to $R \cap TM = 0$. As explained in [15, 16], it follows that

$$R = \mathsf{gr}(\pi^\sharp) := \{(\pi(\alpha, \cdot) + \alpha) \mid \alpha \in \Omega^1(M)\} \subseteq \mathbb{T}M$$

is the graph of a Poisson bivector field $\pi \in \mathfrak{X}^2(M)$. In this way, there is a one-to-one correspondence between morphisms of Manin pairs of the form (20) and Poisson structures on M [14].

Equation (20) is an exact morphism of Manin pairs if $\mathbf{a}|_R : \mathsf{gr}(\pi^\sharp) \to TM$ is a surjection. Equivalently,

$$\pi^\sharp : T^*M \to TM$$

is an isomorphism, or the Poisson structure on M is symplectic. In this way, there is a one-to-one correspondence between exact morphisms of Manin pairs of the form (20) and symplectic structures on M [29].

The Poisson and symplectic structures that appear in this paper will all arise in this way.

Example 25 (E-invariant submanifolds) Let $(\mathbb{E}, E)$ be a Manin pair over a manifold N, and suppose $M \subseteq N$ is an E-invariant submanifold, i.e. $\mathbf{a}(E|_M) \subseteq TM$. Then

$$\mathbf{a}^*|_M : T^*N|_M \to \mathbb{E}$$

descends to a map

$$\mathbf{a}^*|_M : T^*M \cong T^*N/\mathsf{ann}(TM) \to \mathbb{E}.$$

As in [28, Example 1.6], define the Courant relation $R_{E,M} : \mathbb{T}M \dashrightarrow \mathbb{E}$ by

$$\mathbf{a}(e) + i^*\alpha \sim_{R_{E,M}} e + \mathbf{a}^*\alpha, \quad e \in E, \quad \alpha \in T^*N,$$

where $i : M \to N$ denotes the inclusion. Then

$$R_{E,M} : (\mathbb{T}, T)M \dashrightarrow (\mathbb{E}, E) \tag{21}$$

is a morphism of Manin pairs.

Moreover, Eq. (21) is an *exact* morphism of Manin pairs if and only if $\mathbf{a}(E|_M) = TM$ and the Courant algebroid $\mathbb{E} \to N$ is exact along M.

Remark 15 In fact, Eq. (21) is the unique morphism of Manin pairs supported on $\mathsf{gr}(i)$. To see why, suppose

$$R : (\mathbb{T}, T)M \dashrightarrow (\mathbb{E}, E)$$

is such a morphism of Manin pairs. Since R is supported on $\mathsf{gr}(i)$, we must have

$$i^*\alpha \sim_R \mathbf{a}^*\alpha,$$

for any $\alpha \in T^*N$. On the other hand, as explained in [14, Proposition 3.3], for any $e \in E|_M$ there exists a unique $X \in TM$ such that

$$X \sim_R e.$$

Since R is supported on $\mathsf{gr}(i)$, we have $i_*X = \mathbf{a}(e)$, or $X = \mathbf{a}(e)$. Thus $R = R_{E,M}$.

3 Quasi-Hamiltonian Reduction

3.1 Reduction Theorems

Let $\mathfrak{d}$ be a quadratic Lie algebra acting on a manifold N so that all the stabilizers are coisotropic, and let $\mathfrak{h} \subset \mathfrak{d}$ be a Lagrangian Lie subalgebra. We shall consider the following special case of general quasi-Hamiltonian $(\mathbb{E}, E)$-manifolds.

Definition 6 A *quasi-Hamiltonian* $(\mathfrak{d}, \mathfrak{h}) \times N$*-manifold* is a manifold M together with a morphism of Manin pairs

$$(\mathbb{T}, T)M \dashrightarrow (\mathfrak{d}, \mathfrak{h}) \times N.$$

It is *exact* (or *quasi-symplectic*) if the morphism is exact.

Example 26 If $\mathfrak{d} = 0$ and N is a point then (as we saw in Example 24) a quasi-Hamiltonian structure is the same as a Poisson or (in the exact case) symplectic structure. More generally, a quasi-Hamiltonian $(0, 0) \times N$-structure on M is equivalent to a Poisson structure on M and to a map $\mu : M \to N$ such that $\mu^*(C^\infty(N)) \subset C^\infty(M)$ is in the Poisson centre.

If $\mathfrak{g}$ is a quadratic Lie algebra then an *exact* quasi-Hamiltonian $(\mathfrak{g} \oplus \bar{\mathfrak{g}}, \mathfrak{g}_\Delta) \times G$-structure on M is equivalent to a quasi-Hamiltonian G-structure in the sense of Alekseev, Malkin and Meinrenken. If $\mathfrak{h}$ is a Lie algebra then a quasi-Hamiltonian $(\mathfrak{h} \ltimes \mathfrak{h}^*) \times \mathfrak{h}^*$-structure on M is equivalent to a Poisson structure on M together with a moment map $M \to \mathfrak{h}^*$ generating an action of $\mathfrak{h}$. In the exact case the Poisson structure is symplectic.

In this subsection we present a reduction procedure, which will be the main tool used in our study of the moduli spaces of flat connections.

Definition 7 Let M be a quasi-Hamiltonian $(\mathfrak{d}, \mathfrak{h}) \times N$-manifold. *Reductive data* $(\mathfrak{c}, S)$ consists of a coisotropic Lie subalgebra $\mathfrak{c} \subseteq \mathfrak{d}$ together with a $\mathfrak{c}$-invariant submanifold $S \subseteq N$ such that

(r1) the $\mathfrak{c}^\perp$-orbits in S form a regular foliation[7] with quotient $q_N : S \to N_{\mathfrak{c},S}$,
(r2) the graph $\mathsf{gr}(\mu)$, where μ is the underlying map $M \to N$, intersects $S \times M$ cleanly, and the $\mathfrak{h} \cap \mathfrak{c}^\perp$-orbits in $\mu^{-1}(S)$ form a regular foliation with quotient $q_M : \mu^{-1}(S) \to N_{\mathfrak{c},S}$.

Theorem 5 (Quasi-Hamiltonian reduction) *Suppose* $(\mathfrak{c}, S)$ *is reductive data for a morphism of Manin pairs*

$$R : (\mathbb{T}, T)M \dashrightarrow (\mathfrak{d}, \mathfrak{h}) \times N. \tag{22}$$

Then

$$R_{\mathfrak{c},S} : (\mathbb{T}, T)M_{\mathfrak{c},S} \dashrightarrow (\mathfrak{d}_\mathfrak{c}, \mathfrak{h}_\mathfrak{c}) \times N_{\mathfrak{c},S} \tag{23}$$

is a morphism of Manin pairs, where

$$M_{\mathfrak{c},S} = \mu^{-1}(S)/\mathfrak{h} \cap \mathfrak{c}^\perp$$
$$R_{\mathfrak{c},S} := R_2 \circ R \circ R_1^\top,$$

$$R_2 := R_\mathfrak{c} \times (\mathsf{gr}(q_N) \circ \mathsf{gr}(i_N)^\top),$$
$$R_1 := R_{q_M} \circ R_{i_M}^\top,$$

$i_N : S \to N$ *and* $i_M : \mu^{-1}(S) \to M$ *are the canonical inclusions, and* $R_\mathfrak{c} : \mathfrak{d} \dashrightarrow \mathfrak{d}_\mathfrak{c}$ *and* $\mathfrak{h}_\mathfrak{c}$ *are as in Example 23.*

[7] By a *regular foliation*, we mean that the leaf space carries the structure of a smooth manifold for which the quotient map is a surjective submersion.

The first two statements in Theorem 1 follow as consequences of Theorem 5.

Theorem 6 (Quasi-Hamiltonian reduction in the exact case) *If, in the setup of Theorem 5, the following additional assumptions hold:*

- $\mathfrak{c}$ *acts transitively on* S*,*
- *Eq.* (22) *is an exact morphism of Manin pairs,*

then Eq. (23) *is an exact morphism of Manin pairs.*

The third statement in Theorem 1 follows as a consequence of Theorem 6.

We delay the proof of both these theorems to Appendix A.

Notice that if $\mathfrak{c} \subset \mathfrak{d}$ is Lagrangian then the reduced manifold is Poisson or (in the exact case) symplectic, as $\mathfrak{d}_\mathfrak{c} = 0$.

3.2 Bivector Fields and Quasi-Poisson Structures

In this section we shall explain Theorem 5 in more traditional terms, using bivector fields.

Suppose that

$$R : (\mathbb{T}, T)M \dashrightarrow (\mathfrak{d}, \mathfrak{h}) \times N$$

is a morphism of Manin pairs over $\mu : M \to N$, and the subspace $\mathfrak{k} \subseteq \mathfrak{d}$ is a Lagrangian complement to $\mathfrak{h} \subseteq \mathfrak{d}$: that is, $\mathfrak{d} = \mathfrak{h} \oplus \mathfrak{k}$. Then axiom (m1) of Definition 5 implies that R composes transversely with $\mathfrak{k}$, while property (m2) implies that $\mathfrak{k} \circ R \subseteq \mathbb{T}M$ is a Lagrangian complement to TM. Thus there exists a unique bivector field $\pi_\mathfrak{k} \in \mathfrak{X}^2(M)$ such that

$$\mathfrak{k} \circ R = \mathsf{gr}(\pi^\sharp) := \{\big(\pi_\mathfrak{k}(\alpha, \cdot) + \alpha\big) \mid \alpha \in T^*M\}.$$

The triple $(\mathfrak{d}, \mathfrak{h}; \mathfrak{k})$ is called a quasi-Manin triple, and $(M, \pi_\mathfrak{k}, \rho_R)$ is called a Hamiltonian quasi-Poisson $(\mathfrak{d}, \mathfrak{h}; \mathfrak{k})$-space with moment map $\mu : M \to N$ (cf. [14, 23]). The bivector field $\pi_\mathfrak{k}$ is called a quasi-Poisson structure.

Suppose that $(\mathfrak{c}, S)$ is reductive data for the morphism of Manin pairs R. We want to reinterpret Theorem 5 using the language of quasi-Poisson geometry. Thus we will be interested in Lagrangian complements to the reduced Lie algebra $\mathfrak{h}_\mathfrak{c}$.

Lemma 2 *Each Lagrangian complement* $\mathfrak{k}'_\mathfrak{c} \subseteq \mathfrak{d}_\mathfrak{c}$ *to* $\mathfrak{h}_\mathfrak{c}$ *defines an element*

$$\tau \in \wedge^2\big(\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{c}^\perp)\big).$$

Proof Suppose $\mathfrak{k}'_\mathfrak{c} \subseteq \mathfrak{d}_\mathfrak{c} := \mathfrak{c}/\mathfrak{c}^\perp$ is a Lagrangian complement to $\mathfrak{h}_\mathfrak{c}$. Let $R_\mathfrak{c} : \mathfrak{d} \dashrightarrow \mathfrak{d}_\mathfrak{c}$ be the relation described in Example 23, and define $\mathfrak{k}' := \mathfrak{k}'_\mathfrak{c} \circ R_\mathfrak{c} \subseteq \mathfrak{c}$. Now $\mathfrak{k}'$ can be seen as the graph

$$\mathfrak{k}' = \{(\xi + \tau^\sharp(\xi) + \eta) \mid \xi \in \mathfrak{k},\ \langle \xi, \cdot \rangle|_{\mathfrak{h}\cap\mathfrak{k}'} = 0,\ \text{and}\ \eta \in \mathfrak{h} \cap \mathfrak{k}'\}$$

of a map

$$\tau^\sharp : \mathsf{ann}(\mathfrak{h} \cap \mathfrak{k}') \cong \big(\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{k}')\big)^* \to \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{k}'),$$

where $\mathsf{ann}(\mathfrak{h} \cap \mathfrak{k}') := (\mathfrak{h} \cap \mathfrak{k}')^\perp \cap \mathfrak{k}$. Let $\tau \in \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{k}') \otimes \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{k}')$ be the element defined by $\tau^\sharp(\xi) = \tau(\xi, \cdot)$, then the fact that $\mathfrak{k}'$ is Lagrangian forces τ to be skew-symmetric. Finally, since $\mathfrak{k}'_\mathfrak{c} + \mathfrak{h}_\mathfrak{c} = \mathfrak{c}/\mathfrak{c}^\perp$ we have $\mathfrak{h}+\mathfrak{k}' = \mathfrak{h}+\mathfrak{c}$. Hence $\mathfrak{h}\cap\mathfrak{k}' = \mathfrak{h}\cap\mathfrak{c}^\perp$. Thus $\tau \in \wedge^2\big(\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{c}^\perp)\big)$.

Theorem 7 *Suppose that* $(\mathfrak{c}, S)$ *is reductive data for the morphism of Manin pairs*

$$R : (\mathbb{T}, T)M \dashrightarrow (\mathfrak{d}, \mathfrak{h}) \times N, \tag{24}$$

and $(M, \pi_\mathfrak{k}, \rho_R)$ *is the Hamiltonian quasi-Poisson* $(\mathfrak{d}, \mathfrak{h}; \mathfrak{k})$*-space corresponding to the Lagrangian complement* $\mathfrak{k} \subseteq \mathfrak{d}$ *to* $\mathfrak{h}$. *Let*

$$R_{\mathfrak{c},S} : (\mathbb{T}, T)M_{\mathfrak{c},S} \dashrightarrow (\mathfrak{d}_\mathfrak{c}, \mathfrak{h}_\mathfrak{c}) \times N_{\mathfrak{c},S}$$

denote the reduced morphism of Manin pairs described in Theorem 5, and let $(M_{\mathfrak{c},S}, \pi_{\mathfrak{k}'_\mathfrak{c}}, \rho_{R_{\mathfrak{c},S}})$ *be the Hamiltonian* $(\mathfrak{d}_\mathfrak{c}, \mathfrak{h}_\mathfrak{c}; \mathfrak{k}'_\mathfrak{c})$*-quasi-Poisson manifold corresponding to a chosen Lagrangian complement* $\mathfrak{k}'_\mathfrak{c}$ *to* $\mathfrak{h}_\mathfrak{c}$.

As in Lemma 2, let $\tau \in \wedge^2\big(\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{c}^\perp)\big)$ *be the element corresponding to* $\mathfrak{k}'_\mathfrak{c}$. *Then*

$$(\pi_\mathfrak{k} + \rho_R(\tau))|_{\mu^{-1}(S)}$$

is an $\mathfrak{h} \cap \mathfrak{c}^\perp$*-invariant section of* $\wedge^2\big(T(\mu^{-1}(S))/\rho_R(\mathfrak{h} \cap \mathfrak{c}^\perp)\big)$ *which is mapped to* $\pi_{\mathfrak{k}'_\mathfrak{c}}$ *under the surjective submersion* $\mu^{-1}(S) \to M_{\mathfrak{c},S}$.

Proof Let $R_\mathfrak{c} : \mathfrak{d} \dashrightarrow \mathfrak{d}_\mathfrak{c}$ be the relation described in Example 23. Let $\mu : M \to N$ denote the map supporting R, and let

$$R_{i_M} : \mathbb{T}\mu^{-1}(S) \dashrightarrow \mathbb{T}M, \quad R_{q_M} : \mathbb{T}\mu^{-1}(S) \dashrightarrow \mathbb{T}M_{\mathfrak{c},S}$$

denote the standard lifts of the inclusion and projection, respectively. Then

$$\mathsf{gr}(\pi^\sharp_{\mathfrak{k}'_\mathfrak{c}}) = \mathfrak{k}'_\mathfrak{c} \circ R_\mathfrak{c} \circ R \circ R_{i_M} \circ R^\top_{q_M}.$$

Now, as explained in Lemma 2,

$$\mathfrak{k}'_\mathfrak{c} \circ R_\mathfrak{c} = \mathfrak{k}' := \{(\xi + \tau^\sharp(\xi) + \eta) \mid \xi \in \mathfrak{k},\ \langle \xi, \cdot \rangle|_{\mathfrak{h}\cap\mathfrak{c}^\perp} = 0,\ \text{and}\ \eta \in \mathfrak{h} \cap \mathfrak{c}^\perp\}. \tag{25a}$$

Meanwhile, since $\mathsf{gr}(\pi^\sharp_\mathfrak{k}) = \mathfrak{k} \circ R$, it follows that

$$X + \alpha \sim_R \xi + \eta, \quad \xi \in \mathfrak{k}, \eta \in \mathfrak{h} \Leftrightarrow X = \rho_R(\eta) + \pi_{\mathfrak{k}}^{\sharp}\alpha, \text{ and } \xi = j \circ \rho_R^* \alpha, \quad (25b)$$

where $j : \mathfrak{h}^* \to \mathfrak{k}$ inverts the isomorphism $\mathfrak{k} \to \mathfrak{d}/\mathfrak{h} \cong \mathfrak{h}^*$.

Using (25a) and (25b) we compute

$$\mathfrak{k}'_{\mathfrak{c}} \circ R_{\mathfrak{c}} \circ R = \left\{ \left(\rho_R(\eta) + (\pi_{\mathfrak{k}} + \rho(\tau))^{\sharp}(\alpha) + \alpha \right) | \eta \in \mathfrak{h} \cap \mathfrak{c}^{\perp} \text{ and } \alpha \in \mathsf{ann}\left(\rho_R(\mathfrak{h} \cap \mathfrak{c}^{\perp})\right) \right\}.$$

This shows that $\pi_{\mathfrak{k}} + \rho(\tau)$ is $\mathfrak{h} \cap \mathfrak{c}^{\perp}$-invariant. Moreover, $\mathfrak{k}'_{\mathfrak{c}} \circ R_{\mathfrak{c}} \subseteq \mathfrak{c}$ and thus elements of $\mathfrak{k}'_{\mathfrak{c}} \circ R_{\mathfrak{c}}$ act to preserve S. In turn, this implies that

$$(\pi_{\mathfrak{k}} + \rho(\tau))|_{\mu^{-1}(S)} \in \Gamma\left(\wedge^2 \left(TM|_{\mu^{-1}(S)} / \rho_R(\mathfrak{h} \cap \mathfrak{c}^{\perp}) \right) \right)$$

is in fact a section of $\wedge^2(T\mu^{-1}(S)/\rho_R(\mathfrak{h} \cap \mathfrak{c}^{\perp}))$.

Now, $\mathsf{gr}(\pi^{\sharp}_{\mathfrak{k}'_{\mathfrak{c}}}) = \mathfrak{k}'_{\mathfrak{c}} \circ R_{\mathfrak{c}} \circ R \circ R_{i_M} \circ R^{\top}_{q_M}$, i.e.

$$(q_M)_* \left(\pi_{\mathfrak{k}} + \rho(\tau) \right)|_{\mu^{-1}(S)} = \pi_{\mathfrak{k}'_{\mathfrak{c}}}.$$

3.3 Exact Morphisms of Manin Pairs and 2-Forms

In this section, we will examine exact morphisms of Manin pairs in more detail. We recall from [14, 29] that once isotropic splittings are chosen, these are uniquely determined by a map between the underlying spaces, and a 2-form on the domain. This description in terms of 2-forms can be useful for simplifying calculations.

Suppose $\mathbb{E}$ is an exact Courant algebroid over N. That is, the sequence

$$0 \to T^*N \xrightarrow{\mathbf{a}^*} \mathbb{E} \xrightarrow{\mathbf{a}} TN \to 0 \quad (26)$$

is exact. Let $s : TN \to \mathbb{E}$ be a splitting of 26 such that $s(TN) \subseteq \mathbb{E}$ is isotropic (such splittings are called *isotropic splittings*). Then, as explained in [41], the formula

$$\iota_X \iota_Y \iota_Z \gamma := \langle [\![s(X), s(Y)]\!], s(Z) \rangle, \quad X, Y, Z \in \mathfrak{X}(N)$$

defines a closed 3-form, $\gamma \in \Omega^3(N)$, called the *curvature 3-form of the splitting s*. The isomorphism $s \oplus \mathbf{a}^* : TN \oplus T^*N \xrightarrow{\cong} \mathbb{E}$ identifies the metric on $\mathbb{E}$ with

$$\langle X + \alpha, Y + \beta \rangle = \alpha(Y) + \beta(X), \quad X, Y \in TN, \quad \alpha, \beta \in T^*N,$$

and the bracket with

$$[\![X + \alpha, Y + \beta]\!] = [X, Y] + \mathcal{L}_X \beta - \iota_Y d\alpha + \iota_X \iota_Y \gamma, \quad X, Y \in \mathfrak{X}(N), \quad \alpha, \beta. \quad (27)$$

The Courant algebroid with underlying bundle $TN \oplus T^*N$, and bracket given by 27 is called the *γ-twisted exact Courant algebroid* over N, and denoted by $\mathbb{T}_\gamma N$.

Example 27 (The Cartan Courant algebroid [41]*)* The Cartan Courant algebroid

$$(\overline{\mathfrak{g}} \oplus \mathfrak{g}) \times G$$

reviewed in Example 20 is exact. In [1] it is shown that the map $s : TG \to (\overline{\mathfrak{g}} \oplus \mathfrak{g}) \times G$ defined as

$$s : X \to \frac{1}{2}\big(- \iota_X(\mathrm{d}g\, g^{-1}), \iota_X(g^{-1}\mathrm{d}g)\big), \quad X \in TG$$

is a $(\overline{\mathfrak{g}} \oplus \mathfrak{g})$-invariant isotropic splitting of the Cartan Courant algebroid. The corresponding curvature 3-form is computed as $\gamma = \frac{1}{24}\langle [g^{-1}\mathrm{d}g, g^{-1}\mathrm{d}g], g^{-1}\mathrm{d}g\rangle$ (note that the normalization differs from that in [1]).

Suppose $L \subseteq \mathbb{E}$ is a Lagrangian subbundle with support on $S \subseteq N$, such that $\mathbf{a}(L) = TS$, then there is a 2-form $\omega \in \Omega^2(S)$ uniquely determined by the formula

$$\iota_X \iota_Y \omega = \langle s(X), e\rangle, \quad X, Y \in TS, \quad e \in L \text{ and } \mathbf{a}(e) = Y.$$

Thus, we may identify L with

$$L = \mathsf{gr}(\omega^\flat) := \{\big(s(X) + \mathbf{a}^*(\iota_X\omega + \alpha)\big) \mid X \in TS, \alpha \in \mathsf{ann}(TS)\} \subseteq \mathbb{E}.$$

A quick calculation using (27) shows that L is a Dirac structure with support on S if and only if $\mathrm{d}\omega = i^*\gamma$, where $i : S \to N$ is the inclusion (cf. [41] and [14, Proposition 2.8]).

Suppose

$$R : \mathbb{T}M \dashrightarrow \mathbb{E}$$

is a Courant morphism supported on the graph of a map $\mu : M \to N$. Since $\mathbb{E}$ is an exact Courant algebroid, R is *exact* if $\mathbf{a}(R) = T\mathsf{gr}(\mu)$ (cf. Lemma 1), in which case the considerations above show that

$$R = \mathsf{gr}(\omega^\flat) \subseteq \mathbb{E} \times \overline{\mathbb{T}M},$$

where $\omega \in \Omega^2(M) \cong \Omega^2(\mathsf{gr}(\mu))$ satisfies $\mathrm{d}\omega = \mu^*\gamma$ [14, 29]. That is, $R = R_{\mu,\omega}$, where $R_{\mu,\omega}$ is defined by

$$X - \iota_X\omega + \mu^*\alpha \sim_{R_{\mu,\omega}} s(\mu_*X) + \mathbf{a}^*\alpha, \quad X \in TM, \quad \alpha \in T^*N. \tag{28}$$

In particular, if $E \subseteq \mathbb{E}$ is a Dirac structure, once an isotropic splitting $s : TN \to \mathbb{E}$ is chosen, a morphism of Manin pairs

$$R : (\mathbb{T}, T)M \dashrightarrow (\mathbb{E}, E)$$

is entirely determined by the underlying map $\mu : M \to N$ between the spaces, and a 2-form $\omega \in \Omega^2(M)$ (satisfying certain conditions).

Remark 16 (**Twisted quasi-Hamiltonian structures**) Recall the Γ-twisted Cartan-Dirac structure described in Example 21,

$$\big((\bar{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}, \mathfrak{g}_\Gamma\big) \times G^{E_\Gamma}.$$

Suppose we use the splitting $TG^{E_\Gamma} \to \big((\bar{\mathfrak{g}} \oplus \mathfrak{g}) \times G\big)^{E_\Gamma}$ described in Example 27 to identify

$$\big((\bar{\mathfrak{g}} \oplus \mathfrak{g}) \times G\big)^{E_\Gamma} \cong \mathbb{T}_\gamma G^{E_\Gamma},$$

where $\gamma = \frac{1}{24}\langle [g^{-1}\mathrm{d}g, g^{-1}\mathrm{d}g], g^{-1}\mathrm{d}g\rangle$. Then exact morphisms of Manin pairs

$$R : (\mathbb{T}, T)M \dashrightarrow \big((\bar{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}, \mathfrak{g}_\Gamma\big) \times G^{E_\Gamma}$$

are in one-to-one correspondence with quadruples (M, μ, ρ, ω), where

- $\mu : M \to G^{E_\Gamma}$ is a smooth map,
- $\rho : \mathfrak{g}^{V_\Gamma} \to \mathfrak{X}(M)$ is a Lie algebra action, and
- $\omega \in \Omega^2(M)$ is a 2-form,

such that

1. $\mathrm{d}\omega = \mu^*\gamma$,
2. $\mu : M \to G^{E_\Gamma}$ is $\mathfrak{g}^{V_\Gamma}$-equivariant with respect to the $\mathfrak{g}^{V_\Gamma}$ action on G^{E_Γ} given on the $e \in E_\Gamma$th factor by
$$\xi \to -\xi^R_{\mathsf{in}(e)} + \xi^L_{\mathsf{out}(e)}, \quad \xi \in \mathfrak{g}^{V_\Gamma},$$
3. $\mathsf{ker}(\mathrm{d}\mu)_x \cap \mathsf{ker}(\omega^\flat)_x = 0$, for every $x \in M$, and
4. $$\iota_{\rho(\xi)}\omega = \frac{1}{2}\mu^* \sum_{e \in E_\Gamma} \langle g_e^{-1}\mathrm{d}g_e, \xi_{\mathsf{out}(e)}\rangle + \langle \mathrm{d}g_e\, g_e^{-1}, \xi_{\mathsf{in}(e)}\rangle.$$

(Note that conditions (2),(3) and (4) determine ρ uniquely in terms of μ and ω). The quadruple (M, μ, ρ, ω) corresponds to the morphism of Manin pairs (28), supported on the graph of μ.

The quadruples (M, μ, ρ, ω) generalize quasi-Hamiltonian G^{E_Γ}-structures in the sense that they incorporate an automorphism of G^{E_Γ} into their definition. Here the automorphism is simply the permutation of factors described by the permutation graph, Γ. Allowing arbitrary automorphisms leads to a definition of twisted quasi-Hamiltonian spaces along the lines of the one given in [30] for twisted quasi-Poisson structures (cf. [30, Definition3]).

Let $G_{\mathsf{big}} := G^{V_\Gamma} \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts by permuting the factors according to the graph Γ (cf. Remark 13). As explained to the authors by Eckhard Meinrenken, quasi-Hamiltonian $\big((\bar{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}, \mathfrak{g}_\Gamma\big) \times G^{E_\Gamma}$-structures are quasi-Hamiltonian G_{big}-structures in the (original) sense of Alekseev-Malkin-Meinrenken [4] for which the moment map takes values in $G^{V_\Gamma} \times \{-1\} \subset G_{big}$ (cf. Remark 13).

We now examine the behaviour of twisted exact Courant algebroids under the partial reduction procedure described in Theorem 5. Suppose that $\mathfrak{d} \times N$ is an exact Courant algebroid, and a $\mathfrak{d}$-invariant isotropic splitting $s : TN \to \mathfrak{d} \times N$ is chosen, defining an isomorphism

$$\mathfrak{d} \times N \cong \mathbb{T}_\gamma N,$$

We let $E \subseteq \mathbb{T}_\gamma N$ denote the Dirac structure corresponding to $\mathfrak{h} \times N \subseteq \mathfrak{d} \times N$ under this isomorphism. Suppose

$$R_{\mu,\omega} : (\mathbb{T}, T)M \dashrightarrow (\mathbb{T}_\gamma N, E) \cong (\mathfrak{d}, \mathfrak{h}) \times N$$

is a exact morphism of Manin pairs, and $S \subseteq N$ is an orbit of the coisotropic subalgebra $\mathfrak{c} \subseteq \mathfrak{d}$, and the assumptions of both Theorems 5 and 6 hold for the reductive data $(\mathfrak{c}, S)$. Let $i_N : S \to N$ and $i_M : \mu^{-1}(S) \to M$ denote the inclusions, and $q_N : S \to N_{\mathfrak{c},S}$ and $q_M : \mu^{-1}(S) \to M_{\mathfrak{c},S}$ the quotients (by $\mathfrak{c}^\perp$ and $\mathfrak{h} \cap \mathfrak{c}^\perp$, respectively).

We would like to define a splitting of the reduced Courant algebroid $\mathfrak{d}_{\mathfrak{c}^\perp} \times N_{\mathfrak{c},S}$. As explained in [13, Proposition 3.6], unless $s(TS) \subseteq \mathfrak{c}$, this will depend on a choice of a $\mathfrak{c}^\perp$-invariant connection 1-form $\theta \in \Omega^1(S, \mathfrak{c}^\perp)$ for the bundle $q_N : S \to N_{\mathfrak{c},S}$. For any $X \in \mathfrak{X}(N_{\mathfrak{c},S})$, let $X^h \in \mathfrak{X}(S)$ denote its horizontal lift with respect to the chosen connection. The map $X \to s(X^h) \in \Gamma(\mathfrak{d} \times S)$ may not take values in $\Gamma(\mathfrak{c} \times S)$, but

$$s_\theta : X \to s(X^h) + \mathbf{a}^*(\iota_{X^h}\langle \theta, \vartheta_s \rangle)$$

does, where $\vartheta_s \in \Omega^1\big(S, (\mathfrak{c}^\perp)^*\big)$ is defined by $\langle \xi, \vartheta_s \rangle := s^*\xi$, for $\xi \in \mathfrak{c}^\perp$. Note also that $s_\theta(X)$ is $\mathfrak{c}^\perp$-invariant, and hence descends to a unique section of $\Gamma(\mathfrak{c} \times N_{\mathfrak{c},S})$. Thus, the composition

$$\mathfrak{X}(N_{\mathfrak{c},S}) \xrightarrow{s_\theta} \Gamma(\mathfrak{c} \times S) \to \Gamma(\mathfrak{c}/\mathfrak{c}^\perp \times N_{\mathfrak{c},S}) = \Gamma(\mathfrak{d}_\mathfrak{c} \times N_{\mathfrak{c},S})$$

defines an isotropic splitting $\tilde{s}_\theta : TN_{\mathfrak{c},S} \to \mathfrak{d}_\mathfrak{c} \times N_{\mathfrak{c},S}$. We define $\tilde{\gamma}_\theta \in \Omega^3(N_{\mathfrak{c},S})$ to be the associated curvature 3-form. Let $E_{\mathfrak{c},S} \subseteq \mathbb{T}_{\tilde{\gamma}_\theta} N_{\mathfrak{c},S}$ denote the Dirac structure corresponding to $\mathfrak{h}_{\mathfrak{c},S}$ under the isomorphism defined by $\tilde{s}_\theta$.

Proposition 2 (Partial reduction for split exact Courant algebroids) *Suppose that*

$$R_{\mu,\omega} : (\mathbb{T}, T)M \dashrightarrow (\mathbb{T}_\gamma N, E) \cong (\mathfrak{d}, \mathfrak{h}) \times N \tag{29}$$

is an exact morphism of Manin pairs, that the assumptions of both Theorems 5 and 6 hold for the reductive data $(\mathfrak{c}, S)$. *Let* $\theta \in \Omega^1(S, \mathfrak{c}^\perp)$ *be a* $\mathfrak{c}^\perp$*-invariant connection 1-form for the bundle* $q_N : S \to N_{\mathfrak{c},S}$.

Then, under the isomorphism $\mathfrak{d}_\mathfrak{c} \times N_{\mathfrak{c},S} \cong \mathbb{T}_{\tilde{\gamma}_\theta} N_{\mathfrak{c},S}$ *defined by the isotropic splitting* $\tilde{s}_\theta$*, the reduced morphism of Manin pairs (23), described in Theorem 5, is identified with*

$$R_{\tilde{\mu},\tilde{\omega}_\theta} : (\mathbb{T}, T)M_{\mathfrak{c},S} \dashrightarrow (\mathbb{T}_{\tilde{\gamma}_\theta} N_{\mathfrak{c},S}, E_{\mathfrak{c},S}), \tag{30}$$

where $\tilde{\omega}_\theta \in \Omega^2(M_{\mathfrak{c},S})$ *is defined by the equation*

$$q_M^* \tilde{\omega}_\theta = i_M^* \omega - \mu^* \langle \theta, \vartheta_s - \frac{1}{2} s \circ \mathbf{a}(\theta) \rangle, \tag{31}$$

and $\tilde{\mu} : M_{\mathfrak{c},S} \to N_{\mathfrak{c},S}$ *is the unique map such that*

$$\begin{array}{ccc} \mu^{-1}(S) & \xrightarrow{\ \mu\ } & S \\ \downarrow q_M & & \downarrow q_N \\ M_{\mathfrak{c},S} & \xrightarrow{\ \tilde{\mu}\ } & N_{\mathfrak{c},S} \end{array}$$

commutes.

Moreover, the 2-form $\tilde{\omega}_\theta$ *is independent of* θ *if* $s(TS) \subseteq \mathfrak{c}$*, more precisely, the term* $\mu^* \langle \theta, \vartheta_s - \frac{1}{2} s \circ \mathbf{a}(\theta) \rangle$ *in (31) vanishes.*

We refer the proof of Proposition 2 to Appendix A.

Remark 17 In the special case where $\mathfrak{c} = \mathfrak{c}^\perp$ is Lagrangian, then $\mathfrak{c}^\perp$ acts transitively on S, so (31) simplifies to

$$q_M^* \tilde{\omega}_\theta = i_M^* \omega - \frac{1}{2} \mu^* \langle \theta, s \circ \mathbf{a}(\theta) \rangle.$$

Remark 18 Consider the Γ-twisted Cartan-Dirac structure $\big((\overline{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}, \mathfrak{g}_\Gamma\big) \times G^{E_\Gamma}$. The conditions that $\mathfrak{c} \subseteq (\overline{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}$ be symmetric and that $S \subseteq G^{E_\Gamma}$ be the $\mathfrak{c}$-orbit through the identity imply that

$$s(TS) \subseteq \mathfrak{c}. \tag{32}$$

Indeed, Eq. (32) is easily checked at the identity of G^{E_Γ}, and the invariance of s implies that it holds at every other point in the $\mathfrak{c}$-orbit, S.

Thus, Eq. (31) simplifies to

$$q_M^* \tilde{\omega}_\theta = i_M^* \omega,$$

and Theorem 3 follows as a corollary to Proposition 2.

Example 28 (Conjugacy clases and quasi-Hamiltonian geometry) Recall the splitting $s : TG \to (\overline{\mathfrak{g}} \oplus \mathfrak{g}) \times G$ of the Cartan Courant algebroid described in Example 27. Suppose $\mathfrak{c} = \mathfrak{g}_\Delta \subseteq (\overline{\mathfrak{g}} \oplus \mathfrak{g})$ is the diagonal subalgebra, and $S \subseteq G$ is a conjugacy class. Then,

$$s \circ \mathbf{a}(\xi, \xi) = (\xi - \mathbf{Ad}_g \xi, -\mathbf{Ad}_{g^{-1}} \xi + \xi), \quad \xi \in \mathfrak{g}, \quad g \in G.$$

So

$$\iota_{\mathbf{a}(\eta,\eta)} \iota_{\mathbf{a}(\xi,\xi)} \frac{1}{2} \mu^* \langle \theta, s \circ \mathbf{a}(\theta) \rangle = \langle \eta, \mathbf{Ad}_g \xi \rangle - \langle \mathbf{Ad}_g \eta, \xi \rangle.$$

Thus $\frac{1}{2} \langle \theta, s \circ \mathbf{a}(\theta) \rangle$ is proportional to the quasi-Hamiltonian 2-form on the conjugacy class, S, described in [4, Proposition 3.1].

Suppose now that M is a quasi-Hamiltonian G space with moment map $\mu : M \to G$, and 2-form ω. It follows that partial reduction of the morphism of Manin pairs

$$R_{\mu,\omega} : (\mathbb{T}, T)M \dashrightarrow (\overline{\mathfrak{g}} \oplus \mathfrak{g}, \mathfrak{g}_\Delta) \times G$$

by $(\mathfrak{c}, S)$, yields the same symplectic structure as the quasi-Hamiltonian reduction

$$M \circledast S //_1 G,$$

where $M \circledast S$ denotes the fusion of M with S, as described in [4].

3.4 Commutativity of Reductions

Suppose that $S_1, S_2 \subseteq N$ intersect cleanly, and that $(\mathfrak{c}_1, S_1)$ and $(\mathfrak{c}_2, S_2)$ are both reductive data for a morphism of Manin pairs

$$R : (\mathbb{T}, T)M \dashrightarrow (\mathfrak{d}, \mathfrak{h}) \times N. \tag{33}$$

Notice that the Lie algebra $\mathfrak{c}_{2,1} := R_{\mathfrak{c}_1} \circ \mathfrak{c}_2 \subseteq \mathfrak{d}_{\mathfrak{c}_1}$ is coisotropic. Let $S_{2,1}$ denote the image of $S_2 \cap S_1$ under the quotient map $S_1 \to N_{\mathfrak{c}_1, S_1}$. In practise, $(\mathfrak{c}_{2,1}, S_{2,1})$ will often form reductive data for

$$R_{\mathfrak{c}_1, S_1} : (\mathbb{T}, T)M_{\mathfrak{c}_1, S_1} \dashrightarrow (\mathfrak{d}_{\mathfrak{c}_1}, \mathfrak{h}_{\mathfrak{c}_1}) \times N_{\mathfrak{c}_1, S_1}.$$

Similarly, with $\mathfrak{c}_{1,2} := R_{\mathfrak{c}_2} \circ \mathfrak{c}_1 \subseteq \mathfrak{d}_{\mathfrak{c}_2}$ and $S_{1,2}$ the image of $S_1 \cap S_2$ under the quotient map $S_2 \to N_{\mathfrak{c}_2, S_2}$, the pair $(\mathfrak{c}_{1,2}, S_{1,2})$ will often form reductive data for

$$R_{\mathfrak{c}_2, S_2} : (\mathbb{T}, T)M_{\mathfrak{c}_2, S_2} \dashrightarrow (\mathfrak{d}_{\mathfrak{c}_2}, \mathfrak{h}_{\mathfrak{c}_2}) \times N_{\mathfrak{c}_2, S_2}.$$

Proposition 3 (Commutativity of reductions) *Suppose that* $S_1, S_2 \subseteq N$ *intersect cleanly, that* $(\mathfrak{c}_1, S_1)$, $(\mathfrak{c}_2, S_2)$ *and* $(\mathfrak{c}_1 \cap \mathfrak{c}_2, S_1 \cap S_2)$ *all form reductive data for the morphism of Manin pairs*

$$R : (\mathbb{T}, T)M \dashrightarrow (\mathfrak{d}, \mathfrak{h}) \times N,$$

while $(\mathfrak{c}_{2,1}, S_{2,1})$ *and* $(\mathfrak{c}_{1,2}, S_{1,2})$ *form reductive data for*

$$R_{\mathfrak{c}_1,S_1} : (\mathbb{T}, T)M_{\mathfrak{c}_1,S_1} \dashrightarrow (\mathfrak{d}_{\mathfrak{c}_1}, \mathfrak{h}_{\mathfrak{c}_1}) \times N_{\mathfrak{c}_1,S_1}$$

and

$$R_{\mathfrak{c}_2,S_2} : (\mathbb{T}, T)M_{\mathfrak{c}_2,S_2} \dashrightarrow (\mathfrak{d}_{\mathfrak{c}_2}, \mathfrak{h}_{\mathfrak{c}_2}) \times N_{\mathfrak{c}_2,S_2},$$

respectively. Then

$$(R_{\mathfrak{c}_1,S_1})_{\mathfrak{c}_{21},S_{21}} = R_{\mathfrak{c}_1\cap\mathfrak{c}_2,S_1\cap S_2} = (R_{\mathfrak{c}_2,S_2})_{\mathfrak{c}_{12},S_{12}}. \tag{34}$$

The proof of this proposition is deferred to Appendix 5.2.1.

In this paper, Proposition 3 will always be applied as the following corollary:

Corollary 1 *(Reductions of distinct factors commute) Suppose* $\mathfrak{d} \times N$, $\mathfrak{d}' \times N'$, *and* $\mathfrak{d}'' \times N''$ *are all action Courant algebroids,* $S \subseteq N$ *and* $S' \subseteq N'$ *are submanifolds and* $(\mathfrak{c} \oplus \mathfrak{d}' \oplus \mathfrak{d}'', S \times N' \times N'')$, $(\mathfrak{d} \oplus \mathfrak{c}' \oplus \mathfrak{d}'', N \times S' \times N'')$ *and* $(\mathfrak{c} \oplus \mathfrak{c}' \oplus \mathfrak{d}'', S \times S' \times N'')$ *each form reductive data for a morphism of Manin pairs*

$$R : (\mathbb{T}, T)M \dashrightarrow (\mathfrak{d} \oplus \mathfrak{d}' \oplus \mathfrak{d}'', \mathfrak{h}) \times (N \times N' \times N'').$$

Then

$$\begin{aligned}(R_{\mathfrak{c}\oplus\mathfrak{d}'\oplus\mathfrak{d}'',S\times N'\times N''})&_{(\mathfrak{d}_\mathfrak{c}\oplus\mathfrak{c}'\oplus\mathfrak{d}'',N_{\mathfrak{c},S}\times S'\times N'')}\\ &= R_{\mathfrak{c}\oplus\mathfrak{c}'\oplus\mathfrak{d}'',S\times S'\times N''}\\ &= (R_{\mathfrak{d}\oplus\mathfrak{c}'\oplus\mathfrak{d}'',N\times S'\times N''})_{(\mathfrak{c}\oplus\mathfrak{d}'_{\mathfrak{c}'}\oplus\mathfrak{d}'',S\times N'_{\mathfrak{c}',S'}\times N'')}.\end{aligned}$$

4 Quasi-Hamiltonian Structures on Moduli Spaces of Flat Connections

Suppose that (Σ, V) is a marked surface, i.e. Σ is a compact oriented surface and $V \subset \partial\Sigma$ a finite set which intersects each component of Σ and $\partial\Sigma$ non-trivially. Let Γ be the boundary graph of Σ with the vertex set V (see Sect. 1.2.3 for details). In this section we shall prove Theorem 2 using quasi-Hamiltonian reduction. In other words, we want to construct an exact morphism of Manin pairs

$$R_{\Sigma,V} : (\mathbb{T}, T)\mathcal{M}_{\Sigma,V} \dashrightarrow \big((\bar{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}, \mathfrak{g}_\Gamma\big) \times G^{E_\Gamma} \tag{35}$$

over the map

$$\mu : \mathcal{M}_{\Sigma,V} \to G^{E_\Gamma}$$

given by the boundary holonomies.

Let us start with a simple case:

Proposition 4 (Union of polygons) *Let Σ be a disjoint union of discs and $V \subset \partial\Sigma$ a finite subset meeting every boundary circle. Then there is a unique exact morphism*

$$(\mathbb{T}, T)\mathcal{M}_{\Sigma,V} \dashrightarrow \big((\bar{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}, \mathfrak{g}_\Gamma\big) \times G^{E_\Gamma}$$

over μ.

Proof The map μ is in this case an embedding. As explained in Example 25, this implies that there exists a unique exact morphism of Manin pairs

4.1 Sewing Construction

Suppose Σ is a (possibly disconnected) marked surface, and $e_1, e_2 \in \Gamma$ are two distinct edges from the boundary graph, we may 'sew' the surface together along e_1 and e_2 to form a new surface

$$\Sigma' := \frac{\Sigma}{e_1 \sim e_2},$$

as pictured in Fig. 21. In this section, we describe the analogous procedure for the corresponding morphisms of Manin pairs, (35).

First, note that

$$\mathfrak{l}_{sew} := \{\big((\xi, \eta); (\eta, \xi)\big) \mid \xi, \eta \in \mathfrak{g}\} \subseteq (\bar{\mathfrak{g}} \oplus \mathfrak{g}) \bigoplus (\bar{\mathfrak{g}} \oplus \mathfrak{g})$$

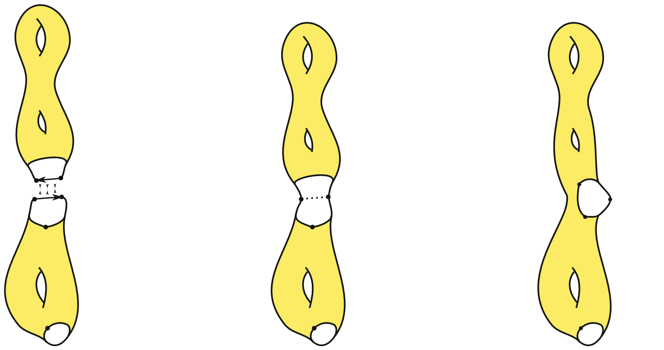
is a Lagrangian subalgebra. Since G is connected, the $\mathfrak{l}_{sew}$-orbit through the identity of $G \times G$ is

$$G_\Delta^\natural := \{(g, g^{-1}) \mid g \in G\}.$$

Definition 8 Suppose that

$$\big((\bar{\mathfrak{g}} \oplus \mathfrak{g}) \bigoplus (\bar{\mathfrak{g}} \oplus \mathfrak{g}) \bigoplus \mathfrak{d}'\big) \times \big(G \times G \times N\big)$$

is the product of two Cartan Courant algebroids with an action Courant algebroid, $\mathfrak{d}' \times N$, and that $\mathfrak{h} \subseteq (\bar{\mathfrak{g}} \oplus \mathfrak{g}) \bigoplus (\bar{\mathfrak{g}} \oplus \mathfrak{g}) \bigoplus \mathfrak{d}'$ is a Lagrangian subalgebra. Further



The initial surface. Sewing. The resulting surface.

Fig. 21 Sewing two edges from Σ. The two edges must have opposite orientation to ensure that the resulting surface is orientable

suppose that

$$R : (\mathbb{T}, T)M \dashrightarrow \Big((\overline{\mathfrak{g}} \oplus \mathfrak{g}) \bigoplus (\overline{\mathfrak{g}} \oplus \mathfrak{g}) \bigoplus \mathfrak{d}', \mathfrak{h}\Big) \times \big(G \times G \times N\big) \tag{36}$$

is a morphism of Manin pairs.

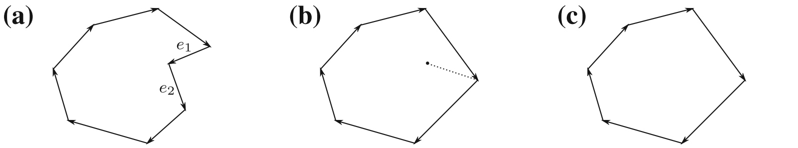
Let $\mathfrak{c}_{sew} = \mathfrak{l}_{sew} \oplus \mathfrak{d}'$ and let

$$S_{sew} = G^{\natural}_{\Delta} \times N \subseteq G \times G \times N.$$

If $(\mathfrak{c}_{sew}, S_{sew})$ is reduction data for (36), then the reduction,

$$R_{\mathfrak{c}_{sew}, S_{sew}} : (\mathbb{T}, T)M_{\mathfrak{c}_{sew}, S_{sew}} \dashrightarrow (\mathfrak{d}', \mathfrak{h}_{\mathfrak{c}_{sew}, S_{sew}}) \times N$$

is called the *sewing* of (36).



The graph Γ', with chosen edges $e_1, e_2 \in E_{\Gamma'}$.

The graph with edges e_1 and e_2 identified with opposite orientation.

Γ is the graph Γ' with edges e_1 and e_2 identified, and the identified edges removed along with any isolated vertices.

In a typical example of sewing, Γ' will be a permutation graph, $e_1, e_2 \subseteq E_{\Gamma'}$ will be two (distinct) edges, and

$$R : (\mathbb{T}, T)M \dashrightarrow \Big((\overline{\mathfrak{g}} \oplus \mathfrak{g})^{E_{\Gamma'}}, \mathfrak{g}_{\Gamma'}\Big) \times G^{E_{\Gamma'}}$$

will be a morphism of Manin pairs.

We let Γ denote the graph with edge set $E_\Gamma := E_{\Gamma'} \setminus \{e_1, e_2\}$ and vertex set

$$V_\Gamma := \frac{\mathsf{out}(E_\Gamma)}{\mathsf{in}(e_1) \sim \mathsf{out}(e_2) \text{ and } \mathsf{in}(e_2) \sim \mathsf{out}(e_1)},$$

as pictured in figure c above.

Let

$$\mathfrak{c}_{sew}^{e_1,e_2} := \{(\xi, \eta) \in (\overline{\mathfrak{g}} \oplus \mathfrak{g})^{E_{\Gamma'}} \mid (\xi, \eta)_{\{e_1,e_2\}} \in \mathfrak{l}_{sew}\}$$
$$S_{sew}^{e_1,e_2} := \{g \in G^{E_{\Gamma'}} \mid g_{\{e_1,e_2\}} \in G_\Delta^\natural\}$$

as in Definition 8. Here we are using the notation $(\xi, \eta)_{\{e_1,e_2\}} := \big((\xi_{e_1}, \eta_{e_1}); (\xi_{e_2}, \eta_{e_2})\big)$, as described in Sect. 1.1. The morphism of Manin pairs

$$R_{\mathfrak{c}_{sew}^{e_1,e_2}, S_{sew}^{e_1,e_2}} : (\mathbb{T}, T)M_{\mathfrak{c}_{sew}^{e_1,e_2}, S_{sew}^{e_1,e_2}} \dashrightarrow ((\overline{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}, \mathfrak{g}_\Gamma) \times G^{E_\Gamma}$$

is called the result of *sewing edges* e_1 and e_2 *together*. Here we have used the following lemma to simplify the right hand side:

Lemma 3 *The Lie subalgebras* $\mathfrak{g}_\Gamma \subseteq (\overline{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}$ *and* $(\mathfrak{g}_{\Gamma'})_{\mathfrak{c}_{sew}^{e_1,e_2}} \subseteq (\overline{\mathfrak{g}} \oplus \mathfrak{g})^{E_{\Gamma'}}$ *are equal.*

Proof Let us recall that

$$\mathfrak{g}_{\Gamma'} = (\mathsf{in} \oplus \mathsf{out})^! \mathfrak{g}_\Delta^{V_{\Gamma'}} \cong \mathfrak{g}^{V_{\Gamma'}}$$

and

$$(\mathfrak{g}_{\Gamma'})_{\mathfrak{c}_{sew}^{e_1,e_2}} = \mathfrak{g}_{\Gamma'} \cap \mathfrak{c}_{sew}^{e_1,e_2} / \mathfrak{g}_{\Gamma'} \cap (\mathfrak{c}_{sew}^{e_1,e_2})^\perp.$$

By definition of $\mathfrak{c}_{sew}^{e_1,e_2}$, $\mathfrak{g}^{V_{\Gamma'}} \cap \mathfrak{c}_{sew}^{e_1,e_2}$ is the subalgebra of $\mathfrak{g}^{V_{\Gamma'}}$ where the components corresponding to identified vertices are equal. After we divide by $\mathfrak{g}_{\Gamma'} \cap (\mathfrak{c}_{sew}^{e_1,e_2})^\perp$ we obtain the Lie algebra $\mathfrak{g}_\Gamma = \mathfrak{g}^{V_\Gamma}$.

The morphism of Manin pairs (35) which we assign to the surface Σ will satisfy the following sewing property:

Sewing property. Let (Σ, V) be obtained out of (Σ', V') by sewing edges e_1 and e_2. There is a canonical isomorphism

$$\mathcal{M}_{\Sigma,V} \xrightarrow{\cong} (\mathcal{M}_{\Sigma',V'})_{\mathfrak{c}_{sew}^{e_1,e_2}, S_{sew}^{e_1,e_2}}, \tag{37}$$

which identifies the following two morphisms of Manin pairs:

$$R_{\Sigma,V} : (\mathbb{T}, T)\mathcal{M}_{\Sigma,V} \dashrightarrow ((\overline{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}, \mathfrak{g}_\Gamma) \times G^{E_\Gamma},$$

and

$$(R_{\Sigma',V'})_{\mathfrak{c}_{sew}^{e_1,e_2},S_{sew}^{e_1,e_2}} : (\mathbb{T},T)\mathcal{M}_{\Sigma,V} \dashrightarrow ((\bar{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}, \mathfrak{g}_\Gamma) \times G^{E_\Gamma}.$$

Notice that

$$\mu^{-1}(S_{sew}^{e_1,e_2}) = \{f \in \mathcal{M}_{\Sigma',V'}(G) \mid f(e_1)f(e_2) = 1\}$$

and that

$$\mu^{-1}(S_{sew}^{e_1,e_2})/\mathfrak{g}^{V'} \cap (\mathfrak{c}_{sew}^{e_1,e_2})^\perp = \mathcal{M}_{\Sigma,V}(G).$$

Thus, the isomorphism (37) is clear. The non-trivial part of the statement is the behaviour of the $R_{\Sigma,V}$'s

4.1.1 Commutativity of Sewing

Suppose $\{e_1, e_2\}, \{e_1', e_2'\} \subseteq E_\Gamma$ are two distinct pairs of distinct edges (i.e. $\{e_1, e_2\} \cap \{e_1', e_2'\} = \emptyset$), and let $i : \{e_1, e_2\} \to E_\Gamma$ and $i' : \{e_1', e_2'\} \to E_\Gamma$ denote the inclusions. We may form the graph Γ'', with edge set $E_{\Gamma''} := E_\Gamma \setminus \{e_1, e_2, e_1', e_2'\}$ and vertex set

$$V_{\Gamma''} := \frac{\mathsf{out}(E_{\Gamma''})}{\mathsf{in}(e_1) \sim \mathsf{out}(e_2), \mathsf{in}(e_2) \sim \mathsf{out}(e_1), \text{ and } \mathsf{in}(e_1') \sim \mathsf{out}(e_2'), \mathsf{in}(e_2') \sim \mathsf{out}(e_1')}.$$

It is the graph obtained from Γ by identifying the oppositely directed edges $e_1 \sim e_2$ and $e_1' \sim e_2'$, and then deleting these newly identified edges and any isolated vertices.

Let

$$\begin{aligned}
\mathfrak{c}_1 := \mathfrak{c}_{sew}^{e_1,e_2} &= i(\mathfrak{l}_{sew}) \oplus (\bar{\mathfrak{g}} \oplus \mathfrak{g})^{\{e_1',e_2'\}} \oplus (\bar{\mathfrak{g}} \oplus \mathfrak{g})^{E_{\Gamma''}},\\
\mathfrak{c}_2 := \mathfrak{c}_{sew}^{e_1',e_2'} &= (\bar{\mathfrak{g}} \oplus \mathfrak{g})^{\{e_1,e_2\}} \oplus i'(\mathfrak{l}_{sew}) \oplus (\bar{\mathfrak{g}} \oplus \mathfrak{g})^{E_{\Gamma''}},\\
\mathfrak{c}_{1,2} := \mathfrak{c}_1 \cap \mathfrak{c}_2 &= i(\mathfrak{l}_{sew}) \oplus i'(\mathfrak{l}_{sew}) \oplus (\bar{\mathfrak{g}} \oplus \mathfrak{g})^{E_{\Gamma''}},
\end{aligned}$$

and

$$\begin{aligned}
S_1 := S_{sew}^{e_1,e_2} &= i(G_\Delta^\natural) \times G^{\{e_1',e_2'\}} \times G^{E_{\Gamma''}},\\
S_2 := S_{sew}^{e_1',e_2'} &= G^{\{e_1,e_2\}} \times i'(G_\Delta^\natural) \times G^{E_{\Gamma''}},\\
S_{1,2} := S_1 \cap S_2 &= i(G_\Delta^\natural) \times i'(G_\Delta^\natural) \times G^{E_{\Gamma''}},
\end{aligned}$$

If the pairs $(\mathfrak{c}_{sew}^{e_1,e_2}, S_{sew}^{e_1,e_2})$, $(\mathfrak{c}_{sew}^{e_1',e_2'}, S_{sew}^{e_1',e_2'})$ and $(\mathfrak{c}_1 \cap \mathfrak{c}_2, S_1 \cap S_2)$ all form reductive data for R, then the assumptions of Corollary 1 are satisfied. Therefore,

$$\left(R_{\mathfrak{c}_{sew}^{e_1,e_2},S_{sew}^{e_1,e_2}}\right)_{\mathfrak{c}_{sew}^{e_1',e_2'},S_{sew}^{e_1',e_2'}} = R_{\mathfrak{c}_{1,2},S_{1,2}} = \left(R_{\mathfrak{c}_{sew}^{e_1',e_2'},S_{sew}^{e_1',e_2'}}\right)_{\mathfrak{c}_{sew}^{e_1,e_2},S_{sew}^{e_1,e_2}}$$

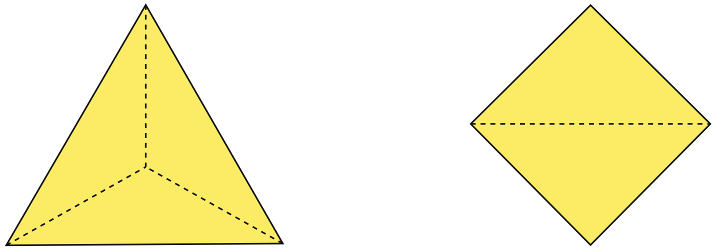


Fig. 22 Triangulations of 3 and 4-gons. The *dashed lines* indicate *internal edges* of the triangulation along which we sew

as morphisms of Manin pairs

$$(\mathbb{T}, T)M'' \dashrightarrow ((\bar{\mathfrak{g}} \oplus \mathfrak{g})^{E_{\Gamma''}}, \mathfrak{g}_{\Gamma''}) \times G^{E_{\Gamma''}},$$

where $M'' = M_{\mathfrak{c}_{1,2}, S_{1,2}}$. That is to say, it makes no difference in which order we sew pairs of edges: the results are all naturally isomorphic.

4.2 The Quasi-Hamiltonian Structure on $\mathcal{M}_{\Sigma,V}(G)$

Theorem 8 *There is a unique way to assign to every marked surface (Σ, V) an exact morphism of Manin pairs*

$$R_\Sigma : (\mathbb{T}, T)\mathcal{M}_{\Sigma,V} \dashrightarrow \left((\bar{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}, \mathfrak{g}_\Gamma\right) \times G^{E_\Gamma}$$

supported on the graph of $\mu : \mathcal{M}_{\Sigma,V} \to G^{E_\Gamma}$ *such that the assignment satisfies the sewing property.*

Proof If Σ' is a disjoint union of disks then $R_{\Sigma'}$ exists and is unique by Proposition 4.

Suppose that Σ is triangulated and $V = \mathcal{T}^0 \cap \partial\Sigma$. Let Σ' be the disjoint union of the triangles and V' the set of its vertices. Then, by sewing the respective edges of $\mathcal{M}_{\Sigma',V'}(G)$ according to the triangulation, we get an exact morphism of Manin pairs

$$R_\mathcal{T} : (\mathbb{T}, T)\mathcal{M}_{\Sigma,V} \dashrightarrow \left((\bar{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}, \mathfrak{g}_\Gamma\right) \times G^{E_\Gamma}.$$

By commutativity of sewing these morphisms satisfy the sewing property. It remains to show that they are independent of the triangulation $\mathcal{T}$.

Thus, we need to prove that $R_\mathcal{T}$ is invariant under Pachner moves applied to $\mathcal{T}$. Pachner moves are, however, simply changes of triangulations of a polygon (either a triangle or a square, see Fig. 22). The independence thus follows from Proposition 4.

5 Poisson Structures on the Moduli Space of Flat Connections

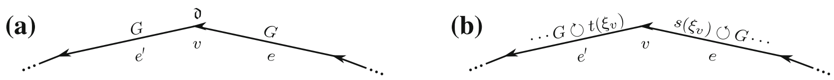
Suppose G is a (connected) Lie group with Lie algebra $\mathfrak{g}$ and $s \in S^2(\mathfrak{g})^G$ is a G-invariant symmetric 2-tensor. We let $\mathfrak{d}$ denote the Drinfel'd double of $\mathfrak{g}$, as in Remark 8. Suppose now that Γ is a permutation graph. Then $\mathfrak{d}^{V_\Gamma}$ acts on G^{E_Γ} with coisotropic stabilizers, as follows: $\mathfrak{d}^{V_\Gamma}$ acts on the $e \in E_\Gamma$th factor G^{E_Γ} via the vector field

$$\rho(\xi)_e = -\mathsf{s}(\xi_{\mathsf{in}(e)})^R + \mathsf{t}(\xi_{\mathsf{out}(e)})^L, \quad \xi \in \mathfrak{d}^{V_\Gamma},$$

(cf. figure b below). Thus

$$\mathfrak{g}^{V_\Gamma} \times G^{E_\Gamma} \subseteq \mathfrak{d}^{V_\Gamma} \times G^{E_\Gamma}$$

is a Dirac structure.



Here $\mathsf{in}(e) = v = \mathsf{out}(e')$, where $v \in V_\Gamma$ is the vertex and $e, e' \in E_\Gamma$ are edges.

The element $\xi_{(\cdot)} \in \mathfrak{d}^{V_\Gamma}$ acts diagonally by ξ_v at the vertex v.

Suppose (Σ, V) is a marked surface (where we now allow components of $\partial\Sigma$ to intersect V trivially), with boundary graph Γ. In this section, we prove there exists a natural quasi-Hamiltonian $(\mathfrak{d}, \mathfrak{g})^{V_\Gamma} \times G^{E_\Gamma}$ structure on $\mathcal{M}_{\Sigma,V}(G)$, i.e a morphism of Manin pairs

$$R_{\Sigma,V} : (\mathbb{T}, T)\mathcal{M}_{\Sigma,V}(G) \dashrightarrow (\mathfrak{d}, \mathfrak{g})^{V_\Gamma} \times G^{E_\Gamma}$$

over the map

$$\mu : \mathcal{M}_{\Sigma,V}(G) \to G^{E_\Gamma}$$

given by the boundary holonomies.

As before, for disjoint unions of polygons the quasi-Hamiltonian structure is uniquely defined.

Proposition 5 *(Union of polygons) Let Σ be a disjoint union of discs and $V \subset \partial\Sigma$ a finite subset. Then there is a unique exact morphism of Manin pairs*

$$(\mathbb{T}, T)\mathcal{M}_{\Sigma,V} \dashrightarrow \left((\overline{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}, \mathfrak{g}_\Gamma\right) \times G^{E_\Gamma}$$

over μ.

Proof As in Proposition 4, μ is an embedding, so this follows from Example 25.

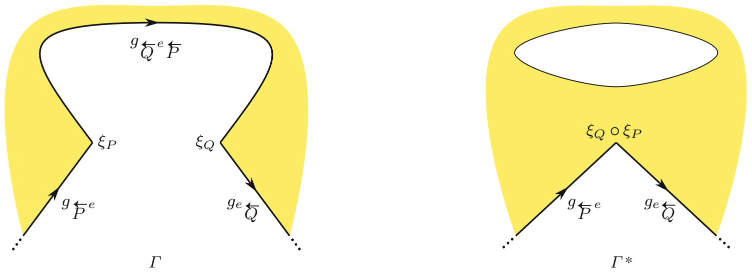


Fig. 23 The graph Γ^* is the permutation graph obtained from Γ by deleting the edge, $_{\overleftarrow{Q}}e_{\overleftarrow{P}}$, passing from P to Q, and identifying the vertices P and Q. (The graphs are isomorphic outside the pictured regions.) Heuristically, we have obtained the graph Γ^* by gluing short sections from both ends of $_{\overleftarrow{Q}}e_{\overleftarrow{P}}$ together (effectively divorcing it from the graph). Meanwhile, the Courant morphism, $R_{(P,Q)}$, is defined by composing the corresponding elements labelling the vertices and by forgetting the element which labelled the deleted edge

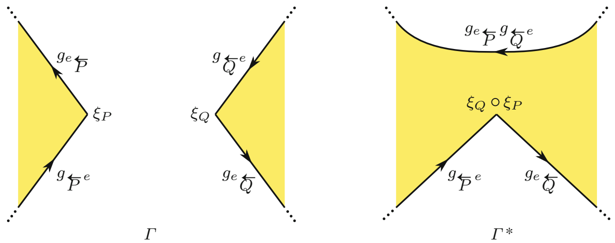


Fig. 24 The graph Γ^* is the permutation graph obtained from Γ by identifying the vertices P and Q and composing the edge entering Q with the edge leaving P. (The graphs are isomorphic outside the pictured regions.) Heuristically, we have obtained the graph Γ^* by gluing a short section of the edges $e_{\overleftarrow{P}}$ and $_{\overleftarrow{Q}}e$ together. Meanwhile, the Courant morphism, $R_{(P,Q)}$, is defined by composing the corresponding elements labelling the edges and vertices

5.1 Fusion

Suppose that $P, Q \in V_\Gamma$ are two distinct vertices. The operation of *fusion* at the ordered pair of vertices (P, Q) described in [30] (following [3, 4]), can be understood in terms of a morphism of Manin pairs

$$R_{(P,Q)} : (\mathfrak{d}, \mathfrak{g})^{V_\Gamma} \times G^{E_\Gamma} \dashrightarrow (\mathfrak{d}, \mathfrak{g})^{V_{\Gamma^*}} \times G^{E_{\Gamma^*}}$$

where the graph Γ^* is a permutation graph constructed from Γ, as we shall now explain:

Let $_{\overleftarrow{P}}e = \mathsf{in}^{-1}(P)$ and $e_{\overleftarrow{P}} := \mathsf{out}^{-1}(P)$ denote the edges entering and exiting P. Similarly, let $_{\overleftarrow{Q}}e = \mathsf{in}^{-1}(Q)$ and $e_{\overleftarrow{Q}} := \mathsf{out}^{-1}(Q)$ denote the edges entering and exiting Q.

Case 1: $\underset{Q}{\leftarrow} e = e_{\underset{P}{\leftarrow}}$ In this case, Γ^* is obtained by discarding the edge $\underset{Q}{\leftarrow} e = e_{\underset{P}{\leftarrow}}$ and identifying the vertices P and Q (cf. Fig. 23).
Meanwhile for $(\xi, g) \in \mathfrak{d}^{V_\Gamma} \times G^{E_\Gamma}$ and $(\xi^*, g^*) \in \mathfrak{d}^{V_{\Gamma^*}} \times G^{E_{\Gamma^*}}$

$$(\xi, g) \sim_{R_{(P,Q)}} (\xi^*, g^*)$$

if and only if $g^*_e = g_e$ for every $e \in E_{\Gamma^*}$ and

$$\xi^*_v = \begin{cases} \xi_Q \circ \xi_P & \text{if } v \text{ is the vertex obtained by identifying } P \text{ and } Q \\ \xi_v & \text{otherwise,} \end{cases} \tag{38}$$

(in particular, we assume that ξ_Q and ξ_P are composable elements of the Lie groupoid $\mathfrak{d}$).

Case 2: $\underset{Q}{\leftarrow} e \neq e_{\underset{P}{\leftarrow}}$ In this case, Γ^* is obtained by identifying the vertices P and Q and composing the edges $e_{\underset{P}{\leftarrow}}$ and $\underset{Q}{\leftarrow} e$ to form a new edge $e_{\widehat{PQ}}$ (cf. Fig. 24).

Meanwhile for $(\xi, g) \in \mathfrak{d}^{V_\Gamma} \times G^{E_\Gamma}$ and $(\xi^*, g^*) \in \mathfrak{d}^{V_{\Gamma^*}} \times G^{E_{\Gamma^*}}$

$$(\xi, g) \sim_{R_{(P,Q)}} (\xi^*, g^*)$$

if and only if

$$g^*_e = \begin{cases} g_{e_{\underset{P}{\leftarrow}}} g_{\underset{Q}{\leftarrow} e} & \text{if } e = e_{\widehat{PQ}} \\ g_e & \text{otherwise,} \end{cases}$$

while ξ^* and ξ satisfy (38), as before.

Now suppose that M is a quasi-Hamiltonian $(\mathfrak{d}, \mathfrak{g})^{V_\Gamma} \times G^{E_\Gamma}$-space defined by the morphism of Manin pairs

$$R : (\mathbb{T}, T)M \dashrightarrow (\mathfrak{d}, \mathfrak{g})^{V_\Gamma} \times G^{E_\Gamma}.$$

Then the morphism of Manin pairs

$$R^* := R_{(P,Q)} \circ R : (\mathbb{T}, T)M \dashrightarrow (\mathfrak{d}, \mathfrak{g})^{V_{\Gamma^*}} \times G^{E_{\Gamma^*}}$$

defines a quasi-Hamiltonian $(\mathfrak{d}, \mathfrak{g})^{V_{\Gamma^*}} \times G^{E_{\Gamma^*}}$-structure on M which we call the *fusion* of R at the ordered pair (P, Q) of vertices.

Remark 19 (**Associativity of Fusion**) Since the Courant morphism $R_{(P,Q)}$ is defined in terms of the groupoid structure on $\mathfrak{d}$ and the group structure on G, it follows that fusion is an associative operation.

5.2 The Quasi-Hamiltonian Structure for Quasi-Triangular Structure Lie Algebras

Let (Σ, V) be a marked surface. If we choose an ordered pair (P, Q) of marked points $(P \neq Q \in V)$ then the corresponding *fused surface* Σ^* is obtained by gluing a short piece of the arc starting at P with a short piece of the arc ending at Q (so that P and Q get identified). The subset $V^* \subset \partial\Sigma^*$ is obtained from V by identifying P and Q. The map

$$M_{\Sigma^*,V^*}(G) \to M_{\Sigma,V}(G),$$

coming from the map $(\Sigma, V) \to (\Sigma^*, V^*)$, is a diffeomorphism.

Theorem 9 *There is unique way to assign to every marked surface (Σ, V) a morphism of Manin pairs*

$$R_{\Sigma,V} : (\mathbb{T}, T)\mathcal{M}_{\Sigma,V}(G) \dashrightarrow (\mathfrak{d}, \mathfrak{g})^{V_\Gamma} \times G^{E_\Gamma}$$

supported on the graph of $\mu : \mathcal{M}_{\Sigma,V}(G) \to G^{E_\Gamma}$ such that if (Σ^*, V^*) is obtained from (Σ, V) by fusion, then R_{Σ^*,V^*} is obtained from $R_{\Sigma,V}$ by the corresponding fusion.

We defer the proof until the next section.

Remark 20 When $s \in S^2(\mathfrak{g})^G$ is non-degenerate (i.e. $\mathfrak{g}$ is quadratic), then $\mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$ is the pair groupoid and

$$(\mathsf{in} \oplus \mathsf{out})^!(\mathsf{s} \oplus \mathsf{t}) : \mathfrak{d}^{V_\Gamma} \to (\bar{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}$$

is an isomorphism. In this case, it is not difficult to convince oneself that the morphisms of Manin pairs

$$R_{\Sigma,V} : (\mathbb{T}, T)\mathcal{M}_{\Sigma,V}(G) \dashrightarrow ((\bar{\mathfrak{g}} \oplus \mathfrak{g})^{E_\Gamma}, \mathfrak{g}_\Delta^{V_\Gamma}) \times G^{E_\Gamma}$$

described in Theorem 9 satisfy the sewing property. That is they are precisely the ones described in Theorem 8.

5.2.1 Quasi-Hamiltonian $(\mathfrak{d}, \mathfrak{g})^{V_\Gamma} \times G^{E_\Gamma}$-Manifolds and Quasi-Poisson Geometry

Using the material in Sect. 3.2, we intend to relate quasi-Hamiltonian $(\mathfrak{d}, \mathfrak{g})^{V_\Gamma} \times G^{E_\Gamma}$-spaces to the quasi-Poisson spaces studied in [3, 30]. A canonical choice of complement to $\mathfrak{g} \subseteq \mathfrak{d}$ is $\mathfrak{g}^*_\Delta := (\mathsf{s} + \mathsf{t})^*(\mathfrak{g}^*) \subseteq \mathfrak{d}$, explicitly

$$\mathfrak{g}^*_{\Delta} = \{(\alpha - \frac{1}{2}s(\alpha, \cdot)) \mid \alpha \in \mathfrak{p} \subseteq \mathfrak{d}\}.$$

Similarly, $\mathfrak{k} = (\mathfrak{g}^*_{\Delta})^{V_\Gamma}$ is a canonical choice of complement to $\mathfrak{g}^{V_\Gamma} \subseteq \mathfrak{d}^{V_\Gamma}$.

Let $\overline{\sigma_\Gamma} : V_\Gamma \to V_\Gamma$ be the permutation given by walking along the graph Γ *against* the direction of each edge. Suppose that (M, ρ, π) is a quasi-Poisson G^{V_Γ}-manifold, in the sense of [3, 30] and $\mu : M \to G^{V_\Gamma}$ is a $\overline{\sigma_\Gamma}^!$-twisted moment map in the sense of [30]. Let $\tilde{\mu} : M \to G^{E_\Gamma}$ be defined by

$$\tilde{\mu}(m)_e = \left(\mu(m)_{\mathsf{in}(e)}\right)^{-1}, \quad m \in M, e \in E_\Gamma.$$

Proposition 6 *There exists a unique morphism of Manin pairs*

$$R : (\mathbb{T}, T)M \dashrightarrow (\mathfrak{d}^{V_\Gamma}, \mathfrak{g}^{V_\Gamma}) \times G^{E_\Gamma}$$

over the map $\tilde{\mu}$ *which is compatible with the* $\mathfrak{g}^{V_\Gamma}$ *action on* M *and such that* $\mathfrak{k} \circ R = \mathsf{gr}(\pi^\sharp)$.

Moreover, the converse holds whenever the action of $\mathfrak{g}^{V_\Gamma}$ *on* M *integrates to an action of* G^{V_Γ}.

Proof This follows from a direct application of [14, Proposition 3.5]. (cf. Sect. 3.2).

In this sense there is a one-to-one correspondence between quasi-Poisson G^{V_Γ}-manifolds with $\overline{\sigma_\Gamma}^!$-twisted moment maps and quasi-Hamiltonian $(\mathfrak{d}^{V_\Gamma}, \mathfrak{g}^{V_\Gamma}) \times G^{E_\Gamma}$-manifolds.

Proposition 7 *Suppose* (M, ρ, π) *is a quasi-Poisson* G^{V_Γ}*-manifold with* $\overline{\sigma_\Gamma}^!$*-twisted moment map, and let*

$$R : (\mathbb{T}, T)M \dashrightarrow (\mathfrak{d}^{V_\Gamma}, \mathfrak{g}^{V_\Gamma}) \times G^{E_\Gamma}$$

be the corresponding quasi-Hamiltonian $(\mathfrak{d}^{V_\Gamma}, \mathfrak{g}^{V_\Gamma}) \times G^{E_\Gamma}$*-structure on* M*. Let* R^* *denote the fusion of* R *at the ordered pair of vertices* $(P, Q) \subset V_\Gamma$*. Then the bivector field for the quasi-Poisson* $G^{V_{\Gamma^*}}$*-structure corresponding to* R^* *is*

$$\pi^* := \pi + \rho(\tau),$$

where $\tau \in \wedge^2(\mathfrak{g}^{V_\Gamma})$ *is the insertion of* $\psi \in \bigwedge^2(\mathfrak{g}^P \oplus \mathfrak{g}^Q)$*,*

$$\psi = \frac{1}{2}\sum_{i,j} s^{ij}\,(\xi_i, 0) \wedge (0, \xi_j)$$

at the P, Q*th factors. Here* $s = \sum_{i,j} s^{ij}\,\xi_i \otimes \xi_j$ *in some basis* ξ_i *of* $\mathfrak{g}$*.*

Thus, (up to a sign difference) fusion in the sense of Sect. 5.1 is precisely the same as fusion in the sense of [3, 30].

Proof Let $\mathfrak{k}_* = (\mathfrak{g}^*_{\bar{\Delta}})^{V_{\Gamma^*}}$. A straightforward computation shows that

$$\mathfrak{k}_* \circ R_{(P,Q)} = \{(\xi + \tau^\sharp(\xi)) \mid \xi \in \mathfrak{k}\}.$$

By definition, $\mathsf{gr}\big((\pi^*)^\sharp\big) = \mathfrak{k}_* \circ R^*$. Thus, we see from

$$\mathfrak{k}_* \circ R_{(P,Q)} \circ R = \mathsf{gr}(\pi) + \mathsf{gr}(\rho(\tau))$$

that $\pi^* = \pi + \rho(\tau)$.

Remark 21 (**Sign differences**) In [30], the bivector field resulting from fusing the ordered pair (P, Q) is defined to be $\pi^* = \pi - \rho(\tau)$. This difference is essentially due to the fact that we orient our boundary graph to agree with the orientation of $\partial\Sigma$, whereas the opposite convention is used in [30].

Proof (Proof of Theorem 9) Propositions 6 and 7 show that it suffices to prove the equivalent statement for quasi-Poisson G^{V_Γ}-structures. However, [30, Theorem 2] and [30, Theorem 4] shows there exists a unique quasi-Poisson G^{V_Γ}-structure on $\mathcal{M}_{\Sigma,V}(G)$ with $\overline{\sigma_\Gamma}^!$-twisted moment map which is compatible with fusion (notice that Proposition 5 implies the first two properties of [30, Theorem 2] are automatically satisfied).

Corollary 2 *The proof of Theorem 9 also shows that the bivector field on $\mathcal{M}_{\Sigma,V}$ is given by* (11).

Appendix

A Proofs of Reduction Theorems

Before proving Theorems 5 and 6 and Proposition 2, we first establish some lemmas.

Lemma 4 *Suppose that $(\eta; Z) \in R_{\mathfrak{c},S}$, where $\eta \in \mathfrak{d}_\mathfrak{c}$, $Z \in TM_{\mathfrak{c},S}$ and $R_{\mathfrak{c},S}$ is as in Theorem 5. Then*

1. *$\eta \in \mathfrak{h}_\mathfrak{c}$, and*
2. *$\eta = 0$ only if $Z = 0$.*

Proof Let $\xi \in \mathfrak{d}$, $X \in TM$ and $\alpha \in T^*M$ be chosen so that $(\xi; X + \alpha) \in R$ and

$$(\xi; X + \alpha) \sim_{(R_2 \times R_1)} (\eta; Z).$$

Since $R_1 = R_{q_M} \circ R_{i_M}^\top$, it follows that $\alpha \in \mathsf{ann}(T\mu^{-1}S)$. Consequently, there exists $\tilde{\alpha} \in \mathsf{ann}(TS)$ such that $\alpha = \mu^*\tilde{\alpha}$. Since S is $\mathfrak{c}$ invariant,

$$\zeta_\alpha := \mathbf{a}^*\tilde{\alpha} \in \mathfrak{c}^\perp.$$

Moreover, since R is supported on the graph of μ, $(\zeta_\alpha, \alpha) = \mathbf{a}^*(-\tilde{\alpha}, \mu^*\tilde{\alpha}) \in R$. Thus

$$(\xi - \zeta_\alpha; X) \in R \tag{39}$$

and

$$(\xi - \zeta_\alpha; X) \sim_{(R_2 \times R_1)} (\eta; Z). \tag{40}$$

Since (22) is a morphism of Manin pairs, axiom (m1) of Definition 5 implies that

$$\xi - \zeta_\alpha \in \mathfrak{h}. \tag{41}$$

Thus (40) implies that $\eta \in \mathfrak{h}_\mathfrak{c}$, establishing the first claim.

Next, suppose $\eta = 0$, then (40) implies that $\xi - \zeta_\alpha \in \mathfrak{c}^\perp$ in addition to (41). That is $\xi - \zeta_\alpha \in \mathfrak{c}^\perp \cap \mathfrak{h}$, and hence Eqs. (19) and (39) imply that X is tangent to the $\mathfrak{c}^\perp \cap \mathfrak{h}$. Therefore $Z = q_M(X) = 0$, establishing the second claim.

Lemma 5 *Under the assumptions of Theorem 5, the Courant relation*

$$R_2 \times R_1 : (\mathfrak{d} \times N) \times \overline{\mathbb{T}M} \dashrightarrow (\mathfrak{d}_\mathfrak{c} \times N_{\mathfrak{c},S}) \times \overline{\mathbb{T}M_{\mathfrak{c},S}}$$

composes cleanly with the Dirac structure $R \subseteq (\mathfrak{d} \times N) \times \overline{\mathbb{T}M}$.

Moreover, the composition

$$R_{\mathfrak{c},S} := R_2 \circ R \circ R_1^\top$$

is a well defined subbundle of $(\mathfrak{d}_\mathfrak{c} \times N_{\mathfrak{c},S}) \times \overline{\mathbb{T}M_{\mathfrak{c},S}}$.

Proof We begin by proving that the composition $(R_2 \times R_1) \circ R$ is clean. For this, it is sufficient to show that

1. the rank of the intersections $\mathsf{ker}(R_2 \times R_1)^\perp \cap R$ and $\mathsf{ker}(R_2 \times R_1) \cap R$ are constant, and
2. the composition of the underlying relation of vector bundle bases,

$$\left(\mathsf{gr}(q_N \times q_M) \circ \mathsf{gr}(i_N \times i_M)^\top\right) \circ \mathsf{gr}(\mu)$$

is clean.

We now show that the rank of $\mathsf{ker}(R_2 \times R_1) \cap R$ is constant. We claim that the sequence

$$0 \to \mathfrak{h} \cap \mathfrak{c}^\perp \xrightarrow{\xi \to (\xi, \rho_R(\xi))} \mathsf{ker}(R_2 \times R_1) \cap R \xrightarrow{(\xi, X+\alpha) \to \alpha} \mathsf{ann}(T\mu^{-1}S) \to 0 \tag{42}$$

is exact, where ρ_R is defined in (19).

First, the second map is surjective: for any $\alpha \in \mathsf{ann}(T\upmu^{-1}S)$, let $\tilde{\alpha} \in \mathsf{ann}(TS)$ be chosen so that $\upmu^*\tilde{\alpha} = \alpha$. Since S is $\mathfrak{c}$ invariant,

$$\zeta_\alpha := \mathbf{a}^*\tilde{\alpha} \in \mathfrak{c}^\perp.$$

Moreover, since R is supported on the graph of $\upmu$,

$$(\zeta_\alpha, \alpha) = \mathbf{a}^*(-\tilde{\alpha}, \upmu^*\tilde{\alpha}) \in \mathsf{ker}(R_2 \times R_1) \cap R.$$

Next, we prove exactness at $\mathsf{ker}(R_2 \times R_1) \cap R$. Suppose $(\xi, X) \in \mathsf{ker}(R_2 \times R_1) \cap R$. Since (22) is a morphism of Manin pairs, $\xi \in \mathfrak{h}$ and $X = \rho_R(\xi - \zeta_\alpha)$. Since $\xi \in \mathsf{ker}(R_2) = \mathfrak{c}^\perp$, we conclude that $\xi \in \mathfrak{h} \cap \mathfrak{c}^\perp$.

This shows that the sequence (42) is exact. Since $\mathfrak{h} \cap \mathfrak{c}^\perp$ and $\mathsf{ann}(T\upmu^{-1}S)$ are both of constant rank, so is $\mathsf{ker}(R_2 \times R_1) \cap R$. Consequently $\big(\mathsf{ker}(R_2 \times R_1) \cap R\big)^\perp = \mathsf{ker}(R_2 \times R_1)^\perp + R$ is also of constant rank, and thus so is $\mathsf{ker}(R_2 \times R_1)^\perp \cap R$.

Next, we need to show that the following composition of relations is clean:

$$\begin{array}{ccc} M & \xrightarrow{\ \mu\ } & N \\ \uparrow{\scriptstyle i_M} & & \uparrow{\scriptstyle i_N} \\ \tilde{S} & & S \\ \downarrow{\scriptstyle q_M} & & \downarrow{\scriptstyle q_N} \\ M_{\mathfrak{c},S} & \overset{\tilde{\mu}}{\dashrightarrow} & N_{\mathfrak{c},S} \end{array}$$

where, for brevity, we have introduced the notation $\tilde{S} := \upmu^{-1}(S)$. Since $\mathsf{gr}(\upmu)$ intersects $S \times M$ cleanly, the composition $\mathsf{gr}(\upmu|_{\tilde{S}}) = \mathsf{gr}(i_N)^\top \circ \mathsf{gr}(\upmu \circ i_M)$ is clean

$$\begin{array}{ccc} M & \xrightarrow{\ \mu\ } & N \\ \uparrow{\scriptstyle i_M} & & \uparrow{\scriptstyle i_N} \\ \tilde{S} & \xrightarrow{\ \mu|_{\tilde{S}}\ } & S \\ \downarrow{\scriptstyle q_M} & & \downarrow{\scriptstyle q_N} \\ M_{\mathfrak{c},S} & \overset{\tilde{\mu}}{\dashrightarrow} & N_{\mathfrak{c},S} \end{array}$$

Next,

$$\mathsf{gr}(q_N \circ \upmu|_{\tilde{S}}) \times \mathsf{gr}(q_M)^\top \subseteq N_{\mathfrak{c},S} \times \tilde{S} \times \tilde{S} \times M_{\mathfrak{c},S}$$

intersects $N_{\mathfrak{c},S} \times \tilde{S}_\Delta \times M_{\mathfrak{c},S}$ transversely, since q_M and $q_N \circ \upmu|_{\tilde{S}}$ are both maps. Moreover, since $M_{\mathfrak{c},S}$ is the set of $\mathfrak{h} \cap \mathfrak{c}^\perp$ orbits in $\tilde{S}$, while $N_{\mathfrak{c},S}$ is the set of $\mathfrak{c}^\perp$ orbits in S, the projection

$$\Big(\mathsf{gr}(q_N \circ \upmu|_{\tilde{S}}) \times \mathsf{gr}(q_M)^\top\Big) \cap \Big(N_{\mathfrak{c},S} \times \tilde{S}_\Delta \times M_{\mathfrak{c},S}\Big) \to \mathsf{gr}(\tilde{\upmu})$$

is a surjective submersion. Thus, by definition, $\mathsf{gr}(q_N \circ \mu|_{\tilde{S}})$ composes cleanly with $\mathsf{gr}(q_M)^\top$.

It follows that $R_2 \times R_1$ composes cleanly with R.

Finally, since $\{(\xi; \rho_R(\xi)) \mid \xi \in \mathfrak{h} \cap \mathfrak{c}^\perp\} \subseteq R$, it follows that R is $\mathfrak{h} \cap \mathfrak{c}^\perp$ invariant. Hence $R_{\mathfrak{c},S} := R_2 \circ R \circ R_1^\top$ is a well defined subbundle of

$$\left(\mathfrak{d}_{\mathfrak{c}} \times D_{\mathfrak{c}}/G_{\mathfrak{c}}\right) \times \overline{\mathbb{T}Q}.$$

We are now ready to prove Theorems 5 and 6.

Proof (Proof of 5) Since $N_{\mathfrak{c},S}$ is the space of $\mathfrak{c}^\perp$-orbits of S, while $M_{\mathfrak{c},S}$ is the space of $\mathfrak{c}^\perp \cap \mathfrak{h}$ orbits of $\mu^{-1}(S)$, the $\mathfrak{h}$-equivariant map

$$\mu : \mu^{-1}(S) \to S$$

descends to a define a unique map

$$\tilde{\mu} : M_{\mathfrak{c},S} \to N_{\mathfrak{c},S}.$$

The composition $R_{\mathfrak{c},S} := R_2 \circ R \circ R_1^\top$ is supported on the graph of $\tilde{\mu}$. Thus Lemma 5 and Proposition 1 shows that

$$R_{\mathfrak{c},S} : \mathbb{T}M_{\mathfrak{c},S} \dashrightarrow \mathfrak{d}_{\mathfrak{c}} \times N_{\mathfrak{c},S}$$

is a Courant morphism.

Finally, Lemma 4 proves that (23) satisfies the defining conditions for a morphism of Manin pairs.

Proof (Proof of 6) We need to show that (23) is a exact morphism of Manin pairs. We do this by first showing that $\mathfrak{d}_{\mathfrak{c}} \times N_{\mathfrak{c},S}$ is an exact Courant algebroid along the image of $\mu(M) \cap S$, and next by showing that the anchor maps $R_{\mathfrak{c},S}$ surjectively onto $T\mathsf{gr}(\mu_{\mathfrak{c},S} : M_{\mathfrak{c},S} \to N_{\mathfrak{c},S})$.

The fact that $\mathfrak{d}_{\mathfrak{c}} \times N_{\mathfrak{c},S}$ is exact follows from [13, Theorem 3.3], but we include a proof here anyways. We must show that

$$0 \to T^*N_{\mathfrak{c},S} \xrightarrow{\mathbf{a}^*} \mathfrak{d}_{\mathfrak{c}} \times N_{\mathfrak{c},S} \xrightarrow{\mathbf{a}} TN_{\mathfrak{c},S} \to 0 \tag{43}$$

is an exact sequence. By assumption, $\mathfrak{c}$ acts transitively on S. Thus $\mathfrak{d}_{\mathfrak{c}} := \mathfrak{c}/\mathfrak{c}^\perp$ acts transitively on $N_{\mathfrak{c},S} := S/\mathfrak{c}^\perp$. It follows that the sequence (43) is exact at $TN_{\mathfrak{c},S}$, and hence, by duality, also at $T^*N_{\mathfrak{c},S}$.

Next, (43) is exact at $\mathfrak{d}_{\mathfrak{c}} \times N_{\mathfrak{c},S}$ if and only if $\mathsf{ker}(\mathbf{a})$ is isotropic. This, in turn, holds if and only if $\mathfrak{c} \cap (\mathsf{ker}(\mathbf{a}_{\mathfrak{d}}) + \mathfrak{c}^\perp)$ is isotropic, where $\mathbf{a}_{\mathfrak{d}}$ denotes the anchor map for $\mathfrak{d} \times N$. But

$$\left(\mathfrak{c} \cap (\mathsf{ker}(\mathbf{a}_{\mathfrak{d}}) + \mathfrak{c}^\perp)\right)^\perp = \mathfrak{c}^\perp + \mathsf{Im}(\mathbf{a}_{\mathfrak{d}}^*) \cap \mathfrak{c}^\perp = \mathfrak{c} \cap (\mathsf{Im}(\mathbf{a}_{\mathfrak{d}}^*) + \mathfrak{c}^\perp).$$

Therefore, (43) is exact at $TN_{\mathfrak{c},S}$ if $\mathsf{Im}(\mathbf{a}_{\mathfrak{d}}^*) = \mathsf{ker}(\mathbf{a}_{\mathfrak{d}})$, which holds whenever $\mathfrak{d} \times N$ is exact (as it is along $\mu(M)$).

Next we need to show that the anchor maps $R_{\mathfrak{c},S}$ surjectively onto $T\mathsf{gr}(\mu_{\mathfrak{c},S} : M_{\mathfrak{c},S} \to N_{\mathfrak{c},S})$. More precisely, for any $Z \in TM_{\mathfrak{c},S}$, we must show there exists $\eta \in \mathfrak{d}_{\mathfrak{c}}$ and $\gamma \in T^*M_{\mathfrak{c},S}$ such that

$$(\eta, Z + \gamma) \in R_{\mathfrak{c},S}.$$

Let $X \in T\mu^{-1}(S)$ be chosen so that it maps to Z under the quotient map $q_M : \mu^{-1}(S) \to M_{\mathfrak{c},S}$. Since (22) is exact, there exists $\alpha \in T^*M$ and $\xi \in \mathfrak{d}$ such that $(\xi, X + \alpha) \in R$. Now R is supported on the graph of μ, so $\mathbf{a}_{\mathfrak{d}}(\xi) = d\mu(X) \in TS$. Since $\mathfrak{c}$ acts transitively on S, we must have $\xi \in \mathfrak{c} + \mathsf{ker}(\mathbf{a}_{\mathfrak{d}})$. Since $\mathfrak{d} \times N$ is exact, $\mathsf{ker}(\mathbf{a}_{\mathfrak{d}}) = \mathsf{Im}(\mathbf{a}_{\mathfrak{d}}^*)$. Thus there exists $\xi' \in \mathfrak{c}$ and $\beta \in T^*N$ such that $\xi = \xi' + \mathbf{a}_{\mathfrak{d}}^*\beta$. Since R is supported on the graph of μ, $(\mathbf{a}_{\mathfrak{d}}^*\beta, \mu^*\beta) \in R$, and thus

$$(\xi', X + \alpha - \mu^*\beta) \in R.$$

Since R is Lagrangian, pairing this element with $(\zeta, \rho_R(\zeta)) \in R$, where $\zeta \in \mathfrak{h} \cap \mathfrak{c}^{\perp}$, we see that

$$0 = \langle \xi', \zeta \rangle = \langle \alpha - \mu^*\beta, \rho_R(\zeta) \rangle.$$

Since $\zeta \in \mathfrak{h} \cap \mathfrak{c}^{\perp}$ was arbitrary, this shows that $\alpha - \mu^*\beta = q_M^*\gamma$ for some $\gamma \in T^*M_{\mathfrak{c},S}$.

Thus, we have shown that

$$(\xi' + \mathfrak{c}^{\perp}, Z + \gamma) \in R_{\mathfrak{c},S},$$

where $\xi' + \mathfrak{c}^{\perp}$ is the image of ξ' under the quotient map $\mathfrak{c} \to \mathfrak{c}/\mathfrak{c}^{\perp}$. Since $Z \in TM_{\mathfrak{c},S}$ was arbitrary, we may conclude that (23) is exact.

Proof (Proof of 2) First we show that the 2-form $\omega_\theta := i_M^*\omega - \mu^*\langle \theta, \vartheta_s - \frac{1}{2} s \circ \mathbf{a}(\theta) \rangle \in \Omega^2\big(\mu^{-1}(S)\big)$ is $\mathfrak{h} \cap \mathfrak{c}^{\perp}$-invariant and basic. To show invariance, note that (by definition) $(\xi, \rho_R(\xi)) \in R$ for any $\xi \in \mathfrak{h}$, and thus R is $\mathfrak{h}$-invariant. Since $s : TN \to \mathfrak{d} \times N$ is an $\mathfrak{d}$-invariant splitting, it follows that $R_{\mu,\omega}$ and hence $\omega \in \Omega^2(M)$ is $\mathfrak{h}$-invariant. Additionally, s, $\mathbf{a}$ and ϑ_s are $\mathfrak{d}$-equivariant, while θ is $\mathfrak{c}^{\perp}$-equivariant. Hence $\langle \theta, \vartheta_s - \frac{1}{2} s \circ \mathbf{a}(\theta) \rangle$ is $\mathfrak{c}^{\perp}$-invariant. It follows that the sum, ω_θ is $\mathfrak{h} \cap \mathfrak{c}^{\perp}$-invariant.

The isomorphism $\mathfrak{d} \times N \to \mathbb{T}_\gamma N$ is given by $\xi \to \mathbf{a}(\xi) + s^*(\xi)$, for all $\xi \in \mathfrak{d}$. Thus, for $\xi, \eta \in \mathfrak{d}$, we have

$$\langle \xi, \eta \rangle = \langle \mathbf{a}(\xi) + s^*(\xi), \mathbf{a}(\eta) + s^*(\eta) \rangle = \langle s \circ \mathbf{a}(\xi), \eta \rangle + \langle \xi, s \circ \mathbf{a}(\eta) \rangle.$$

Thus, since $\mathfrak{c}^{\perp}$ is coisotropic, the assignment $\xi, \eta \to \langle \xi, s \circ \mathbf{a}\eta \rangle$ defines a skew-symmetric form on $\mathfrak{c}^{\perp}$.

Now suppose $\xi \in \mathfrak{c}^{\perp}$, then

$$\begin{aligned}\iota_{\mathbf{a}(\xi)}\langle\theta,\vartheta_s-\frac{1}{2}s\circ\mathbf{a}(\theta)\rangle &= \langle\xi,\vartheta_s-\frac{1}{2}s\circ\mathbf{a}(\theta)\rangle-\langle\theta,s\circ\mathbf{a}(\xi)-\frac{1}{2}s\circ\mathbf{a}(\xi)\rangle\\ &= s^*\xi-\frac{1}{2}\langle\xi,s\circ\mathbf{a}(\theta)\rangle-\langle\theta,s\circ\mathbf{a}(\xi)-\frac{1}{2}s\circ\mathbf{a}(\xi)\rangle\\ &= s^*\xi, \end{aligned}\tag{44}$$

where the last line follows from the skew-symmetry of the assignment $\xi,\eta\to\langle\xi,s\circ\mathbf{a}(\eta)\rangle$. Now, for $\xi\in\mathfrak{h}$, we have $\mu_*\rho_R(\xi)=\mathbf{a}(\xi)$, and thus (28) implies that

$$\rho_R(\xi)-\iota_{\rho_R(\xi)}\omega+\iota_{\rho_R(\xi)}\mu^*\langle\theta,\vartheta_s-\frac{1}{2}s\circ\mathbf{a}(\theta)\rangle\sim_{R_{\mu,\omega}}\mathbf{a}(\xi)+s^*(\xi),\quad \xi\in\mathfrak{h}\cap\mathfrak{c}^\perp.\tag{45}$$

Since (29) is a morphism of Manin pairs, and since $\mathbf{a}(\xi)+s^*(\xi)\in E$ for any $\xi\in\mathfrak{h}$, it follows that the left hand side of (45) lies in TM. That is, $\iota_{\rho_R(\xi)}\omega_\theta=0$ for any $\xi\in\mathfrak{h}\cap\mathfrak{c}^\perp$. We conclude that there is a unique 2-form $\tilde{\omega}_\theta\in\Omega^2(M_{\mathfrak{c},S})$ such that $q_M^*\tilde{\omega}_\theta=\omega_\theta$.

We define the Courant relation

$$R_{\mathfrak{c},s,\theta}:=\mathsf{gr}(\mathbf{a}\oplus\tilde{s}_\theta^*)\circ R_{\mathfrak{c}}\circ\mathsf{gr}(\mathbf{a}\oplus s^*)^\top:\mathbb{T}_\gamma N\dashrightarrow\mathbb{T}_{\gamma_\theta}N_{\mathfrak{c},S},$$

so that

$$\mathsf{gr}(\mathbf{a}\oplus\tilde{s}_\theta^*)\circ R_{\mathfrak{c},S}=R_{\mathfrak{c},s,\theta}\circ R_{\mu,\omega}\circ\mathsf{gr}(i_M)\circ\mathsf{gr}(q_M)^\top.$$

Now the definition of s_θ shows that for any $X\in\mathfrak{X}(N_{\mathfrak{c},S})$, we have

$$X^h+\iota_{X^h}\langle\theta,\vartheta_s\rangle\sim_{R_{\mathfrak{c},s,\theta}}X.$$

Since X^h is horizontal with respect to θ and $X=(q_N)_*X^h$, we also have

$$X^h+\iota_{X^h}\langle\theta,\vartheta_s-\frac{1}{2}s\circ\mathbf{a}(\theta)\rangle\sim_{R_{\mathfrak{c},s,\theta}}(q_N)_*X^h.$$

On the other hand, for $\xi\in\mathfrak{c}^\perp$, we have $\mathbf{a}(\xi)+s^*(\xi)\sim_{R_{\mathfrak{c},s,\theta}}0$, so (44) shows that

$$\mathbf{a}(\xi)+\iota_{\mathbf{a}(\xi)}\langle\theta,\vartheta_s-\frac{1}{2}s\circ\mathbf{a}(\theta)\rangle\sim_{R_{\mathfrak{c},s,\theta}}(q_N)_*\mathbf{a}(\xi).$$

Therefore, for any $X\in TS$,

$$X+\iota_X\langle\theta,\vartheta_s-\frac{1}{2}s\circ\mathbf{a}(\theta)\rangle\sim_{R_{\mathfrak{c},s,\theta}}(q_N)_*X,$$

since this relation holds for both horizontal and vertical vector fields on the bundle $q_N:S\to N_{\mathfrak{c},S}$. This implies $R_{\mathfrak{c},s,\theta}\circ R_{\mu,\omega}\circ\mathsf{gr}(i_M)=R_{\mu,\omega_\theta}$. Finally, since $\omega_\theta=q_M^*\tilde{\omega}_\theta$, we conclude that

$$R_{\mathfrak{c},s,\theta} \circ R_{\mu,\omega} \circ \mathsf{gr}(i_M) \circ \mathsf{gr}(q_M)^\top = R_{\tilde{\mu},\tilde{\omega}_\theta},$$

which proves the first part of the proposition.

Now suppose that $s(TS) \subseteq \mathfrak{c}$, then $s^*(\mathfrak{c}^\perp) \subseteq \mathsf{ann}(TS)$, so $\vartheta_s = 0$ and $s^*\theta = 0$. Thus $\langle \theta, \vartheta_s - \frac{1}{2} s \circ \mathbf{a}(\theta)\rangle = 0$, and $\tilde{\omega}_\theta$ is independent of θ. This concludes the proof of the proposition.

Proof of the Commutativity of Partial Reduction

Proof (Proof of Theorem 3) By symmetry, it is sufficient to prove the first equality in (34). The key fact is that when $(\mathfrak{c}_1 \cap \mathfrak{c}_2, S_1 \cap S_2)$ is reductive data then $\mathfrak{c}_1 \cap \mathfrak{c}_2$ is coisotropic. As a result,

$$\mathfrak{c}_1^\perp \subseteq (\mathfrak{c}_1 \cap \mathfrak{c}_2)^\perp \subseteq \mathfrak{c}_1 \cap \mathfrak{c}_2 \subseteq \mathfrak{c}_2. \tag{46}$$

Thus,

$$\mathfrak{c}_{2,1} = (\mathfrak{c}_1 \cap \mathfrak{c}_2)/(\mathfrak{c}_1^\perp \cap \mathfrak{c}_2) = (\mathfrak{c}_1 \cap \mathfrak{c}_2)/\mathfrak{c}_1^\perp,$$

and

$$\mathfrak{c}_{2,1}^\perp = (\mathfrak{c}_1^\perp + \mathfrak{c}_2^\perp)/\mathfrak{c}_1^\perp.$$

Hence $\xi \in \mathfrak{d}$, $\xi' \in \mathfrak{d}_{\mathfrak{c}_1}$ and $\xi'' \in (\mathfrak{d}_{\mathfrak{c}_1})_{\mathfrak{c}_{21}}$ satisfy

$$\xi \sim_{R_{\mathfrak{c}_1}} \xi' \sim_{R_{\mathfrak{c}_{2,1}}} \xi''$$

if and only if

$$\xi \in \mathfrak{c}_1, \quad \xi' = \xi + \mathfrak{c}_1^\perp, \quad \xi' \in (\mathfrak{c}_1 \cap \mathfrak{c}_2) + \mathfrak{c}_1^\perp, \text{ and } \xi'' = \xi' + \mathfrak{c}_1^\perp + \mathfrak{c}_2^\perp.$$

Equivalently,

$$\xi \in \mathfrak{c}_1 \cap \mathfrak{c}_2, \quad \xi'' = \xi + (\mathfrak{c}_1 \cap \mathfrak{c}_2)^\perp, \text{ and } \xi' = \xi + \mathfrak{c}_1.$$

So

$$R_{\mathfrak{c}_{2,1}} \circ R_{\mathfrak{c}_1} = R_{\mathfrak{c}_1 \cap \mathfrak{c}_2}. \tag{47}$$

Before continuing, we introduce some notation. Let $\mu_{\mathfrak{c}_1,S_1} : M_{\mathfrak{c}_1,S_1} \to N_{\mathfrak{c}_1,S_1}$ be the function whose graph is the support of $R_{\mathfrak{c}_1,S_1}$, let

$$\begin{aligned} i_{N_1} &: S_1 \to N, & i_{M_1} &: \mu^{-1}(S_1) \to M, \\ i_N &: S_1 \cap S_2 \to N, & i_M &: \mu^{-1}(S_1 \cap S_2) \to M, \\ i_{N_{21}} &: S_{2,1} \to N_{\mathfrak{c}_1,S_1}, & i_{M_{21}} &: \mu^{-1}_{\mathfrak{c}_1,S_1}(S_{2,1}) \to M_{\mathfrak{c}_1,S_1} \end{aligned}$$

denote the inclusions, and

$$\begin{aligned} q_{N_1} &: S_1 \to N_{\mathfrak{c}_1,S_1}, & q_{M_1} &: \mu^{-1}(S_1) \to M_{\mathfrak{c}_1,S_1}, \\ q_N &: S_1 \cap S_2 \to N_{\mathfrak{c}_1\cap\mathfrak{c}_2,S_1\cap S_2}, & q_M &: \mu^{-1}(S_1 \cap S_2) \to M_{\mathfrak{c}_1\cap\mathfrak{c}_2,S_1\cap S_2}, \\ q_{N_{2\,1}} &: S_{2,1} \to (N_{\mathfrak{c}_1,S_1})_{\mathfrak{c}_{2\,1},S_{2\,1}}, & q_{M_{2\,1}} &: \mu^{-1}_{\mathfrak{c}_1,S_1}(S_{2,1}) \to (M_{\mathfrak{c}_1,S_1})_{\mathfrak{c}_{2\,1},S_{2\,1}} \end{aligned}$$

denote the quotient maps.

Now

$$\begin{aligned} (R_{\mathfrak{c}_1,S_1})_{\mathfrak{c}_{2\,1},S_{2\,1}} = \Big(&\big(R_{\mathfrak{c}_{2,1}} \circ R_{\mathfrak{c}_1}\big) \times \mathsf{gr}(q_{N_{2\,1}}) \circ \mathsf{gr}(i_{N_{2\,1}})^\top \circ \mathsf{gr}(q_{N_1}) \circ \mathsf{gr}(i_{N_1})^\top \Big) \\ &\circ R \circ \big(R_{q_{N_{2\,1}}} \circ R^\top_{i_{N_{2\,1}}} \circ R_{q_{N_1}} \circ R^\top_{i_{N_1}}\big)^\top \end{aligned}$$

Now

$$x \sim_{\mathsf{gr}(q_{N_1})\circ\mathsf{gr}(i_{N_1})^\top} y \sim_{\mathsf{gr}(q_{N_{21}})\circ\mathsf{gr}(i_{N_{21}})^\top} z \tag{48}$$

if and only if

$$x \in S_1, \quad y = q_{N_1}(x), \quad y \in S_{2,1}, \quad \text{and } z = q_{N_{21}}(y).$$

Eq. (46) implies that the $\mathfrak{c}_1 \cap \mathfrak{c}_2$ invariant manifold $S_1 \cap S_2$ is also $\mathfrak{c}_1^\perp$ invariant. Since $S_{2,1}$ is the set of $\mathfrak{c}_1^\perp$ orbits in $S_1 \cap S_2$, it follows that $y \in S_{2,1}$ if and only if $x \in S_1 \cap S_2$. Thus, (48) holds if and only if

$$x \in S_1 \cap S_2, \quad z = q_{N_{21}} \circ q_{N_1}(x), \text{ and } y = q_{N_1}(x).$$

But

$$q_{N_{21}} \circ q_{N_1}|_{S_1\cap S_2} : S_1 \cap S_2 \to N_{\mathfrak{c}_1\cap\mathfrak{c}_2,S_1\cap S_2}$$

is the quotient map for the $\mathfrak{c}_1^\perp + \mathfrak{c}_2^\perp$ action, i.e. $q_{N_{21}} \circ q_{N_1}|_{S_1\cap S_2} = q_N$. Therefore

$$\mathsf{gr}(q_{N_{21}}) \circ \mathsf{gr}(i_{N_{21}})^\top \circ \mathsf{gr}(q_{N_1}) \circ \mathsf{gr}(i_{N_1})^\top = \mathsf{gr}(q_N) \circ \mathsf{gr}(i_N)^\top. \tag{49}$$

A similar calculation shows that

$$\mathsf{gr}(q_{M_{21}}) \circ \mathsf{gr}(i_{M_{21}})^\top \circ \mathsf{gr}(q_{M_1}) \circ \mathsf{gr}(i_{M_1})^\top = \mathsf{gr}(q_M) \circ \mathsf{gr}(i_M)^\top,$$

and since the composition is clean we also have

$$R_{q_{N_{21}}} \circ R_{i_{N_{21}}}^{\top} \circ R_{q_{N_1}} \circ R_{i_{N_1}}^{\top} = R_{q_M} \circ R_{i_M}^{\top} \tag{50}$$

Combining Eqs. (47), (49) and (50) shows that

$$(R_{\mathfrak{c}_1,S_1})_{\mathfrak{c}_{21},S_{21}} = \big(R_{\mathfrak{c}_1\cap\mathfrak{c}_2} \times (\mathsf{gr}(q_N) \circ \mathsf{gr}(i_N)^{\top})\big) \circ R \circ (R_{q_M} \circ R_{i_M}^{\top})^{\top} = R_{\mathfrak{c}_1\cap\mathfrak{c}_2,S_1\cap S_2}.$$

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Groupoids, Frobenius Algebras and Poisson Sigma Models

Ivan Contreras

Abstract This note is devoted to report some results proven in [5, 8] and some work in progress [6] concerning the relation between groupoids and Frobenius algebras specialized in the case of Poisson sigma models with boundary. We prove a correspondence between groupoids in **Set** and relative Frobenius algebras in **Rel**, as well as an adjunction between a special type of semigroupoids and relative H*-algebras. The connection between groupoids and Frobenius algebras is made explicit by introducing what we called weak monoids and relational symplectic groupoids, in the context of Poisson sigma models with boundary and in particular, describing such structures in the extended symplectic category and the category of Hilbert spaces. This is part of a joint work with Alberto Cattaneo and Chris Heunen.

1 Introduction

As we know, groupoid structures appear in several scenarios: Lie theory as generalization of Lie groups, in noncommutative geometry, foliation theory, Poisson geometry, the study of stacks, among others. On the other hand, Frobenius algebras appear, for example, as an equivalent way to understand two dimensional topological quantum field theories (2-TQFT) and it is possible to define them in more generality in monoidal dagger categories.

In [8], the connection between groupoids and Frobenius algebras is made precise. Namely, there is a way to understand groupoids in the category **Set** as what we called *Relative Frobenius algebras*, a special type of dagger Frobenius algebra in the category **Rel**, where the objects are sets and the morphisms are relations.

In addition, there exists an adjunction between a special type of semigroupoids (a more relaxed version of groupoids where the identities or inverses do not necessarily exist) and H^*—algebras, a structure similar to Frobenius algebras but without unitality conditions and a more relaxed Frobenius relation.

I. Contreras (✉)
Department of Mathematics, University of California, Berkeley, CA 94305, USA
e-mail: ivan.contreras@berkeley.edu

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In particular, this correspondence between groupoids and relative Frobenius algebras can be studied in the context of Poisson sigma models (PSM), a particular 2-dimensional topological field theory, where the reduced phase space, for an integrable Poisson manifold M as target space, has the structure of a symplectic groupoid integrating M. In [5], we study the non reduced phase space of PSM with boundary and we construct what we call a *relational symplectic groupoid*, that is, roughly speaking, a symplectic groupoid up to algebroid homotopy, where the space of morphisms is allowed to be an infinite dimensional weak symplectic manifold and the structure maps of the groupoid are replaced by immersed canonical relations, which are morphisms in the extended symplectic "category", denoted by $\mathbf{Symp}^{ext}$.[1]

The study of the non reduced phase space is relevant for the description of general Lagrangian field theories with boundary, following the work of Cattaneo, Mnëv and Reshetikhin in [4]. The interesting features of the relational symplectic groupoids could be useful to describe similar constructions in other types of gauge theories.

In addition, it turns out that relational symplectic groupoids in the category **Hilb** of Hilbert spaces correspond to a special type of Frobenius algebras, whereas usual symplectic groupoids in **Hilb** are in correspondence with relative Frobenius algebras. This would correspond to the quantized version of the relational symplectic groupoid associated to the classical PSM with boundary, assuming that the quantization procedure is functorial.

2 Groupoids and Relative Frobenius Algebras

In this section, we consider a groupoid in **Set** as a category internal to the category **Set** of sets as objects and functions as morphisms. Now, consider the category **Rel** with sets and relations. In addition, this category carries an involution $\dagger : \mathbf{Rel}^{op} \to \mathbf{Rel}$ given by the transpose of relations; this is a contravariant involution and is the identity on objects, therefore, **Rel** is a dagger symmetric monoidal category that contains **Set** as a subcategory. For details on dagger monoidal categories, see e.g. [1, 2]. In **Rel** we define what we call *relative Frobenius algebra*, a special dagger Frobenius algebra.[2]

Definition 1 A morphism $m : X \times X \nrightarrow X$ in **Rel**[3] is called a special dagger Frobenius algebra or shortly, relative Frobenius algebra, if it satisfies the following axioms (see also Fig. 1)

- (*F*) $(1_X \times m) \circ (m^\dagger \times 1_X) = m^\dagger \circ m = (m \times 1_X) \circ (1_X \times m^\dagger)$,

[1] $\mathbf{Symp}^{ext}$ is not properly speaking a category, since the composition of canonical relations is not in general a smooth manifold; some transversality conditions are required. For our purposes, the smoothness of the composition of canonical relations will be guaranteed from the defining axioms of the relational symplectic groupoid.

[2] A dagger Frobenius algebra on the category **Hilb** of finite dimensional Hilbert spaces corresponds to the usual notion of Frobenius algebra.

[3] The symbol $\nrightarrow$ denotes that we are considering relations instead of maps as morphisms.

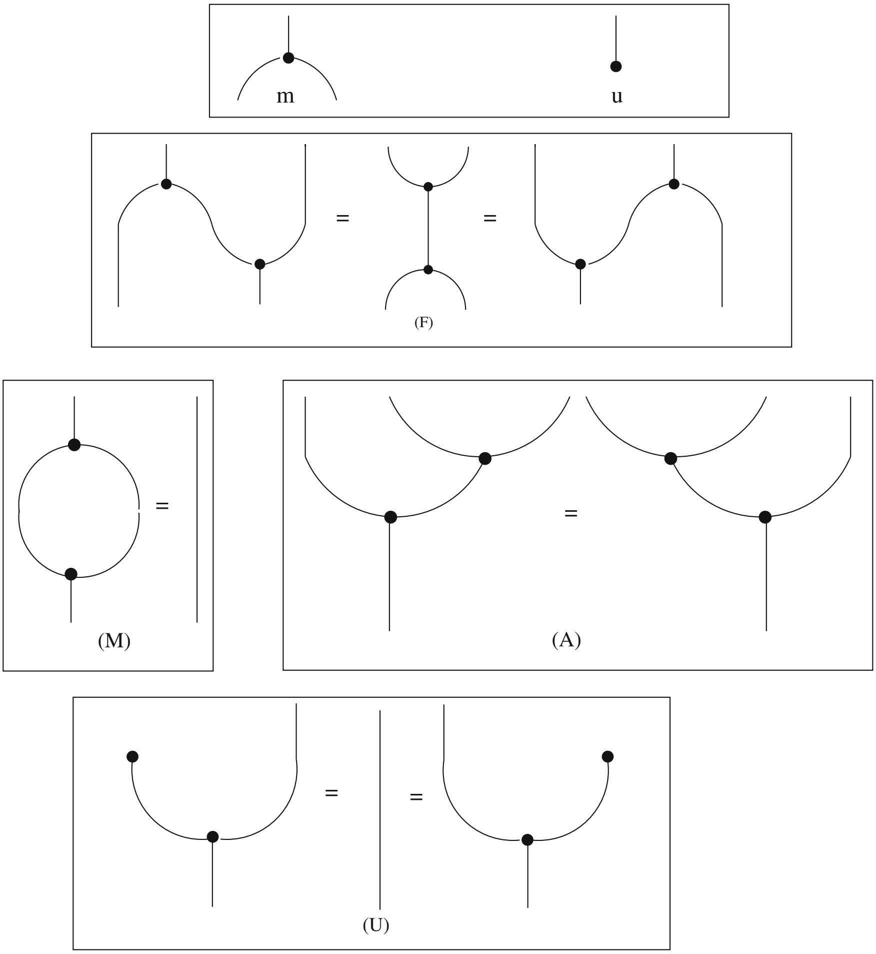

Fig. 1 Relative Frobenius algebra: diagrammatics

- (M) $m \circ m^\dagger = 1_X$,
- (A) $m \circ (1_X \times m) = m \circ (m \times 1_X)$,
- (U) $\exists u : 1 \nrightarrow X | m \circ (u \times 1_X) = 1 = m \circ (1_X \times u)$.

Remark 1 If such u exists, it is unique.

2.1 From Relative Frobenius Algebras to Groupoids

Here, from a given relative Frobenius algebra we construct a groupoid, but first of all, we give precise meaning of the axioms defined above. We will use the notation

$f = hg$ when $((h, g), f) \in m$ and we say that g and h are *composable*. First of all, observe that axiom (M) implies that m is single valued and that

$$\forall f \in X\, \exists g, \quad h \in X | f = hg.$$

The axiom (F) means that for all $a, b, c, d \in X$

$$ab = cd \iff \exists\, e \in X | b = ed, \quad c = ae \iff \exists e \in X | d = eb, \quad a = ce.$$

The axiom (A) is associativity, i.e. $(fg)h = f(gh)$. For the last axiom, after identifying the morphism $u : 1 \nrightarrow X$ with a subset $U \subseteq X$, we get that (U) is equivalent to the following assertions

$$\begin{aligned}
&\forall f \in X \exists\, u \in U | fu = f \\
&\forall f \in X \exists\, u \in U | uf = f \\
&\forall f \in X \forall u \in U | f \text{ and } u \text{ are composable} \Longrightarrow fu = f \\
&\forall f \in X \forall u \in U | u \text{ and } f \text{ are composable} \Longrightarrow uf = f.
\end{aligned}$$

From this data, we are able to give explicitly a groupoid in **Set**.

Definition 2 Given a relative Frobenius algebra (X, m), we define the following objects and morphisms in **Rel**:

$$\begin{aligned}
G_1 &= X, \\
G_2 &= \{(g, f) \in X^2 | g \text{ and } f \text{ are composable}\}, \\
G_0 &= U, \\
\varepsilon &= U \times U : G_0 \nrightarrow G_1, \\
s &= \{(f, x) \in G_1 \times G_0 | f \text{ and } x \text{ are composable}\} : G_1 \nrightarrow G_0 \\
t &= \{(f, y) \in G_1 \times G_0 | y \text{ and } f \text{ are composable}\} : G_1 \nrightarrow G_0 \\
\iota &= \{(g, f) \in G_2 | gf \in G_0,\ fg \in G_0\} : G_1 \nrightarrow G_1.
\end{aligned}$$

Using this description of the axioms, it is possible to prove the following

Proposition 1 *The data*

$$G_2 \xrightarrow{\ m\ } G_1 \xrightarrow{\ \iota\ } G \overset{s}{\underset{t}{\overset{\xleftarrow{\ \varepsilon\ }}{\rightrightarrows}}} G_0$$

correspond to a groupoid in ***Set****.*

2.2 From Groupoids to Relative Frobenius Algebras

Here we fix a groupoid

$$G_2 \xrightarrow{m} G_1 \xrightarrow{\iota} G \begin{smallmatrix}\xrightarrow{\;s\;}\\ \xleftarrow{\;\varepsilon\;}\\ \xrightarrow{\;t\;}\end{smallmatrix} G_0$$

in **Set**.

Definition 3 For a groupoid G_1, define $X = G_1$, and let $m\colon G_1 \times G_1 \nrightarrow G_1$ be the graph of the function m.

We can prove

Proposition 2 *(X, m) is a relative Frobenius algebra.*

Furthermore, under an appropriate choice for morphisms in the corresponding categories, it is possible to prove.

Theorem 1 *There is an isomorphism of categories* $\mathbf{Frob(Rel)}^{ext} \cong \mathbf{Gpd}^{ext}$.

The category $\mathbf{Gpd}^{\mathrm{ext}}$ has groupoids as objects. Morphisms $\mathbf{G} \to \mathbf{H}$ are subgroupoids of $\mathbf{G} \times \mathbf{H}$. The category $\mathbf{Frob}(\mathbf{Rel})^{\mathrm{ext}}$ has relative Frobenius algebras as objects and the choice of the morphisms is natural with respect to the choice of morphisms for $\mathbf{Gpd}^{\mathrm{ext}}$, for details see [8].

3 Relative H*-Algebras and Semigroupoids

Definition 4 A relative H*-algebra is a morphism $m\colon X \times X \nrightarrow X$ in **Rel** satisfying (M), (A), and

$$\begin{gathered} \textit{there is an involution } *\colon \mathbf{Rel}(1, X) \to \mathbf{Rel}(1, X) \textit{ such that} \\ m \circ (1 \times x^*) = (1 \times x) \circ m^\dagger \text{ and } m \circ (x^* \times 1) = (x \times 1) \circ m^\dagger \\ \textit{for all } x\colon 1 \nrightarrow X. \end{gathered} \tag{H}$$

On the other hand, we have a more relaxed version of groupoids in **Set**. A *semigroupoid* consists of a diagram

$$G_0 \underset{t}{\overset{s}{\leftleftarrows}} G_1 \xleftarrow{\;m\;} G_1 \times_{G_0} G_1$$

(in the category **Set** of sets and functions) such that

$$m(m \times 1) = m(1 \times m).$$

A *pseudoinverse* of $f \in G_1$ is an element $f^* \in G_1$ satisfying ($s(f) = t(f^*)$ and $t(f) = s(f^*)$ and) $f = ff^*f$ and $f^* = f^*ff^*$. A semigroupoid is *regular* when every $f \in G_1$ has a pseudoinverse. Finally, a semigroupoid is *locally cancellative* when $fhh^* = gh^*$ implies $fh = g$, and $h^*hf = h^*g$ implies $hf = g$, for any $f, g, h \in G_1$ and any pseudoinverse H* of h.

3.1 From Semigroupoids to Relative H*-Algebras

Definition 5 Given a locally cancellative regular semigroupoid **G**, define

$$\begin{aligned} X &= G_1, \\ m &= \{(g, f, gf) \mid s(g) = t(f)\}\colon G_1 \times G_1 \nrightarrow G_1, \\ A^* &= \{a^* \in X \mid a^*aa^* = a^* \text{ and } aa^*a = a \text{ for all } a \in A\}. \end{aligned}$$

Theorem 2 *If* **G** *is a locally cancellative regular semigroupoid, then m is a relative H*-algebra.*

3.2 From H*-Algebras to Semigroupoids

Definition 6 Given a relative H*-algebra $m\colon X \times X \nrightarrow X$, define **G** by

$$\begin{aligned} G_0 &= \{f \in X \mid m(f, f) = f\}, \\ G_1 &= X, \\ s &= \{(f, f^*f) \mid f^* \textit{ is a pseudoinverse of } f\}\colon G_1 \nrightarrow G_0 \\ t &= \{(f, ff^*) \mid f^* \textit{ is a pseudoinverse of } f\}\colon G_1 \nrightarrow G_0. \end{aligned}$$

Theorem 3 *If m is a relative H*-algebra, then* **G** *is a locally cancellative regular semigroupoid.*

The category $\mathbf{LRSgpd}^{\mathrm{ext}}$ has locally cancellative regular semigroupoids as objects. Morphisms $\mathbf{G} \to \mathbf{H}$ are locally cancellative regular subsemigroupoids of $\mathbf{G} \times \mathbf{H}$. In the other hand, the category $\mathbf{Hstar}(\mathbf{Rel})^{\mathrm{ext}}$ has relative H*-algebras as objects and a morphism $(X, m_X) \to (Y, m_Y)$ is a morphism $r\colon X \nrightarrow Y$ in **Rel**, natural with respect to the choice of morphisms in $\mathbf{LRSgpd}^{\mathrm{ext}}$ [8]. In a similar way as before it can be proven that

Theorem 4 *There is an adjunction between* $\mathbf{LRSgpd}^{\mathrm{ext}}$ *and* $\mathbf{Hstar}(\mathbf{Rel})^{\mathrm{ext}}$.

4 Groupoids and Poisson Sigma Models

In this section, we describe briefly the construction of groupoids as a way to integrate Poisson manifolds, through the phase space of a 2-dimensional topological field theory, the Poisson sigma model (PSM). This model was first introduced by Ikeda [9] and independently by Schaller and Strobl [12], while understanding the connection between 2-dimensional gravity and Yang-Mills theories. The geometric interpretation of the reduced phase space of the PSM was introduced by Cattaneo and Felder in [3] and gives explicitly a Lie groupoid $G \rightrightarrows M$ (for which G_i and G are smooth finite dimensional manifolds and the structure maps of the groupoid are smooth), if M is an integrable Poisson manifold. In addition, there is a symplectic structure ω in G that is compatible with the multiplication map m; such compatibility turns G into a *symplectic groupoid* integrating the manifold M. More precisely,

Definition 7 A groupoid is called *symplectic* if there is a symplectic structure ω on G_1 such that the graph of m is Lagrangian in $(G_1, \omega) \times (G_1, \omega) \times (G_1, -\omega)$.

The second part of the section is devoted to describe a generalization of such construction, defining what we call a *relational symplectic groupoid*, which lives in the extended symplectic category $\mathbf{Symp}^{ext}$, where the objects are (possibly weak) symplectic manifolds and the morphisms are immersed canonical relations.[4]

This construction turns out to be a way to *integrate* any Poisson manifold.

Definition 8 A Poisson sigma model (PSM) corresponds to the following data:

1. A compact surface Σ, possibly with boundary, called the *source*.
2. A finite dimensional Poisson manifold (M, Π), called the *target*.

The space of fields for this theory is denoted with Φ and corresponds to the space of vector bundle morphisms between $T\Sigma$ and T^*M. This space can be parametrized by a pair (X, η), where $X \in \mathcal{C}^{k+1}(\Sigma, M)$ and $\eta \in \Gamma^k(\Sigma, T^*\Sigma \otimes X^*T^*M)$, and $k \in \{0, 1, \ldots\}$ denotes the regularity type of the map, that we choose to work with.

On Φ, the following first order action is defined:

$$S(X, \eta) := \int_{\Sigma} \langle \eta, dX \rangle + \frac{1}{2} \langle \eta, (\Pi^{\#} \circ X)\eta \rangle,$$

where,

$$\Pi^{\#} : T^*M \to TM \tag{1}$$

$$\psi \mapsto \Pi(\psi, \cdot). \tag{2}$$

[4] More precisely, in $\mathbf{Symp}^{ext}$, by a morphism between two symplectic manifolds (M, ω_M) and (N, ω_N) we mean a pair (X, p) where X is a smooth manifold, p is a smooth map from X to $M \times N$, such that dp is surjective and $T_x(\mathfrak{Im}(p))$ is a Lagrangian subspace of $T(p(x))((M, \omega_M) \times (N, -\omega_N))$, $\forall x \in X$.

Here, dX and η are regarded as elements in $\Omega^1(\Sigma, X^*(TM))$, $\Omega^1(\Sigma, X^*(T^*M))$, respectively and $\langle\, , \rangle$ is the pairing between $\Omega^1(\Sigma, X^*(TM))$ and $\Omega^1(\Sigma, X^*(T^*M))$ induced by the natural pairing between T_xM and T^*_xM, for all $x \in M$.

The integrand, called the Lagrangian, will be denoted by $\mathcal{L}$. Associated to this action, the corresponding variational problem $\delta S = 0$ induces the following space

$$\mathrm{EL} = \{\text{Solutions of the Euler-Lagrange equations}\} \subset \Phi,$$

which is the space of (X, η) satisfying the following equations (up to boundary contributions).

$$\frac{\delta \mathcal{L}}{\delta X} = dX + (\Pi^{\#} \circ X)\eta = 0 \tag{3}$$

$$\frac{\delta \mathcal{L}}{\delta \eta} = d\eta + \frac{1}{2}\langle(\partial \Pi^{\#} \circ X)\eta, \eta\rangle = 0. \tag{4}$$

Now, if we restrict to the boundary, the general space of boundary fields corresponds to

$$\Phi_\partial := \{\text{vector bundle morphisms between } T(\partial\Sigma) \text{ and } T^*M\}.$$

Following the program of classical Lagrangian field theories with boundary [4], Φ_∂ is endowed with a symplectic form and a surjective submersion $p : \Phi \to \Phi_\partial$. We can define

$$L_\Sigma := p(EL)$$

and also C_Π as the set of fields in Φ_∂ which can be completed to a field in $L_{\Sigma'}$, with $\Sigma' := \partial\Sigma \times [0, \varepsilon]$, for some ε.

It turns out that Φ_∂ can be identified with $T^*(PM)$, the cotangent bundle of the path space on M and that

$$C_\Pi := \{(X, \eta) | dX = \pi^{\#}(X)\eta,\ X : \partial\Sigma \to M,\ \eta \in \Gamma(T^*I \otimes X^*(T^*M))\}.$$

It can be proven that C_Π is a coisotropic submanifold of finite codimension of Φ_∂.The proof of the fact that it is coisotropic can be found in [13], and an additional discussion on the manifold structure can be found in [3].

4.1 Geometric Interpretation of EL and Symplectic Reduction

There is a geometric meaning for the equations of motions of PSM in terms of Lie algebroids that will be useful to understand the reduced phase space in terms of Poisson geometry. In order to do that, we recall some basic notions about Lie algebroids.

Definition 9 A Lie algebroid is a triple $(A, [,]_A, \rho)$, where $\pi : A \to M$ is a vector bundle over M, $[,]_A$ is a Lie bracket on $\Gamma(A)$ and ρ (called the anchor map) is a vector bundle morphism from A to TM satisfying the following property
Leibniz property:

$$[X, fY]_A = f[X, Y] + \rho_*(X)(f)Y, \ \forall \ X, Y \in \Gamma(A), \ f \in \mathcal{C}^\infty(M).$$

In our case, a basic example of a Lie algebroid is the cotangent bundle of a Poisson manifold T^*M, where $[,]_{T^*M}$ is the Koszul bracket for 1-forms, that is defined by

$$[df, dg] := d\{f, g\}, \ \forall f, \ g \in \mathcal{C}^\infty(M),$$

in the case of exact forms and is extended for general 1-forms by Leibniz. The anchor map in this example is given by $\Pi^\# : T^*M \to TM$.

Definition 10 To define a morphism of Lie algebroids we consider the complex $\Lambda^\bullet A^*$, where A^* is the dual bundle and a differential δ_A is defined by

$$\begin{aligned} \delta_A f \, : &= \rho^* df, \ \forall f \in \mathcal{C}^\infty(M). \\ \langle \delta_A \alpha, X \wedge Y \rangle \, : &= -\langle \alpha, \ [X, Y]_A \rangle + \langle \delta_A \langle \alpha, X \rangle, Y \rangle \\ & \quad - \langle \delta_A \langle \alpha, Y \rangle, X \rangle, \ \forall X, Y \in \Gamma(A), \alpha \in \Gamma(A^*), \end{aligned}$$

where $\langle \, , \rangle$ is the natural pairing between $\Gamma(A)$ and $\Gamma(A^*)$.

A vector bundle morphism $\varphi : A \to B$ is a Lie algebroid morphism if

$$\delta_A \varphi^* = \varphi^* \delta_B.$$

This condition gives rise to some PDE's that the anchor maps and the structure functions for $\Gamma(A)$ and $\Gamma(B)$ should satisfy. For the case of PSM, regarding T^*M as a Lie algebroid, we can prove that

$$C_\Pi := \{\text{Lie algebroid morphisms between } T(\partial \Sigma) \text{ and } T^*M\},$$

where the Lie algebroid structure on the left is given by the Lie bracket of vector fields on $T(\partial\Sigma)$ with identity anchor map.

Since C_Π is a coisotropic submanifold, it is possible to perform symplectic reduction, that is, when it is smooth, a symplectic finite dimensional manifold. In the case of Σ being a rectangle and with vanishing boundary conditions for η (see [3]), following the notation in [7, 11], we could also redefine the reduced phase space $\underline{C_\Pi}$ as

$$\underline{C_\Pi} := \left\{ \frac{T^*M\text{-paths}}{T^*M\text{-homotopy}} \right\}.$$

In the smooth case, it was proven in [3] that

Theorem 5 *The following data*

$$\begin{aligned}
G_0 &= M \\
G_1 &= \underline{C_\Pi} \\
G_2 &= \{[X_1, \eta_1], [X_2, \eta_2] | X_1(1) = X_2(0)\} \\
m : G_2 \to G &:= ([X_1, \eta_1], [X_2, \eta_2]) \mapsto [(X_1 * X_2, \eta_1 * \eta_2)] \\
\varepsilon : G_0 \to G_1 &:= x \mapsto [X \equiv x, \eta \equiv 0] \\
s : G_1 \to G_0 &:= [X, \eta] \mapsto X(0) \\
t : G_1 \to G_0 &:= [X, \eta] \mapsto X(1) \\
\iota : G_1 \to G_1 &:= [X, \eta] \to [i^* \circ X, i^* \circ \eta] \\
i : [0, 1] \to [0, 1] &:= t \to 1 - t,
\end{aligned}$$

correspond to a symplectic groupoid that integrates the Lie algebroid T^*M.[5]

4.2 Categorical Extensions

The objective in this section is to introduce several constructions for more general categories (not just $\mathbf{Symp}^{ext}$), which resemble the construction of symplectic groupoids and relative Frobenius algebras. More precisely, in the case of Poisson manifolds, the study of the phase space *before reduction* yields to the construction of what we will denote as relational symplectic groupoids. In the sequel we consider a category $\mathcal{C}$ which admits products and with a special object pt.

Definition 11 A weak monoid in $\mathcal{C}$ corresponds to the following data:

1. An object X.
2. A morphism $L_1 : pt \to X$
3. A morphism $L_3 : X \times X \to X$,

satisfying the following axioms

- (Associativity).
$$L_3 \circ (L_3 \times Id) = L_3 \circ (Id \times L_3)$$
- (Weak unitality).
$$L_3 \circ (L_1 \times Id) = L_3 \circ (Id \times L_1) =: L_2$$
  and $L_2 \circ L_2 = L_2$.

We call L_1 a weak unit and L_2 a projector.

[5] Here $*$ denotes path concatenation.

Example 1 Any monoid object in $\mathcal{C}$ is a weak monoid with L_1 being the unit and L_2 being the identity morphism.

Example 2 Any relative Frobenius algebra X in **Rel** is by definition a weak monoid.

Example 3 A commutative monoid $(X, m, 1)$ equipped with a projector p, that means, $p^2 = 1$, can be made into a weak monoid. In this case, $L_3 = m$, $L_1 = p$ and $L_2 : x \mapsto m(p, x)$. Since in general L_2 is not the identity morphism, this is not an example of an usual monoid, but for a commutative monoid in **Set** it can be checked that the quotient X/L_2 is a monoid.

Remark 2 The last example does not yield in general to a monoid if we start with a commutative monoid in a category different from **Set**. For instance, if we take the monoid $\mathbb{R}, \cdot, 1$ and the projector $p = -1$, the quotient space $\underline{X} = [0, \infty)$ is a monoid object in **Set** but it is not an object in **Man**, the category of smooth manifolds and smooth maps.

Example 4 It follows from the definition that when X is a vector space, a weak monoid yields into an associative algebra with a preferred central element that induces a projection. This could be called a *prounital associative algebra* [6].

Definition 12 Let $\mathcal{C}$ be a dagger category with products and adjoints. A weak *-monoid in $\mathcal{C}$ consists of the following data:

1. An object X
2. A morphism $\psi : X \to X^\dagger$
3. A morphism $L_3 : X \times X \to X$

such that the following axioms hold

- (Associativity).
$$L_3 \circ (L_3 \times Id) = L_3 \circ (Id \times L_3)$$
- (Involutivity). $\psi^\dagger \psi = Id$
- Defining ψ_R the (unique) induced morphism $\psi_R : pt \to X \times X$, then
$$L_1 := L_3 \circ \psi_R$$
determines a weak monoid (X, L_1, L_3).

Example 5 Consider $\mathcal{C}$ the category $\mathbf{Vect}^{Ext}$ of vector spaces (possibly infinite dimensional) whose morphisms are linear subspaces. The dagger structure is the identity in objects and the relational converse for morphisms. Let ϕ be a involutive diffeomorphism of M. If $X = \mathcal{C}^\infty(M)$, then $(X, +, \phi^*)$ is a weak *-monoid. To check this, first observe that

$$\begin{aligned} L_1 &= \{f + \phi^*(f),\ f \in X\} \\ L_2 &= \{(g, g + h + \phi^* h),\ g, h \in X\} \\ L_2 \circ L_2 &= \{(g, g + h + h' + \phi^* h + \phi^* h''),\ g, h, h' \in X\}. \end{aligned}$$

Setting $h' \equiv 0$ we get that $L_2 \subset L_2 \circ L_2$ and by linearity of ϕ $L_2 \circ L_2 \subset L_2$. Associativity and unitality follow from the additive structure of X.

Example 6 (Deformation quantization) Let $\mathcal{C} = \mathbf{Vect}^{Ext}$ and consider a Poisson manifold M. Let $X = \mathcal{C}^\infty(M, \mathbb{C})$ be the algebra of smooth complex valued functions on M. By deformation quantization for Poisson manifolds (see, for example, [10]), given a Poisson structure Π on M, there exists an associative $\mathbb{C}[\varepsilon]]$-linear product in $X[[\varepsilon]]$,[6] denoted by $\star$, such that

1. $1 \star f = f \star 1 = f, \forall f \in X[[\varepsilon]]$
2. $$f \star g = fg + \varepsilon B_1(f, g) + \varepsilon^2 B_2(f, g) + \cdots,$$

with $f,\ g \in X \subset X[[\varepsilon]]$ and B_i are bidifferential operators, where

$$\Pi(df, dg) = \frac{f \star g - g \star f}{\varepsilon}.$$

It can be checked [6] that $(X[[\varepsilon]], \star, \bar{\cdot})$ is a weak-* monoid, where $\bar{\cdot}$ denotes complex conjugation.

Definition 13 Let $\mathcal{C}$ be a dagger category with products and adjoints. A cyclic weak *-monoid in $\mathcal{C}$ consists of the following data:

1. An object X
2. A morphism $\psi : X \to X^\dagger$
3. A morphism $L : X \times X \to X^\dagger$

such that

- (Cyclicity). For the associated morphism $L_R : pt \to X^3$

$$L_R = \sigma \circ L_R = \sigma \circ \sigma \circ L_R$$

where

$$\sigma : X^3 \to X^3 \tag{5}$$
$$(a, b, c) \mapsto (c, a, b) \tag{6}$$

- If $L_3 := \psi^\dagger \circ L$, then (X, ψ, L_3) is a weak *-monoid.

Example 7 (Frobenius algebras) Consider $(X, \langle\ ,\ \rangle)$ a Hilbert space over $\mathbb{C}$. We set

$$\psi : X \to X^\dagger$$
$$v \mapsto \langle \bar{v}, \cdot \rangle.$$

[6] In this case that we are considering complex valued functions we set $\varepsilon = i\hbar/2$.

Let $m : X \times X \to X$ be a commutative and associative $\mathbb{C}$-bilinear map. It can be checked that the cyclicity condition for

$$L := \psi \circ m$$

is equivalent to saying that

$$(X, m, (\cdot\,,\cdot))$$

is a Frobenius algebra, where

$$(v, w) := \langle \bar{v}, w \rangle.$$

In this case, if we fix a basis $\{e_i\}_{i\in I}$ for X, L_1 corresponds to a central element

$$L_1 : e = \sum_{i\in I} m(e_i, \bar{e_i}),$$

$$\begin{aligned} L_2 : X &\to X \\ v &\mapsto m(e, v) \end{aligned}$$

and $L_3 = m$.

Remark 3 For the case of a unimodular Poisson manifold (M, Π), it can be proven, following Example 6 that deformation quantization gives rise to a cyclic weak *-monoid, where $\psi = \phi^*$, with ϕ being a Π-invariant diffeomorphism (see [6]).

Example 8 (Relational symplectic groupoids) Following [5], we consider $\mathcal{C} =$ **Symp**^{ext} and M an arbitrary Poisson manifold.

Proposition 3 *The following data*

$$\begin{aligned} X &:= T^*(PM) \\ \psi &: (x, \eta) \mapsto (i^* \circ x, i^* \circ \eta) \\ &\quad i : t \mapsto 1 - t \\ L &:= \{(x_1, \eta_1), (x_1, \eta_1), (x_3, \eta_3) | (x_1 * x_2, \eta_1 * \eta_2) \sim \psi((x_3, \eta_3))\}, \end{aligned}$$

where $\sim$ denotes the equivalence relation by T^*M-homotopy of T^*M-paths, corresponds to a cyclic weak $*$-monoid. In this case,

$$\begin{aligned} L_1 &= \{(x, \eta) \in X | (x, \eta) \sim (x \equiv x_0, \eta \equiv 0), x_0 \in M\} \\ L_2 &= \{(x_1, \eta_1), (x_2, \eta_2) \in X \times X | (x_1, \eta_1) \sim (x_2, \eta_2)\}. \end{aligned}$$

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Part IV
Algebraic Aspects of Locality

We have already noticed in Chaps. “A Derived and Homotopical View on Field Theories”, “Lectures on Mathematical Aspects of (Twisted) Supersymmetric Gauge Theories”, and “Factorization Homology in 3-Dimensional Topology” the emergence of the notion of factorization algebras in the mathematical treatment of Quantum Field Theory: local observables naturally carry the structure of a factorization algebra, which is locally constant in the case of a topological field theory. Moreover, factorization homology is nothing but the partition function.

Part IV consists of a single chapter (written by Grégory Ginot) about the mathematical foundations of Factorization Algebras and Factorization Homology. It mainly emphasizes the axiomatic framework as well as applications to algebraic topology (such as string topology, mapping spaces, higher Hochschild homology, Deligne conjecture, etc.).

Notes on Factorization Algebras, Factorization Homology and Applications

Grégory Ginot

Abstract These notes are an expanded version of two series of lectures given at the winter school in mathematical physics at les Houches and at the Vietnamese Institute for Mathematical Sciences. They are an introduction to factorization algebras, factorization homology and some of their applications, notably for studying E_n-algebras. We give an account of homology theory for manifolds (and spaces), which give invariant of manifolds but also invariant of E_n-algebras. We particularly emphasize the point of view of factorization algebras (a structure originating from quantum field theory) which plays, with respect to homology theory for manifolds, the role of sheaves with respect to singular cohomology. We mention some applications to the study of mapping spaces, in particular in string topology and for (iterated) Bar constructions and study several examples.

1 Introduction and Motivations

These notes are an introduction to factorization algebras and factorization homology in the context of topological spaces and manifolds. The origin of factorization algebras and factorization homology, as defined by Lurie [71] and Costello-Gwilliam [24], are to be found in topological quantum field theories and conformal field theories. Indeed, they were largely motivated and influenced by the pioneering work of Beilinson–Drinfeld [7] and also of Segal [85, 86]. *Factorization homology* is a catchword to describe homology theories *specific* to say oriented manifolds of a fixed dimension n. There are also variant specific to many other classes of structured manifold of fixed dimension. Typically the structure in question would be a framing[1] or a spin structure or simply no structure at all.

[1] that is a trivialization of the tangent bundle.

G. Ginot (✉)
Institut de Mathématiques de Jussieu - Paris Rive gauche, Université Pierre et Marie Curie - Sorbonne Universités,
4 Place Jussieu, 75252 Paris Cedex 05, France
e-mail: gregory.ginot@imj-prg.fr

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Factorization algebras are algebraic structures which shed many similarities with (co)sheaves and were introduced to describe Quantum Field Theories much in the same way as the structure of a manifold or scheme is described by its sheaf of functions [7, 24]. They are related to factorization homology in the same way as singular cohomology is related to sheaf cohomology.

Unlike classical singular homology for which any abelian group can be used as coefficient of the theory, in order to define factorization homology, one needs a more complicated piece of algebraic data: that of an E_n-algebra.[2] These algebras have been heavily studied in algebraic topology ever since the seventies where they were introduced to study iterated loop spaces and configuration spaces [12, 76, 84]. They have been proved to also have deep significance in mathematical physics [24, 64], string topology [17, 20] and (derived) algebraic geometry [7, 33, 72, 77]. E_1-algebras are essentially the same thing as A_∞-algebras, that is homotopy associative algebras. On the other hand, E_∞-algebras are homotopy commutative algebras. In general the E_n-structures form a hierarchy of more and more homotopy commutative algebra structures. In fact, an E_n-algebra is an homotopy associative algebra whose multiplication μ_0 is commutative up to an homotopy operator μ_1. This operator is itself commutative up to an homotopy operator μ_2 and so on up to μ_{n-1} which is no longer required to be homotopy commutative.

Since factorization homology depends on (some class of) both manifold and E_n-algebra, they also give rise to invariants of E_n-algebras. These invariants have proven useful as we illustrate in Sect. 7. For instance, in dimension $n = 1$, factorization homology evaluated on a circle is the usual Hochschild homology of algebras (together with its circle action inducing cyclic homology as well). For $n = \infty$, factorization homology gives rise to an invariant of topological spaces[3] (sometimes called higher Hochschild homology [79]) which we recall in Sect. 2. It is easier to study and interesting in its own since it is closely related to mapping spaces, their derived analogues and observables of classical topological field theories.

We give the precise axioms of homology theory of manifolds in Sect. 3. Factorization homology can be computed using Čech complexes of factorization algebras, which, as previously alluded to, are a kind of "multiplicative, non-commutative" analogue of cosheaves. Definitions, properties and many examples of factorization algebras are discussed in Sect. 4. Factorization algebras were introduced to describe observables of Quantum Field Theories [7, 24] but they also are a very convenient way to encode and study many algebraic structures which arose in algebraic topology and mathematical physics as we illustrate in Sect. 4. In particular, in Sect. 6 we study in depth locally constant factorization algebras on stratified spaces and their link with various categories of modules over E_n-algebras, giving many examples. We also give a detail account of various operations and properties of factorization algebras in Sect. 5. We then (Sect. 7) review several applications of the formalism of

[2] More accurately, E_n-algebras are the piece of data needed in the case of framed manifolds. For other structured manifolds, one needs E_n-algebras equipped with additional structure; for instance an invariance under their natural $SO(n)$-action in the oriented manifold case.

[3] And not just manifolds of a fix dimension.

factorization algebras and homology. Notably to cohomology and deformations of E_n-algebras, (higher) Deligne conjecture and also in (higher) string topology and for Bar constructions of iterated loop spaces (and more generally to obtain models for iterated Bar constructions *with* their algebraic structure). In Sect. 8, we consider the case of commutative factorization algebras and prove their theory reduces to the one of cosheaves. In particular, we cover the pedagogical example of classical homology (with twisted coefficient) viewed as factorization homology.

1.1 Eilenberg-Steenrod Axioms for Homology Theory of Spaces

Factorization homology and factorization algebras generalize ideas from the axiomatic approach to classical homology of spaces (and (co)sheaf theory) which we now recall. We then explain how they can be generalized. The usual (co)homology groups of topological spaces are uniquely determined by a set of axioms. These are the *Eilenberg-Steenrod axioms* which were formulated in the 40s [29].

Classically they express that an homology theory for spaces is uniquely determined by (ordinary) functors from the category of pairs (X, A) $(A \subset X)$ of spaces to the category of $(\mathbb{N})$-graded abelian groups satisfying some axioms. Such a functor splits as the direct sum $H_*(X, A) = \bigoplus_{i\geq 0} H_i(X, A)$ where $H_i(X, A)$ is the degree i homology groups of the pair. This homology group can in fact be defined as the homology of the mapping cone $cone(A \hookrightarrow X)$ of the inclusion of the pair. Further, the long exact sequence in homology relating the homology of the pair to the homology of A and X is induced by a short exact sequence of chain complexes $C_*(A) \hookrightarrow C_*(X) \twoheadrightarrow C_*(X, A)$ (where C_* is the singular chain complex). Similarly, the Mayer-Vietoris exact sequence is induced by a short exact sequence of chain complexes.

This suggests that the classical Eilenberg-Steenrod axioms can be lifted at the *chain complex* level. That is, we can characterize classical homology as a functor from the category of spaces (up to homotopy) to the category of chain complexes (up to quasi-isomorphism).

Let us formalize a bit this idea. A *homology theory* $\mathcal{H}$ for *spaces* is a functor $\mathcal{H} : \mathrm{Top} \to Chain(\mathbb{Z})$ from the category Top of topological spaces[4] to the category $Chain(\mathbb{Z})$ of chain complexes over $\mathbb{Z}$ (in other words differential graded abelian groups). This functor has to satisfy the following three axioms.

1. *(homotopy invariance)* The functor $\mathcal{H}$ shall send homotopies between maps of topological spaces to homotopies between maps of chain complexes.
2. *(monoidal)* The functor $\mathcal{H}$ shall be defined by its value on the connected components of a space. Hence we require it sends disjoint unions of topological spaces

[4] For simplicity we assume that we consider only spaces homotopy equivalent to CW-complexes.

to direct sum, that is the canonical map $\oplus_{\alpha\in I}\mathcal{H}(X_\alpha) \to \mathcal{H}(\coprod_{\alpha\in I} X_\alpha)$ is an homotopy equivalence[5] (here I is any set).

3. *(excision)* There is another additional property encoding (given the other ones) the classical excision property as well as the Mayer-Vietoris principle. The additional property essentially stipulates the effect of gluing together two CW-complexes along a sub-complex. Let us formulate it this way: assume $i : Z \hookrightarrow X$ and $j : Z \hookrightarrow Y$ are inclusions of closed sub CW-complex of X and Y. Let $X \cup_Z Y \cong X \coprod Y/(i(z) = j(z),\ z \in Z)$ be the pushout of X, Y along Z. The functoriality of $\mathcal{H}$ gives maps $\mathcal{H}(Z) \stackrel{i_*}{\to} \mathcal{H}(X)$ and $\mathcal{H}(Z) \stackrel{j_*}{\to} \mathcal{H}(Y)$; hence a chain complex morphism $\mathcal{H}(Z) \stackrel{i_*-j_*}{\to} \mathcal{H}(X) \oplus \mathcal{H}(Y)$. Functoriality also yields a natural map $\mathcal{H}(X) \oplus \mathcal{H}(Y) \to \mathcal{H}(X \cup_Z Y)$ whose composition with $i_* - j_*$ is null.
   The *excision axioms requires that the canonical map*
   $$cone\Big(\mathcal{H}(Z) \stackrel{i_*-j_*}{\to} \mathcal{H}(X) \oplus \mathcal{H}(Y)\Big) \longrightarrow \mathcal{H}(X \cup_Z Y)$$
   is an homotopy equivalence. Here cone(f) is the mapping cone[6] (in $Chain(\mathbb{Z})$) of the map f of chain complexes.

We can state the following theorem (which follows from Corollary 20 and is the (pre-)dual of a result of Mandell [74] for cochains).

Theorem 1 (Eilenberg-Steenrod) *Let G be an abelian group. Up to natural homotopy equivalence, there is a unique homology theory for spaces, that is functor $\mathcal{H} : Top \to Chain(\mathbb{Z})$ satisfying axioms 1, 2 and 3 and further the* dimension *axiom:*

$$\mathcal{H}(pt) \stackrel{\simeq}{\to} G.$$

The functor in Theorem 1 is of course given by the usual singular chain complex with value in G. We can even assume in the theorem that G is any chain complex, in which case we recover extraordinary homology theories.[7]

Theorem 1 implies that the category of functors satisfying axioms 1, 2, 3 is (homotopy) equivalent to the category of chain complexes; the equivalence being given by the evaluation of a functor at the point. To assign to a chain complex V_* an homology theory, one consider the functor $X \mapsto C_*(X, \mathbb{Z}) \otimes V_*$.

For a CW-complex X, the singular cohomology $H^*(X, G)$ can be computed as sheaf cohomology of X with value in the constant sheaf G_X of locally constant functions on X with values in G. In particular, the singular cochain complex is naturally quasi-isomorphic to the derived functor $\mathbb{R}\Gamma(G_X)$ of sections of G_X. Replacing G_X

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5 Sometimes this map is required to be an actual isomorphism but this is not needed.

6 If we know that $f : C_* \to D_*$ is injective, then $cone(f)$ is quasi-isomorphic to the quotient chain complex D_*/C_*. See for instance [101] for mapping cones of general chain maps.

7 In this case, the uniqueness is not necessarily true if one works at the homology level instead of chain complexes.

by a locally constant sheaf (with germs G) yields cohomology with local coefficient in G.[8] This point of view realizes singular cohomology (with local coefficient) as a special case of the theory of sheaf/Čech cohomology which also has other significance and applications in geometry when allowing more general sheaves.

Note that the homotopy invariance axiom can be reinterpreted as saying that the functor $\mathcal{H}$ is continuous. Indeed, there are natural topologies on the morphism sets of both categories. For instance, one can consider the compact-open topology on the set of maps $\mathrm{Hom}_{\mathrm{Top}}(X, Y)$ (see Example 61 for $Chain(\mathbb{Z})$). Any continuous functor, that is a functor $\mathcal{H}$ such that the maps $\mathrm{Hom}_{\mathrm{Top}}(X, Y) \to \mathrm{Hom}_{\mathbf{Chain}(\mathbb{Z})}(\mathcal{H}(X), \mathcal{H}(Y))$ are continuous, sends homotopies to homotopies (and homotopies between homotopies to homotopies between homotopies and so on).

Note also that excision axiom really identifies $\mathcal{H}(X \cup_Z Y)$ with a homotopy colimit. It is precisely the homotopy coequalizer $hocoeq\left(\mathcal{H}(Z) \overset{i_*}{\underset{j_*}{\rightrightarrows}} \mathcal{H}(X) \oplus \mathcal{H}(Y)\right)$ which is computed by the mapping cone $cone(i_* - j_*)$. Further, in this axiom, we do not need $\mathcal{H}$ to be precisely the cone but any natural chain complex quasi-isomorphic to it will do the job. This suggests to actually use a more flexible model than topological categories. A convenient way to *deal simultaneously with topological categories, homotopy colimits* (in particular homotopy quotients) *and identification of chain complexes up to quasi-isomorphism* is to consider the ∞-categories associated to topological spaces and chain complexes and ∞-functors between them (see Appendix A, Examples 60 and 61). The passage from topological categories to ∞-categories essentially allows to work in categories in which (weak) homotopy equivalences have been somehow "inverted" but which still retain enough information of the topology of the initial categories.

Furthermore, in the monoidal axiom, we can replace the direct sum of chain complexes by any symmetric monoidal structure, for instance by the tensor product $\otimes$ of chain complexes. This yields the notion of homology theory for spaces with values in $(Chain(\mathbb{Z}), \otimes)$ see Sect. 2.1.1. The latter are not determined by a mere chain complex but by a (homotopy) commutative algebra A. This theory is called factorization homology for spaces and commutative algebras and its main properties are detailled in Sect. 2. In fact, already at this level, we see that one needs to replace the cone construction in the excision axiom by an appropriate derived functor.

To produce invariant of manifolds which are not invariant of spaces, one needs to replace Top by another topological category of manifolds. For instance, fixing $n \in \mathbb{N}$, one can consider the category Mfld_n^{fr} whose objects are *framed manifolds of dimension n* and whose morphisms are framed embeddings. In that case, an homology theory is completely determined by an E_n-algebra. The precise definitions and variants of homology theories for various classes of structured manifolds (including the local coefficient ones) and the appropriate notion of coefficient is the content of Sect. 3.

[8] If G is a linear representation of a group H and X is the classifying space of H, then one recovers this way the group (co)homology of H with value in G.

The (variants of) E_n-algebras which arise as coefficient of homology theory for manifolds can be seen as a special case of factorization algebras, and more precisely as locally constant factorization algebras, which are to factorization algebras what (acyclic resolutions of) locally constant sheaves are to sheaves. This point of view is detailed in Sect. 4.2 and extended to stratified spaces in Sect. 6. The latter case gives simple[9] description of several categories of modules over E_n-algebras as well as categories of E_n-algebras acting on E_m-algebras, which is used in the many applications of Sect. 7.

1.2 Notation and Conventions

1. Let k be a commutative unital ring. The ∞-category of differential graded k-modules (i.e. chain complexes) will be denoted **Chain**(k). The (*derived*) tensor product over k will be denoted $\otimes$. The k-linear dual of $M \in$ **Chain**(k) will be denoted $M^\vee$.
2. All manifolds are assumed to be Hausdorff, second countable, paracompact and thus metrizable.
3. We write **Top** for the ∞-category of topological spaces (up to homotopy) and **Top**^{f} for its ∞-subcategory spanned by the (spaces with the homotopy type of) finite CW-complexes. We also denote **Top**$_*$, resp. **sSet**$_*$, the ∞-categories of *pointed* topological spaces and simplicial sets. We write $C_*(X)$ *and* $C^*(X)$ *for the singular chain and cochain complex* of a space X. We write **sSet** for the ∞-category of simplicial sets (up to homotopy) which is equivalent to **Top**.
4. The ∞-categories of unital commutative differential graded algebras (up tho homotopy) will be denoted by **CDGA**. We simply refer to unital commutative differential graded algebras as CDGAs.
5. Let $n \in \mathbb{N} \cup \{\infty\}$. By an E_n-algebra we mean an algebra over an E_n-operad. We write E_n**-Alg** for the ∞-category of (unital) E_n-algebras (in **Chain**(k)). See Appendix 10.2. We also write E_n**-Mod**$_A$, resp. E_1**-LMod**$_A$, resp. E_1**-RMod**$_A$ the ∞-categories of E_n-A-modules, resp. left A-modules, resp. right A-modules.
6. The ∞-category of (small) ∞-categories will be denoted ∞**-Cat**.
7. We work with a cohomological grading (unless otherwise stated) for all our (co)homology groups and graded spaces, even when we use subscripts to denote the grading (so that our chain complexes have a funny grading). In particular, all differentials are of degree $+1$, of the form $d : A^i \to A^{i+1}$ and the homology groups $H_i(X)$ of a space X are concentrated in non-positive degree. If $(C^*, d_C) \in$ **Chain**(k), we denote $C^*[n]$ the chain complex given by $(C^*[n])^i := C^{i+n}$ with differential $(-1)^n d_C$.

[9] In the sense that the cosheaf condition satisfied by factorization algebras encodes some topology which, from the classical E_n-operad point of view necessitates an heavier homotopical machinery.

8. We will denote $\mathbf{PFac}_X$, resp. $\mathbf{Fac}_X$, resp. $\mathbf{Fac}_X^{lc}$ the ∞-categories of prefactorization algebras, resp. factorization algebras, resp. locally constant factorization algebras over X. See Definition 15.
9. Usually, if C a (topological or simplicial or model) category, we will use the boldface letter $\mathbf{C}$ to denote the ∞-category associated to C (see Sect. 10). This is for instance the case for the categories of topological spaces or chain complexes or CDGAs mentionned above.
10. Despite their names, the values of Hochschild or factorization (co)homology will be (co)chain complexes (up to equivalences), i.e. objects of $\mathbf{Chain}(k)$, or objects of another ∞-category such as E_∞-**Alg**.

These notes deal mainly with applications of factorization algebras in algebraic topology and homotopical algebra. However, there are very interesting applications to mathematical physics as described in the work of Costello et al. [22–24, 52, 54] and also beautiful applications in algebraic geometry and geometric representation theory, for instance see [7, 33, 41, 42].

We almost always refer to the existing literature for proofs; though there are some exceptions to this rule, mainly in Sects. 5, 6 and 8, where we treat several new (or not detailed in the literature) examples and results related to factorization algebras. To help the reader browsing through the examples in Sects. 5 and 6, the longer proofs are postponed to a dedicated appendix, namely Sect. 9. Some other references concerning factorization algebras and factorization homology include [3, 5, 14, 34, 35, 47, 48, 70, 71, 91].

1.2.1 About ∞-Categories

We use ∞-categories as a convenient framework for homotopical algebra and in particular as higher categorical *derived categories*. In our context, they will typically arise when one considers a topological category or a category $\mathcal{M}$ with a notion of (weak homotopy) equivalence. The ∞-category associated to that case will be a lifting of the homotopy category $Ho(\mathcal{M})$ (the category obtained by formally inverting the equivalences). It has *spaces* of morphisms and composition and associativity laws are defined up to coherent homotopies. We recall some basic examples and definitions in Appendix 10.1.

Topological categories and continuous functors between them are actually a model for ∞-categories and (∞-) functors between them. By a topological category we mean a category $\mathcal{C}$ endowed with a *space* of morphisms $\mathrm{Map}_{\mathcal{C}}(x, y)$ between objects such that the composition $\mathrm{Map}_{\mathcal{C}}(x, y) \times \mathrm{Map}_{\mathcal{C}}(y, z) \to \mathrm{Map}_{\mathcal{C}}(x, z)$ is continuous. A continuous functor $F : \mathcal{C} \to \mathcal{D}$ between topological categories is a functor (of the underlying categories) such that for all objects x, y, the map $\mathrm{Map}_{\mathcal{C}}(x, y) \stackrel{F}{\to} \mathrm{Map}_{\mathcal{D}}(F(x), F(y))$ is continuous. In fact, every ∞-category admits a strict model, in other word is equivalent to a topological category (though finding a strict model can be hard in practice). One can also replace, in the previous paragraph, topological categories by simplicially enriched categories, which are the same thing as topological categories where spaces are replaced by simplicial sets (and continuous maps by maps of simplicial sets). In practice many topological categories we consider are geometric realization of simplicially enriched categories.

The reader can thus substitute topological category to ∞-category in every statement of these notes (or also simplicially enriched or even differential graded[10]), but modulo the fact that one may have to replace the topological or ∞-category in question by another equivalent topological one. The same remark applies to functors between ∞-categories. Furthermore, many constructions involving factorization algebras are actually carried out in (topological) categories (which provide concrete models to homotopy equivalent (derived) ∞-category of some algebraic structures).

If $\mathcal{C}$ is an ∞-category, we will denote $\mathrm{Map}_{\mathcal{C}}(x, y)$ its space of morphisms from x to y while we will simply write $\mathrm{Hom}_{\mathcal{D}}(x, y)$ for the morphism *set* of an ordinary category $\mathcal{D}$ (that is a topological category whose space of morphisms are discrete).

Many derived functors of homological algebra have natural extensions to the setting of ∞-categories. In that case we will use the usual derived functor notation to denote their canonical lifting to ∞-category and to emphasize that they can be computed using the *usual resolutions* of homological algebra. For instance, we will denote $(M, N) \mapsto M \otimes^{\mathbb{L}}_A N$ for the functor $E_1\textbf{-RMod}_A \times E_1\textbf{-LMod}_A \to \textbf{Chain}(k)$ lifting the usual tensor product of left and right modules to their ∞-categories.

There is a slight exception to this notational rule. We denote $M \otimes N$ the *derived* tensor products of complexes in $\textbf{Chain}(k)$. We do not use a derived tensor product notation since it will be too cumbersome and since in practice it will often be applied in the case where k is a field or M, N are projective over k.

2 Factorization Homology for Commutative Algebras and Spaces and Derived Higher Hochschild Homology

Factorization homology restricted to commutative algebras is also known as *higher Hochschild homology* and has been studied (in various guise) since at least the end of the 90s (see the approach of [30, 73] to topological Hochschild homology, or, the work of Pirashvili [79] which is closely related to Γ-homology). Though its axiomatic description is an easy corollary of the description of **Top** as a symmetric monoidal category with pushouts, it has a lot of nice properties and appealing combinatorial description in characteristic zero (related to rational homotopy theory à la Sullivan). We review some of its main properties in this Section.

[10] In this case, we refer to [93] for the needed homotopy categorical framework on dg-categories.

2.1 Homology Theory for Spaces and Derived Hochschild Homology

2.1.1 Axiomatic Presentation

Let us first start by defining the axioms of an homology theory for spaces with values in the symmetric monoidal ∞-category $(\mathbf{Chain}(k), \otimes)$ (instead of $(Chain(\mathbb{Z}), \oplus)$). The (homotopy) commutative monoids in $(\mathbf{Chain}(k), \otimes)$ are the E_∞-algebras (Definition 34). In characteristic zero, one can restrict to differential graded commutative algebras since the natural functor $\mathbf{CDGA} \to E_\infty\textbf{-Alg}$ is an (homotopy) equivalence.

The ∞-category **Top** has a symmetric monoidal structure given by disjoint union of spaces $X \coprod Y$, which is also the coproduct of X and Y in **Top**. The identity map $id_X : X \to X$ yields a canonical map $X \coprod X \stackrel{\coprod id_X}{\longrightarrow} X$ which is associative and commutative in the ordinary category of topological spaces. Hence X is *canonically* a commutative algebra object in $(\mathbf{Top}, \coprod)$. And so is its image by a symmetric monoidal functor. We thus have:

Lemma 1 *Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal ∞-category. Any symmetric monoidal functor $F : \mathbf{Top} \to \mathcal{C}$ has a canonical lift $\tilde{F} : \mathbf{Top} \to E_\infty$-**Alg**$(\mathcal{C})$.*

In particular, any homology theory $\mathbf{Top} \to \mathbf{Chain}(k)$ shall have a canonical factorization $\mathbf{Top} \to E_\infty\textbf{-Alg}$. This motivates the following definition.

Definition 1 An homology theory for spaces with values in the symmetric monoidal ∞-category $(\mathbf{Chain}(k), \otimes)$ is an ∞-functor $\mathcal{CH} : \mathbf{Top} \times E_\infty\textbf{-Alg} \to E_\infty\textbf{-Alg}$ (denoted $(X, A) \mapsto CH_X(A)$ on the objects), satisfying the following axioms:

(i) **(value on a point)** there is a natural equivalence $CH_{pt}(A) \stackrel{\simeq}{\to} A$ in E_∞**-Alg**;
(ii) **(monoidal)** the canonical maps (induced by universal property of coproducts)

$$\bigotimes_{i\in I} CH_{X_i}(A) \stackrel{\simeq}{\longrightarrow} CH_{\coprod_{i\in I} X_i}(A)$$

are equivalences (for any set I);
(iii) **(excision)** The functor $\mathcal{CH}$ commutes with homotopy pushout of spaces, i.e., the canonical maps (induced by the universal property of derived tensor product)

$$CH_X(A) \mathop{\otimes}^{\mathbb{L}}_{CH_Z(A)} CH_Y(A) \stackrel{\simeq}{\longrightarrow} CH_{X\cup^h_Z Y}(A)$$

are natural equivalences.

Remark 1 If one replace **Top** by $\mathbf{Top}^f$ then axiom (ii) is equivalent to saying that, the functors $X \mapsto CH_X(A)$ are symmetric monoidal.

Remark 2 (**Homology theory for spaces in an arbitrary symmetric monoidal ∞-category**) If $(\mathcal{C}, \otimes)$ is a symmetric monoidal ∞-category, we define an homology theory with values in $(\mathcal{C}, \otimes)$ in the same way, simply replacing **Chain**(k) by $\mathcal{C}$ (and thus E_∞-**Alg** by E_∞-**Alg**$(\mathcal{C})$).

All results (in particular the existence and uniqueness Theorem 2) in Sects. 2.1, 2.2 and 2.3 still hold by just replacing the monoidal structure of **Chain**(k) by the one of $\mathcal{C}$, provided that $\mathcal{C}$ has colimits and that its monoidal structure commutes with geometric realization.

Theorem 2

1. *There is an unique*[11] *homology theory for spaces (in the sense of Definition 1).*
2. *This homology theory is given by* derived Hochschild chains, i.e., *there are natural equivalences*

$$A \boxtimes X \cong CH_X(A) \tag{1}$$

where $A \boxtimes X$ is the tensor of the E_∞-algebra A with the space X (see Remark 37). In particular,

$$\mathrm{Map}_{\mathbf{Top}}\big(X, \mathrm{Map}_{E_\infty\text{-}\mathbf{Alg}}(A, B)\big) \cong \mathrm{Map}_{E_\infty\text{-}\mathbf{Alg}}(CH_X(A), B). \tag{2}$$

3. ***(generalized uniqueness)*** *Let $F : E_\infty$-**Alg** $\to E_\infty$-**Alg** be a functor. There is an unique functor* **Top** $\times\, E_\infty$-***Alg*** $\to E_\infty$-***Alg*** *satisfying axioms **ii), iii)** in Definition 1 and whose value on a point is $F(A)$. This functor is $(X, A) \mapsto CH_X(F(A))$.*

Remark 3 Theorem 2 still holds with $\mathbf{Top}^f$ instead of **Top** (and where in axiom (ii) one restricts to finite sets I). In that case, it can be rephrased as follows:

Proposition 1 *The functor $F \mapsto F(pt)$ from the category of symmetric monoidal functors $\mathbf{Top}^f \to$ **Chain**(k) satisfying excision*[12] *to the category of E_∞-algebras is a natural equivalence.*

Similarly, Theorem 2 can be rephrased in the following way:

> *The functor $F \mapsto F(pt)$ from the category of functors* **Top** $\to$ **Chain**(k) *preserving arbitrary coproducts and satisfying excision to the category of E_∞-algebras is an natural equivalence.*

An immediate consequence of $A \boxtimes X \cong CH_X(A)$ and the identity (2) is the following natural equivalence

$$CH_{X\times Y}(A) \cong CH_X\big(CH_Y(A)\big) \tag{3}$$

in E_∞-**Alg**. This is the "exponential law" for derived Hochschild homology.

[11] Up to contractible choices.

[12] By Lemma 1, the excision axiom makes sense for any such functor.

Another interesting consequence of (2) is that, for any spaces K and X and E_∞-algebra A, the identity map in $\mathrm{Map}_{E_\infty\text{-}\mathbf{Alg}}(CH_{X\times K}(A), CH_{X\times K}(A))$ yields a canonical element in $\mathrm{Map}_{\mathbf{Top}}\big(K, \mathrm{Map}_{E_\infty\text{-}\mathbf{Alg}}(CH_X(A), CH_{X\times K}(A))\big)$ hence a canonical map of chain complexes

$$\mathrm{tens}:\ C_*\big(K\big)\otimes CH_X(A) \longrightarrow CH_{K\times X}(A). \tag{4}$$

Similarly, let $f : K \times X \to Y$ be a map of topological spaces, then we get a canonical continuous map $K \longrightarrow Map_{E_\infty\text{-}Alg}(CH_X(A), CH_Y(A))$ or equivalently a chain map $f_* : C_*(K) \otimes CH_X(A) \longrightarrow CH_Y(A)$ in **Chain**(k) which is just the composition

$$C_*(K)\otimes CH_X(A) \stackrel{\mathrm{tens}}{\longrightarrow} CH_{K\times X}(A) \stackrel{f_*}{\longrightarrow} CH_Y(A)$$

where the last map is by functoriality of $\mathcal{CH}$ with respect to maps of topological spaces.

Remark 4 (**Group actions on derived Hochschild homology**) Since $\mathcal{CH}$ is a functor of both variables, $CH_X(A)$ has a natural action of the topological monoid $Map_{\mathrm{Top}}(X, X)$ (and thus of the group $Homeo(X)$), *i.e.*, there is a monoid[13] map $Map_{\mathbf{Top}}(X, X) \to Map_{E_\infty\text{-}\mathbf{Alg}}(CH_X(A), CH_X(A))$. By adjunction, we get a chain map[14] $C_*\big(Map_{\mathbf{Top}}(X, X)\big) \otimes CH_X(A) \to CH_X(A)$ which exhibits $CH_X(A)$ as a module over $Map_{\mathbf{Top}}(X, X)$ in E_∞-**Alg**.

2.1.2 Derived Functor Interpretation

We now explain a derived functor interpretation of derived Hochschild homology. Recall (Example 65) that the singular chain functor of a space X has a natural structure of E_∞-coalgebras. In other words, it is an object (abusively denoted $C_*(X)$) of $\mathbf{Fun}^{\otimes}(\mathbf{Fin}^{op}, \mathbf{Chain}(k))$ the category of *contravariant* symmetric monoidal functor from finite sets to chain complexes.

We can identify an E_∞-coalgebra C, resp. an E_∞-algebra A, respectively, with a *right* module, resp. *left* module over the (∞-)operad $\mathbb{E}_\infty$; or equivalently with contravariant, resp. covariant, symmetric monoidal functors from **Fin** to **Chain**(k)). We can thus form their (derived) tensor products $C \otimes^{\mathbb{L}}_{\mathbb{E}_\infty} A \in \mathbf{Chain}(k)$ which is computed as a (homotopy) coequalizer:

$$C \underset{\mathbb{E}_\infty^{\otimes}}{\overset{\mathbb{L}}{\otimes}} A \cong hocoeq\Big(\coprod_{f:\{1,\dots,q\}\to\{1,\dots,p\}} C^{\otimes p}\otimes \mathbb{E}_\infty(q,p)\otimes A^{\otimes q} \rightrightarrows \coprod_n C^{\otimes n}\otimes A^{\otimes n}\Big)$$

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[13] Here, monoid means an homotopy monoid, that is an E_1-algebra in the symmetric monoidal category (**Top**, $\times$.)

[14] And higher homotopy coherences.

where the maps $f : \{1, \dots, q\} \to \{1, \dots, p\}$ are maps of sets. The upper map in the coequalizer is induced by the maps $f^* : C^{\otimes p} \otimes \mathbb{E}_\infty(q, p) \otimes A^{\otimes q} \to C^{\otimes q} \otimes A^{\otimes q}$ obtained from the coalgebra structure of C and the lower map is induced by the maps $f_* : C^{\otimes p} \otimes \mathbb{E}_\infty^{\otimes}(q, p) \otimes A^{\otimes q} \to C^{\otimes p} \otimes A^{\otimes p}$ induced by the algebra structure. One can define similarly $C \overset{\mathbb{L}}{\underset{\mathbf{Fin}}{\otimes}} A$ the derived tensor product of a covariant and contravariant **Fin**-modules.

Proposition 2 *Let X be a space and A be an E_∞-algebra. There is a natural equivalence (in $\mathbf{Chain}(k)$)*

$$CH_X(A) \cong C_*(X) \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty}{\otimes}} A.$$

If A has a structure of CDGA, then we further have $CH_X(A) \cong C_*(X) \overset{\mathbb{L}}{\underset{\mathbf{Fin}}{\otimes}} A$

Proof Note that the E_∞-coalgebra structure on $C_*(X)$ is given by the functor $\mathbf{Fin}^{op} \to \mathbf{Chain}(k)$ defined by $I \mapsto k\big[Hom_{Fin}(I, X_\bullet)\big]$. The rest of the proof is the same as in [47, Proposition 4]. □

Remark 5 (**Factorization homology of commutative algebras as derived mapping stacks**) There is another nice interpretation of derived Hochschild homology in terms of derived (or homotopical) algebraic geometry. Let $\mathbf{dSt}_k$ be the ∞-category of derived stacks over the ground ring k described in details in [95, Section 2.2]. This category admits internal Hom's that we denote by $\mathbb{R}\, Map(F, G)$ following [95, 96] and further is also an enrichment of the homotopy category of spaces. Indeed, any simplicial set $X_{\cdot}$ yields a constant simplicial presheaf $E_\infty\text{-}Alg \to sSet$ defined by $R \mapsto X_{\cdot}$ which, in turn, can be stackified. We denote $\mathfrak{X}$ the associated stack, *i.e.* the stackification of $R \mapsto X_{\cdot}$, which depends only on the (weak) homotopy type of $X_\bullet$. For a (derived) stack $\mathfrak{Y} \in \mathbf{dSt}_k$, we denote $\mathcal{O}_{\mathfrak{Y}}$ its functions, i.e., $\mathcal{O}_{\mathfrak{Y}} := \mathbb{R}\underline{Hom}(\mathfrak{Y}, \mathbb{A}^1)$, (see [95]). A direct application of Theorem 2 is:

Corollary 1 ([47]) *Let $\mathfrak{R} = \mathbb{R}\, Spec(R)$ be an affine derived stack (for instance an affine stack)* [95] *and $\mathfrak{X}$ be the stack associated to a space X. Then the Hochschild chains over X with coefficients in R represent the mapping stack $\mathbb{R}\, Map(\mathfrak{X}, \mathfrak{R})$. That is, there are canonical equivalences*

$$\mathcal{O}_{\mathbb{R}\, Map(\mathfrak{X},\mathfrak{R})} \cong CH_X(R), \quad \mathbb{R}\, Map(\mathfrak{X}, \mathfrak{R}) \cong \mathbb{R}\, Spec\left(CH_X(R)\right)$$

If a group G acts on X, the natural action of G on $CH_X(A)$ (Remark 4) identifies with the natural one on $\mathbb{R}\, Map(\mathfrak{X}, \mathfrak{R})$ under the equivalence given by Corollary 1.

2.2 Pointed Spaces and Higher Hochschild Cohomology

In order to have a dual and relative versions of the construction of Sect. 2.1, we consider the (∞-)category $\mathbf{Top}_*$ of *pointed* spaces. Let $\tau : pt \to X$ be a base point of $X \in \mathbf{Top}_*$. The map τ yields a map of E_∞-algebras $A \cong CH_{pt}(A) \xrightarrow{\tau_*} CH_X(A)$ and thus makes $CH_X(A)$ an A-module. Let M be an E_∞-module over A; for instance, take M to be a module over a CDGA A. Note that M has induced left and right modules structures[15] over A.

Definition 2 Let A be an E_∞-algebra and M be an E_∞-module over A.

- The (derived) *Hochschild cochains* of A with values in M over (a pointed topological space) X is given by

$$CH^X(A, M) := \mathbb{R}\mathrm{Hom}_A^{left}(CH_X(A), M),$$

  the (derived) chain complex of homomorphisms of underlying left E_1-modules over A (Definition 36).
- The (derived) *Hochschild homology* of A with values in M over (a pointed space) X is defined as

$$CH_X(A, M) := M \overset{\mathbb{L}}{\underset{A}{\otimes}} CH_{X_\bullet}(A) \tag{5}$$

  the relative tensor product of (a left and a right) E_1-modules over A.

The two definitions above depend on the choice of the base point even though we do not write it explicitly in the definition.

Remark 6 One can also use the relative tensor products of E_∞-modules over A (as defined, for instance, in [66, 71]) for defining the Hochschild homology $CH_X(A, M)$. This does not change the computation (and makes Lemma 2 below trivial) according to Proposition 40 (or [66, 71]). The same remark applies to the definition of derived Hochschild cohomology.

Since the based point map $\tau_* : A \to CH_X(A)$ is a map of E_∞-algebras, the canonical module structure of $CH_X(A)$ over itself induces a $CH_X(A)$-module structure on $CH_X(A, M)$ after tensoring by A (see [66, PartV], [71]):

Lemma 2 *Let M be in E_∞-**Mod**$_A$, that is, M is an E_∞-A-module. Then $CH_X(A, M)$ is canonically a E_∞-module over $CH_X(A)$.*

The Lemma is obvious when A is a CDGA.

We have the ∞-category E_∞-**Mod** of pairs (A, M) with A an E_∞-algebra and M an A-module (Definition 35). Let $\pi_{E_\infty} : E_\infty$-**Mod** $\to E_\infty$-**Alg** be the canonical functor.

---

[15] Note also that there is an equivalence of ∞-categories E_1-**LMod**$_A \cong E_1$-**RMod**$_A$ if $A \in E_\infty$-**Alg**.

Proposition 3 ([47, 48])

- *The derived Hochschild chains (Definition 2) induces a functor of ∞-categories $CH : (X, M) \mapsto CH_X(\pi_{E_\infty}(M), M)$ from* $\mathbf{Top}_* \times E_\infty$*-**Mod** to E_∞-**Mod** which fits into a commutative diagram*

$$\begin{array}{ccc} \mathbf{Top}_* \times E_\infty\text{-}\boldsymbol{Mod} & \xrightarrow{CH} & E_\infty\text{-}\boldsymbol{Mod} \\ \Big\downarrow{\scriptstyle for \times \pi_{E_\infty}} & & \Big\downarrow{\scriptstyle \pi_{E_\infty}} \\ \mathbf{Top} \times E_\infty\text{-}\boldsymbol{Alg} & \xrightarrow{\mathcal{CH}} & E_\infty\text{-}\boldsymbol{Alg}. \end{array}$$

Here for : $\mathbf{Top}_* \to \mathbf{Top}$ forget the base point.
- *The derived Hochschild cochains (Definition 2) induces a functor of ∞-categories $(X, M) \mapsto CH^{X_\bullet}(A, M)$ from $(\mathbf{Top}_*)^{op} \times E_\infty$-**Mod**$_A$ to E_∞-**Mod**$_A$, which is further contravariant with respect to A.*

In particular, if $M = A$, then we have an natural equivalence $CH_X(A, A) \cong CH_X(A)$ in E_∞-**Mod**.[16]

Remark 7 (**Functor homology point of view**) There is also a derived functor interpretation of the above functors as in Sect. 2.1.2. Let $\mathbf{Fin}_*$ be the ∞-category associated to the category of pointed finite sets (Example 57). If X is pointed, then we have a functor $\tilde{C}_*(X) : \mathbf{Fin}_*{}^{op} \to \mathbf{Chain}(k)$ which sends a finite pointed set I to $C_*(\mathrm{Map}_{pointed}(I, X))$ the singular chain on the space of pointed maps from I to X. Further, let M be an E_∞-module. Similarly to Sect. 2.1.2 we find a symmetric monoidal functor $\tilde{M} : \mathbf{Fin}_* \to \mathbf{Chain}(k)$. When M is a module over a CDGA A, denoting $*$ the base point, this is simply the functor $\tilde{M}(\{*\} \coprod J) = M \otimes A^{\otimes J}$, see [45]. This functor actually factors through E_1-$\mathbf{LMod}_A$.

We have a dual version of $\tilde{M}$, that we denote $\mathcal{H}(A, M) : \mathbf{Fin}_*{}^{op} \to \mathbf{Chain}(k)$, defined as $\mathcal{H}(A, M)(J) := \mathrm{Hom}_A(\tilde{A}(J), M)$ (where $J \mapsto \tilde{A}(J)$ is the functor $\mathbf{Fin}_* \to E_1$-$\mathbf{LMod}_A$ defined by the canonical E_∞-module structure of A). See [45] for an explicit construction when A is a CDGA and M a module.

A proof similar to the one of Proposition 2 yields:

Proposition 4 *There are natural equivalences*

$$CH_X(A, M) \cong \tilde{C}_*(X) \overset{\mathbb{L}}{\underset{\mathbf{Fin}_*}{\otimes}} \tilde{M}, \quad CH^X(A, M) \cong \mathbb{R}Hom_{\mathbf{Fin}_*}(\tilde{C}_*(X), \mathcal{H}(A, M)).$$

[16] Here we implicitly use the canonical functor E_∞-**Alg** $\to E_\infty$-**Mod** which sees an A-algebra as a module over itself.

2.3 Explicit Model for Derived Hochschild Chains

Following Pirashvili [79], one can construct rather simple explicit chain complexes computing derived Hochschild chains when the input is a CDGA. We mainly deal with the unpointed case, the pointed one being similar and left to the reader.

In this section, we consider only CDGAs. Note that, if we assume k is of characteristic zero, E_∞-algebras are always homotopy equivalent to a CDGA so that we do not loose much generality. This construction, using simplicial sets as models for topological spaces, provides explicit semi-free resolutions for $CH_X(A)$ which makes them combinatorially appealing.

Let $(A = \bigoplus_{i\in\mathbb{Z}} A^i, d, \mu)$ be a differential graded, associative, commutative algebra and let n_+ be the set $n_+ := \{0, \dots, n\}$. We define $CH_{n_+}(A) := A^{\otimes n+1} \cong A^{\otimes n_+}$. Let $f : k_+ \to \ell_+$ be any set map, we denote by $f_* : A^{\otimes k_+} \to A^{\otimes \ell_+}$, the linear map given by

$$f_*(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = (-1)^\epsilon \cdot b_0 \otimes b_1 \otimes \cdots \otimes b_\ell, \tag{6}$$

where $b_j = \prod_{i\in f^{-1}(j)} a_i$ (or $b_j = 1$ if $f^{-1}(j) = \emptyset$) for $j = 0, \dots, \ell$. The sign ϵ in Eq. (4) is determined by the usual Koszul sign rule of $(-1)^{|x|\cdot|y|}$ whenever x moves across y. In particular, $n_+ \mapsto CH_{n_+}(A)$ is functorial. Extending the construction by colimit we obtain a well-defined functor

$$Y \mapsto CH_Y(A) := \varinjlim_{Fin\ni K\to Y} CH_K(A) \tag{7}$$

from sets to differential graded commutative algebras (since the tensor products of CDGAs is a CDGA). Now, if $Y_\bullet$ is a simplicial set, we get a simplicial CDGA $CH_{Y_\bullet}(A)$ and by the Dold-Kan construction a CDGA whose product is induced by the *shuffle product* which is defined (in simplicial degree p, q) as the composition

$$sh : CH_{Y_p}(A) \otimes CH_{Y_q}(A) \xrightarrow{sh^\times} CH_{Y_{p+q}}(A) \otimes CH_{Y_{p+q}}(A) \cong CH_{Y_{p+q}}(A \otimes A) \xrightarrow{\mu_*} CH_{Y_{p+q}}(A). \tag{8}$$

Here $\mu : A \otimes A \to A$ denotes the multiplication in A (which is a map of algebras) and, denoting s_i the degeneracies of the simplicial structure in $CH_{Y_\bullet}(A)$,

$$sh^\times(v \otimes w) = \sum_{(\mu,\nu)} sgn(\mu, \nu)(s_{\nu_q} \cdots s_{\nu_1}(v) \otimes s_{\mu_p} \cdots s_{\mu_1}(w)),$$

where (μ, ν) denotes a (p, q)-shuffle, i.e. a permutation of $\{0, \dots, p+q-1\}$ mapping $0 \le j \le p-1$ to μ_{j+1} and $p \le j \le p+q-1$ to ν_{j-p+1}, such that $\mu_1 < \cdots < \mu_p$ and $\nu_1 < \cdots < \nu_q$. The differential $D : CH_{Y_\bullet}(A) \to CH_{Y_\bullet}(A)[1]$ is given as follows. The tensor products of chain complexes $A^{\otimes Y_i}$ have an internal differential which we abusively denote as d since it is induced by the differential $d : A \to A[1]$. Then, the differential on $CH_{Y_\bullet}(A)$ is given by the formula:

$$D\Big(\bigotimes_{i\in Y_i} a_i\Big) := (-1)^i d\Big(\bigotimes_{i\in Y_i} a_i\Big) + \sum_{r=0}^{i} (-1)^r (d_r)_*\Big(\bigotimes_{i\in Y_i} a_i\Big),$$

where the $(d_r)_* : CH_{Y_i}(A) \to CH_{Y_{i-1}}(A)$ are induced by the corresponding faces $d_r : Y_i \to Y_{i-1}$ of the simplicial set $Y_\bullet$.

Definition 3 Let $Y_\bullet$ be a simplicial set. The Hochschild chains over $Y_\bullet$ of A is the commutative differential graded algebra $(CH_{Y_\bullet}(A), D, sh)$.

The rule $(Y_\bullet, A) \mapsto (CH_{Y_\bullet}(A), D, sh)$ is a bifunctor from the ordinary discrete categories of simplicial sets and CDGA to the ordinary discrete category of CDGA.

If $Y_\bullet$ is a *pointed* simplicial set, we have a canonical CDGA map $A \overset{\sim}{\to} CH_{pt_\bullet}(A) \to CH_{Y_\bullet}(A)$. This allows to mimick Definition 2:

Definition 4 Let $Y_\bullet$ be a simplicial set, A a CDGA and M an A-module (viewed as a symmetric bimodule).

- The *Hochschild chains* of A with values in M over $Y_\bullet$ are:

$$CH_{X_\bullet}(A, M) := M \underset{A}{\otimes} CH_{X_\bullet}(A).$$

- The *Hochschild cochains* of A with values in M over $Y_\bullet$ are:

$$CH^{X_\bullet}(A, M) = Hom_A(CH_{X_\bullet}(A), M).$$

The above definition computes the derived Hochschild homology of Theorem 2. Indeed, we have the adjunction $|-| : sSet \underset{\sim}{\overset{\sim}{\rightleftarrows}} \text{Top} : \Delta_\bullet(-)$ given by the geometric realization $|Y_\bullet|$ of a simplicial set and the singular set functor: $n \mapsto \Delta_n(X) := \text{Hom}_{\text{Top}}(\Delta^n, X)$ (where $\Delta^n \in \text{Top}$ is the standard n-simplex). This adjunction is a Quillen adjunction hence induces an equivalence of ∞-categories. Further (by unicity Theorem 2) we have a commutative diagram (in Fun(**sSet** $\times$ **CDGA**, E_∞**-Alg**))

$$\begin{array}{ccccc} \mathbf{sSet} \times \mathbf{CDGA} & & \xrightarrow{(Y_\bullet, A) \to CH_{Y_\bullet}(A)} & & \mathbf{CDGA} \\ {\scriptstyle |-|}\downarrow {\scriptstyle \simeq} & & & & \downarrow \\ \mathbf{Top} \times \mathbf{CDGA} & \longrightarrow & \mathbf{Top} \times E_\infty\text{-}\mathbf{Alg} & \xrightarrow{\mathcal{CH}} & E_\infty\text{-}\mathbf{Alg}, \end{array} \qquad (9)$$

see [47, 48] for more details. From there, we get

Proposition 5 *One has natural equivalences* $CH_{X_\bullet}(A) \cong CH_{|X_\bullet|}(A)$ *of* E_∞*-algebras as well as equivalences*

$$CH_{X_\bullet}(A, M) \cong CH_{|X_\bullet|}(A, M), \quad CH^{X_\bullet}(A, M) \cong CH^{|X_\bullet|}(A, M)$$

of $CH_{|X_\bullet|}(A)$*-modules.*

We now demonstrate the above combinatorial definitions in a few examples (in which we assume, for simplicity, that A has a trivial differential).

Example 1 (*The point and the interval*) The point has a trivial simplicial model given by the constant simplicial set $pt_n = \{pt\}$. Hence

$$(CH_{pt_\bullet}(A), D) := A \overset{0}{\leftarrow} A \overset{id}{\leftarrow} A \overset{0}{\leftarrow} A \overset{id}{\leftarrow} A \cdots$$

which is a deformation retract of A (as a CDGA). A (pointed) simplicial model for the interval $I = [0, 1]$ is given by $I_n = \{\underline{0}, 1 \dots, n+1\}$, hence in simplicial degree n, $CH_{I_n}(A, M) = M \otimes A^{\otimes n+1}$ and the simplicial face maps are

$$d_i(a_0 \otimes \cdots a_{n+1}) = a_0 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_{n+1}.$$

An easy computation shows that $CH_{I_\bullet}(A, M) = Bar(M, A, A)$ is the standard Bar construction[17] which is quasi-isomorphic to M.

Example 2 (*The circle*) The circle $S^1 \cong I/(0 \sim 1)$ has (by Example 1) a simplicial model $S^1_\bullet$ which is the quotient $S^1_n = I_n/(0 \sim n+1) \cong \{0, \dots, n\}$. One computes that the face maps $d_i : S^1_n \to S^1_{n-1}$, for $0 \le i \le n-1$ are given by $d_i(j)$ is equal to j or $j-1$ depending on $j = 0, \dots, i$ or $j = i+1, \dots, n$ and $d_n(j)$ is equal to j or 0 depending on $j = 0, \dots, n-1$ or $j = n$. For $i = 0, \dots, n$, the degeneracies $s_i(j)$ is equal to j or $j+1$ depending on $j = 0, \dots, i$ or $j = i+1, \dots, n$. This is the standard simplicial model of S^1 cf. [67, 6.4.2]. Thus, $CH_{S^1_\bullet}(A) = \bigoplus_{n\ge 0} A \otimes A^{\otimes n}$ and the differential agrees with the *usual one on the Hochschild chain complex* $C_\bullet(A)$ of A (see [67]).

It can be proved that the S^1 action on $CH_{S^1}(A)$ given by Remark 4 agrees with the canonical mixed complex structure of $CH_{S^1_\bullet}(A)$ (see [97]).

Example 3 (*The torus*) The torus $\mathbb{T}$ is the product $S^1 \times S^1$. Thus, by Example 2, it has a simplicial model given by $(S^1 \times S^1)_\bullet$ the diagonal simplicial set associated to the bisimplicial set $S^1_\bullet \times S^1_\bullet$, i.e. $(S^1 \times S^1)_k = S^1_k \times S^1_k = \{0, \dots, k\}^2$. We may write $(S^1 \times S^1)_k = \{(p, q) \mid p, q = 0, \dots, k\}$ which we equipped with the lexicographical ordering. The face maps $d_i : (S^1 \times S^1)_k \to (S^1 \times S^1)_{k-1}$ and degeneracies $s_i : (S^1 \times S^1)_k \to (S^1 \times S^1)_{k+1}$, for $i = 0, \dots, k$, are given as the products of the differentials and degeneracies of $S^1_\bullet$, i.e. $d_i(p, q) = (d_i(p), d_i(q))$ and $s_i(p, q) = (s_i(p), s_i(q))$.

[17] The two-sided one, with values in the two A-modules A and M.

We obtain $CH_{(S^1\times S^1)_\bullet}(A,A)=\bigoplus_{k\geq 0} A\otimes A^{\otimes(k^2+2k)}$. The face maps d_i can be described more explicitly, when placing the tensor $a_{(0,0)}\otimes\cdots\otimes a_{(k,k)}$ in a $(k+1)\times(k+1)$ matrix. For $i=0,\dots,k-1$, we obtain $d_i(a_{(0,0)}\otimes\cdots\otimes a_{(k,k)})$ by multiplying the ith and $(i+1)$th rows and the ith and $(i+1)$th columns simultaneously, i.e., $d_i(a_{(0,0)}\otimes\cdots\otimes a_{(k,k)})$ is equal to:

$$\begin{array}{ccccc}
a_{(0,0)} & \cdots & (\mathbf{a_{(0,i)}a_{(0,i+1)}}) & \cdots & \mathbf{a_{(0,k)}} \\
\vdots & & \vdots & & \vdots \\
\mathbf{a_{(i-1,0)}} & \cdots & (a_{(i-1,i)}a_{(i-1,i+1)}) & \cdots & \otimes a_{(i-1,k)} \\
(\mathbf{a_{(i,0)}a_{(i+1,0)}}) & \cdots & (a_{(i,i)}a_{(i,i+1)}a_{(i+1,i)}a_{(i+1,i+1)}) & \cdots & (a_{(i,k)}a_{(i+1,k)}) \\
\mathbf{a_{(i+2,0)}} & \cdots & (a_{(i+2,i)}a_{(i+2,i+1)}) & \cdots & a_{(i+2,k)} \\
\vdots & & \vdots & & \vdots \\
\mathbf{a_{(k,0)}} & \cdots & (a_{(k,i)}a_{(k,i+1)}) & \cdots & a_{(k,k)}
\end{array}$$

The differential d_k is obtained by multiplying the kth and 0th rows and the kth and 0th columns simultaneously, i.e., $d_k(a_{(0,0)}\otimes\cdots\otimes a_{(k,k)})$ equals

$$\begin{array}{cccc}
(a_{(0,0)}a_{(0,k)}a_{(k,0)}a_{(k,k)}) & (\mathbf{a_{(0,1)}a_{(k,1)}}) & \cdots & (\mathbf{a_{(0,k-1)}a_{(k,k-1)}}) \\
(\mathbf{a_{(1,0)}a_{(1,k)}}) & a_{(1,1)} & \cdots & a_{(1,k-1)} \\
\vdots & \vdots & & \vdots \\
(\mathbf{a_{(k-1,0)}a_{(k-1,k)}}) & a_{(k-1,1)} & \cdots & a_{(k-1,k-1)}
\end{array}$$

Example 4 (*The Riemann sphere* S^2) The sphere S^2 has a simplicial model $S^2_\bullet=I^2_\bullet/\partial I^2_\bullet$ i.e. $S^2_n=\{(0,0)\}\coprod\{1\cdots n\}^2$. Thus $CH_{S^2_\bullet}(A)=\bigoplus_{n\geq 0} A\otimes A^{\otimes n^2}$.

Here the face and degeneracies maps are the diagonal ones as for $(S^1\times S^1)_\bullet$ in Example 3. In particular, the ith differential is also obtained from the previous examples by setting $d_i^{S^2_\bullet}(p,q)=(0,0)$ in the case that $d_i(p)=0$ or $d_i(q)=0$ (where d_i is the ith-face map of $S^1_\bullet$), or setting otherwise $d_i(p,q)=(d_i(p),d_i(q))$. For $i\leq n-1$, we obtain $d_i(a_{(0,0)}\otimes\cdots\otimes a_{(k,k)})$ is equal to:

$$\begin{array}{cccc}
a_{(0,0)} & & & \\
 & (a_{(i-1,i)}a_{(i-1,i+1)}) & \cdots & a_{(i-1,n)} \\
 & (a_{(i,i)}a_{(i,i+1)}a_{(i+1,i)}a_{(i+1,i+1)}) & \cdots & (a_{(i,n)}a_{(i+1,n)}) \\
 & (a_{(i+2,i)}a_{(i+2,i+1)}) & \cdots & a_{(i+2,n)} \\
 & \vdots & & \vdots \\
 & (a_{(n,i)}a_{(n,i+1)}) & \cdots & a_{(n,n)}
\end{array}$$

which is similar to the one of Example 3 without the "boldface" tensors.

Example 5 (*Higher spheres*) Similarly to S^2, we have the *standard model* $S^d_\bullet:=(I_\bullet)^d/\partial(I_\bullet)^d\cong S^1_\bullet\wedge\cdots\wedge S^1_\bullet$ (d-factors) for the sphere S^d. Hence $S^d_n\cong\{\underline{0}\}\coprod\{1\cdots n\}^d$ and $CH_{S^d_\bullet}(A)=\bigoplus_{n\geq 0} A\otimes A^{\otimes n^d}$. The face operators are similar to those of

Example 4 (except that, instead of a matrix, we have a dimension d-lattice) and face maps are obtained by simultaneously multiplying each ith-hyperplane with $(i+1)$th-hyperplane in each dimension. The last face d_n is obtained by multiplying all tensors of all nth-hyperplanes with a_0.

We also have the *small model* $S^d_{sm_\bullet}$ which is the simplicial set with exactly two *non-degenerate* simplices, one in degree 0 and one in degree d. Then $S^d_{sm\,n} \cong \{1, \dots, \binom{n}{d}\}$. Using this model, it is straightforward to check the following computation of the first homology groups of $CH_{S^d}(A)$:

$$H_n(CH_{S^d}(A)) \cong H_n(CH_{S^d_{sm_\bullet}}(A)) = \begin{cases} = A & \text{if } n = 0 \\ = 0 & \text{if } 0 < n < d \\ = \Omega^1_A & \text{if } n = d \end{cases}$$

where Ω^1_A is the A-module of Kähler differentials (see [67, 101]).

Example 6 (*Hochschild-Kostant-Rosenberg*) Let A be a smooth commutative algebra. The classical Hochschild-Kostant-Rosenberg Theorem states that its (standard) Hochschild homology is given by the algebra of Kähler forms $\wedge^\bullet_A(\Omega^1_A) \cong S^\bullet_A(\Omega^1_A[1])$, where Ω^1_A is the A-module of Kähler differentials; here a i-form is viewed as having cohomological degree $-i$ and $S^\bullet_A$ is the free graded commutative algebra functor (in the category of graded A-modules). This theorem extends to Hochschild homology over all spheres:

Theorem 3 (Generalized HKR) *Let A be a smooth algebra and X be an affine smooth scheme or a smooth manifold. Let $n \geq 1$ and Σ^g be a genus g surface.*

1. ***(Pirashvili** [79]) There is a quasi-isomorphism of CDGAs: $CH_{S^n}(A) \cong S^\bullet_A\big(\Omega^1_A[n]\big)$.*
2. *([46]) There is an equivalence $CH_{\Sigma^g}(A) \cong S^\bullet_A\big(\Omega^1_A[2] \oplus (\Omega^1_A[1])^{\oplus 2g}\big)$ of CDGAs.*
3. *There are equivalences $CH_{S^n}(\mathcal{O}_X) \cong S^\bullet_{\mathcal{O}_X}\big(\Omega^1_X[n]\big)$ and*

$$CH_{\Sigma^g}(\mathcal{O}_X) \cong S^\bullet_{\mathcal{O}_X}\big(\Omega^1_X[2] \oplus (\Omega^1_X[1])^{\oplus 2g}\big)$$

of sheaves of CDGAs.[18]

The third assertion in Theorem 3 follows from 1 and 2 after sheafifying in an appropriate way the derived Hochschild chains.

2.4 Relationship with Mapping Spaces

We have seen the relationship between derived Hochschild chains and derived mapping spaces (Remark 5). It is also classical that the usual Hochschild homology of de

[18] Here the differentials on the right hand sides are zero; they are *not* the de Rham differential.

Rham forms of a simply connected manifold M is a model for the de Rham forms on the free loop space $LM := \mathrm{Map}(S^1, M)$ of M (see [18]). There is a generalization of this result for spaces where the forms are replaced by the singular cochains with their E_∞-algebra structure ([62]). These two results extend to derived Hochschild chains in general to provide algebraic models of mapping spaces.

First, we sketch a generalization of Chen iterated integrals (studied in [46]). Let M be a compact, oriented manifold, denote by $\Omega^\bullet_{dR}(M)$ the differential graded algebra of differential forms on M, and let $Y_\bullet$ be a simplicial set with geometric realization $Y := |Y_\bullet|$. Denote $M^Y := \mathrm{Map}_{sm}(Y, M)$ the space of continuous maps from Y to M, which are smooth on the interior of each simplex in Y. Recall from Chen [18, Definition1.2.1], that a differentiable structure on M^Y is specified by the set of plots $\phi : U \to M^Y$, where $U \subset \mathbb{R}^n$ for some n, which are those maps whose adjoint $\phi_\sharp : U \times Y \to M$ is continuous on $U \times Y$, and smooth on the restriction to the interior of each simplex of Y, i.e. $\phi_\sharp|_{U\times(\text{simplex of }Y)^\circ}$ is smooth. Following [18, Definition1.2.2], a p-form $\omega \in \Omega^p_{dR}(M^Y)$ on M^Y is given by a p-form $\omega_\phi \in \Omega^p_{dR}(U)$ for each plot $\phi : U \to M^Y$, which is invariant with respect to smooth transformations of the domain.

We now define the space of Chen (generalized) *iterated integrals* $Chen(M^Y)$ of the mapping space M^Y. Let $\eta : Y_\bullet \to \Delta_\bullet|Y_\bullet|$ be the canonical simplicial map (induced by adjunction) which is given for $i \in Y_k$ by maps $\eta(i) : \Delta^k \to Y$ in the following way,

$$\eta(i)(t_1 \le \cdots \le t_k) := [(t_1 \le \cdots \le t_k) \times \{i\}] \in \Big(\coprod \Delta^\bullet \times Y_\bullet / \sim\Big) = Y. \quad (10)$$

The map η allows to define, for any plot $\phi : U \to M^Y$, a map $\rho_\phi := ev \circ (\phi \times id)$,

$$\rho_\phi : U \times \Delta^k \xrightarrow{\phi\times id} M^Y \times \Delta^k \xrightarrow{ev} M^{Y_k}, \quad (11)$$

where ev is defined as the evaluation map,

$$ev(\gamma : Y \to M, \underline{t})(i) = \gamma\big(\eta(i)(\underline{t})\big). \quad (12)$$

Now, if we are given a form $\bigotimes_{y\in Y_k} a_y \in \big(\Omega_{dR}(M)\big)^{\otimes Y_k}$ (with only finitely many $a_i \neq 1$), the pullback $(\rho_\phi)^*\big(\bigotimes_{y\in Y_k} a_y\big) \in \Omega^\bullet(U \times \Delta^k)$, may be integrated along the fiber Δ^k, and is denoted by

$$\Bigg(\int_{\mathcal{C}} \bigotimes_{y\in Y_k} a_y\Bigg)_\phi := \int_{\Delta^k} (\rho_\phi)^*\big(\underset{y\in Y_k}{\otimes} a_y\big) \quad \in \Omega^\bullet_{dR}(U).$$

The resulting $p = (\sum_i deg(a_i) - k)$-form $\int_{\mathcal{C}} \Big(\otimes_{y\in Y_k} a_y\Big) \in \Omega^p_{dR}(M^Y$ is called the (generalized) *iterated integral* of $a_0, \ldots, a_{y_k}$. The subspace of the space of De Rham forms $\Omega^\bullet(M^Y)$ generated by all iterated integrals is denoted by $\mathcal{C}hen(M^Y)$. In short, we may picture an iterated integral as the pullback composed with the integration along the fiber Δ^k of a form in M^{Y_k},

$$M^Y \xleftarrow{\int_{\Delta^k}} M^Y \times \Delta^k \xrightarrow{ev} M^{Y_k}$$

Definition 5 We define $\mathcal{I}t^{Y_\bullet} : CH_{Y_\bullet}(\Omega^\bullet_{dR}(M)) \cong (\Omega^\bullet_{dR}(M))^{\otimes Y_\bullet} \to \mathcal{C}hen(M^Y)$ by

$$\mathcal{I}t^{Y_\bullet}\left(\bigotimes_{y\in Y_k} a_y\right) := \int_{\mathcal{C}} \Big(\bigotimes_{y\in Y_k} a_y \Big). \tag{13}$$

Interesting applications of iterated integrals to study gerbes and higher holonomy are given in [1, 99].

Theorem 4 ([46]) *The iterated integral map $\mathcal{I}t^{Y_\bullet} : CH_{Y_\bullet}(\Omega^\bullet_{dR}(M)) \to \Omega^\bullet_{dR}(M^Y)$ is a (natural) map of CDGAs.*

Further, assume that $Y = |Y_\bullet|$ is n-dimensional, i.e. the highest degree of any non-degenerate simplex is n, and assume that M is n-connected. Then, $\mathcal{I}t^{Y_\bullet}$ is a quasi-isomorphism.

There is also a purely topological and characteristic free analogue of this result using singular cochains instead of forms.

Theorem 5 ([35, 48]) *Let X, Y be topological spaces. There is a natural map of E_∞-algebras*

$$CH_Y(C^*(X)) \longrightarrow C^*(\mathrm{Map}(Y, X))$$

which is an equivalence when $Y = |Y_\bullet|$ is n-dimensional and X is connected, nilpotent with finite homotopy groups in degree less or equal to n (for instance when X is n-connected).

Example 7 We compute the iterated integral map (13) in the case of $S^1_\bullet$ (Example 2) and $\mathbb{T}$ (Example 3). Since S^1 is the interval $I = [0, 1]$ where the endpoints 0 and 1 are identified, the map $\eta(i) : S^1_k = \{0, 1 \dots, k\} \to \Delta_k(S^1) = \mathrm{Map}(\Delta^k, S^1)$ defined via (10) is given by $\eta(i)(0 \le t_1 \le \cdots \le t_k \le 1) = t_i$, where we have set $t_0 = 0$. Thus, the evaluation map (12) becomes

$$ev(\gamma : S^1 \to M, t_1 \le \cdots \le t_k) = (\gamma(0), \gamma(t_1), \dots, \gamma(t_k)) \in M^{k+1}.$$

Furthermore, this recovers the classical Chen iterated integrals $It^{S^1_\bullet} : CH_\bullet(A, A) \to \Omega^\bullet(M^{S^1})$ as follows. For a plot $\phi : U \to M^{S^1}$ we have,

$$
\begin{aligned}
It^{S^1_\bullet}(a_0\otimes\cdots\otimes a_k)_\phi &= \left(\int_{\mathcal{C}} a_0\cdots a_k\right)_\phi\\
&= \int_{\Delta^k}(\rho_\phi)^*(a_0\otimes\cdots\otimes a_k)\\
&= (\pi_0)^*(a_0)\wedge\int_{\Delta^k}(\widetilde{\rho_\phi})^*(a_1\otimes\cdots\otimes a_k)\\
&= (\pi_0)^*(a_0)\wedge\int a_1\cdots a_k,
\end{aligned}
$$

where $\widetilde{\rho_\phi}: U\times\Delta^k \xrightarrow{\phi\times id} M^{S^1}\times\Delta^k \xrightarrow{\widetilde{ev}} M^k$ is the classical Chen integral $\int a_1\cdots a_k$ from [18] and $\pi_0: M^{S^1}\to M$ is the evaluation at the base point $\pi_0:\gamma\mapsto\gamma(0)$.

In the case of the torus $\mathbb{T}=S^1\times S^1$, the map $\eta(p,q):(S^1\times S^1)_k\to \mathrm{Map}(\Delta^k, S^1\times S^1)$ is given by $\eta(p,q)(0\le t_1\le\cdots\le t_k\le 1)=(t_p,t_q)\in S^1\times S^1$, for $p,q=0,\dots,k$ and $t_0=0$. Thus, the evaluation map (12) becomes

$$
ev(\gamma:\mathbb{T}\to M, t_1\le\cdots\le t_k)=\begin{pmatrix}\gamma(0,0),\ \gamma(0,t_1),\ \dots,\ \gamma(0,t_k),\\ \gamma(t_1,0),\gamma(t_1,t_1),\dots,\gamma(t_1,t_k),\\ \dots\\ \gamma(t_k,0),\gamma(t_k,t_1),\dots,\gamma(t_k,t_k)\end{pmatrix}\in M^{(k+1)^2}
$$

According to Definition 5, the iterated integral $It^{S^1\times S^1}(a_{(0,0)}\otimes\cdots\otimes a_{(k,k)})$ is given by a pullback under the above map $M^{S^1\times S^1}\times\Delta^k\xrightarrow{ev} M^{(k+1)^2}$, and integration along the fiber Δ^k.

2.5 The wedge Product *of higher Hochschild cohomology*

Let $A\xrightarrow{f}B$ be a map of CDGAs. Note that it makes B into an A-algebra as well as an $A\otimes A$-algebra (since the multiplication $A\otimes A\to A$ is an algebra morphism). The excision axiom (Theorem 2) implies

Lemma 3 *Let M be an A-module and X, Y be pointed topological spaces. There is a natural equivalence*

$$
\mu:\ Hom_{A\otimes A}\left(CH_X(A)\otimes CH_Y(A), M\right)\xrightarrow{\simeq} CH^{X\vee Y}(A,M)
$$

We use Lemma 3 to obtain

Definition 6 ([45]) The *wedge product* of (derived) Hochschild cochains is the linear map

$$\mu_\vee : CH^X(A, B) \otimes CH^Y(A, B) \longrightarrow Hom_{A\otimes A}\Big(CH_X(A) \otimes CH_Y(A), B \otimes B\Big) \stackrel{(m_B)_*}{\longrightarrow} Hom_{A\otimes A}\Big(CH_X(A) \otimes CH_Y(A), B\Big) \cong CH^{X\vee Y}(A, B) \tag{14}$$

where the first map is the obvious one: $f \otimes g \mapsto \big(x \otimes y \mapsto f(x) \otimes g(y)\big)$.

Example 8 If $X_\bullet, Y_\bullet$ are finite simplicial sets models of X, Y, the map $\mu_\vee$ can be combinatorially described as the composition of the linear map $\tilde{\mu}$ given, for any $f \in CH^{X_n}(A, B) \cong Hom_A(A^{\otimes \# X_n}, B)$, $g \in CH^{X_n}(A, B) \cong Hom_A(A^{\otimes \# Y_n}, B))$ by

$$\tilde{\mu}(f, g)(a_0, a_2, \dots a_{\# X_n}, b_2, \dots, b_{\# Y_n}) = a_0.f(1, a_2, \dots a_{\# X_n}).g(1, b_2, \dots, b_{\# Y_n})$$

(where a_0 corresponds to the element indexed by the base point of $X_n \vee Y_n$) with the Eilenberg-Zilber quasi-isomorphism from $CH^{X_\bullet}(A, B) \otimes CH^{Y_\bullet}(A, B)$ to the chain complex associated to the diagonal cosimplicial space $\big(CH^{X_n}(A, B) \otimes CH^{Y_n}(A, B)\big)_{n\in\mathbb{N}}$.

Proposition 6 *The wedge product (of Definition 6) is associative.*[19] *In particular, if there is a diagonal* $X \stackrel{\delta}{\to} X \vee X$ *making* X *an* E_1*-coalgebra (in* $(\mathbf{Top}_*, \vee)$*), then* $(CH^X(A, B), \delta^* \circ \mu_\vee)$ *is an* E_1*-algebra.*

Example 9 A standard example of space with a diagonal is a sphere S^d. For $d = 1$, we obtain a cup product on the usual Hochschild cochain complex which is (homotopy) equivalent to the standard cup-product for Hochschild cochains from [44].

The little d-dimensional little cubes operad Cube_d acts continuously on S^d by the pinching map

$$pinch : \mathrm{Cube}_d(r) \times S^d \longrightarrow \bigvee_{i=1\cdots r} S^d. \tag{15}$$

given, for any $c \in \mathrm{Cube}_d(r)$, by the map $pinch_c : S^d \to \bigvee S^d$ collapsing the complement of the interiors of the r rectangles to the base point. We thus get a map

$$\tilde{pinch} : \mathrm{Cube}_d(r) \longrightarrow \mathrm{Map}_{\mathbf{CDGA}}(CH_{S^d}(A, B), CH_{\bigvee S^d}(A, B)) \tag{16}$$

Applying the contravariance of Hochschild cochains and the wedge product (Definition 6), we get, for all $d \geq 1$, a morphism

$$pinch^*_{S^d, r} : C_*\big(\mathrm{Cube}_d(r)\big) \otimes \Big(CH^{S^d}(A, B)\Big)^{\otimes r} \stackrel{(\mu_\vee)^{(d-1)}}{\longrightarrow} C_*\big(\mathrm{Cube}_d(r)\big) \otimes CH^{\bigvee_{i=1}^r S^d}\big(A, B\big) \stackrel{\tilde{pinch}^*}{\longrightarrow} CH^{S^d}(A, B). \tag{17}$$

[19] Precisely, it means that $\mu_\vee$ makes $X \mapsto CH^X(A, B)$ into a lax monoidal functor $((\mathbf{Top}_*)^{op}, \vee) \to (\mathbf{Chain}(k), \otimes)$.

The map (17) has a canonical extension to the case of E_∞-algebras. We find

Proposition 7 ([45, 48]) *Let $A \stackrel{f}{\to} B$ be a CDGA (or E_∞-algebra) map. The collection of maps $(pinch_{S^d,k})_{k\geq 1}$ makes $CH^{S^d}(A, B)$ an E_d-algebra.*

The algebra structure is natural with respect to CDGA maps, meaning that given a commutative diagram the canonical map $h' \mapsto \varphi \circ h' \circ \psi$ is an E_d-algebras morphism $CH^{S^d}(A', B') \to CH^{S^d}(A, B)$.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} \\ A' & \xrightarrow{f'} & B' \end{array}$$

$$CH^{S^d}(A', B') \to CH^{S^d}(A, B).$$

Remark 8 If $f : A \to B$ is a CDGA map, it is possible to describe this E_d-algebra structure by giving an explicit action of the filtration F_dBE of the Barrat-Eccles operad on $CH^{S^d_\bullet}(A, B)$ using the standard simplicial model of S^d (Example 5).

Example 10 If $A = k$, $CH^{S^n}(k, B) \cong B$ (viewed as E_n-algebras). If $B = k$, then the E_n-algebra structure of $CH^{S^n}(A, k)$ is the dual of the E_n-coalgebra structure given by the n-times iterated Bar construction $Bar^{(n)}(A)$, see Sect. 7.4.

3 Homology Theory for Manifolds

3.1 Categories of Structured Manifolds and Variations on E_n-Algebras

In order to specify what is a homology theory for manifolds, we need to specify an interesting category of manifolds.

Definition 7 Let Mfld_n be the ∞-category associated[20] to the topological category with objects topological manifolds of dimension n and with morphism space

$$\mathrm{Map}_{\mathrm{Mfld}_n}(M, N) := \mathrm{Emb}(M, N)$$

[20] By Example 58.

the space of all embeddings of M into N (viewed as a subspace of the space $\mathrm{Map}(M, N)$ of all continuous maps from M to N endowed with the compact-open topology).

In the above definition, the manifolds can be closed or open, but have no boundary.[21]

Remark 9 It is important to consider embeddings instead of smooth maps. Indeed, the category of all manifolds and all (smooth) maps is weakly homotopy equivalent to Top so that, in that case, one would obtain a homology theory which extends to spaces.

Remark 10 (**Smooth manifolds**) One can also restrict to *smooth* manifolds in which case it makes sense to equip $\mathrm{Emb}(M, N)$ with the weak Whitney C^∞-topology; this gives us *the ∞-category* Mfld_n^{un} *of smooth manifolds of dimension n*. This latter category embeds in Mfld_n and this embedding is an equivalence onto the full subcategory of Mfld_n spanned by the smooth manifolds.

One can also consider categories of more structured manifolds, such as oriented, spin or framed manifolds, as follows. Let $E \to X$ be a topological n-dimensional vector bundle, which is the same as a space X together with a (homotopy class of) map $e : X \to B\mathrm{Homeo}(\mathbb{R}^n)$ from X to the classifying space of the group of homeomorphisms of $\mathbb{R}^n$. *An (X, e) -structure on a manifold M* $\in \mathrm{Mfld}_n$ is a map $f : M \to X$ such that TM is the pullback $f^*(E)$ which is the same as a factorization $M \stackrel{f}{\to} X \stackrel{e}{\to} B\mathrm{Homeo}(\mathbb{R}^n)$ of the map $M \xrightarrow{e_M} B\mathrm{Homeo}(\mathbb{R}^n)$ classifying the tangent (micro-)bundle of M.

Definition 8 Let $\mathrm{Mfld}_n^{(X,e)}$ be the (homotopy) pullback (in ∞-**Cat**)

$$\mathrm{Mfld}_n^{(X,e)} := \mathrm{Mfld}_n \times^h_{\mathbf{Top}_{/B\mathrm{Homeo}(\mathbb{R}^n)}} \mathbf{Top}_{/X}.$$

In other words $\mathrm{Mfld}_n^{(X,e)}$ is the ∞-category with objects n-dimensional topological manifolds with an (X, e)-structure and with morphism the embeddings preserving the (X, e)-structure. The latter morphisms are made into a topological space by identifying them with the homotopy pullback space

$$\begin{aligned}\mathrm{Map}_{\mathrm{Mfld}_n^{(X,e)}}(M, N) &:= \mathrm{Emb}^{(X,e)}(M, N)\\ &\cong \mathrm{Emb}(M, N) \times^h_{\mathrm{Map}_{/B\mathrm{Homeo}(\mathbb{R}^n)}(M,N)} \mathrm{Map}_{/X}(M, N).\end{aligned}$$

Example 11 We list our main examples of study.

- Let $X = pt$, then E is trivial (here e is induced by the canonical base point of $B\mathrm{Homeo}(\mathbb{R}^n)$) and an (X, e)-structure on M is a framing, that is, a trivialization of the tangent (micro-)bundle of M. In that case, *we denote* $\mathrm{Mfld}_n^{fr} := \mathrm{Mfld}_n^{(pt,e)}$

[21] Though homology theory for manifolds can be extended to stratified manifolds, see [5].

the ∞-category of framed manifolds. Note that this ∞-category is equivalent to the one associated to the topological category with objects the framed manifolds of dimension n and morphism spaces from M to N the framed embeddings, that is the pairs (f, h) where $f \in \mathrm{Emb}(M, N)$ and h is an homotopy between the two trivialisation of TM induced by the framing of M and the framing of N pulled-back along f.

- Let $X = BO(n)$ and $BO(n) \stackrel{e}{\to} B\mathrm{Homeo}(\mathbb{R}^n)$ be the canonical map. Then $\mathrm{Mfld}_n^{(BO(n),e)}$ is (equivalent to) the *∞-category of smooth n-manifolds* of Remark 10. This essentially follows because the map $O(n) \to \mathrm{Diffeo}(\mathbb{R}^n)$ is a deformation retract and the characterization of smooth manifolds in terms of their micro-bundle structure [63].
- Let $X = BSO(n)$ and $BSO(n) \stackrel{e}{\to} B\mathrm{Homeo}(\mathbb{R}^n)$ be the canonical map induced by the inclusion of $SO(n) \hookrightarrow \mathrm{Homeo}(\mathbb{R}^n)$. Then a $(BSO(n), e)$-structure on M is an orientation of M. We denote $\mathrm{Mfld}_n^{or} := \mathrm{Mfld}_n^{(BSO(n),e)}$ the *∞-category of oriented smooth n-manifolds*. Similarly to the framed case, it has a straightforward description as the ∞-category associated a topological category with morphisms the space of oriented embeddings.
- If X is a n-dimensional manifold, we can take $e_X : X \to B\mathrm{Homeo}(\mathbb{R}^n)$ to be the map corresponding to the tangent bundle $TX \to X$ of X. We simply denote $\mathrm{Mfld}_n^{(X,TX)}$ the associated ∞-category of manifolds. Every open subset of X is canonically an object of $\mathrm{Mfld}_n^{(X,TX)}$.

The (topological) coproduct $M \coprod N$ (that is disjoint union) of two (X, e)-manifolds M, N has a canonical structure of (X, e)-manifold (given by $M \coprod N \stackrel{f \coprod g}{\to} X$ where $M \stackrel{f}{\to} X$ and $N \stackrel{g}{\to} X$ define the (X, e)-structures). Note that in general there are no embeddings $M \coprod M \to M$ so that the disjoint union of manifolds is *not* a coproduct (in the sense of category theory) in $\mathrm{Mfld}_n^{(X,e)}$. Nevertheless

Lemma 4 *$(Mfld_n^{(X,e)}, \coprod)$ is a symmetric monoidal ∞-category.*

There is a canonical choice of framing of $\mathbb{R}^n$ which induces a canonical (X, e)-structure on $\mathbb{R}^n$ for any pointed space X. Unlike other manifolds, there are interesting framed embeddings $\coprod \mathbb{R}^i \to \mathbb{R}^i$ for any integer i. Indeed, in view of Example 11 and Definition 51 the space of embeddings $\mathrm{Emb}^{fr}(\coprod_{\{1,\dots,r\}} \mathbb{R}^n, \mathbb{R}^n) = \mathrm{Disk}_n^{fr}(r, 1)$ is homotopy equivalent to $\mathrm{Cube}_n(r)$ the arity r space of the little cube operad and thus is homotopy equivalent to the configuration space of r unordered points in $\mathbb{R}^n$.

This motivates the following (X, e)-structured version of E_n-algebras.

Definition 9 Let $\mathrm{Disk}_n^{(X,e)}$ be the full subcategory of $\mathrm{Mfld}_n^{(X,e)}$ spanned by disjoints union of standard euclidean disks $\mathbb{R}^n$. The ∞-category of $\mathrm{Disk}_n^{(X,e)}$-algebras[22] is the category

$$\mathbf{Fun}^{\otimes}(\mathrm{Disk}_n^{(X,e)}, \mathbf{Chain}(k)) \tag{18}$$

[22] Which we also referred to as the category of (X, e)-structured E_n-algebras.

of symmetric monoidal (∞-)functors from $(\mathrm{Disk}_n^{(X,e)}, \coprod)$ to $(\mathbf{Chain}(k), \otimes)$.

The *underlying object* of a $\mathrm{Disk}_n^{(X,e)}$-algebra A is its value $A(\mathbb{R}^n)$ on a single disk $\mathbb{R}^n$. We will often abusively denote in the same way the $\mathrm{Disk}_n^{(X,e)}$-algebra and its underlying object.

We denote $\mathrm{Disk}^{(X,e)}$**-Alg** *the* ∞*-category of* $\mathrm{Disk}_n^{(X,e)}$*-algebras* and $\mathrm{Disk}^{(X,e)}$**-Alg**$(\mathcal{C})$ the one of $\mathrm{Disk}_n^{(X,e)}$-algebras with values in a symmetric monoidal category $\mathcal{C}$ (whose definition are the same as Definition 9 with $(\mathbf{Chain}(k), \otimes)$ replaced by $(\mathcal{C}, \otimes)$). The underlying object induces a functor

$$\mathrm{Disk}^{(X,e)}\textbf{-Alg}(\mathcal{C}) \longrightarrow \mathcal{C} \tag{19}$$

Example 12
- For $X = pt$, the category of $\mathrm{Disk}_n^{(X,e)}$-algebras will be *denoted* Disk_n^{fr}**-Alg**. It is equivalent to the usual category of E_n-algebras and corresponds to the case of framed manifolds.
- The category of $\mathrm{Disk}_n^{(BSO(n),e)}$-algebras is equivalent to the category of algebras over the operad $\left(\mathrm{Cube}_n(r) \ltimes SO(n)^r\right)_{r\geq 1}$ introduced in [82] (and called the framed little disk operad). Since these algebras corresponds to the case of oriented manifolds, we call them *oriented* E_n*-algebras* and we simply write Disk_n^{or} for $\mathrm{Disk}_n^{(BSO(n),e)}$. It can be shown that Disk_n^{or}-algebras are homotopy fixed points of the E_n-algebras with respect to the action of $SO(n)$ on the operad Disk_n^{fr}.
- Similarly, the category of $\mathrm{Disk}_n^{(BO(n),e)}$-algebras is equivalent to the category of algebras over the operad $\left(\mathrm{Cube}_n(r) \ltimes O(n)^r\right)_{r\geq 1}$. We also call them *unoriented* E_n*-algebras* and simply write Disk_n^{un} for $\mathrm{Disk}_n^{(BO(n),e)}$.
- Let $U \cong \mathbb{R}^n$ be a disk in X. By restriction to sub-disks of U, we have a canonical functor $\mathrm{Disk}_n^{(X,TX)}$**-Alg** $\to$ $\mathrm{Disk}_n^{(\mathbb{R}^n,T\mathbb{R}^n)}$**-Alg** $\cong$ E_n**-Alg** (see Theorem 9). It follows that a $\mathrm{Disk}_n^{(X,TX)}$-algebra is simply a family of E_n-algebras over X.

Example 13 (Commutative algebras as $\mathrm{Disk}_n^{(X,e)}$*-algebras)* The canonical functor $\mathrm{Disk}_n^{(X,e)} \to$ **Fin** (where **Fin** is the ∞-category associated to the category of finite sets) shows that any E_∞-algebra (Definition 34) has a canonical structure of $\mathrm{Disk}_n^{(X,e)}$-algebras. Thus we have canonical functors

$$\mathbf{CDGA} \longrightarrow E_\infty\textbf{-Alg} \longrightarrow \mathrm{Disk}_n^{(X,e)}\textbf{-Alg}.$$

For A a differential graded commutative algebra, this structure is the symmetric monoidal functor defined by $A(\coprod_{i\in I}\mathbb{R}^n) := A^{\otimes I}$ and, for an (X,e)-preserving embedding $\coprod_I \mathbb{R}^n \hookrightarrow \mathbb{R}^n$, by the (iterated) multiplication $A^{\otimes I} \to A$.

Example 14 (Opposite of an E_n*-algebra)* There is a canonical $\mathbb{Z}/2\mathbb{Z}$-action on E_n**-Alg** induced by the antipodal map $\tau : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto -x$ acting on the source of $\mathbf{Fun}^{\otimes}(\mathrm{Disk}_n^{fr}, \mathbf{Chain}(k))$. If A is an E_n-algebra, then the result of this action $A^{op} := \tau^*(A)$ is its opposite algebra. If $n = \infty$, the antipodal map is homotopical to the identity so that A^{op} is equivalent to A as an E_∞-algebra.

3.2 Factorization Homology of Manifolds

We now explain what is a Homology Theory for Manifolds (Definition 10) in a way parallel to the presentation of the Eilenberg-Steenrod axioms. We first need an analogue of Lemma 1 for monoidal functors out of manifolds (instead of spaces) in order to formulate the correct excision axiom.

Observe that $\mathbb{R}^n$ is canonically an E_n-algebra object in Mfld_n. Let N be an $(n-s)$-dimensional manifold such that $N\times\mathbb{R}^s$ has an (X,e)-structure. Then, similarly, $N\times\mathbb{R}^s$ is also an E_s-algebra object in $\mathrm{Mfld}_n^{(X,e)}$. Let us describe more precisily this structure: for finite sets I, J, we have continuous maps

$$\gamma^N_{I,J} : \mathrm{Emb}^{fr}\Big(\coprod_I \mathbb{R}^s, \coprod_J \mathbb{R}^s\Big) \to \mathrm{Emb}^{(X,e)}\Big(\coprod_I (N\times\mathbb{R}^s), \coprod_J (N\times\mathbb{R}^s)\Big)$$

induced by the composition

$$\coprod_I (N\times\mathbb{R}^s) \cong N\times\Big(\coprod_I \mathbb{R}^s\Big) \overset{id_N\times f}{\to} N\times\Big(\coprod_J \mathbb{R}^s\Big) \cong \coprod_J (N\times\mathbb{R}^s)$$

for any $f\in \mathrm{Emb}^{fr}\big(\coprod_I \mathbb{R}^s, \coprod_J \mathbb{R}^s\big)$. In particular, taking $s=n$, $N=pt$, the above maps induce a canonical map of operads

$$\gamma : \mathrm{Disk}_n^{fr} \to \mathrm{Disk}_n^{(X,e)}$$

and thus we have an underlying functor $\gamma^* : \mathrm{Disk}_n^{(X,e)}\textbf{-Alg} \longrightarrow E_n\textbf{-Alg}$. And more generally we obtain functors: $(\gamma^N)^* : \mathrm{Disk}_n^{(N\times\mathbb{R}^s, T(N\times\mathbb{R}^s))}\textbf{-Alg} \longrightarrow E_s\textbf{-Alg}$.

The main consequence is that any symmetric monoidal functor (from $\mathrm{Mfld}_n^{(X,e)}$) maps $N\times\mathbb{R}^s$ to an E_s-algebra object of the target category. More precisely:

Lemma 5 *Let* $\big(Mfld_n^{(X,e)}, \coprod\big) \overset{\mathcal{F}}{\longrightarrow} (\textbf{\textit{Chain}}(k), \otimes)$ *be a symmetric monoidal functor.*

1. *For any manifold* $N\times\mathbb{R}^s$ *with an* (X,e)*-structure,* $\mathcal{F}(N\times\mathbb{R}^s)$ *has a canonical* E_s*-algebra structure.*
2. *Let* M *be an* (X,e)*-structured manifold with an end*[23] *trivialized as* $N\times\mathbb{R}$ *(where* N *is of codimension 1 and the open part of* M *lies in the neighborhood of* $N\times\{-\infty\}$*, see Fig. 1). Then* $\mathcal{F}(M)$ *has a canonical left*[24] *module structure over the* E_1*-algebra* $\mathcal{F}(N\times\mathbb{R})$.
3. $\mathcal{F}(\mathbb{R}^n)$ *has a natural structure of* $\mathrm{Disk}_n^{(X,e)}$*-algebra.*

[23] i.e. open boundary component.

[24] If $N\times\mathbb{R}$ is trivialized so that the open part of M is in the neighborhood of $N\times\{+\infty\}$, then $\mathcal{F}(M)$ has a canonical *right* module structure.

Proof Endow $\mathbb{R}^n$ with its canonical framing. It automatically inherits an (X, e)-structure for every connected component[25] of X (since every vector bundle is locally trivial). Now, since $\mathcal{F}$ is symmetric monoidal, then $\mathcal{F}(\mathbb{R}^n)$ also has an induced structure of $\mathrm{Disk}_n^{(X,e)}$-algebra. Let us describe the structure mentioned in 1. and 2.

1. For any manifold $N \times \mathbb{R}^s$ with an (X, e)-structure, the E_s-algebra structure on $\mathcal{F}(N \times \mathbb{R}^i)$ is given by the structure maps

$$\mathrm{Emb}^{fr}\Big(\coprod_I \mathbb{R}^s, \mathbb{R}^s\Big) \times \big(\mathcal{F}(N \times \mathbb{R}^s)\big)^I \xrightarrow{\gamma^N_{I,pt}} \mathrm{Emb}^{(X,e)}\Big(\coprod_I (N \times \mathbb{R}^i), N \times \mathbb{R}^s\Big) \times \big(\mathcal{F}(N \times \mathbb{R}^s)\big)^I$$
$$\longrightarrow \mathrm{Emb}^{(X,e)}\Big(\coprod_I (N \times \mathbb{R}^s), N \times \mathbb{R}^s\Big) \times \mathcal{F}\Big(\coprod_I (N \times \mathbb{R}^s)\Big) \xrightarrow{\mathcal{F}(\mathrm{Emb}^{(X,e)}(-,-))} \mathcal{F}(N \times \mathbb{R}^s).$$

The fact that $\mathcal{F}$ is monoidal ensures it defines an E_s-algebra structure.
From the definition, it is clear that $\gamma^*(\mathcal{F}(\mathbb{R}^n)) \cong \mathcal{F}(\mathbb{R}^n)$ as an E_n-algebra (where the two structures are given by 1. and 3.).

2. Now, let M be an (X, e)-structured manifold with an end trivialized as $N \times \mathbb{R}$; $\mathcal{F}(N \times \mathbb{R})$ is an E_1-algebra by 1. The left module structure of $\mathcal{F}(M)$ is given by the maps

$$\mathrm{Emb}^{fr}\Big(\big(\coprod_I \mathbb{R}\big) \coprod (0, 1], (0, 1]\Big) \times \Big(\mathcal{F}(N \times \mathbb{R})\Big)^I \times \mathcal{F}(M)$$
$$\xrightarrow{\gamma^N_{I \coprod \{*\}, pt}} \mathrm{Emb}^{(X,e)}\Big(\big(\coprod_I N \times \mathbb{R}\big) \coprod M, M\Big) \times \big(\mathcal{F}(N \times \mathbb{R}^i)\big)^I \times \mathcal{F}(M)$$
$$\longrightarrow \mathrm{Emb}^{(X,e)}\Big(\coprod_I (N \times \mathbb{R}) \coprod M, M\Big) \times \mathcal{F}\Big(\big(\coprod_I (N \times \mathbb{R})\big) \coprod M\Big)$$
$$\xrightarrow{\mathcal{F}(\mathrm{Emb}^{(X,e)}(-,-))} \mathcal{F}(N \times \mathbb{R}^i).$$

The above lemma is crucial in order to formulate the excision property.

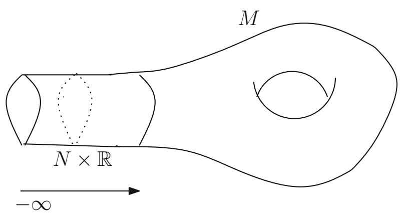

Fig. 1 A manifold M with a trivialization $N \times \mathbb{R}$ of its open boundary

[25] In practice, X will almost always be connex so that the structure will be canonical.

Definition 10 An homology theory for (X, e)-manifolds (with values in the symmetric monoidal ∞-category $(\mathbf{Chain}(k), \otimes)$) is a functor

$$\mathcal{F} : \mathrm{Mfld}_n^{(X,e)} \times \mathrm{Disk}_n^{(X,e)}\textbf{-Alg} \to \mathbf{Chain}(k)$$

(denoted $(M, A) \mapsto \mathcal{F}_M(A)$) satisfying the following axioms:

(i) **(dimension)** there is a natural equivalence $\mathcal{F}_{\mathbb{R}^n}(A) \cong A$ in $\mathbf{Chain}(k)$;

(ii) **(monoidal)** the functor $M \mapsto \mathcal{F}_M(A)$ is symmetric lax-monoidal and, for any set I, the following induced maps are equivalences (naturally in A)

$$\bigotimes_{i \in I} \mathcal{F}_{M_i}(A) \xrightarrow{\simeq} \mathcal{F}_{\coprod_{i\in I} M_i}(A). \tag{20}$$

(iii) **(excision)** Let M be an (X, e)-manifold. Assume there is a codimension 1 submanifold N of M with a trivialization $N \times \mathbb{R}$ of its neighborhood such that M is decomposable as $M = R \cup_{N\times\mathbb{R}} L$ where R, L are submanifolds of M glued along $N \times \mathbb{R}$. By Lemma 5, $\mathcal{F}_{N\times\mathbb{R}}(A)$ is an E_1-algebra and $\mathcal{F}_R(A)$, $\mathcal{F}_L(A)$ are respectively right and left modules. The *excision axiom*[26] is that the canonical map

$$\mathcal{F}_L(A) \overset{\mathbb{L}}{\underset{\mathcal{F}_{N\times\mathbb{R}}(A)}{\otimes}} \mathcal{F}_R(A) \xrightarrow{\simeq} \mathcal{F}_M(A)$$

(induced by the universal property of the right hand side) is an equivalence.

Remark 11 The symmetric lax-monoidal condition in axiom (ii) means that there are natural (in A, M) transformations like (20) compatible with composition for any finite I and invariant under the action of permutations. The axiom (ii) thus implies that $M \mapsto \mathcal{F}_M(A)$ is symmetric monoidal. When I is not finite, the right hand side in (20) is the colimit $\varinjlim_{F\to I} \bigotimes_{j\in F} \mathcal{F}_{M_j}(A)$ over all finite sets F and the map is induced by the universal property of the colimit and the lax monoidal property.

Theorem 6 (Francis [35]) *There is an unique*[27] *homology theory for* (X, e)*-manifolds (in the sense of Definition 10), which is called factorization homology.*[28]

Factorization homology is defined in [71] and *its value on a* (X, e)*-manifold* M *and* $\mathrm{Disk}_n^{(X,e)}$*-algebra* A *is denoted* $\int_M A$.

Remark 12 (**Other coefficients**) In Definition 10 and Theorem 6, one can replace the symmetric monoidal category $(\mathbf{Chain}(k), \otimes)$ by any symmetric monoidal ∞-category $(\mathcal{C}, \otimes)$ which has all colimits and whose monoidal structure commutes with geometric realization and filtered colimits, see [5, 34, 35].

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[26] Or Mayer-Vietoris principle.

[27] Up to contractible choices.

[28] The name comes from the fact that it satisfies the factorization property (Remark 20). Another name is topological chiral homology.

Remark 13 (**Finite variant**) If one restricts to the full subcategory of $\mathrm{Mfld}_n^{(X,e)}$ spanned by the manifolds which have finitely many connected components which are the interior of closed manifolds, then axiom (ii) becomes equivalent to asking $\mathcal{F}$ to be naturally symmetric monoidal and Theorem 6 stills holds in this context.

Note that factorization homology depends on the (X, e)-structure not the underlying topological manifold structure of M in general. For instance, if $M = \mathbb{R}$ is equipped with its standard framing and A is an associative algebra (hence E_1), then $\int_{\mathbb{R}} A \cong A$ as an E_1-algebra. However, if $N = \mathbb{R}$ is equipped with the opposite framing (pointing toward $-\infty$), then $\int_N A \cong A^{op}$ (where A^{op} is the algebra with opposite multiplication) as an E_1-algebra (see [34, 71] for more general statements).

Remark 14 In particular, Theorem 6 implies that the functor $\mathcal{F} \mapsto \mathcal{F}_{\mathbb{R}^n}$ from, the category of symmetric monoidal functors $\mathrm{Mfld}_n^{(X,e)} \to \mathbf{Chain}(k)$ satisfying excision,[29] to the category of $\mathrm{Disk}_n^{(X,e)}$-algebras (which is a well defined functor by Lemma 5) is a natural equivalence.

Definition 11 Let $A \in \mathrm{Disk}_n^{(X,e)}$**-Alg**. The homology theory for (X, e)-manifolds defined[30] by A will be called factorization homology (or homology theory) with coefficient in A.

Example 15 (*Hochschild homology*) Let A be a differential graded associative algebra (or even an A_∞-algebra) and choose a framing of $S^1 = SO(2)$ induced by its Lie group structure. We can use excision to evaluate the factorization homology with value in A on the framed manifold S^1. Here, we see the circle as being obtained by gluing two intervals: $S^1 = \mathbb{R} \cup_{\{1,-1\}\times\mathbb{R}} \mathbb{R}$, see Fig. 2

Note that the induced framing on $\{1, -1\} \times \mathbb{R}$ correspond to the standard framing of $\mathbb{R}$ on the component $\{1\} \times \mathbb{R}$ and the opposite framing on the component $\{-1\} \times \mathbb{R}$ so that $\int_{\{-1\}\times\mathbb{R}} A = A^{op}$ (see Example 27). Thus, by excision we find that

$$\int_{S^1} A \cong \int_{\mathbb{R}} A \overset{\mathbb{L}}{\underset{\int_{\{1,-1\}\times\mathbb{R}} A}{\otimes}} \int_{\mathbb{R}} A \cong A \overset{\mathbb{L}}{\underset{A\otimes A^{op}}{\otimes}} A \cong HH(A) \tag{21}$$

where $HH(A)$ is the usual Hochschild homology[31] of A with value in itself.

Example 16 Let Free_n be the free E_n-algebra on k, which is naturally a Disk_n^{un}-algebra (Example 12). It can thus be evaluated on any manifold.

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[29] i.e. axiom (iii) in Definition 10.

[30] In the sense of Remark 14.

[31] At least if A is projective over k; if A is not projective over k, there are several variants of Hochschild homology, the one we are considering is the derived version and correspond to what is sometimes called shukla homology [80, 87].

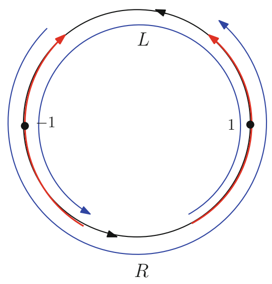


Fig. 2 The decomposition of the circle S^1 into 2 intervals (pictured in *blue* just across the *circle*) $L \cong \mathbb{R}$ and $R \cong \mathbb{R}$ along a trivialization $\{1, -1\} \times \mathbb{R}$ (pictured in *red*). The *arrows* are indicating the orientations/framing of the *circle* and other various pieces of the decomposition

Proposition 8 ([5]) *Let M be a manifold. Then* $\int_M Free_n \cong C_*\big(\coprod_{n\in\mathbb{N}} Conf_n(M)\big)$ *where* $Conf_n(M)$ *is the space of configurations of n -unordered points in M.*

In particular, factorization homology is not an homotopy invariant of manifolds (since configurations spaces of unordered points are not, see [68]). By considering configuration spaces of points with labels, one has a similar result for the free E_n-algebra $\mathrm{Free}_n(V)$ associated to $V \in \mathbf{Chain}(k)$, see [5].

Example 17 (*Non-abelian Poincaré duality*) Let us now mention another important example of computation of factorization homology. Let (Y, y_0) be a pointed space and $\Omega^n(Y) := \{f : [0,1]^n \to Y,\ f(\partial[0,1]^n) = y_0\}$ be its n-fold based loop space. Then the singular chains $C_*(\Omega^n(Y))$ has a natural structure of unoriented E_n-algebra.

Theorem 7 (Non-abelian Poincaré duality, Lurie [71]) *If M is a manifold of dimension n and Y an* $n-1$*-connective pointed space, then*

$$\int_M C_*(\Omega^n(Y)) \cong C_*(\mathrm{Map}_c(M, Y))$$

where $\mathrm{Map}_c(M, Y)$ *is the space of compactly supported maps from M to Y.*

If $n = 1$ and Y is connected, Theorem 7 reduces to Goodwillie's quasi-isomorphism [50] $HH(C_*(\Omega(Y))) \cong C_*(LY)$ where $LY = \mathrm{Map}(S^1, Y)$ is the free loop space of Y.

Remark 15 (**Derived functor definition**) One possible way for defining factorization homology is similar to the one of Sect. 2.1.2. Indeed, let A be a $\mathrm{Disk}_n^{(X,e)}$-algebra. Then A defines a covariant functor $\mathrm{Disk}_n^{(X,e)} \to \mathbf{Chain}(k)$. Similarly, if M is in $\mathrm{Mfld}_n^{(X,e)}$, then it defines a contravariant functor $E_M^{(X,e)} : \big(\mathrm{Disk}_n^{(X,e)}\big)^{op} \to \mathbf{Top}$, given by the formula

$$E_M^{(X,e)}\Big(\coprod_{i\in I}\mathbb{R}^n\Big) := \mathrm{Emb}^{(X,e)}\Big(\coprod_{i\in I}\mathbb{R}^n, M\Big).$$

The data of A and M thus gave a functor

$$E_M^{(X,e)} \otimes A : \left(\mathrm{Disk}_n^{(X,e)}\right)^{op} \times \mathrm{Disk}_n^{(X,e)} \stackrel{E_M^{(X,e)} \times A}{\longrightarrow} \mathrm{Top} \times \mathbf{Chain}(k) \stackrel{\otimes}{\longrightarrow} \mathbf{Chain}(k).$$

Here $\mathrm{Top} \times \mathbf{Chain}(k) \stackrel{\otimes}{\to} \mathbf{Chain}(k)$ means the tensor of a space with a chain complex which is equivalent to $(X, D_*) \mapsto C_*(X) \otimes D_*$ where $C_*(X)$ is the singular chain functor of X (with value in k).

Proposition 9 ([35]) *The factorization homology $\int_M A$ is the (homotopy) coend of $E_M^{(X,e)} \otimes A$. In other words:*

$$\begin{aligned}\int_M A &\cong E_M^{(X,e)} \overset{\mathbb{L}}{\underset{\mathrm{Disk}_n^{(X,e)}}{\otimes}} A \\ &\cong hocolim \Bigg(\coprod_{f:\{1,\dots,q\}\to\{1,\dots,p\}} C_*\big(E_M^{(X,e)}(\mathbb{R}^n)\big)^{\otimes p} \otimes \mathrm{Disk}_n^{(X,e)}(q,p) \otimes A^{\otimes q} \\ &\qquad \rightrightarrows \coprod_m C_*\big(E_M^{(X,e)}(\mathbb{R}^n)\big)^{\otimes m} \otimes A^{\otimes m} \Bigg)\end{aligned}$$

The Proposition remains true with $(\mathbf{Chain}(k), \otimes)$ replaced by any symmetric monoidal ∞-category satisfying the assumptions of Remark 12.

4 Factorizations Algebras

In this section we will give a Čech type construction of Factorization homology which plays for Factorization homology the same role as sheaf cohomology plays for singular cohomology.[32] This analogue of cosheaf theory is given by factorization algebras which we describe in length here.

4.1 The Category of Factorization Algebras

We start by describing various categories of (pre)factorization algebras (including the locally constant ones).

Following Costello Gwilliam [24], given a topological space X, a *prefactorization* algebra over X is an algebra over the colored operad whose objects are open subsets

[32] Singular cohomology of a paracompact space X can be computed as the cohomology of the constant sheaf $\mathbb{Z}_X$ on X while singular cohomology with twisted coefficient is computed by sheaf cohomology with value in a locally constant sheaf.

of X and whose morphisms from $\{U_1, \ldots, U_n\}$ to V are empty unless when U_i's are mutually disjoint subsets of U, in which case they are singletons. Unfolding the definition, we find

Definition 12 A prefactorization algebra on X (with value in chain complexes) is a rule that assigns to any open set U a chain complex $\mathcal{F}(U)$ and, to any finite family of pairwise disjoint open sets $U_1, \ldots, U_n \subset V$ included in an open V, a chain map

$$\rho_{U_1,\ldots,U_n,V} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(V).$$

These structure maps are required to satisfy obvious *associativity and symmetry* conditions (see [24]): the map $\rho_{U_1,\ldots,U_n,V}$ is invariant with respect to the action of the symmetric group S_n by permutations of the factors on its domain (in other words, the map $\rho_{U_1,\ldots,U_n,V}$ depends only of the collection $U_1, \ldots, U_n, V$ not on the particular choice of ordering of the open sets) and $\rho_{U,U}$ is the identity[33] of $\mathcal{F}(U)$. Further, the associativity condition is that: for any finite collection of pairwise disjoint open subsets $(V_j)_{j\in J}$ lying in an open subset W together with, for all $j \in J$, a finite collections $(U_{i,j})_{i\in I_j}$ of pairwise disjoint open subset lying in V_j, the following diagram

$$\begin{array}{ccc} \bigotimes_{(i,j)\in\coprod_r I_r} \mathcal{F}(U_{ij}) & \xrightarrow{\rho_{(U_{ij}),W}} & \mathcal{F}(W) \\ & \searrow^{\bigotimes_j \rho_{(U_{ij})_{i\in I_j},V_j}} \quad \nearrow_{\rho_{(V_j)_{j\in J},W}} & \\ & \bigotimes_{j\in J} \mathcal{F}(V_j) & \end{array} \tag{22}$$

is commutative.

If $\mathcal{U}$ is an open cover of X, we define a *prefactorization algebra on* $\mathcal{U}$, also denoted a $\mathcal{U}$ *-prefactorization algebra*, to be the same thing as a prefactorization algebra except that $\mathcal{F}(U)$ is defined only for $U \in \mathcal{U}$.

Remark 16 One can define a prefactorization algebra with value in any symmetric monoidal category $(\mathcal{C}, \otimes)$ by replacing chain complexes by objects of $\mathcal{C}$.

Remark 17 Prefactorization algebras are *pointed* since the inclusion $\emptyset \hookrightarrow U$ of the empty set in any open induces a canonical map $\mathcal{F}(\emptyset) \to \mathcal{F}(U)$. Further, the structure maps of a prefactorization algebra exhibit $\mathcal{F}(\emptyset)$ as a commutative algebra in $(\mathcal{C}, \otimes)$ (non necessarily unital) and $\mathcal{F}(U)$ as a $\mathcal{F}(\emptyset)$-module.

[33] We could weaken this condition to be only a weak-equivalence or actually just a chain map. In the latter case, we will obtain a (homotopy) strictly weaker notion of prefactorization algebras; however, this will *not* change the notion of factorization algebras since the condition of being a factorization algebra (Definition 13) will imply that $\rho_{U,U}$ is an equivalence as U is always a factorizing cover of itself; since it is idempotent by associativity, we will get that it is homotopy equivalent to the identity.

There is a *Čech-complex* associated to a cover $\mathcal{U}$ of an open set U. Denoting $P\mathcal{U}$ the set of finite pairwise disjoint open subsets $\{U_1, \dots, U_n \mid U_i \in \mathcal{U}\}$ (n is not fixed), it is, by definition the realization of the simplicial chain complex

$$\check{C}_\bullet(\mathcal{U}, \mathcal{F}) = \bigoplus_{\alpha \in P\mathcal{U}} \Bigg(\bigotimes_{U \in \alpha} \mathcal{F}(U) \Bigg) \leftleftarrows \bigoplus_{(\alpha,\beta) \in P\mathcal{U} \times P\mathcal{U}} \Bigg(\bigotimes_{(U,V) \in \alpha \times \beta} \mathcal{F}(U \cap V) \Bigg) \Lleftarrow \cdots$$

where the horizontal arrows are induced by the natural inclusions as for the usual Čech complex of a cosheaf (see [24]).

Let us describe the simplicial structure more precisely. In simplicial degree i, we get the chain complex $\check{C}_i(\mathcal{U}, \mathcal{F}) := \bigoplus_{\alpha \in P\mathcal{U}^{i+1}} \mathcal{F}(\alpha)$ where, for $\alpha = (\alpha_0, \dots, \alpha_i) \in P\mathcal{U}^i$, we denote $\mathcal{F}(\alpha)$ the tensor product of chain complexes (with its natural differential) :

$$\mathcal{F}(\alpha) = \bigotimes_{U_j \in \alpha_j} \mathcal{F}\Big(\bigcap_{j=0}^{i} U_j \Big). \tag{23}$$

We write $d_{in} : \bigoplus_{\alpha \in P\mathcal{U}^{i+1}} \mathcal{F}(\alpha) \to \bigoplus_{\alpha \in P\mathcal{U}^{i+1}} \mathcal{F}(\alpha)$ the induced differential. The face maps $\partial_s : \bigoplus_{\alpha \in P\mathcal{U}^{n+1}} \mathcal{F}(\alpha) \longrightarrow \bigoplus_{\beta \in P\mathcal{U}^n} \mathcal{F}(\beta)$ ($s = 0 \cdots n$) are the direct sum of maps $\widehat{\rho}^s_\alpha : \mathcal{F}(\alpha) \to \mathcal{F}(\widehat{\alpha}^s)$ where $\widehat{\alpha}^s = (\alpha_0, \dots, \alpha_{s-1}, \alpha_{s+1}, \dots, \alpha_n)$ is obtained by discarding the sth-collection of opens in $P\mathcal{U}^{n+1}$. Precisely $\widehat{\rho}^s_\alpha$ is the tensor product

$$\bigotimes_{U_j \in \alpha_j} \mathcal{F}\Big(\bigcap_{j=0}^{n} U_j \Big) \longrightarrow \bigotimes_{\substack{U_k \in \alpha_k, \\ k \neq s}} \mathcal{F}\Big(\bigcap_{\substack{k=0 \\ k \neq s}}^{n} U_k \Big) \tag{24}$$

of the structure maps associated to the inclusion of opens $\bigcap_{j=0 \dots n} U_j$ into $\bigcap_{j \neq s} U_j$. The degeneracies are similarly given by operations $(\alpha_0, \dots, \alpha_n) \mapsto (\alpha_0, \dots, \alpha_j, \alpha_j, \dots, \alpha_n)$ doubling a set α_j.

The simplicial chain-complex $\check{C}_\bullet(\mathcal{U}, \mathcal{F})$ can be made into a chain complex (which is the total complex of a bicomplex):

$$\check{C}(\mathcal{U}, \mathcal{F}) = \bigoplus_{\alpha \in P\mathcal{U}} \mathcal{F}(\alpha) \leftarrow \bigoplus_{\beta \in P\mathcal{U} \times P\mathcal{U}} \mathcal{F}(\beta)[1] \leftarrow \cdots$$

where the horizontal arrows are induced by the alternating sum of the faces ∂_j in the standard way. In other words, $\check{C}(\mathcal{U}, \mathcal{F}) = \bigoplus_{i \geq 0} \Big(\bigoplus_{\alpha \in P\mathcal{U}^{i+1}} \mathcal{F}(\alpha)[i] \Big)$ with differential the sum of $\mathcal{F}(\alpha)[i] \overset{(-1)^i d_{in}}{\longrightarrow} \mathcal{F}(\alpha)[i]$ and

$$\sum_{j=0}^{n}(-1)^j\partial_j : \bigoplus_{\alpha\in P\mathcal{U}^{n+1}} \mathcal{F}(\alpha)[n] \to \bigoplus_{\beta\in P\mathcal{U}^{n}} \mathcal{F}(\beta)[n-1].$$

Remark 18 If a cover $\mathcal{U}$ is stable under finite intersections, we only need $\mathcal{F}$ to be a prefactorization algebra on $\mathcal{U}$, to define the Čech-complex $\check{C}(\mathcal{U},\mathcal{F})$.

If $\mathcal{U}$ is a cover of an open set U, then the structure maps of $\mathcal{F}$ yield canonical maps $\mathcal{F}(\alpha) \to \mathcal{F}(U)$ which commute with the simplicial maps. Thus, we get a natural map of simplicial chain complexes $\big(\check{C}_i(\mathcal{U},\mathcal{F}) \to \mathcal{F}(U)\big)_{i\geq 0}$ to the constant simplicial chain complex $\big(\mathcal{F}(U)\big)_{i\geq 0}$. Passing to geometric realization, we obtain a canonical chain complex homomorphism:

$$\check{C}(\mathcal{U},\mathcal{F}) \longrightarrow \mathcal{F}(U). \tag{25}$$

Remark 19 (**Čech complexes in $(\mathcal{C},\otimes)$**) If $(\mathcal{C},\otimes)$ is a symmetric monoidal category with coproducts, we define the Čech complex of a prefactorization algebra with values in $\mathcal{C}$ in the same way, replacing the direct sum by the coproduct in order to get a simplicial object $\check{C}_\bullet(\mathcal{U},\mathcal{F})$ in $\mathcal{C}$. If further, $\mathcal{C}$ has a geometric realization, then we obtain the Čech complex $\check{C}(\mathcal{U},\mathcal{F}) \in \mathcal{C}$ exactly as for chain complexes above and the canonical map (25) is also well defined.

Definition 13 An open cover of $\mathcal{U}$ is *factorizing* if, for all finite collections $x_1,\dots,x_n$ of distinct points in U, there are pairwise disjoint open subsets $U_1,\dots,U_k$ in $\mathcal{U}$ such that $\{x_1,\dots,x_n\} \subset \bigcup_{i=1}^k U_i$.

A prefactorization algebra $\mathcal{F}$ on X is said to be a *homotopy*[34] *factorization algebra* if, for all open subsets $U \in Op(X)$ and for every factorizing cover $\mathcal{U}$ of U, the canonical map $\check{C}(\mathcal{U},\mathcal{F}) \to \mathcal{F}(U)$ is a quasi-isomorphism (see [22, 24]).

Remark 20 (**Factorization property**) If $\mathcal{F}$ is a factorization algebra and $U_1,\dots,U_i$ are disjoint open subsets of X, the factorization condition implies that the structure map

$$\mathcal{F}(U_1)\otimes\cdots\otimes\mathcal{F}(U_i) \longrightarrow \mathcal{F}(U_1\cup\cdots\cup U_i) \tag{26}$$

is a quasi-isomorphism.

In particular $\mathcal{F}(\emptyset) \cong k$ (or, more generally, is the unit of the symmetric monoidal category $\mathcal{C}$ if $\mathcal{F}$ has values in $\mathcal{C}$).

The fact that the map (26) is an equivalence is called the *factorization property* in the terminology of Beilinson–Drinfeld [7], in the sense that the value of $\mathcal{F}$ on disjoint opens factors through its value on each connected component.

[34] We can also say *derived factorization algebra*. Unless otherwise specified, the word factorization algebra will always mean a homotopy factorization algebra in these notes.

Example 18 (*The trivial factorization algebra*) The ***trivial*** prefactorization algebra k is the *constant* prefactorization algebra given by the rule $U \mapsto k(U) := k$, with structure maps given by multiplication. It is a (homotopy) factorization algebra. It is in particular locally constant over any stratified space X (Definitions 14 and 21).

One defines similarly the trivial factorization algebra over X with values in a symmetric monoidal ∞-category $(\mathcal{C}, \otimes)$ by the rule $U \mapsto \mathbf{1}_{\mathcal{C}}$ where $\mathbf{1}_{\mathcal{C}}$ is the unit of the monoidal structure.

Remark 21 (**Genuine factorization algebras**) The notion of homotopy (or derived) factorization algebra in Definition 13 is a homotopy version of a more naive, underived, version of factorization algebra. This version is a prefactorization algebra such that the following sequence

$$\Big(\bigoplus_{\alpha \in P\mathcal{U}^2} \mathcal{F}(\alpha)\Big) \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} \Big(\bigoplus_{\beta \in P\mathcal{U}} \mathcal{F}(\beta)\Big) \to \mathcal{F}(U)$$

is (right) exact for any factorizing cover $\mathcal{U}$ of U. In other words we ask for a similar condition as in Definition 13 but with the truncated Čech complex. We refer to prefactorization algebras satisfying this condition as *genuine factorization algebras* (they are also called strict in [24]). Note that a genuine factorization algebra is *not* a (homotopy) factorization algebra in general. Homotopy factorization algebras are to genuine factorization algebras what homotopy cosheaves are to cosheaves; that is they are obtained by replacing the naive version by an acyclic resolution.

When X is a manifold we have the notion of locally constant factorization algebra which roughly means that the structure maps do not depend on the size of the open subsets but only their relative shapes:

Definition 14 Let X be a topological manifold of dimension n. We say that an open subset U of X is a *disk* if U is homeomorphic to a standard euclidean disk $\mathbb{R}^n$. A (pre-)factorization algebra over X is *locally constant* if for any inclusion of open disks $U \hookrightarrow V$ in X, the structure map $\mathcal{F}(U) \to \mathcal{F}(V)$ is a quasi-isomorphism.

Let us mention that a locally constant prefactorization algebra is automatically a (homotopy) factorization algebra, see Remark 24.

Definition 15 A *morphism* $\mathcal{F} \to \mathcal{G}$ of (pre)factorization algebras over X is the data of chain complexes morphisms $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open set $U \subset X$ which commute with the structures maps; that is the following diagram

$$\begin{array}{ccc}
\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_i) & \xrightarrow{\rho_{U_1,\dots,U_i,V}} & \mathcal{F}(V) \\
\downarrow{\scriptstyle \otimes\phi_j(U_j)} & & \downarrow{\scriptstyle \phi_V} \\
\mathcal{G}(U_1) \otimes \cdots \otimes \mathcal{G}(U_i) & \xrightarrow{\rho_{U_1,\dots,U_i,V}} & \mathcal{G}(V)
\end{array}$$

is commutative for any pairwise disjoint finite family $U_1, \ldots, U_i$ of open subsets of an open set V. Morphisms of (pre)factorization algebras are naturally enriched over topological space. Indeed, we have *mapping spaces* $\mathrm{Map}(\mathcal{F}, \mathcal{G})$ defined as the geometric realization of the simplicial set

$$n \mapsto \mathrm{Map}(\mathcal{F}, \mathcal{G})_n := \{\text{prefactorization algebras morphisms from } \mathcal{F} \text{ to } C^*(\Delta^n)\otimes\mathcal{G}\}$$

where $C^*(\Delta^n) \otimes \mathcal{G}$ is the prefactorization algebra whose value on an open set U is $C^*(\Delta^n) \otimes \mathcal{G}(U)$. We obtain in this way ∞-categories of (pre)factorization algebras (as in Appendix 10.2, Example 58).

The ∞-*category of prefactorization algebras over* X *is denoted* $\mathbf{PFac}_X$ and similarly we write $\mathbf{Fac}_X$ *for the* ∞ *-categories of factorization algebras over* X and $\mathbf{Fac}_X^{lc}$ *for the locally constant ones* (which is a full subcategory). Also if $(\mathcal{C}, \otimes)$ is a symmetric monoidal (∞-)category (with coproducts and geometric realization), we will denote $\mathbf{Fac}_X(\mathcal{C})$, $\mathbf{Fac}_X^{lc}(\mathcal{C})$ the ∞-categories of *factorization algebras in* $\mathcal{C}$.

Note that the embedding $\mathbf{Fac}_X \to \mathbf{PFac}_X$ is a fully faithful embedding.

The underlying tensor product[35] of chain complexes induces a tensor product of factorization algebras which is computed pointwise: for $\mathcal{F}, \mathcal{G} \in \mathbf{PFac}_X$ and an open set U, we have

$$\big(\mathcal{F} \otimes \mathcal{G}\big)(U) := \mathcal{F}(U) \otimes \mathcal{G}(U) \tag{27}$$

and the structure maps are just the tensor product of the structure maps. If $(\mathcal{C}, \otimes)$ is symmetric monoidal, the same construction yields a monoidal structure on $\mathbf{PFac}_X(\mathcal{C})$. Its unit is the trivial factorization algebra with values in $\mathcal{C}$ (Example 18).

Proposition 10 (Costello-Gwilliam [24]) *The* (∞-)*categories* $\boldsymbol{PFac}_X(\mathcal{C})$, $\boldsymbol{Fac}_X(\mathcal{C})$, $\boldsymbol{Fac}_X^{lc}(\mathcal{C})$ [36] *are symmetric monoidal with tensor product given by* (27).

Remark 22 (**Restrictions**) If $Y \subset X$ is an open subspace, then we have natural restriction functors $\mathbf{PFac}_X(\mathcal{C}) \to \mathbf{PFac}_Y(\mathcal{C})$, $\mathbf{Fac}_X(\mathcal{C}) \to \mathbf{Fac}_Y(\mathcal{C})$. When X is a manifold, the same holds for locally constant factorization algebras.

Definition 16 If U is an open subset of X and $\mathcal{A} \in \mathbf{PFac}(X)$, we write $\mathcal{A}_{|U} \in \mathbf{PFac}_U$ *for the restriction of* $\mathcal{A}$ *to* U and similarly for (possibly locally constant) factorization algebras.

A homeomorphism $f : X \overset{\simeq}{\to} Y$ induces isomorphisms $\mathbf{PFac}_X \cong \mathbf{PFac}_Y$ and $\mathbf{Fac}_X \cong \mathbf{Fac}_Y$ (or $\mathbf{Fac}_X^{lc} \cong \mathbf{Fac}_Y^{lc}$ when X is a manifold) realized by the functor f_* see Sect. 5.1.

[35] recall our convention that if k is not a field, the tensor product really means derived tensor product.

[36] The latter is defined when X is a manifold.

4.2 Factorization Homology and Locally Constant Factorization Algebras

We now explain the relationship between the Čech complex of a factorization algebras and factorization homology. We first start to express $\mathrm{Disk}_n^{(X,TX)}$-algebras in terms of factorization algebras. For simplicity, we assume in Sect. 4.2 that manifolds are smooth. For topological manifolds one obtains the same result as below by replacing geodesic convex neighborhoods by families of embeddings $\mathbb{R}^n \to M$ wich preserves the (M, TM)-structure and whose images form a basis of open of M.

Let M be a manifold with an (X, e)-structure. Every open subset U of M inherits a canonical (X, e)-structure given by the factorization $U \hookrightarrow M \xrightarrow{f} X \xrightarrow{e} B\mathrm{Homeo}(\mathbb{R}^n)$ of the map $e_U : U \to B\mathrm{Homeo}(\mathbb{R}^n)$ classifying the tangent bundle of U. This construction extends canonically into a functor

$$f_* : \mathrm{Disk}_n^{(M,TM)} \longrightarrow \mathrm{Disk}_n^{(X,e)}$$

and (by Definition 9) we have

Lemma 6 *An* (X, e)*-structure* $M \xrightarrow{f} X \xrightarrow{e} B\mathrm{Homeo}(\mathbb{R}^n)$ *on a manifold* M *induces a functor* $f^* : \mathrm{Disk}_n^{(X,e)}$***-Alg*** $\to \mathrm{Disk}_n^{(M,TM)}$***-Alg***.

Now let A be a $\mathrm{Disk}_n^{(X,e)}$-algebra and choose a metric on M. A family of pairwise disjoint open convex geodesic neighborhoods $U_1, \dots, U_i$ which lies in a convex geodesic neighborhood[37] V, defines an (X, e)-structure preserving embedding $i_{U_1,\dots,U_i,V} \in \mathrm{Emb}^{(X,e)}\big(\coprod_{\{1,\dots,i\}} \mathbb{R}^n, \mathbb{R}^n\big)$ so that the $\mathrm{Disk}_n^{(X,e)}$-algebra structure of A yields a structure map

$$\mu_{U_1,\dots,U_i,V} : A^{\otimes i} \xrightarrow{i_{U_1,\dots,U_i,V}} \mathrm{Emb}^{(X,e)}\Big(\coprod_{\{1,\dots,i\}} \mathbb{R}^n, \mathbb{R}^n\Big) \otimes A^{\otimes i} \to A.$$

This allows us to define a *prefactorization algebra* $\mathcal{F}_A$ *on open convex geodesic subsets* by the formula $\mathcal{F}_A(V) := A$. Since, the convex geodesic neighborhoods form a basis of open which is stable by intersection, for any open set $U \subset M$, we have the Čech complex[38]

$$\check{C}(\mathcal{CV}(U), \mathcal{F}_A) \tag{28}$$

where $\mathcal{CV}(U)$ is the factorizing cover of U given by the geodesic convex open subsets of U. The following result shows that $\mathcal{F}_A$ is actually (the restriction of) a factorization algebra and computes factorization homology.

[37] Which is thus canonically homeomorphic to an euclidean disk.

[38] The construction is actually the extension of a factorization algebra on $\mathcal{CV}(U)$ as in Sect. 5.

Theorem 8 ([47]) *Let A be a* $\mathrm{Disk}_n^{(X,e)}$*-algebra.*

- *The rule* $M \mapsto \check{C}(\mathcal{CV}(M), \mathcal{F}_A)$ *is a homology theory for* (X, e)*-manifolds. In particular the Čech complex is independent of the choice of the metric and computes factorization homology of* M*:*

$$\check{C}(\mathcal{CV}(M), \mathcal{F}_A) \simeq \int_M A.$$

- *The functor* $(U, A) \mapsto \check{C}(\mathcal{CV}(U), \mathcal{F}_A)$ *induces an equivalence of* ∞*-categories* $\mathit{Disk}_n^{(M,TM)}\textbf{-}\textit{\textbf{Alg}} \xrightarrow{\simeq} \textit{\textbf{Fac}}_M^{lc}$.

Since we have a preferred choice of framing for $\mathbb{R}^n$, the projection map $\mathbb{R}^n \to pt$ induces an equivalence of ∞-categories $\mathrm{Disk}_n^{(\mathbb{R}^n, T\mathbb{R}^n)} \xrightarrow{\simeq} \mathrm{Disk}_n^{(pt,e)}$ and thus equivalences $\mathrm{Disk}_n^{(X,TX)}\textbf{-Alg} \cong \mathrm{Disk}_n^{fr}\textbf{-Alg} \cong E_n\textbf{-Alg}$ (see Example 12). Hence Theorem 8 is a slight generalization of the following beautiful result.

Theorem 9 (Lurie [71]) *There is a natural equivalence of* ∞*-categories*

$$E_n\textbf{-}\textit{\textbf{Alg}} \;\cong\; \textit{\textbf{Fac}}_{\mathbb{R}^n}^{lc}.$$

The functor $\textit{\textbf{Fac}}_{\mathbb{R}^n}^{lc} \to E_n\textbf{-}\textit{\textbf{Alg}}$ *is given by the global section (i.e. the pushforward* p_* *where* $p : \mathbb{R}^n \to pt$*, see Sect. 5.1) and the inverse functor is precisely given by factorization homology.*

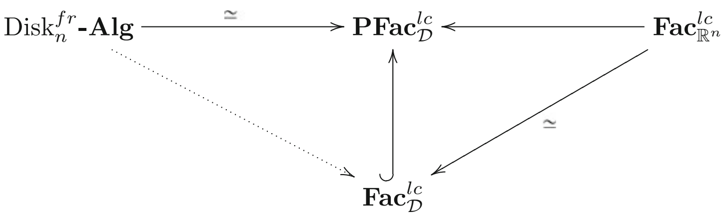
Locally constant factorization algebras on $\mathbb{R}^n$ are thus *a model for* E_n*-algebras*. More generally, locally constant factorization algebras are a model for $\mathrm{Disk}_n^{(X,TX)}$-algebras in which the cosheaf property replaces[39] some of the higher homotopy machinery needed for studying these algebras (at the price of working with “lax” algebras).

Remark 23 Theorem 9 is the key example of the relationship between factorization algebras and factorization homology so we now explain the equivalence in more depth. Recall that $E_n\textbf{-Alg}$ is the ∞-category of algebras over the operad Cube_n of little cubes. It is equivalent to the ∞-category $\mathrm{Disk}_n^{fr}\textbf{-Alg}$ since we have an equivalence of operads $\mathrm{Cube}_n \xrightarrow{\simeq} \mathrm{Disk}_n^{fr}$ induced by a choice of diffeomorphism $\theta : (0, 1)^n \cong \mathbb{R}^n$. Consider the open cover $\mathcal{D}$ of $\mathbb{R}^n$ consisting of all open disks and denote $\textbf{PFac}_{\mathcal{D}}^{lc}$ [40] the category of $\mathcal{D}$-prefactorization algebras which satisfy the locally constant condition (Definitions 14 and 12). Evaluation of a Disk_n^{fr}-algebra on an open disk yields a functor $\mathrm{Disk}_n^{fr}\textbf{-Alg} \to \textbf{PFac}_{\mathcal{D}}^{lc}$ which is an equivalence by [71, Sect. 5.2.4].

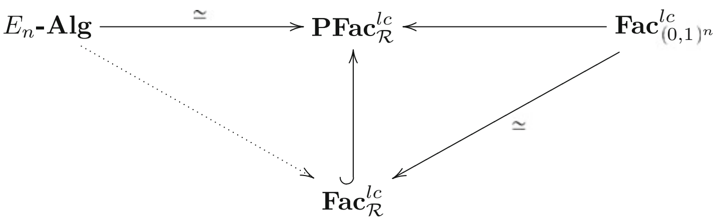
[39] Note that factorization algebras are described by operads in *discrete* space together with the Čech condition, see Remark 24.

[40] Note that the category of $\mathcal{D}$-prefactorization algebras is the category of algebras over the colored operad $N(\mathrm{Disk}(\mathbb{R}^n))$, see Remark 24.

Similarly, let $\mathcal{R}$ be the cover of $(0,1)^n$ by open rectangles and $\mathbf{PFac}^{lc}_{\mathcal{R}}$ be the category of locally constant $\mathcal{R}$-prefactorization algebras. Evaluation of a Cube_n-algebra on rectangles yields a functor $E_n\textbf{-Alg} = \text{Cube}_n\textbf{-Alg} \to \mathbf{PFac}^{lc}_{\mathcal{R}}$. Denote $\mathbf{Fac}^{lc}_{\mathcal{D}}, \mathbf{Fac}^{lc}_{\mathcal{R}}$ the category of locally constant factorization algebras over the covers $\mathcal{D}$, $\mathcal{R}$ respectively (see Sect. 5.2). We have two commutative diagram and an equivalence between them induced by the diffeomorphism θ:



(29)



(30)

where the dotted arrows exists by Theorem 8 and the diagonal right equivalences are given by Proposition 17. Since the embedding of factorization algebras in prefactorization algebras (over any cover or space) is fully faithful, we obtain that all maps in Diagrams (29) and (30) are equivalences so that we recover the equivalence $E_n\textbf{-Alg} \cong \mathbf{Fac}^{lc}_{\mathbb{R}^n} \cong \mathbf{Fac}^{lc}_{(0,1)^n}$ of Theorem 9, also see [14].

Example 19 (Constant factorization algebra on framed manifolds) Let M be a framed manifold of dimension n. By Theorem 8 or [47, 71], any E_n-algebra A yields a locally constant factorization algebra $\mathcal{A}$ on M which is defined by assigning to any geodesic disk D the chain complex $\mathcal{A}(D) \cong A$. We call such a factorization algebra the *constant factorization algebra* on M associated to A since it satisfies the property that there is a (globally defined) E_n-algebra A together with natural (with respect to the structure map of the factorization algebra) quasi-isomorphism $\mathcal{A}(D) \stackrel{\simeq}{\to} A$ for every disk D.

In particular, for $n = 0, 1, 3, 7$, there is a faithful embedding of E_n-algebras into constant factorization algebras over the n-sphere S^n.

If a manifold X is not framable, we can obtain constant factorization algebras on X by using (un)oriented E_n-algebras instead of plain E_n-algebras:

Example 20 (*Constant factorization algebra on (oriented) smooth manifolds*) Let A be an unoriented E_n-algebra (i.e. a Disk_n^{un}-algebra, Example 12). Then A yields a (locally) constant factorization $\mathcal{A}$ algebra on any smooth manifold of dimension n which is defined by assigning to any geodesic disk D the chain complex $\mathcal{A}(D) \cong A$.

Similarly, an oriented E_n-algebra (i.e. a Disk_n^{or}-algebra) yields a (locally) constant factorization algebra on any oriented dimension n manifold.

Example 21 (*Commutative factorization algebras*) The canonical functor E_∞**-Alg**$\rightarrow \mathrm{Disk}_n^{(X,e)}$**-Alg** (see Example 13) shows that any E_∞-algebras induces a canonical structure of (locally) constant factorization algebra on any (topological) manifold M. In that case, the factorization homology reduces to the derived Hochschild chains according to Theorems 8 and 10 below. See Sect. 8.2 for more details.

Theorem 10 *If A is an E_∞-algebra*[41]*, then, for every topological manifold M, there is an natural equivalence $CH_M(A) \cong \int_M A$. In particular, factorization homology of E_∞-algebras extends uniquely as an homology theory for spaces (see Definition 1).*

Proof This is proved in [47], also see [34]. The result essentially follows by uniqueness of the homology theories (Theorem 6). Namely, if $\mathcal{H}_A$ is an homology theory for spaces whose value on a point is A, then $\mathcal{H}_A(\mathbb{R}^n) = A$ ($\mathbb{R}^n$ is contractible) and further $\mathcal{H}_A$ satisfies the monoidal and excision axioms of a homology theory for manifolds. □

Example 22 (*Pre-cosheaves*) Let $\mathcal{P}$ be a pre-cosheaf on X (with values in vector spaces or chain complexes). For any open $U \subset X$, set $\mathcal{F}(U) := S^\bullet(\mathcal{P}(U)) = \bigoplus_{n\geq 0}(\mathcal{P}(U)^{\otimes n})_{S_n}$ where $S^\bullet$ is the free (differential graded) commutative algebra functor. Then $\mathcal{F}$ is a prefactorization algebra with structure maps given by the algebra structure of $S^\bullet(\mathcal{P}(V))$:

$$\begin{aligned}\mathcal{F}(U_1)\otimes\cdots\otimes\mathcal{F}(U_i) \cong & S^\bullet(\mathcal{P}(U_1))\otimes\cdots\otimes S^\bullet(\mathcal{P}(U_i)) \\ & \stackrel{\otimes S^\bullet(\mathcal{P}(U_i\to V))}{\longrightarrow} S^\bullet(\mathcal{P}(V))\otimes\cdots\otimes S^\bullet(\mathcal{P}(V)) \longrightarrow S^\bullet(\mathcal{P}(V)) = \mathcal{F}(V).\end{aligned}$$

Proposition 11 (*cf.* [24]) *If $\mathcal{F}$ is a homotopy cosheaf, then $\mathcal{F}$ is a factorization algebra (not necessarily locally constant).*

In characteristic zero, if $\mathcal{P}$ is a homotopy cosheaf, then $\mathcal{F}$ is a factorization algebra.

Example 23 (*Observables*) Several examples of (pre-)factorization algebras arising from theoretical physics (more precisely from perturbative quantum field theories) are described in the beautiful work [22, 24]. They arose as deformations of those obtained as in the previous Example 22. For instance, let $E \to X$ be a (possibly graded) vector bundle over a smooth manifold X. Let $\mathcal{E}$ be the sheaf of smooth sections of E (which may be endowed with a differential which is a differential operator) and $\mathcal{E}'$ be its associated distributions. The above construction yields a (homotopy)

[41] For instance a (differential graded) commutative algebra.

factorization algebra $U \mapsto S(\mathcal{E}'(U)) = \bigoplus_{n\geq 0} \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{E}(U)^{\otimes n}, \mathbb{R})$. In [24], Costello-Gwilliam have been refining this example to equip the classical observables of a classical field theory with the structure of a factorization algebra (with values in P_0-algebras, Example 64). Their construction is a variant of the classicalAKSZ formalism [2]. Related constructions are studied in [78].

Further, the quantum observables of a quantization of the classical field theory, when they exist, also form a factorization algebra (not necessarily locally constant). A very nice example of this procedure arises when X is an elliptic curve, see [22].

These factorization algebras (with values in lax P_0-algebras) encode the algebraic structure governing observables of the field theories (in the same way as the observables of classical mechanics are described by the algebra of smooth functions on a manifold together with its Poisson bracket). Very roughly speaking, the *locally constant factorization algebras correspond to observables of topological field theories*.

Example 24 (*Enveloping factorization algebra of a dg-Lie algebra*) Let $\mathcal{L}$ be a homotopy cosheaf of differential graded Lie algebras on a Hausdorff space X, over a characteristic zero ring. For instance, $\mathcal{L}$ can be the cosheaf of *compactly supported* forms $\Omega^{\bullet,c}_{dR,M} \otimes \mathfrak{g}$ with value in $\mathfrak{g}$ where $\mathfrak{g}$ is a differential graded Lie algebra and $\Omega^{\bullet}_{dR,M}$ is the (complex of) sheaf on a manifold M given by the de Rham complex. If M is a complex manifold, another interesting example is obtained by substituting the Dolbeaut complex to the de Rham complex.

For any open $U \subset X$, we can form the Chevalley-Eilenberg chain complex $C^{CE}_{\bullet}\big(\mathcal{L}(U)\big)$ of the (dg-)Lie algebra $\mathcal{L}(U)$. Its underlying k-module (see [101]) is given by

$$C^{CE}_{\bullet}\big(\mathcal{L}(U)\big) := S^{\bullet}\big(\mathcal{L}(U)[1]\big)$$

and its differential is induced by the Lie bracket and inner differential of $\mathcal{L}$. The structure maps of Example 22 (applied to $\mathcal{F} = \mathcal{L}[1]$) are maps of chain complexes (since $\mathcal{L}$ is a precosheaf of dg-Lie algebras), hence make $C^{CE}_{\bullet}\big(\mathcal{L}(-)\big)$ a prefactorization algebra over X, which we denote $C^{CE}(\mathcal{L})$ (note that this construction only requires $\mathcal{L}$ to be a precoheaf of dg-Lie algebras). As a corollary of Proposition 11, one obtains

Corollary 2 (Theorem 4.5.3, [54]) *If $\mathcal{L}$ is a homotopy cosheaf of dg-Lie algebras, the prefactorization algebra $C^{CE}(\mathcal{L})$ is a (homotopy) factorization algebra.*

The above corollary extends to homotopy cohseaves of L_∞-algebras as well.

This construction actually generalizes the construction of the *universal enveloping algebra* of a Lie algebra which corresponds to the case $\mathcal{L} = \Omega^{\bullet,c}_{dR,\mathbb{R}} \otimes \mathfrak{g}$ ([54]).

More generally the observables of *Free Field Theories* can be obtained this way, see [54] for many examples.

Remark 24 (**Algebras over disks in X**) Assume X has a cover by euclidean neighborhoods. One can define a colored operad whose objects are open subsets of X that are homeomorphic to $\mathbb{R}^n$ and whose morphisms from $\{U_1, \dots, U_n\}$ to V are empty except when the U_i's are mutually disjoint subsets of V, in which case they are singletons. We can take the monoidal envelope of this operad (as in Appendix 10.2 or [71, Sect. 2.4]) to get a symmetric monoidal ∞-category $\mathrm{Disk}(X)$ (also see [71], Remark 5.2.4.7). For any symmetric monoidal ∞-category $\mathcal{C}$, we thus get the ∞-category $\mathrm{Disk}(X)\textbf{-Alg} := \mathbf{Fun}^{\otimes}(\mathrm{Disk}(X), \mathcal{C})$ of $\mathrm{Disk}(X)$-algebras.

Unfolding the definition we find that a $\mathrm{Disk}(X)$-algebra is precisely a $\mathcal{D}_{isk}$-prefactorization algebra over X where $\mathcal{D}_{isk}$ is the set of all open disks in X.

A $\mathrm{Disk}(X)$-algebra is *locally constant* if for any inclusion of open disks $U \hookrightarrow V$ in X, the structure map $\mathcal{F}(U) \to \mathcal{F}(V)$ is a quasi-isomorphism (see [71]). By Theorem 8 and [71, Sect. 5.2.4], locally constant $N(\mathrm{Disk}(M))$-algebras are the same as locally constant factorization algebras. Hence we have

Proposition 12 *A locally constant $\mathcal{D}_{isk}$-prefactorization algebra has an unique extension as a locally constant homotopy factorization algebra. In fact, the functor $\boldsymbol{Fac}_X^{lc} \to \boldsymbol{PFac}_{\mathcal{D}_{isk}}^{lc} \cong Disk(X)\boldsymbol{\text{-}Alg}^{lc}$ is an equivalence.*

In particular, a *locally constant prefactorization algebra $\mathcal{F}$ has an unique extension as a locally constant factorization algebra $\mathcal{F}^{\circ}$* taking the same values as $\mathcal{F}$ on any disk.

Example 25 The unique extension $\mathcal{F}^{\circ}$ of $\mathcal{F}$ as a factorization algebra can have different values than $\mathcal{F}$ and two different prefactorization algebras on X can have the same values on the open cover $\mathcal{D}_{isk}$. As a trivial example, let $X = \{x, y\}$ be a two points discrete set. Then a $\mathcal{D}_{isk}$-prefactorization algebra is given by two pointed chain complexes $\mathcal{F}(\{x\}) = V_x, \mathcal{F}(\{y\}) = V_y$. It is locally constant and the associated factorization algebra is given by $\mathcal{F}^{\circ}(\{x, y\}) \cong V_x \otimes V_y$. However, if W is any chain complex with pointed maps $V_x \to W$, $V_y \to W$ then we have a prefactorization algebras $\mathcal{G}$ defined by $\mathcal{G}(X) = W$ and is otherwise the same as $\mathcal{F}^{\circ}$. In particular, $\mathcal{G}$ is a prefactorization algebra on X with the same values as $\mathcal{F}^{\circ}$ on the disks of X but is different from $\mathcal{F}^{\circ}$.

The notion of being locally constant for a *factorization algebra* is indeed a local property (though its definition is about all disks) as proved by the following result.

Proposition 13 *Let M be a topological manifold and $\mathcal{F}$ be a factorization algebra on M. Assume that there is an open cover $\mathcal{U}$ of M such that for any $U \in \mathcal{U}$ the restriction $\mathcal{F}_{|U}$ is locally constant. Then $\mathcal{F}$ is locally constant on M.*

See Sect. 9.1 for a proof.

Remark 25 (**Ran space**) Factorization algebras on X can be seen as a certain kind of cosheaf on the Ran space of X. This definition is actually the correct one to deal with factorization algebras in the algebraic geometry context (see [7, 33]). The *Ran space* $\mathrm{Ran}(X)$ *of a manifold* X is the space of finite non-empty subsets of X. Its topology[42] is the coarsest topology on $\mathrm{Ran}(X)$ for which the sets $\mathrm{Ran}(\{U_i\}_{i\in I})$ are open for

[42] This topology is closely related (but different) to the final topology on $\mathrm{Ran}(X)$ making the canonical applications $X^n \to \mathrm{Ran}(X)$ ($n > 0$) continuous.

every non-empty finite collection of pairwise disjoint opens subsets U_i $(i \in I)$ of X. Here, the set $\mathrm{Ran}(\{U_i\}_{i\in I})$ is the collection of all finite subsets $\{x_j\}_{j\in J} \subset X$ such that $\{x_j\}_{j\in J} \cap U_i$ is non empty for every $i \in I$.

Two subsets U, V of $\mathrm{Ran}(X)$ are said to be independent if the two subsets $\left(\bigcup_{S\in U} S\right) \subset X$ and $\left(\bigcup_{T\in V} T\right) \subset X$ are disjoint (as subsets of X). If U, V are subsets of $\mathrm{Ran}(X)$, one denotes $U \star V$ the subset $\{S \cup T,\ S \in U,\ T \in V\}$ of $\mathrm{Ran}(X)$.

It is proved in [71] that a factorization algebra on X is the same thing as a constructible cosheaf F on $\mathrm{Ran}(X)$ which satisfies in addition the *factorizing condition*, that is, that satisfies that, for every family of pairwise independent open subsets, the canonical map

$$F(U_1) \otimes \cdots \otimes F(U_n) \longrightarrow F(U_1 \star \cdots \star U_n)$$

is an equivalence (the condition is similar to Remark 20).

This characterization explain why there is a similarity between cosheaves and factorization algebras. However, the factorization condition is *not* a purely cosheaf condition and is not compatible with every operations on cosheaves.

5 Operations for Factorization Algebras

In this section we review many properties and operations available for factorization algebras.

5.1 Pushforward

If $\mathcal{F}$ is a prefactorization algebra on X, and $f : X \to Y$ is a continuous map, one can define the *pushforward* $f_*(\mathcal{F})$ by the formula $f_*(\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$. If $\mathcal{F}$ is an (homotopy) factorization algebra then so is $f_*(\mathcal{F})$, see [24].

Proposition 14 *The pushforward is a symmetric monoidal functor $f_* : \boldsymbol{Fac}_X \to \boldsymbol{Fac}_Y$ and further $(f \circ g)_* = f_* \circ g_*$.*

Let us abusively *denote $\mathcal{G}$ for the global section $\mathcal{G}(pt)$ of a factorization algebra over the point pt*. Let $p : X \to pt$ be the canonical map. By Theorem 8, when X is a manifold and $\mathcal{F}_A$ is a locally constant factorization algebra associated to a $\mathrm{Disk}_n^{(X,TX)}$-algebra, then the factorization homology of M is

$$\int_X A \cong \mathcal{F}_A(A) \cong p_*(\mathcal{F}_A). \tag{31}$$

This analogy legitimates to call $p_*(\mathcal{F})$, that is the (derived) global sections of $\mathcal{F}$, its factorization homology:

Definition 17 The *factorization homology*[43] *of a factorization algebra* $\mathcal{F} \in \mathbf{Fac}_X$ is $p_*(\mathcal{F})$ and is also denoted $\int_X \mathcal{F}$.

Proposition 15 *Let* $f : X \to Y$ *be a locally trivial fibration between smooth manifolds. If* $\mathcal{F} \in \boldsymbol{Fac}_X$ *is locally constant, then* $f_*(\mathcal{F}) \in \boldsymbol{Fac}_Y$ *is locally constant.*

Proof Let $U \hookrightarrow V$ be an inclusion of an open sub-disk U inside an open disk $V \subset Y$. Since V is contractible, it can be trivialized so we can assume $f^{-1}(V) = V \times F$ with F a smooth manifold. Taking a stable by finite intersection and factorizing cover $\mathcal{V}$ of F by open disks, we have a factorizing cover $\{V\} \times \mathcal{V}$ of $f^{-1}(V)$ consisting of open disks in X. Similarly $\{U\} \times \mathcal{V}$ is a factorizing cover of $f^{-1}(U)$ consisting of open disks. In particular for any $D \in \mathcal{V}$, the structure map $\mathcal{F}(U \times D) \to \mathcal{F}(V \times D)$ is a quasi-isomorphism since $\mathcal{F}$ is locally constant. Thus the induced map $\check{C}(\{U\} \times \mathcal{V}, \mathcal{F}) \to \check{C}(\{V\} \times \mathcal{V}, \mathcal{F})$ is a quasi-isomorphism as well which implies that $f_*(\mathcal{F})(U) \to f_*(\mathcal{F})(V)$ is a quasi-isomorphism. □

Example 26 (*Locally constant factorization algebras induced on a submanifold*) Let $i : X \hookrightarrow \mathbb{R}^n$ be an embedding of a manifold X into $\mathbb{R}^n$ and NX be an open tubular neighborhood of X in $\mathbb{R}^n$. We write $q : NX \to X$ for the bundle map. If A is an E_n-algebra, then it defines a factorisation algebra $\mathcal{F}_A$ on $\mathbb{R}^n$. Then the pushforward $q_*(\mathcal{F}_{\mathcal{A}|NX})$ is a locally constant (by Proposition 15) factorization algebra on X, which is *not* constant in general if the normal bundle NX is not trivialized.

Since a continuous map $f : X \to Y$ yields a factorization $X \xrightarrow{f} Y \to pt$ of $X \to pt$, Proposition 14 and the equivalence (31) imply the following pushforward formula of factorization homology.

Proposition 16 (Pushforward formula) *Let* $X \xrightarrow{f} Y$ *be continuous and* $\mathcal{F}$ *be in* $\boldsymbol{Fac}_X$. *The factorization homology of* $\mathcal{F}$ *over* X *is the same as the factorization homology of* $f_*(\mathcal{F})$ *over* Y*:*

$$\int_X \mathcal{F} \cong p_*(\mathcal{F}) \cong p_*\big(f_*(\mathcal{F})\big) \cong \int_Y f_*(\mathcal{F}). \tag{32}$$

5.2 *Extension from a Basis*

Let $\mathcal{U}$ be a *basis* stable by finite intersections for the topology of a space X and which is also a *factorizing cover*. Let $\mathcal{F}$ be a (homotopy) $\mathcal{U}$ *-factorization algebra*, that is a

[43] Recall that we are considering homotopy factorization algebras, which are already derived objects. If $\mathcal{G}$ is a genuine factorization algebra (Remark 21), then its factorization homology would be $\mathbb{L}p_*(\mathcal{G}) := p_*(\tilde{G})$ where $\tilde{G}$ is an acyclic resolution of $\mathcal{G}$ (as genuine factorization algebra).

$\mathcal{U}$-prefactorization algebra (Definition 12) such that, for any $U \in \mathcal{U}$ and factorizing cover $\mathcal{V}$ of U consisting of open sets in $\mathcal{U}$, the canonical map $\check{C}(\mathcal{V}, \mathcal{F}) \to \mathcal{F}(U)$ is a quasi-isomorphism.

Proposition 17 (Costello-Gwilliam [24]) *There is an unique*[44] *(homotopy) factorization algebra $i_*^{\mathcal{U}}(\mathcal{F})$ on X extending $\mathcal{F}$ (that is equipped with a quasi-isomorphism of $\mathcal{U}$-factorization algebras $i_*^{\mathcal{U}}(\mathcal{F}) \to \mathcal{F}$).*

Precisely, for any open set $V \subset X$, one has

$$i_*^{\mathcal{U}}(\mathcal{F})(V) := \check{C}(\mathcal{U}_V, \mathcal{F})$$

where $\mathcal{U}_V$ is the open cover of V consisting of all open subsets of V which are in $\mathcal{U}$.

Note that the uniqueness is immediate since, if $\mathcal{G}$ is a factorization algebra on X, then for any open V the canonical map $\check{C}(\mathcal{U}_V, \mathcal{G}) \to \mathcal{G}(V)$ is a quasi-isomorphism and, further, the Čech complex $\check{C}(\mathcal{U}_V, \mathcal{G})$ is computed using only open subset in $\mathcal{U}$.

Proposition 17 gives a way to construct (locally constant) factorization algebras as we now demonstrate.

Example 27 By Example 19, we know that an associative unital algebra (possibly differential graded) gives a locally constant factorization algebra on the interval $\mathbb{R}$. It can be explicitly given by using extension along a basis. Indeed, the collection $\mathcal{I}$ of intervals (a, b) $(a < b)$ is a factorizing basis of opens, which is stable by finite intersections. Then one can set a $\mathcal{I}$-prefactorization algebra $\mathcal{F}_A$ by setting $\mathcal{F}_A((a, b)) := A$. For pairwise disjoints open interval $I_1, \dots, I_n \subset I$, where the indices are chosen so that $\sup(I_i) \le \inf(I_{i+1})$, the structure maps are given by

$$A^{\otimes n} = \mathcal{F}_A(I_1) \otimes \cdots \otimes \mathcal{F}_A(I_n) \longrightarrow \mathcal{F}_A(I) = A \tag{33}$$

$$a_1 \otimes \cdots \otimes a_n \longmapsto a_1 \cdots a_n\,. \tag{34}$$

To extend this construction to a full homotopy factorization algebra on $\mathbb{R}$, one needs to check that $\mathcal{F}_A$ is a $\mathcal{I}$-factorization algebra which is the content of Proposition 27.

In the construction, we have chosen an implicit orientation of $\mathbb{R}$; namely, in the structure map (33), we have decided to multiply the elements (a_i) by choosing to order the intervals in increasing order from left to right.

Let $\tau : \mathbb{R} \to \mathbb{R}$ be the antipodal map $x \mapsto -x$ reversing the orientation. One can check that $\tau_*(\mathcal{F}_A) = \mathcal{F}_{A^{op}}$ where A^{op} is the algebra A with opposite multiplication $(a, b) \mapsto b \cdot a$. In other words, *choosing the opposite orientation* (that is the decreasing one) *of* $\mathbb{R}$ amounts to *replacing the algebra by its opposite algebra.*

Example 28 (*Back to the circle*) The circle S^1 also has a (factorizing) basis given by the open (embedded) intervals (of length less than half of the perimeter of the circle in order to be stable by intersection). Choosing an orientation on the circle, one can define a (homotopy) factorization algebra $\mathcal{S}_A$ on the circle using again the

[44] Up to contractible choices.

structure maps (33). This gives an explicit construction of the factorization algebra associated to a framing of S^1 from Example 19 (since they agree on a stable by finite intersection basis of open sets). The global section of $\mathcal{S}_A$ are thus the Hochschild chains of A by the computation (21).

Similarly to Example 27, choosing the opposite framing on the circle amounts to considering the factorization algebra $\mathcal{S}_{A^{op}}$. However, *unlike on* $\mathbb{R}$, there is an equivalence $\mathcal{S}_A \cong \mathcal{S}_{A^{op}}$ induced by the fact that there is an orientation preserving diffeomorphism between the two possible orientations of the circle and that further, the value of $\mathcal{S}_A$ on any interval is constant.

5.3 Exponential Law: Factorization Algebras on a Product

Let $\pi_1 : X \times Y \to X$ be the canonical projection. By Proposition 15, we have the *pushforward* functor $\pi_{1*} : \mathbf{Fac}^{lc}_{X\times Y} \to \mathbf{Fac}^{lc}_X$. This functor has an natural lift into $\mathbf{Fac}^{lc}(Y)$. Indeed, if U is open in X and V is open in Y, we have a chain complex:

$$\underline{\pi_1}_*(\mathcal{F})(U, V) := \mathcal{F}(\pi_1{}^{-1}_{|X\times V}(U)) = \mathcal{F}(U \times V).$$

Let $V_1, \dots, V_i$ be pairwise disjoint open subsets in an open set $W \subset Y$ and consider

$$\underline{\pi_1}_*(\mathcal{F})(U, V_1) \otimes \cdots \otimes \underline{\pi_1}_*(\mathcal{F})(U, V_i) \cong \mathcal{F}(U \times V_1) \otimes \cdots \otimes \mathcal{F}(U \times V_i) \stackrel{\rho_{U\times V_1,\dots,U\times V_i,U\times W}}{\longrightarrow} \mathcal{F}(U \times W) = \underline{\pi_1}_*(\mathcal{F})(U, W). \tag{35}$$

The map (35) makes $\underline{\pi_1}_*(\mathcal{F})(U)$ a prefactorization algebra on Y. If $U_1, \dots, U_j$ are pairwise disjoint open inside an open $O \subset X$, the collection of structure maps

$$\underline{\pi_1}_*(\mathcal{F})(U_1, V) \otimes \cdots \otimes \underline{\pi_1}_*(\mathcal{F})(U_j, V) \cong \mathcal{F}(U_1 \times V) \otimes \cdots \otimes \mathcal{F}(U_j \times V) \stackrel{\rho_{U_1\times V,\dots,U_j\times V,O\times V}}{\longrightarrow} \mathcal{F}(O \times V) = \underline{\pi_1}_*(\mathcal{F})(O, V) \tag{36}$$

indexed by opens $V \subset Y$ is a map $\underline{\pi_1}_*(\mathcal{F})(U_1) \otimes \cdots \otimes \underline{\pi_1}_*(\mathcal{F})(U_j) \longrightarrow \underline{\pi_1}_*(\mathcal{F})(O)$ of prefactorization algebras over Y.

Combining the two constructions, we find that the structure maps (35) and (36) make $\pi_{1*}(\mathcal{F})$ a prefactorization algebra over X with values in the category of prefactorization algebras over Y. In other words we have just defined a functor:

$$\underline{\pi_1}_* : \mathbf{PFac}_{X\times Y} \longrightarrow \mathbf{PFac}_X(\mathbf{PFac}_Y) \tag{37}$$

fitting into a commutative diagram

$$\begin{array}{ccc} \mathbf{PFac}_{X\times Y} & \xrightarrow{\underline{\pi_1}_*} & \mathbf{PFac}_X(\mathbf{PFac}_Y) \\ & \searrow^{(\pi_1)_*} & \downarrow{\scriptstyle \mathbf{PFac}_X(p_*)} \\ & & \mathbf{PFac}_X \end{array}$$

where p_* is given by Definition 17.

Proposition 18 *Let $\pi_1 : X \times Y \to X$ be the canonical projection. The pushforward (37) by π_1 induces a functor*

$$\underline{\pi_1}_* : \boldsymbol{Fac}_{X\times Y} \longrightarrow \boldsymbol{Fac}_X(\boldsymbol{Fac}_Y)$$

and, if X, Y are smooth manifolds, an equivalence $\underline{\pi_1}_* : \boldsymbol{Fac}^{lc}_{X\times Y} \xrightarrow{\simeq} \boldsymbol{Fac}^{lc}_X(\boldsymbol{Fac}^{lc}_Y)$ of ∞-categories .

See Sect. 9.1.2 for a proof.

The above Proposition is a slight generalization of (and relies on) the following ∞-category version of the beautiful Dunn Theorem [27] proved under the following form by Lurie [71] (see [47] for the pushforward interpretation):

Theorem 11 (Dunn Theorem) *There is an equivalence of ∞-categories*

$$E_{m+n}\text{-}\boldsymbol{Alg} \xrightarrow{\simeq} E_m\text{-}\boldsymbol{Alg}(E_n\text{-}\boldsymbol{Alg}).$$

Under the equivalence E_n-$\boldsymbol{Alg} \cong \boldsymbol{Fac}^{lc}_{\mathbb{R}^n}$ (Theorem 9), the above equivalence is realized by the pushforward $\underline{\pi}_* : \boldsymbol{Fac}^{lc}_{\mathbb{R}^m\times\mathbb{R}^n} \to \boldsymbol{Fac}^{lc}_{\mathbb{R}^m}(\boldsymbol{Fac}^{lc}_{\mathbb{R}^n})$ associated to the canonical projection $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$.

Example 29 (*PTVV construction*) There is a derived geometry variant of the AKSZ formalism introduced recently in [77] which leads to factorization algebras similarly to Example 23. We briefly sketch it. The main input is a (derived Artin) stack X (over a characteristic zero field) which is assumed to be compact and equipped with an orientation (write d_X for the dimension of X) and a stack Y with an n-shifted symplectic structure ω (for instance take Y to be the shifted cotangent complex $Y = T^*[n]Z$ of a scheme Z). The natural evaluation map $ev : X \times \mathbb{R}\,Map(X, Y)$ allows to pullback the symplectic structure on the space of fields $\mathbb{R}\,Map(X, Y)$. Precisely, $\mathbb{R}\,Map(X, Y)$ carries an natural $(n - d_X)$-shifted symplectic structure roughly given by the integration $\int_{[X]} ev^*(\omega)$ of the pullback of ω on the fundamental class of X. It is expected that the observables $\mathcal{O}_{\mathbb{R}\,Map(X,Y)}$ carries a structure of P_{1+n-d_X}-algebra. Assume further that X is a Betti stack, that is in the essential image of $\mathbf{Top} \to \mathbf{dSt}_k$. It will then follow from Corollary 1 and Example 21 that $\mathcal{O}_{\mathbb{R}\,Map(X,Y)}$ belongs to $\mathbf{Fac}^{lc}_X(P_{1+n-d_X}\text{-}\mathbf{Alg})$. Using the formality of the little disks operads in dimension ≥ 2, Proposition 18 then will give $\mathcal{O}_{\mathbb{R}\,Map(X,Y)}$ the structure of a locally constant factorization algebra on $X \times \mathbb{R}^{1+n-d_X}$ when $n > d_X$. It is also expected that the quantization of such shifted symplectic stacks shall carries canonical locally constant factorization algebras structures.

Note that the Pantev-Toën-Vaquié-Vezzosi construction was recently extended by Calaque [13] to add boundary conditions. The global observables of such *relative* mapping stacks shall be naturally endowed with the structure of locally constant factorization algebra on stratified spaces (as defined in Sect. 6).

Proposition 18 and Theorem 8 have the following consequence

Corollary 3 (Fubini formula [47]) *Let M, N be manifolds of respective dimension m, n and let A be a* $\mathrm{Disk}_{n+m}^{(M\times N, T(M\times N))}$*-algebra. Then, $\int_N A$ has a canonical structure of* $\mathrm{Disk}_m^{(M,TM)}$*-algebra and further,*

$$\int_{M\times N} A \;\cong\; \int_M \Big(\int_N A\Big).$$

Example 30 Let A be a smooth commutative algebra. By Hochschild-Kostant-Rosenberg theorem (see Example 6), $CH_{S^1}(A) \xrightarrow{\simeq} S^\bullet_A(\Omega^1(A)[1])$; this algebra is also smooth. Since A is commutative it defines a factorization algebra on the torus $S^1 \times S^1$. By Corollary 3, we find that

$$\int_{S^1\times S^1} A \cong \int_{S^1} \big(S^\bullet_A(\Omega^1(A)[1])\big) \;\cong\; S^\bullet_A\Big(\Omega^1(A)[1] \oplus \Omega^1(A)[1] \oplus \Omega^1(A)[2]\Big).$$

5.4 *Pullback Along Open Immersions and Equivariant Factorization Algebras*

Let $f : X \to Y$ be an open immersion and let $\mathcal{G}$ be a factorization algebra on Y. Since $f : X \to Y$ is an open immersion, the set

$$\mathcal{U}_f := \{U \text{ open in } X \text{ such that } f_{|U} : U \to Y \text{ is an homeomorphism}\}$$

is an open cover of X as well as a factorizing basis. For $U \in \mathcal{U}_f$, we define $f^*(\mathcal{G})(U) := \mathcal{G}(f(U))$. The structure maps of $\mathcal{G}$ make $f^*(\mathcal{G})$ a $\mathcal{U}_f$-factorization algebra in a canonical way. Thus by Proposition 17, $i_*^{\mathcal{U}_f}(f^*(\mathcal{G}))$ is the factorization algebra on X extending $f^*(\mathcal{G})$. We (abusively) denote $f^*(\mathcal{G}) := i_*^{\mathcal{U}_f}(f^*(\mathcal{G}))$ and call it the pullback along f of the factorization algebra $\mathcal{G}$.

Proposition 19 ([24]) *The pullback along open immersion is a functor $f^* : \boldsymbol{Fac}_Y \to \boldsymbol{Fac}_X$. If $f : X \to Y$ and $g : Y \to Z$ are open immersions, then $(g \circ f)^* = f^* \circ g^*$.*

If $U \in \mathcal{U}_f$, then U is an open subset of the open set $f^{-1}(f(U))$. Thus if $\mathcal{F}$ is a factorization algebra on X, we have the natural map

$$\mathcal{F}(U) \overset{\rho_{U,f^{-1}(f(U))}}{\longrightarrow} \mathcal{F}(f^{-1}(f(U))) \cong f^*(f_*(\mathcal{F}))(U) \tag{38}$$

which is a map of $\mathcal{U}_f$-factorization algebras. Since $\mathcal{F}$ and $f^*(f_*(\mathcal{F}))$ are factorization algebras on X, the above map extends uniquely into a map of factorization algebras on X. We have proved:

Proposition 20 *Let $f : X \to Y$ be an open immersion. There is an natural transformation $Id_{\mathbf{Fac}_X} \to f^* f_*$ induced by the maps (38).*

Example 31 Let $X = \{c, d\}$ be a discrete space with two elements and consider the projection $f : X \to pt$. A factorization algebra $\mathcal{G}$ on pt is just the data of a chain complex G with a distinguished cycle g_0 while a factorization algebra $\mathcal{F}$ on X is given by two chain complexes C, D (with distinguished cycles c_0, d_0) and the rule $\mathcal{F}(\{c\}) = C$, $\mathcal{F}(\{d\}) = D$, $\mathcal{F}(\{c\}) = C \overset{id\otimes\{d_0\}}{\longrightarrow} C \otimes D = \mathcal{F}(X)$ and $\mathcal{F}(\{d\}) = C \overset{\{c_0\}\otimes id}{\longrightarrow} C \otimes D = \mathcal{F}(X)$. In that case we have that

$$f^*(f_*(\mathcal{F})(\{x\}) = C \otimes D = f^*(f_*(\mathcal{F})(\{y\}) = f^*(f_*(\mathcal{F})(X)$$

while $f_*(f^*(\mathcal{G}))(pt) = F \otimes F$. Note that there are no natural transformation of chain complexes $F \otimes F \to F$ in general; in particular f^* *and* f_* *are not adjoint*.

In fact f_* does *not* have any adjoint in general; indeed as the above example of $F : \{c, d\} \to pt$ demonstrates, f_* does not commute with coproducts nor products.

We now turn on to a descent property of factorization algebras. Let G be a discrete group acting on a space X. For $g \in G$, we write $g : X \to X$ the homeomorphism $x \mapsto g \cdot x$ induced by the action.

Definition 18 A G-equivariant factorization algebra on X is a factorization algebra $\mathcal{G} \in \mathbf{Fac}_X$ together with, for all $g \in G$, (quasi-)isomorphisms of factorization algebras

$$\theta_g : g^*(\mathcal{G}) \overset{\simeq}{\to} \mathcal{G}$$

such that $\theta_1 = \text{id}$ and

$$\theta_{gh} = \theta_h \circ h^*(\theta_g) : h^*(g^*(\mathcal{G})) \to \mathcal{G}.$$

We write $\mathbf{Fac}_X^G$ for the category of G-equivariant factorization algebras over X.

Assume G acts properly discontinuously and X is Hausdorff so that the quotient map $q : X \to X/G$ is an open immersion. If $\mathcal{F}$ is a factorization algebra over X/G, then $q^*(\mathcal{F})$ is G-equivariant (since $q(g \cdot x) = q(x)$). We thus have a functor $q^* : \mathbf{Fac}_{X/G} \longrightarrow \mathbf{Fac}_X^G$.

Proposition 21 *(*Costello-Gwilliam [24]*) If the discrete group G acts properly discontinuously on X, then the functor $q^* : \mathbf{Fac}_{X/G} \longrightarrow \mathbf{Fac}_X^G$ is an equivalence of categories.*

The proof essentially relies on considering the factorization basis given by trivialization of the principal G-bundle $X \to X/G$ to define an inverse to q^*.

Proposition 22 *If the discrete group G acts properly discontinuously on a smooth manifold X, then the equivalence $q^* : \mathbf{Fac}_{X/G} \longrightarrow \mathbf{Fac}_X^G$ factors as an equivalence $q^* : \mathbf{Fac}^{lc}_{X/G} \longrightarrow (\mathbf{Fac}^{lc}_X)^G$ between the subcategories of locally constant factorization algebras.*

Proof Let U be an open set such that $q_{|U} : U \to X/G$ is an homeomorphism onto its image. Then, for every open subset V of U, $q^*(\mathcal{F})(V) = \mathcal{F}(q(V))$. Thus, if $\mathcal{F}$ satisfies the condition of being locally constant for disks included in U, then so does $q^*(\mathcal{F})$ for disks included in $q(U)$. Hence, by Proposition 13, $q^*(\mathcal{F})$ is locally constant if $\mathcal{F}$ is locally constant.

Now, assume $\mathcal{G} \in \mathbf{Fac}_X^G$ is locally constant. Then $(q^*)^{-1}(\mathcal{G})$ is the factorization algebra defined on every section $X/G \supset U \to \sigma(U) \subset X$ of q (with U open) by $(q^*)^{-1}(\mathcal{G})(U) = \mathcal{G}(\sigma(U))$. Since every disk D is contractible, we always have a section $D \to X$ of $q_{|D}$. Thus, if $\mathcal{G}$ is locally constant, then so is $(q^*)^{-1}(\mathcal{G})$. □

Remark 26 (**General definition of equivariant factorization algebras**) Definition 18 can be easily generalized to topological groups as follows. Indeed, if G acts continuously on X, then the rule $(g, \mathcal{F}) \mapsto g^*(\mathcal{F})$ induces a right action of G on $\mathbf{Fac}_X$.[45]

The ∞-category of G-equivariant factorization algebras is the ∞-category of homotopy G-fixed points of $\mathbf{Fac}_X$:

$$\mathbf{Fac}_X^G := (\mathbf{Fac}_X)^{hG}.$$

This ∞-category is equivalent to the one of Definition 18 for discrete groups. It is the ∞-category consisting of a factorization algebra $\mathcal{G}$ on X together with quasi-isomorphisms of factorization algebras $\theta_g : g^*(\mathcal{G}) \to \mathcal{G}$ (inducing a ∞-functor $BG \to \mathbf{Fac}_X$, where BG is the ∞-category associated to the topological category with a single object and mapping space of morphisms given by G) and equivalences $\theta_{gh} \sim \theta_h \circ h^*(\theta_g)$ satisfying some higher coherences.

5.5 *Example: Locally Constant Factorization Algebras over the Circle*

Let $q : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ be the universal cover of S^1 and let $\mathcal{F}$ be a locally constant factorization algebra on S^1. By Proposition 22, $\mathcal{F}$ is equivalent to the data of a $\mathbb{Z}$-equivariant locally constant factorization algebra on $\mathbb{R}$ which is the same as a locally

[45] That is a map of E_1-algebras (in **Top**) from G^{op} to $\mathbf{Fun}(\mathbf{Fac}_X, \mathbf{Fac}_X)$.

constant factorization algebra over $\mathbb{R}$ together with an equivalence of factorization algebras, the equivalence being given by $\theta_1 : 1^*(q^*(\mathcal{F})) \stackrel{\simeq}{\to} q^*(\mathcal{F})$. By Theorem 9, the category of locally constant factorization algebras on $\mathbb{R}$ is the same as the category of E_1-algebras, and thus is equivalent to its full subcategory of *constant* factorization algebras. It follows that we have a canonical equivalence $1^*(\mathcal{G}) \cong \mathcal{G}$ for $\mathcal{G} \in \mathbf{Fac}^{lc}_{\mathbb{R}}$ (in particular, for any open interval I, the structure map $1 * (\mathcal{G}(I)) = \mathcal{G}(1+I) \to \mathcal{G}(\mathbb{R})$ is a quasi-isomorphism).

Definition 19 We denote $mon : q^*(\mathcal{F}) \cong 1^*(q^*(\mathcal{F})) \stackrel{\theta_1}{\to} q^*(\mathcal{F})$ the self-equivalence of $q^*(\mathcal{F})$ induced by θ_1 and call it the *monodromy* of $\mathcal{F}$.

We thus get the following result

Corollary 4 *The category* $\boldsymbol{Fac}^{lc}_{S^1}$ *of locally constant factorization algebras on the circle is equivalent to the* ∞*-category* $\boldsymbol{Aut}(E_1\text{-}\boldsymbol{Alg})$ *of* E_1*-algebras equipped with a self-equivalence.*

Remark 27 Using Proposition 18, it is easy to prove similarly that $\mathbf{Fac}^{lc}_{S^1\times S^1}$ is equivalent to the category of E_2-algebras equipped with two commuting monodromies (i.e. self-equivalences).

It seems harder to describe the categories of locally constant factorization algebras over the spheres S^3, S^7 in terms of E_3 and E_7-algebras (due to the complicated homotopy groups of the spheres). However, for $n = 3, 7$, there shall be an embedding of the categories of E_n-algebras equipped with an n-gerbe[46] into $\mathbf{Fac}^{lc}_{S^n}$.

Let $\mathcal{F}$ be a locally constant factorization algebra on S^1 (identified with the unit sphere in $\mathbb{C}$). We wish to compute the *global section of* $\mathcal{F}$ (i.e. its factorization homology $\int_{S^1} \mathcal{F}$). Let $\mathcal{B} \cong \mathcal{F}(S^1 \setminus \{1\})$ be its underlying E_1-algebra (with monodromy $mon : \mathcal{B} \stackrel{\simeq}{\to} \mathcal{B}$). We use the orthogonal projection $\pi : S^1 \to [-1, 1]$ from S^1 to the real axis. The equivalence (32) yields $\mathcal{F}(S^1) \cong \pi_*(\mathcal{F})([-1, 1])$ and by Propositions 15 and 27, we are left to compute the E_1-algebra $\pi_*(\mathcal{F})\big((-1, 1)\big)$ and left and right modules $\pi_*(\mathcal{F}\big((-1, 1]\big), \pi_*(\mathcal{F})\big([-1, 1)\big)$. From Example 27, we get $\pi_*(\mathcal{F})\big((-1, 1)\big) \cong \mathcal{B} \otimes \mathcal{B}^{op}$. Further,

$$\pi_*(\mathcal{F})\big([-1, 1)\big) \cong \mathcal{F}(S^1 \setminus \{1\}) = \mathcal{B} \quad \text{(as a}\mathcal{B} \otimes \mathcal{B}^{op}\text{-module).}$$

Similarly $\pi_*(\mathcal{F})\big((-1, 1]\big) \cong \mathcal{B}^{mon}$, that is $\mathcal{B}$ viewed as a $\mathcal{B} \otimes \mathcal{B}^{op}$-module through the monodromy. When $\mathcal{B}$ is actually a differential graded algebra, then the bimodule stucture of $\mathcal{B}^{mon}$ boils down to $a \cdot x \cdot b = mon(b) \cdot m \cdot a$. This proves the following which is also asserted in [71, Sect. 5.3.3].

Corollary 5 *Let* $\mathcal{B}$ *be a locally constant factorization algebra on* S^1*. Let* B *be a differential graded algebra and* $mon : B \stackrel{\simeq}{\to} B$ *be a quasi-isomorphism of algebras so that* (B, mon) *is a model for the underlying* E_1*-algebra of* $\mathcal{B}$ *and its monodromy. Then the factorization homology*

[46] By an n-gerbe over $A \in E_n$**-Alg**, we mean a monoid map $\mathbb{Z} \to \Omega^{n-1}\mathrm{Map}_{E_n\text{-}\mathbf{Alg}}(A, A)$.

$$\int_{S^1} \mathcal{B} \cong B \overset{\mathbb{L}}{\underset{B\otimes B^{op}}{\otimes}} B^{mon} \cong HH(B, B^{mon})$$

is computed by the (standard) Hochschild homology[47] $HH(B)$ *of* B *with value in* B *twisted by the monodromy.*

Example 32 (*The circle again*) Let $p : \mathbb{R} \to S^1$ be the universal cover of S^1. By Example 27, an unital associative algebra A defines a locally constant factorization algebra, denoted $\mathcal{A}$, on $\mathbb{R}$. By Proposition 15, the pushforward $p_*(\mathcal{A})$ is a locally constant factorization algebra on S^1, which, on any interval $I \subset S^1$ is given by $p_*(\mathcal{A})(I) = \mathcal{A}(I \times \mathbb{Z}) = A^{\otimes\mathbb{Z}}$. It is however *not a constant* factorization algebra since the global section of $p_*(\mathcal{A})$ is different from the Hochschild homology of A:

$$p_*(\mathcal{A})(S^1) = \mathcal{A}(\mathbb{R}) \cong A \not\cong HH(A^{\otimes\mathbb{Z}})$$

(for instance if A is commutative the Hochschild homology of $A^{\otimes\mathbb{Z}}$ is $A^{\otimes\mathbb{Z}}$ in degree 0.) Indeed, the monodromy of $\mathcal{A}$ is given by the automorphism σ of A which sends the element a_i in the tensor index by an integer i into the tensor factor indexed by $i+1$, that is $\sigma\big(\bigotimes_{i\in\mathbb{Z}} a_i\big) = \bigotimes_{i\in\mathbb{Z}} a_{i-1}$.

However, by Corollary 5, we have that, for any E_1-algebra A,

$$HH\big(A^{\otimes\mathbb{Z}}, \big(A^{\otimes\mathbb{Z}}\big)^{mon}\big) \cong A.$$

5.6 Descent

There is a way to glue together factorization algebras provided they satisfy some descent conditions which we now explain.

Let $\mathcal{U}$ be an open cover of a space X (which we assume to be equipped with a factorizing basis). We also assume that all intersections of infinitely many different opens in $\mathcal{U}$ are empty. For every finite subset $\{U_i\}_{i\in I}$ of $\mathcal{U}$, let $\mathcal{F}_I$ be a factorization algebra on $\bigcap_{i\in I} U_i$. For any $i \in I$, we have an inclusion $s_i : \bigcap_{i\in I} U_i \hookrightarrow \bigcap_{j\in I\setminus\{i\}} U_j$.

Definition 20 A gluing data is a collection, for all finite subset $\{U_i\}_{i\in I} \subset \mathcal{U}$ and $i \in I$, of quasi-isomorphisms $r_{I,i} : \mathcal{F}_I \longrightarrow \big(\mathcal{F}_{I\setminus\{i\}}\big)_{|\mathcal{U}_I}$ such that, for all $I, i, j \in I$, the following diagram commutes:

[47] In particular by the standard Hochschild complex (see [67]) $C_*(B, B^{mon})$ when B is flat over k.

$$\begin{array}{ccc} \mathcal{F}_I & \xrightarrow{r_{I,j}} & \left(\mathcal{F}_{I\setminus\{j\}}\right)_{|\mathcal{U}_I} \\ \downarrow{\scriptstyle r_{I,i}} & & \downarrow{\scriptstyle r_{I\setminus\{j\},i}} \\ \left(\mathcal{F}_{I\setminus\{i\}}\right)_{|\mathcal{U}_I} & \xrightarrow{r_{I\setminus\{i\},j}} & \left(\mathcal{F}_{I\setminus\{i,j\}}\right)_{|\mathcal{U}_I}. \end{array}$$

Given a gluing data, one can define a factorizing basis $\mathcal{V}_{\mathcal{U}}$ given by the family of all opens which lies in some $U \in \mathcal{U}$. For any $V \in \mathcal{V}_{\mathcal{U}}$, set $\mathcal{F}(V) = \mathcal{F}_{I_V}(V)$ where I_V is the largest subset of I such that $V \in \bigcap_{j\in I_V} U_j$. The maps $R_{I,i}$ induce a structure of $\mathcal{V}_{\mathcal{U}}$-prefactorization algebra.

Proposition 23 ([24]) *Given a gluing data, the $\mathcal{V}_{\mathcal{U}}$-prefactorization $\mathcal{F}$ extends uniquely into a factorization algebra $\mathcal{F}$ on X whose restriction $\mathcal{F}_{|\mathcal{U}_I}$ on each $\mathcal{U}_I$ is canonically equivalent to $\mathcal{F}_I$.*

Note that if the $\mathcal{F}_I$ are the restrictions to $\mathcal{U}_I$ of a factorization algebra $\mathcal{F}$, then the collection of the $\mathcal{F}_I$ satisfy the condition of a gluing data.

6 Locally Constant Factorization Algebras on Stratified Spaces and Categories of Modules

There is an interesting variant of locally constant factorization algebras over (topologically) stratified spaces which can be used to *encode categories of E_n-algebras and their modules* for instance. Note that by Remark 20, all our categories of modules will be pointed, that is coming with a preferred element. We gave the definition and several examples in this Section. An analogue of Theorem 8 for stratified spaces shall provide the link between the result in this section and results of [5].

6.1 Stratified Locally Constant Factorization Algebras

In this paper, by a *stratified space* of dimension n, we mean a Hausdorff paracompact topological space X, which is filtered as the union of a sequence of closed subspaces $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that any point $x \in X_i \setminus X_{i-1}$ has a neighborhood $U_x \overset{\phi}{\simeq} \mathbb{R}^i \times C(L)$ in X where $C(L)$ is the (open) cone on a stratified space of dimension $n-i-1$ and the homeomorphism preserves the filtration.[48] We further require that $X \setminus X_{n-1}$ is dense in X. In particular, a stratified space of

[48] That is $\phi(U_x \cap X_{i+j+1}) = \mathbb{R}^i \times C(L_j)$ for $0 \le j \le n-i-1$.

dimension 0 is simply a topological manifold of dimension 0 and $X_i \setminus X_{i-1}$ is a topological manifold of dimension i (possibly empty or non-connected).

The connected components of $X_i \setminus X_{i-1}$ are called the dimension i-strata of X. We always assume that X has at most countable strata.

Definition 21 An open subset D of X is called a (stratified) *disk* if it is homeomorphic to $\mathbb{R}^i \times C(L)$ with L stratified of dimension $n-i-1$, the homeomorphism preserves the filtration and further $D \cap X_i \neq \emptyset$ and $D \subset X \setminus X_{i-1}$. We call i the *index of* the (stratified disk) D. It is the *smallest integer* j such that $D \cap X_j \neq \emptyset$

We say that a (stratified) disk D is a *good neighborhood at* X_i if i is the index of D and D intersects only one connected component of $X_i \setminus X_{i-1}$.

A *factorization algebra* $\mathcal{F}$ over a stratified space X is called *locally constant* if for any inclusion of (stratified) disks $U \hookrightarrow V$ such that both U and V are good neighborhoods at X_i (for the same $i \in \{0, \ldots, n\}$),[49] the structure map $\mathcal{F}(U) \to \mathcal{F}(V)$ is a quasi-isomorphism.

The underlying space of almost all examples of stratified spaces X arising in these notes will be a manifold (with boundary or corners). In that cases, all (stratified) disk are homeomorphic to to a standard euclidean (half-)disk $\mathbb{R}^{n-j} \times [0, +\infty)^j$.

Let X be a manifold (without boundary) and let X^{str} be the same manifold endowed with some stratification. A locally constant factorization algebra on X is also locally constant with respect to the stratification. Thus, we have a fully faithful embedding

$$\mathbf{Fac}_X^{lc} \longrightarrow \mathbf{Fac}_{X^{str}}^{lc}. \tag{39}$$

Several general results on locally constant factorization algebras from Sect. 4 have analogues in the stratified case. We now list three useful ones.

Proposition 24 *Let X be a stratified manifold and $\mathcal{F}$ be a factorization algebra on X such that there is an open cover $\mathcal{U}$ of X such that for any $U \in \mathcal{U}$ the restriction $\mathcal{F}_{|U}$ is locally constant. Then $\mathcal{F}$ is locally constant on X.*

The functor (39) generalizes to inclusion of any stratified subspace.

Proposition 25 *Let $i : X \hookrightarrow Y$ be a stratified (that is filtration preserving) embedding of stratified spaces in such a way that $i(X)$ is a reunion of strata of Y. Then the pushforward along i preserves locally constantness, that is lift as a functor*

$$\boldsymbol{Fac}_X^{lc} \to \boldsymbol{Fac}_Y^{lc}.$$

Proof Let $\mathcal{F}$ be in $\mathbf{Fac}_X^{lc}$ and $U \subset D$ be good disks of index j at a neigborhood of a strata in $i(X)$. The preimage $i^{-1}(U) \cong i(X) \cap U$ is a good disk of index j in X and so is $i^{-1}(D)$. Hence $i_*(\mathcal{F}(U)) \to i_*(\mathcal{F}(D))$ is a quasi-isomorphism. On the other hand, if $V \subset Y \setminus i(X)$ is a good disk, then $i_*(\mathcal{F}(V)) \cong k$. Since the constant factorization algebra with values k (Example 18) is locally constant on every stratified space, the result follows.

[49] In other words are good neighborhoods of same index.

Let $f : X \to Y$ be a locally trivial fibration between stratified spaces. We say that f is *adequatly stratified* if Y has an open cover by trivializing (stratified) disks V which are good neighborhoods satisfying that:

- $f^{-1}(V) \overset{\psi}{\simeq} V \times F$ has a cover by (stratified) disks of the form $\psi^{-1}(V \times D)$ which are good neighborhoods in X;
- for sub-disks $T \subset U$ which are good neighborhoods (in V) with the same index, then $\psi^{-1}(T \times D)$ is a good neighborhood of X of same index as $\psi^{-1}(U \times D)$.

Obvious examples of adequatly stratified maps are given by locally trivial stratified fibrations; in particular by proper stratified submersions according to Thom first isotopy lemma [51, 92].

Proposition 26 *Let $f : X \to Y$ be adequatly stratified. If $\mathcal{F} \in \boldsymbol{Fac}_X$ is locally constant, then $f_*(\mathcal{F}) \in \boldsymbol{Fac}_Y$ is locally constant.*

Proof Let $U \hookrightarrow V$ be an inclusion of open disks which are both good neighborhoods at Y_i; by Proposition 24, we may assume V lies in one of the good trivializing disk in the definition of an adequatly stratified map so that we have a cover by opens homeomorphic to $(\psi^{-1}(V \times D_j))_{j\in J}$ which are good neigborhood such that $(\psi^{-1}(U \times D_j))_{j\in J}$ is also a good neighborhood of same index. This reduces the proof to the same argument as the one of Proposition 15.

If X, Y are stratified spaces with finitely many strata, there is a natural stratification on the product $X \times Y$, given by $(X \times Y)_k := \bigcup_{i+j=k} X_i \times Y_j \subset X \times Y$. The natural projections on X and Y are adequatly stratified.

Corollary 6 *Let X, Y be stratified spaces with finitely many strata. The push-forward $\underline{\pi_1}_* : \boldsymbol{Fac}_{X\times Y} \longrightarrow \boldsymbol{Fac}_X(\boldsymbol{Fac}_Y)$ (see Proposition 18) induces a functor $\underline{\pi_1}_* : \boldsymbol{Fac}^{lc}_{X\times Y} \longrightarrow \boldsymbol{Fac}^{lc}_X(\boldsymbol{Fac}^{lc}_Y)$.*

We conjecture that $\underline{\pi_1}_*$ is an equivalence under rather weak conditions on X and Y. We will give a couple of examples.

Proof Since the projections are adequatly stratified, the result follows from Proposition 18 together with Proposition 26 applied to both projections (on X and Y).

Remark 28 Let $\mathcal{F}$ be a stratified locally constant factorization algebra on X. Let $U \subset V$ be stratified disks of same index i, but not necessarily good neighborhoods at X_i. Assume all connected component of $V \cap X_i$ contains exactly one connected component of $U \cap X_i$. Then the structure maps $\mathcal{F}(U) \to \mathcal{F}(V)$ is a quasi-isomorphism. Indeed, we can take a factorizing cover $\mathcal{V}$ of V by good neighborhoods D such that $D \cap U$ are good neigborhoods. Then, $\mathcal{F}(D \cap U) \to \mathcal{F}(D)$ is a quasi-isomorphism and thus we get a quasi-isomorphism $\check{C}(\mathcal{F}, \mathcal{V} \cap U) \overset{\simeq}{\to} \check{C}(\mathcal{F}, \mathcal{V})$.

6.2 Factorization Algebras on the Interval and (bi)modules

Let us consider an important example: the closed interval $I = [0, 1]$ viewed as a stratified space[50] with two dimension 0-strata given by $I_0 = \{0, 1\}$.

The disks at I_0 are the half-closed intervals $[0, s)$ $(s < 1)$ and $(t, 1]$ $(0 < t)$ and the disks at I_1 are the open intervals (t, u) $(0 < t < u < 1)$. The disks of (the stratified space) I form a (stable by finite intersection) factorizing basis denoted $\mathcal{I}$.

An example of stratified locally constant factorization algebra on I is obtained as follows. Let A be a differential graded associative unital algebra, M^r a pointed differential graded right A-module (with distinguished element denoted $m^r \in M^r$) and M^ℓ a pointed differential graded left A-module (with distinguished element $m^\ell \in M^\ell$). We define a $\mathcal{I}$-prefactorization algebra by setting, for any interval $J \in \mathcal{I}$

$$\mathcal{F}(J) := \begin{cases} M^r \text{ if } 0 \in J \\ M^\ell \text{ if } 1 \in J \\ A \text{ else.} \end{cases}$$

We define its structure maps to be given by the following[51]:

- $\mathcal{F}(\emptyset) \to \mathcal{F}([0, s))$ is given by $k \ni 1 \mapsto m^r$, $\mathcal{F}(\emptyset) \to \mathcal{F}((t, s))$ is given by $1 \mapsto 1_A$ and $\mathcal{F}(\emptyset) \to \mathcal{F}((t, 1]))$ is given by $k \ni 1 \mapsto m^\ell$;
- For $0 < s < t_1 < u_1 < \cdots < t_i < u_i < v < 1$ one sets

$$\begin{aligned} M^r \otimes A^{\otimes i} = \mathcal{F}([0, s)) \otimes \mathcal{F}((t_1, u_1)) \otimes \cdots \otimes \mathcal{F}((t_i, u_i)) &\longrightarrow \mathcal{F}([0, v)) = M^r \\ m \otimes a_1 \otimes \cdots \otimes a_i &\longmapsto m \cdot a_1 \cdots a_i\ ; \end{aligned}$$

$$\begin{aligned} A^{\otimes i} \otimes M^\ell = \mathcal{F}((t_1, u_1)) \otimes \cdots \otimes \mathcal{F}((t_i, u_i)) \otimes \mathcal{F}((v, 1]) &\longrightarrow \mathcal{F}((s, 1]) = M^\ell \\ a_1 \otimes \cdots \otimes a_i \otimes n &\longmapsto a_1 \cdots a_i \cdot n\ ; \end{aligned}$$

and also

$$\begin{aligned} A^{\otimes i} = \mathcal{F}((t_1, u_1)) \otimes \cdots \otimes \mathcal{F}((t_i, u_i)) &\longrightarrow \mathcal{F}((s, v)) = A \\ a_1 \otimes \cdots \otimes a_i &\longmapsto a_1 \cdots a_i\ . \end{aligned}$$

It is straightforward to check that $\mathcal{F}$ is a $\mathcal{I}$-prefactorization algebra and, by definition, it satisfies the locally constant condition.

Proposition 27 shows that $\mathcal{F}$ is indeed a locally constant factorization algebra on the closed interval I. Further, *any* locally constant factorization algebra on I is (homotopy) equivalent to such a factorization algebra.

[50] Note that this stratification is just given by looking at [0, 1] as a manifold with boundary.

[51] As in Example 27, we use the implicit orientation of I given by increasing numbers.

Also note that the $\mathcal{I}$-prefactorization algebra induced by $\mathcal{F}$ on the open interval $(0, 1)$ is precisely the $\mathcal{I}$-prefactorization algebra constructed in Example 27 (up to an identification of $(0, 1)$ with $\mathbb{R}$). We denote it $\mathcal{F}_A$.

Proposition 27 *Let $\mathcal{F}$, $\mathcal{F}_A$ be defined as above.*

1. *The $\mathcal{I}$- prefactorization algebra $\mathcal{F}$ is an $\mathcal{I}$-factorization algebra hence extends uniquely into a factorization algebra (still denoted) $\mathcal{F}$ on the stratified closed interval $I = [0, 1]$;*
2. *in particular, $\mathcal{F}_A$ also extends uniquely into a factorization algebra (still denoted) $\mathcal{F}_A$ on $(0, 1)$.*
3. *There is an equivalence $\int_{[0,1]} \mathcal{F} = \mathcal{F}([0, 1]) \cong M^r \overset{\mathbb{L}}{\underset{A}{\otimes}} M^\ell$ in $\textbf{\textit{Chain}}(k)$.*
4. *Moreover, any locally constant factorization algebra $\mathcal{G}$ on $I = [0, 1]$ is equivalent[52] to $\mathcal{F}$ for some A, M^ℓ, M^r, that is, it is uniquely determined by an E_1-algebra $\mathcal{A}$ and pointed left module $\mathcal{M}^\ell$ and pointed right module $\mathcal{M}^r$ satisfying*

$$\mathcal{G}([0, 1)) \cong \mathcal{M}^r, \quad \mathcal{G}((0, 1]) \cong \mathcal{M}^\ell, \quad \mathcal{G}((0, 1)) \cong \mathcal{A}$$

with structure maps given by the E_1-structure similarly to those of $\mathcal{F}$.

For a proof, see Sect. 9.2.1. The last statement restricted to the open interval $(0, 1)$ is just Theorem 9 (in the case $n = 1$).

Example 33 We consider the closed half-line $[0, +\infty)$ as a stratified manifold, with strata $\{0\} \subset [0, +\infty)$ given by its boundary. Namely, it has a 0-dimensional strata given by $\{0\}$ and thus one dimension 1 strata $(0, +\infty)$. Similarly, there is a stratified closed half-line $(-\infty, 0]$. From Proposition 27 (and its proof) we also deduce

Proposition 28 *There is an equivalence of ∞-categories between locally constant factorization algebra on the closed half-line $[0, +\infty)$ and the category E_1**-RMod** of (pointed) right modules over E_1-algebras.*[53] *This equivalence sits in a commutative diagram*

$$\begin{array}{ccc}
\textbf{\textit{Fac}}^{lc}_{[0,+\infty)} & \xrightarrow{\ \cong\ } & E_1\textbf{\textit{-RMod}} \\
\downarrow & & \downarrow \\
\textbf{\textit{Fac}}^{lc}_{(0,+\infty)} & \xrightarrow{\ \cong\ } & E_1\textbf{\textit{-Alg}}
\end{array}$$

[52] More precisely, taking A, M^ℓ, M^r be strictification of $\mathcal{A}$, $\mathcal{M}^\ell$ and $\mathcal{M}^r$, there is a quasi-isomorphism of factorization algebras from $\mathcal{F}$ (associated to A, M^ℓ, M^r) to $\mathcal{G}$.

[53] Which, informally, is the category of pairs (A, M^r) where A is an E_1-algebra and M^r a (pointed) right A-module.

where the left vertical functor is given by restriction to the open line and the lower horizontal functor is given by Theorem 9.

*There is a similar equivalence (and diagram) of ∞-categories between locally constant factorization algebra on the closed half-line $(-\infty, 0]$ and the category E_1-**LMod** of (pointed) left modules over E_1-algebras.*

Let X be a manifold and consider the stratified manifold $X \times [0, +\infty)$, with a $\dim(X)$ open strata $X \times \{0\}$. Using Corollary 6 and Proposition 28 one can get

Corollary 7 *The pushforward along the projection $X \times [0, +\infty) \to [0, +\infty)$ induces an equivalence*

$$\mathbf{Fac}^{lc}_{X\times[0,+\infty)} \xrightarrow{\simeq} E_1\text{-}\mathbf{RMod}(\mathbf{Fac}^{lc}_X).$$

6.3 Factorization Algebras on Pointed Disk and E_n-modules

In this section we relate E_n-modules[54] and factorization algebras over the pointed disk.

Let $\mathbb{R}^n_*$ *denote the pointed disk* which we see as a stratified manifold with one 0-dimensional strata given by the point $0 \in \mathbb{R}^n$ and n-dimensional strata given by the complement $\mathbb{R}^n \setminus \{0\}$.

Definition 22 We denote $\mathbf{Fac}^{lc}_{\mathbb{R}^n_*}$ the ∞*-category of locally constant factorization algebras on the pointed disk* $\mathbb{R}^n_*$ (in the sense of Definition 21).

Recall the functor (39) giving the obvious embedding $\mathbf{Fac}^{lc}_{\mathbb{R}^n} \longrightarrow \mathbf{Fac}^{lc}_{\mathbb{R}^n_*}$.

Locally constant factorization algebras on $\mathbb{R}^n_*$ are related to those on the closed half-line (Sect. 6.2) as follows: let $N : \mathbb{R}^n \to [0, +\infty)$ be the euclidean norm map $x \mapsto \|x\|$. We have the pushforwards $N_* : \mathbf{Fac}_{\mathbb{R}^n_*} \to \mathbf{Fac}_{[0,+\infty)}$ and $(-N)_* : \mathbf{Fac}_{\mathbb{R}^n_*} \to \mathbf{Fac}_{(-\infty,0]}$.

Lemma 7 *If $\mathcal{F} \in \mathbf{Fac}^{lc}_{\mathbb{R}^n_*}$, then $N_*(\mathcal{F}) \in \mathbf{Fac}^{lc}_{[0,+\infty)}$ and $(-N)_*(\mathcal{F}) \in \mathbf{Fac}^{lc}_{(-\infty,0]}$.*

Proof For $0 < \varepsilon < \eta$, the structure map $N_*(\mathcal{F})\big([0,\varepsilon)\big) \cong \mathcal{F}\big(N^{-1}([0,\varepsilon))\big) \to \mathcal{F}\big(N^{-1}([0,\eta))\big) \cong N_*(\mathcal{F})\big([0,\eta)\big)$ is an equivalence since $\mathcal{F}$ is locally constant and $N^{-1}([0,\alpha)$ is a euclidean disk centered at 0. Further, by Proposition 15, $N_*(\mathcal{F}_{|\mathbb{R}^n\setminus\{0\}})$ is locally constant from which we deduce that $N_*(\mathcal{F})$ is locally constant on the stratified half-line $[0, +\infty)$. The case of $(-N)_*$ is the same. □

Our next task is to define a functor E_n-**Mod** $\to \mathbf{Fac}^{lc}_{\mathbb{R}^n_*}$ from (pointed) E_n-modules (see Appendix 10.2) to (locally constant) factorization algebras on the pointed disk. It is enough to associate (functorially), to any $M \in E_n$-**Mod**, a $\mathcal{CV}(\mathbb{R}^n)$-factorization algebra $\mathcal{F}_M$ where $\mathcal{CV}(\mathbb{R}^n)$ is the (stable by finite intersection) factorizing basis of

[54] According to our convention in Appendix 10.2, all E_n-modules are pointed by definition.

$\mathbb{R}^n$ of convex open subsets. It turns out to be easy: since any convex subset C is canonically an embedded framed disk, we can set $\mathcal{F}_M(C) := M(C)$. In other words, we assign the module to a convex neighborhood of 0 and the algebra to a convex neigborhood which does not contain the origin.

Then set the structure maps $\mathcal{F}_M(C_1)\otimes\cdots\otimes\mathcal{F}_M(C_i) \to \mathcal{F}_M(D)$, for any pairwise disjoint convex subsets C_k of D, to be given by the map $M(C_1) \otimes \cdots \otimes M(C_i) \to M(D)$ associated to the framed embedding $\coprod_{k=1\cdots i} \mathbb{R}^n \cong \bigcup_{k=1\cdots i} C_k \hookrightarrow D \hookrightarrow \mathbb{R}^n$.

Theorem 12 *The rule $M \mapsto \mathcal{F}_M$ induces a fully faithful functor $\psi : E_n$-**Mod** $\to$ **Fac**$^{lc}_{\mathbb{R}^n_*}$ which fits in a commutative diagram*

$$\begin{array}{ccc} E_n\text{-}\boldsymbol{Alg} & \xrightarrow{\simeq} & \boldsymbol{Fac}^{lc}_{\mathbb{R}^n} \\ \Big\downarrow{\scriptstyle can} & & \Big\downarrow \\ E_n\text{-}\boldsymbol{Mod} & \xrightarrow{\psi} & \boldsymbol{Fac}^{lc}_{\mathbb{R}^n_*}. \end{array}$$

*Here $can : E_n$-**Alg** $\to E_n$-**Mod** is given by the canonical module structure of an algebra over itself.*

We will now identify E_n-modules in terms of factorization algebras on $\mathbb{R}^n_*$; that is the essential image of the functor $\psi : E_n\text{-}\mathbf{Mod} \to \mathbf{Fac}^{lc}_{\mathbb{R}^n_*}$ given by Theorem 12. Recall the functor $\pi_{E_n} : E_n\text{-}\mathbf{Mod} \to E_n\text{-}\mathbf{Alg} \cong \mathbf{Fac}^{lc}_{\mathbb{R}^n}$ which, to a module $M \in E_n\text{-}\mathbf{Mod}_A$, associates $\pi_{E_n}(M) = A$.

By restriction to the open set $\mathbb{R}^n \setminus \{0\}$ we get that the two compositions of functors

$$E_n\text{-}\mathbf{Mod} \xrightarrow{\psi} \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \to \mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}} \quad \text{and} \quad E_n\text{-}\mathbf{Mod} \xrightarrow{\pi_{E_n}} E_n\text{-}\mathbf{Alg} \cong \mathbf{Fac}^{lc}_{\mathbb{R}^n} \to \mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}$$

are equivalent. Hence, we get a factorization of (ψ, π_{E_n}) to the pullback

$$(\psi, \pi_{E_n}) : E_n\text{-}\mathbf{Mod} \longrightarrow \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times^h_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \mathbf{Fac}^{lc}_{\mathbb{R}^n}.$$

Informally, $\mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times^h_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \mathbf{Fac}^{lc}_{\mathbb{R}^n}$ is simply the (∞-)category of pairs $(\mathcal{A}, \mathcal{M}) \in \mathbf{Fac}^{lc}_{\mathbb{R}^n} \times \mathbf{Fac}^{lc}_{\mathbb{R}^n_*}$ together with a quasi-isomorphism $f : \mathcal{A}_{|\mathbb{R}^n\setminus\{0\}} \to \mathcal{M}_{|\mathbb{R}^n\setminus\{0\}}$ of factorization algebras.

Corollary 8 *The functor E_n-**Mod** $\xrightarrow{(\psi,\pi_{E_n})}$ $\boldsymbol{Fac}^{lc}_{\mathbb{R}^n_*} \times^h_{\boldsymbol{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \boldsymbol{Fac}^{lc}_{\mathbb{R}^n}$ is an equivalence.*

Now, if A is an E_n-algebra, we can see it as a factorization algebra on $\mathbb{R}^n$ (Theorem 9) and taking the (homotopy) fiber at $\{A\}$ of the right hand side of the equivalence in Corollary 8, we get

Corollary 9 *The functor E_n-**Mod**$_A \xrightarrow{\psi}$ **Fac**$^{lc}_{\mathbb{R}^n_*} \times^h_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \{A\}$ is an equivalence.*

Example 34 (*Locally constant factorization algebra on the pointed line $\mathbb{R}_*$*) The case of the pointed line $\mathbb{R}_*$ is slightly special since it has two (and not one) strata of maximal dimension. The two embeddings $j_+ : (0, +\infty) \hookrightarrow \mathbb{R}$ and $j_- : (-\infty, 0) \hookrightarrow \mathbb{R}$ yields two restrictions functors: $j^*_\pm : \mathbf{Fac}^{lc}_{\mathbb{R}_*} \to \mathbf{Fac}^{lc}_{\mathbb{R}}$. Hence $\mathcal{F} \in \mathbf{Fac}^{lc}_{\mathbb{R}_*}$ determines two E_1-algebras $R \cong \mathcal{F}\big((0, +\infty)\big)$ and $L \cong \mathcal{F}\big((-\infty, 0)\big)$.

By Lemma 7, the pushforward $(-N)_* : \mathbf{Fac}^{lc}_{\mathbb{R}_*} \to \mathbf{Fac}^{lc}_{(-\infty,0]}$ along $-N : x \mapsto -|x|$ is well defined. Then Proposition 28 implies that $(-N)_*(\mathcal{F})$ is determined by a left module M over the E_1-algebra $A \cong (-N)_*(\mathcal{F})\big((0, +\infty)\big) \cong L \otimes R^{op}$, i.e. by a (L, R)-bimodule.

We thus have a functor $(j^*_\pm, (-N)_*) : \mathbf{Fac}^{lc}_{\mathbb{R}_*} \to$ **BiMod** where **BiMod** is the ∞-category of bimodules (in **Chain**(k)) defined in [71, Sect. 4.3] (i.e. the ∞-category of triples (L, R, M) where L, R are E_1-algebras and M is a (L, R)-bimodule).

Proposition 29 *The functor $(j^*_\pm, (-N)_*) : \mathbf{Fac}^{lc}_{\mathbb{R}_*} \cong$ **BiMod** is an equivalence.*

Example 35 (*From pointed disk to the n-dimensional annulus $S^{n-1} \times \mathbb{R}$*) Let $\mathcal{M}$ be a locally constant factorization algebra on the pointed disk $\mathbb{R}^n_*$. By the previous results in this Section (specifically Lemma 7 and Proposition 28), for any $\mathcal{A} \in \mathbf{Fac}^{lc}_{S^{n-1}\times\mathbb{R}}$, the pushforward along the euclidean norm $N : \mathbb{R}^n \to [0, +\infty)$ factors as a functor

$$N_* : \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \{\mathcal{A}\} \longrightarrow E_1\textbf{-RMod}_{\mathcal{A}(S^{n-1}\times\mathbb{R})}$$

from the category of locally constant factorization algebras on the pointed disk whose restriction to $\mathbb{R}^n \setminus \{0\}$ is (quasi-isomorphic to) $\mathcal{A}$ to the category of right modules over the E_1-algebra[55] $\mathcal{A}(S^{n-1} \times \mathbb{R}) \cong \int_{S^{n-1}\times\mathbb{R}} \mathcal{A}$. Similarly, we have the functor $(-N)_* : \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \{\mathcal{A}\} \longrightarrow E_1\textbf{-LMod}_{\mathcal{A}(S^{n-1}\times\mathbb{R})}$.

Proposition 30 *Let $\mathcal{A}$ be in $\mathbf{Fac}^{lc}_{S^{n-1}\times\mathbb{R}}$.*

- *The functor $N_* : \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \{\mathcal{A}\} \longrightarrow E_1\textbf{-RMod}_{\mathcal{A}(S^{n-1}\times\mathbb{R})}$ is an equivalence of ∞-categories.*
- *$(-N)_* : \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \{\mathcal{A}\} \longrightarrow E_1\textbf{-LMod}_{\mathcal{A}(S^{n-1}\times\mathbb{R})}$ is an equivalence.*

See Sect. 9.3 for a Proof.

The Proposition allows to reduce a category of "modules" over $\mathcal{A}$ to a category of modules over an (homotopy) dg-*associative* algebra.

We will see in Sect. 7.1 that it essentially gives a concrete description of the enveloping algebra of an E_n-algebra.

[55] The E_1-structure is given by Lemma 5.

6.4 More Examples

We now give, without going into any details, a few more examples of locally constant factorization algebras on stratified spaces which can be studied along the same way as we did previously.

Example 36 (*Towards quantum mechanics*) One can make the following variant of the interval example of Proposition 27 (see [24] for details). Fix a one parameter group $(\alpha_t)_{t\in[0,1]}$ of invertible elements in an associative (topological) algebra A. Now, we define a factorization algebra on the basis $\mathcal{I}$ of open disks of $[0, 1]$ in the same way as in Proposition 27 except that we add the element α_d to any hole of length d between two intervals (corresponding to the inclusion of the empty set: $\mathcal{F}(\emptyset) \to \mathcal{F}(I)$ into any open interval). Precisely, we set:

- the structure map $k = \mathcal{F}(\emptyset) \to \mathcal{F}\big(s, t)\big) = A$ is given by the element α_{t-s}, the structure map $k = \mathcal{F}(\emptyset) \to \mathcal{F}\big[0, t)\big) = M^r\ m^r \cdot \alpha_t$ and the structure map $k = \mathcal{F}(\emptyset) \to \mathcal{F}\big(u, 1]\big) = M^\ell\ \alpha_{1-u} \cdot m^\ell$.
- For $0 < s < t_1 < u_1 < \cdots < t_i < u_i < v < 1$ one sets the map

$$M^r \otimes A^{\otimes i} = \mathcal{F}([0, s)) \otimes \mathcal{F}((t_1, u_1)) \otimes \cdots \mathcal{F}((t_i, u_i)) \longrightarrow \mathcal{F}([0, v)) = M^r$$

  to be given by

$$m \otimes a_1 \otimes \cdots \otimes a_i \longmapsto m \cdot \alpha_{t_1-s} a_1 \alpha_{t_2-u_1} a_2 \cdots \alpha_{t_i-u_{i-1}} a_i \alpha_{v-u_i},$$

  the map

$$A^{\otimes i} \otimes M^\ell = \mathcal{F}((t_1, u_1)) \otimes \cdots \mathcal{F}((t_i, u_i)) \otimes \mathcal{F}((v, 1]) \longrightarrow \mathcal{F}((s, 1]) = M^\ell$$

  to be given by $a_1 \otimes \cdots \otimes a_i \otimes n \longmapsto \alpha_{t_1-s} a_1 \alpha_{t_2-u_1} a_2 \cdots \alpha_{t_i-u_{i-1}} a_i \alpha_{v-u_i} n$, and also the map $A^{\otimes i} = \mathcal{F}((t_1, u_1)) \otimes \cdots \otimes \mathcal{F}((t_i, u_i)) \longrightarrow \mathcal{F}((s, v)) = A$ to be given by

$$a_1 \otimes \cdots \otimes a_i \longmapsto \alpha_{t_1-s} a_1 \alpha_{t_2-u_1} a_2 \cdots \alpha_{t_i-u_{i-1}} a_i \alpha_{v-u_i}.$$

One can check that these structure maps define a $\mathcal{I}$-factorization algebra and thus a factorization algebra on I.

One can replace chain complexes with topological $\mathbb{C}$-vector spaces (with monoidal structure the completed tensor product) and take V to be an Hilbert space. Then one can choose $A = \big(End^{cont}(V)\big)^{op}$ and $M^r = V = M^\ell$ where the left A-module structure on M^ℓ is given by the action of adjoint operators. Let $\alpha_t := e^{it\varphi}$ where $\varphi \in End^{cont}(V)$. One has $\mathcal{F}([0, 1]) = \mathbb{C}$. One can think of A as the algebra of observables where V is the space of states and $e^{it\varphi}$ is the time evolution operator. Now, given two consecutives measures O_1, O_2 made during the time intervals $]s, t[$ and $]u, v[$ $(0 < s < t < u < v < 1)$, the probability amplitude that the system goes from an initial state v^r to a final state v^ℓ is the image of $O_1 \otimes O_2$ under the structure map

$$A \otimes A = \mathcal{F}(]s, t[) \otimes \mathcal{F}(]u, v[) \longrightarrow \mathcal{F}([0, 1]) = \mathbb{C},$$

and is usually denoted $\langle v^\ell | e^{i(1-v)\varphi} O_2 e^{i(u-t)\varphi} O_1 e^{is\varphi} | v^r \rangle$.

Example 37 (*The upper half-plane*) Let $H = \{z = x + iy \in \mathbb{C}, y \geq 0\}$ be the closed upper half-plane, viewed as a stratified space with $H_0 = \emptyset$, $H_1 = \mathbb{R}$ the real line as dimension 1 strata and $H_2 = H$. The orthogonal projection $\pi : H \to \mathbb{R}$ onto the imaginary axis $\mathbb{R}i \subset \mathbb{C}$ induces, by Propositions 6 and 28), an equivalence

$$\mathbf{Fac}^{lc}_H \xrightarrow{\simeq} \mathbf{Fac}^{lc}_{[0,+\infty)}(\mathbf{Fac}^{lc}_{\mathbb{R}}) \cong E_1\text{-}\mathbf{RMod}(E_1\text{-}\mathbf{Alg}).$$

Using Dunn Theorem 11, we have (sketched a proof of the fact) that

Proposition 31 *The ∞-category of stratified locally constant factorization algebra on H is equivalent to the ∞-category of algebras over the swiss cheese operad, that is the ∞-category consisting of triples (A, B, ρ) where A is an E_2-algebra, B an E_1-algebra and $\rho : A \to \mathbb{R}Hom_B^{E_1}(B, B)$ is an action of A on B compatible with all the multiplications.*[56]

One can also consider another stratification $\tilde{H}$ on $\mathbb{R} \times [0, +\infty)$ given by adding a 0-dimensional strata to H, given by the point $0 \in \mathbb{C}$. There is now 4 kinds of (stratified) disks in $\tilde{H}$: the half-disk containing 0, the half disk with a connected boundary component lying on $(-\infty, 0)$, the half-disk containing 0, the half disk with a connected boundary component lying on $(0, +\infty)$ and the open disks in the interior $\{x + iy,\ y > 0\}$ of H.

One proves similarly

Proposition 32 *Locally constant factorization algebras on the stratified space $\tilde{H}$ are the same as the category given by quadruples $(M, A, B, \rho_A, \rho_B, E)$ where E is an E_2-algebra, (A, ρ_A), (B, ρ_B) are E_1-algebras together with a compatible action of E and (M, ρ_M) is a (A, B)-bimodule together with a compatible action*[57] *of E.*

Examples of such factorization algebras occur in deformation quantization in the presence of two branes, see [15].

Note that the norm is again adequatly stratified so that, if $\mathcal{F} \in \mathbf{Fac}^{lc}_{\tilde{H}}$, then $(-N)_*(\mathcal{F}) \in \mathbf{Fac}^{lc}\big((-\infty, 0]\big) \cong E_1\text{-}\mathbf{LMod}$. Using the argument of Proposition 30 and Proposition 27, we see that if $\mathcal{F}$ is given by a tuple $(M, A, B, \rho_A, \rho_B, E)$, then the underlying E_1-algebra of $(-N)_*(\mathcal{F})$ is (the two-sided Bar construction) $A \otimes^{\mathbb{L}}_E B^{op}$.

---

[56] That is the map ρ is a map of E_2-algebras where $\mathbb{R}\mathrm{Hom}_B^{E_1}(B, B)$ is the E_2-algebra given by the (derived) center of B. In particular, ρ induces a map $\rho(1_B) : A \to B$.

[57] Precisely, this means the choice of a factorization $A \otimes B^{op} \longrightarrow A \overset{\mathbb{L}}{\underset{E}{\otimes}} B^{op} \xrightarrow{\rho_M} \mathbb{R}\mathrm{Hom}(M, M)$ (in E_1-**Alg**) of the (A, B)-bimodule structure of M.

Example 38 (*The unit disk in* $\mathbb{C}$) Let $D = \{z \in \mathbb{C},\ |z| \leq 1\}$ be the closed unit disk (with dimension 1 strata given by its boundary). By Lemma 7, the norm gives us a pushforward functor $N_* : \mathbf{Fac}^{lc}_D \to \mathbf{Fac}^{lc}_{[0,1]}$. A proof similar to the one of Corollary 7 and Theorem 12 shows that the two restrictions of N_* to $D \setminus \partial D$ and $D \setminus \{0\}$ induces an equivalence

$$\mathbf{Fac}^{lc}_D \xrightarrow{\simeq} \mathbf{Fac}^{lc}_{D\setminus\partial D} \times^h_{\mathbf{Fac}^{lc}_{(0,1)}} E_1\text{-}\mathbf{RMod}\big(\mathbf{Fac}^{lc}_{S^1}\big).$$

By Corollary 4 and Theorem 9, one obtains:

Proposition 33 *The ∞-category of locally constant factorization algebra on the (stratified) closed unit disk D is equivalent to the ∞-category consisting of quadruples (A, B, ρ, f) where A is an E_2-algebra, B an E_1-algebra, $\rho : A : \mathbb{R}Hom_{E_1\text{-}\boldsymbol{Alg}}(B, B)$ is an action of A on B compatible with all the multiplications and $f : B \to B$ is a monodromy compatible with ρ.*

One can also consider a variant of this construction with dimension 0 strata given by the center of the disk. In that case, one has to add a E_2-A-module to the data in Proposition 33.

Example 39 (*The closed unit disk* $\mathbb{R}^n$) Let D^n be the closed unit disk of $\mathbb{R}^n$ which is stratified with a single strata of dimension $n-1$ given by its boundary $\partial D^n = S^{n-1}$. We have restriction functors $\mathbf{Fac}^{lc}_{D^n} \longrightarrow \mathbf{Fac}^{lc}_{D^n\setminus\partial D^n} \cong E_n\text{-}\mathbf{Alg}$ (Theorem 9),

$$E_n\text{-}\mathbf{Alg} \cong \mathbf{Fac}^{lc}_{D^n\setminus\partial D^n} \to \mathbf{Fac}^{lc}_{S^{n-1}\times(0,1)} \cong E_1\text{-}\mathbf{Alg}(\mathbf{Fac}^{lc}_{S^{n-1}}) \quad \text{(by Proposition 18)}$$

and $\mathbf{Fac}^{lc}_{D^n} \longrightarrow \mathbf{Fac}^{lc}_{D^n\setminus\{0\}}$. From Corollary 7, we deduce

Proposition 34 *The above restriction functors induce an equivalence*

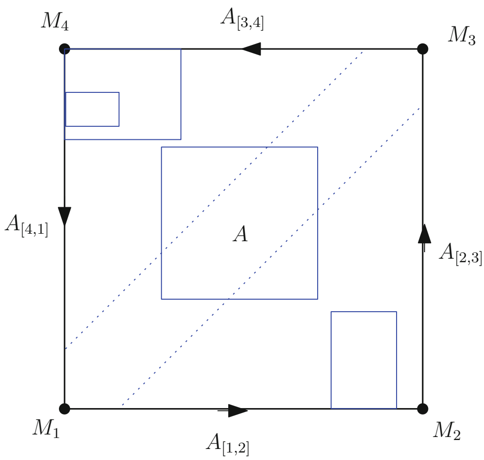
$$\boldsymbol{Fac}^{lc}_{D^n} \xrightarrow{\simeq} E_n\text{-}\boldsymbol{Alg} \times_{E_1\text{-}\boldsymbol{Alg}\left(\boldsymbol{Fac}^{lc}_{S^{n-1}}\right)} E_1\text{-}\boldsymbol{LMod}\big(\boldsymbol{Fac}^{lc}_{S^{n-1}}\big).$$

Let $f : A \to B$ be an E_n-algebra map. Since $S^{n-1} \times \mathbb{R}$ has a canonical framing, both A and B carries a structure of locally constant factorization algebra on S^{n-1}, which is induced by pushforward along the projection $q : S^{n-1} \times \mathbb{R} \to S^{n-1}$ (see Example 26). Further, B inherits an E_n-module structure over A induced by f and, by restriction, $q_*(B_{|S^{n-1}\times\mathbb{R}})$ is an E_1-module over $q_*(A_{|S^{n-1}\times\mathbb{R}})$. Thus, Proposition 34 yields a factorization algebra $\omega_{D^n}(A \stackrel{f}{\to} B) \in \mathbf{Fac}^{lc}_{D^n}$.

Proposition 35 *The map ω_{D^n} induces a faithful functor $\omega_{D^n} : Hom_{E_n\text{-}\boldsymbol{Alg}} \longrightarrow \boldsymbol{Fac}^{lc}_{D^n}$ where $Hom_{E_n\text{-}\boldsymbol{Alg}}$ is the ∞-groupoid of E_n-algebras maps.*

By the relative higher Deligne conjecture (Theorem 16), we also have the E_n-algebra $HH_{E_n}(A, B_f)$ and an E_n-algebra map $HH_{E_n}(A, B_f) \to B$ given as the composition

Fig. 3 The stratification of the *square*. The (oriented) edges and vertices are decorated with their associated algebras and modules. Also 4 *rectangles* which are good neighborhoods at respectively 1, [4, 1], [1, 2] and the dimension 2-strata are depicted in *blue*. The band D_{13} is delimited by the *dotted lines* while the annulus T is the complement of the interior *blue* square



$$HH_{E_n}(A, B_f) \cong \mathfrak{z}(f) \stackrel{id\otimes 1_A}{\longrightarrow} \mathfrak{z}(f) \otimes \mathfrak{z}(k \stackrel{1_A}{\to} A) \longrightarrow \mathfrak{z}(k \stackrel{1_B}{\to} B) \cong B.$$

Thus by Proposition 35, the pair $(HH_{E_n}(A, B_f), B)$ also defines a factorization algebra on D^n.

Let $\tau : D^n \to \mathbb{R}^n \cup \{\infty\}$ be the map collapsing ∂D^n to the point ∞. It is an adequatly stratified map if we see $\mathbb{R}^n \cup \{\infty\}$ as being stratified with one dimension 0 strata given by ∞. Denote $\widehat{\mathbb{R}^n}$ this stratified manifold. The composition of τ_* with ω_{D^n} gives us:

Corollary 10 *There is a faithful functor* $\widehat{\omega} : Hom_{E_n\text{-}\boldsymbol{Alg}} \longrightarrow \boldsymbol{Fac}^{lc}_{\widehat{\mathbb{R}^n}}$.

Example 40 (*The square*) Consider the square $I^2 := [0, 1]^2$, the product of the interval (with its natural stratification of Sect. 6.2) with itself. This stratification agrees with the one given by seeing I^2 as a manifold with corners. Thus, there are four 0-dimensional strata corresponding to the vertices of I^2 and four 1-dimensional strata corresponding to the edges. Let us denote $\{1, 2, 3, 4\}$ the set of vertices and $[i, i+1]$ the corresponding edge linking the vertices i and $i+1$ (ordered cyclically[58]). See Fig. 3.

We can construct a factorization algebra on I^2 as follows. For every edge $[i, j]$ of I^2, let $A_{[i,j]}$ be an E_1-algebra; also let A be an E_2-algebra. For every vertex i, let M_i be a $(A_{[i-1,i]}, A_{[i,i+1]})$-bimodule. Finally, assume that A acts on each $A_{[i,j]}$ and M_k in a compatible way with algebras and module structures. Compatible, here, means that for every vertex i, the data $(M_i, A_{[i-1,i]}, A_{[i,i+1]}, A)$ (together with the various module structures) define a locally constant stratified factorization algebra on a neighborhood[59] of the vertex i as given by Proposition 32.

[58] In other words we have the 4 edges [1, 2], [2, 3], [3, 4] and [4, 1].

[59] For instance, take the complement of the closed sub-triangle of I^2 given by the three other vertices, which is isomorphic as a stratified space to the pointed half plane $\tilde{H}$ of Example 37.

Using Propositions 27 and 29 (and their proofs) and Remark 23, we see that we obtain a locally constant factorization algebra $\mathcal{I}$ on I^2 whose value on an open rectangle $R \subset I^2$ is given by

$$\mathcal{I}(R) := \begin{cases} M_i & \text{if } R \text{ is a good neighborhood at the vertex } i; \\ A_{[i,i+1]} & \text{if } R \text{ is a good neighborhood at the edge } [i, i+1]; \\ A & \text{if } R \text{ lies in } I^2 \setminus \partial I^2. \end{cases} \tag{40}$$

The structure maps being given by the various module and algebras structure.[60]

By Corollary 6, we get the functor $\underline{\pi_1}_* : \mathbf{Fac}^{lc}_{[0,1]^2} \to \mathbf{Fac}^{lc}_{[0,1]}(\mathbf{Fac}^{lc}_{[0,1]})$. Combining the proof of Propositions 18 and 27 (similarly to the proof of Corollary 7), we get:

Proposition 36 *The functor $\underline{\pi_1}_* : \boldsymbol{Fac}^{lc}_{[0,1]^2} \to \boldsymbol{Fac}^{lc}_{[0,1]}(\boldsymbol{Fac}^{lc}_{[0,1]})$ is an equivalence of ∞-categories. Moreover, any stratified locally constant factorization algebra on I^2 is quasi-isomorphic to a factorization algebra associated to a tuple $(A, A_{[i,i+1]}, M_i, i = 1 \dots 4)$ as in the rule (40) above.*

Let us give some examples of computations of the global sections of $\mathcal{I}$ on various opens.

- The band $I^2 \setminus ([4, 1] \cup [2, 3])$ is isomorphic (as a stratified space) to $(0, 1) \times [0, 1]$. Then from the definition of factorization homology and Proposition 27(3), we get

$$\mathcal{I}\big(I^2 \setminus ([4, 1] \cup [2, 3])\big) \cong p_*\big(\mathcal{I}_{|I^2\setminus([4,1]\cup[2,3])}\big) \cong p_*(\underline{\pi_1}_*(\mathcal{I}_{|I^2\setminus([4,1]\cup[2,3])})$$
$$= \int_{[0,1]} \underline{\pi_1}_*(\mathcal{I}_{|I^2\setminus([4,1]\cup[2,3])}) \cong A_{[1,2]} \overset{\mathbb{L}}{\underset{A}{\otimes}} (A_{[3,4]})^{op}.$$

  In view of Corollary 7 in the case $X = \mathbb{R}$ (and Theorem 9); this is an equivalence of E_1-algebras.
- Consider a tubular neighborhood of the boundary, namely the complement[61] $T := I^2 \setminus [1/4, 3/4]^2$ (see Fig. 3). The argument of Proposition 27 and Example 15 shows (by projecting the square on $[0, 1] \times \{1/2\}$) that

$$\mathcal{I}(T) \cong \Big(M_4 \overset{\mathbb{L}}{\underset{A_{[4,1]}}{\otimes}} M_1\Big) \overset{\mathbb{L}}{\underset{A_{[1,2]}\otimes(A_{[3,4]})^{op}}{\bigotimes}} \Big(M_2 \overset{\mathbb{L}}{\underset{A_{[2,3]}}{\otimes}} M_3\Big).$$

- Now, consider a diagonal band, say a tubular neigborhood D_{13} of the diagonal linking the vertex 1 to the vertex 3 (see Fig. 3). Then projecting onto the diagonal $[1, 3]$ and using again Proposition 27, we find,

[60] Note that we orient the edges accordingly to the ordering of the edges.

[61] Note that the obvious radial projection $T \to \partial I^2$ is *not* adequatly stratified.

$$\mathcal{I}\big(D_{13}\big) \cong M_1 \overset{\mathbb{L}}{\underset{A}{\otimes}} M_3.$$

Iterating the above constructions to higher dimensional cubes, one finds

Proposition 37 *Let* $[0,1]^n$ *be the stratified cube. The pushforward along the canonical projections is an equivalence* $\boldsymbol{Fac}^{lc}_{[0,1]^n} \overset{\simeq}{\to} \boldsymbol{Fac}^{lc}_{[0,1]}\big(\cdots(\boldsymbol{Fac}^{lc}_{[0,1]})\cdots\big)$.

In other words, $\mathbf{Fac}^{lc}_{[0,1]^n}$ is a *tractable model* for an ∞-category consisting of the data of an E_n-algebra A_n together with E_{n-1}-algebras $A_{n-1,i_{n-1}}$ $(i_{n-1} = 1\dots 2n)$ equipped with an action of A_n, E_{n-2}-algebras $A_{n2,i_{n-2}}$ $(i_{n-2} = 1\dots 2n(n-1))$ each equipped with a structure of bimodule over 2 of the $A_{n-1,j}$ compatible with the A actions, …, E_k-algebras A_{k,i_k} $(i_k = 1\dots 2^{n-k}\binom{n}{k})$ equipped with structure of $n-k$-fold modules over ($n-k$ many of) the $A_{k+1,j}$ algebras, compatible with the previous actions, and so on ….

Similarly to the previous Example 39, we also have a faithful functor

$$E_n\textbf{-Alg} \cong \mathbf{Fac}^{lc}_{\mathbb{R}^n} \longrightarrow \mathbf{Fac}^{lc}_{[0,1]^n}.$$

Example 41 (*Iterated categories of (bi)modules*) We consider the n-fold product $(\mathbb{R}_*)^n$ of the pointed line (see Example 34) with its induced stratification. It has one 0-dimensional strata given by the origins, $2n$ many 1-dimensional strata given by half of the coordinate axis, …, and 2^n open strata.

Locally constant (stratified) factorization algebras on $(\mathbb{R}_*)^n$ are a model for iterated categories of bimodules objects. Indeed, by Corollary 6, the iterated first projections on $\mathbb{R}_*$ yields a functor $\underline{\pi}_* : \mathbf{Fac}^{lc}_{(\mathbb{R}_*)^n} \longrightarrow \mathbf{Fac}^{lc}_{\mathbb{R}_*}\Big(\mathbf{Fac}^{lc}_{\mathbb{R}_*}\big(\dots(\mathbf{Fac}^{lc}_{\mathbb{R}_*})\dots\big)\Big)$. From Proposition 29 (and its proof) combined with the arguments of the proofs of Corollary 7 and Proposition 30, we get

Corollary 11 *The functor*

$$\underline{\pi}_* : \boldsymbol{Fac}^{lc}_{(\mathbb{R}_*)^n} \longrightarrow \boldsymbol{Fac}^{lc}_{\mathbb{R}_*}\Big(\boldsymbol{Fac}^{lc}_{\mathbb{R}_*}\big(\dots(\boldsymbol{Fac}^{lc}_{\mathbb{R}_*})\dots\big)\Big) \cong \boldsymbol{BiMod}\Big(\boldsymbol{BiMod}\big(\dots(\boldsymbol{BiMod})\dots\big)\Big)$$

is an equivalence.

An interesting consequence arise if we assume that the restriction to $\mathbb{R}^n \setminus \{0\}$ of $\mathcal{F} \in \mathbf{Fac}^{lc}_{(\mathbb{R}_*)^n}$ is constant, that is in the essential image of the functor $\mathbf{Fac}^{lc}_{\mathbb{R}^n} \to \mathbf{Fac}^{lc}_{(\mathbb{R}_*)^n}$. In that case, $\mathcal{F}$ belongs to the essential image of $\mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \to \mathbf{Fac}^{lc}_{(\mathbb{R}_*)^n}$. Thus Corollaries 9 and 11 imply

Corollary 12 *There is a natural equivalence*

$$E_n\textbf{-}\boldsymbol{Mod} \cong E_1\textbf{-}\boldsymbol{Mod}\Big(E_1\textbf{-}\boldsymbol{Mod}\big(\dots(E_1\textbf{-}\boldsymbol{Mod})\dots\big)\Big)$$

inducing an equivalence

$$E_n\text{-}\boldsymbol{Mod}_A \cong E_1\text{-}\boldsymbol{Mod}_A\Big(E_1\text{-}\boldsymbol{Mod}_A\big(\dots(E_1\text{-}\boldsymbol{Mod}_A)\dots\big)\Big)$$

between the relevant ∞*-subcategories.*[62]

Let us describe now informally $\mathbf{Fac}^{lc}_{(\mathbb{R}_*)^2}$. A basis of (stratified) disks is given by the convex open subsets. Such a subset is a good disk of index 0 if it is a neigborhood of the origin, of index 1 if it is in $\mathbb{R}^2 \setminus \{0\}$ and intersects one and only one half open coordinate axis. It is a good disk of index 2 if it lies in the complement of the coordinate axis. We can construct a factorization algebra on $(\mathbb{R}_*)^2$ as follows. Let E_i, $i = 1\dots 4$ be four E_2-algebras (labelled by the cyclically ordered four quadrants of $\mathbb{R}^2$). Let $A_{1,2}$, $A_{2,3}$, $A_{3,4}$ and $A_{4,1}$ be four E_1-algebras endowed, for each $i \in \mathbb{Z}/4\mathbb{Z}$, with a compatible (E_i, E_{i+1})-bimodule structure[63] on $A_{i,i+1}$. Also let M be a E_2-module over each E_i in a compatible way. Precisely, this means that M is endowed with a right action[64] of the E_1-algebra $\big(A_{4,1} \overset{\mathbb{L}}{\underset{E_1}{\otimes}} A_{1,2}\big) \overset{\mathbb{L}}{\underset{E_2\otimes(E_4)^{op}}{\bigotimes}} \big(A_{2,3} \overset{\mathbb{L}}{\underset{E_3}{\otimes}} A_{3,4}\big)$.

Similarly to previous examples (in particular Example 40), we see that we obtain a locally constant factorization algebra $\mathcal{M}$ on $(\mathbb{R}_*)^2$ whose value on an open rectangle is given by

$$\mathcal{M}(R) := \begin{cases} M & \text{if } R \text{ is a good neighborhood of the origin;} \\ A_{i,i+1} & \text{if } R \text{ is a good neighborhood of index 1} \\ & \quad \text{intersecting the quadrant labelled } i \text{ and } i+1; \\ E_i & \text{if } R \text{ lies in the interior of the } ith\text{-quadrant.} \end{cases} \tag{41}$$

The structure maps are given by the various module and algebras structures. As in Example 40, we get

Proposition 38 *Any stratified locally constant factorization algebra on* $(\mathbb{R}_*)^2$ *is quasi-isomorphic to a factorization algebra associated to a tuple* $(M, A_{i,i+1}, E_i,$ $i = 1\dots 4)$ *as in the rule* (41) *above.*

Example 42 (*Butterfly*) Let us give one of the most simple singular stratified example. Consider the "(semi-open) butterfly" that is the subspace $B := \{(x, y) \in \mathbb{R}^2 |\, |y| < |x|\} \cup \{(0, 0)\}$ of $\mathbb{R}^2$. B has a dimension 0 strata given by the origin and two open strata B_+, B_- of dimension 2 given respectively by restricting to those points $(x, y) \in B$ such that $x > 0$, resp. $x < 0$.

62 The A-E_1-module structure on the right hand side are taken along the various underlying E_1-structures of A obtained by projecting on the various component $\mathbb{R}$ of $\mathbb{R}^n$.

63 That is a map of E_2-algebras $E_i \otimes (E_{i+1})^{op} \to \mathbb{R}\mathrm{Hom}^{E_1}_{A_{i,i+1}}(A_{i,i+1}, A_{i,i+1})$.

64 In particular, it implies that the tuple $(M, A_{i-1,i}, A_{i,i+1}, E_i)$ defines a stratified locally constant factorization algebra on $\tilde{H}$ by Proposition 32.

The restriction to B_+ of a stratified locally constant factorization algebra on B is locally constant factorization over $B_+ \cong \mathbb{R}^2$ (hence is determined by an E_2-algebra). This way, we get the restriction functor $\mathrm{res}^* : \mathbf{Fac}^{lc}_B \to \mathbf{Fac}^{lc}_{\mathbb{R}^2} \times \mathbf{Fac}^{lc}_{\mathbb{R}^2}$.

Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection $(x, y) \mapsto x$ on the first coordinate. Then, by Corollary 6, we get a functor $\pi_* : \mathbf{Fac}^{lc}_B \to \mathbf{Fac}^{lc}_{\mathbb{R}_*}$ where $\mathbb{R}_*$ is the pointed line (Example 34). The restriction of π_* to $B_+ \cup B_-$ thus yields a the functor $\mathbf{Fac}^{lc}_{\mathbb{R}^2} \times \mathbf{Fac}^{lc}_{\mathbb{R}^2} \to \mathbf{Fac}^{lc}_{\mathbb{R}\setminus\{0\}}$.

A proof similar (and slightly easier) to the one of Proposition 45 shows that the induced functor

$$(\pi_*, \mathrm{res}^*) : \mathbf{Fac}^{lc}_B \longrightarrow \mathbf{Fac}^{lc}_{\mathbb{R}_*} \times^h_{\mathbf{Fac}^{lc}_{\mathbb{R}\setminus\{0\}}} (\mathbf{Fac}^{lc}_{\mathbb{R}^2} \times \mathbf{Fac}^{lc}_{\mathbb{R}^2})$$

is an equivalence. From Proposition 29, we then deduce

Proposition 39 *There is an equivalence*

$$\boldsymbol{Fac}^{lc}_B \cong \boldsymbol{BiMod} \times^h_{(E_1\text{-}\boldsymbol{Alg}\times E_1\text{-}\boldsymbol{Alg})} \big(E_2\text{-}\boldsymbol{Alg} \times E_2\text{-}\boldsymbol{Alg}\big).$$

In other words, locally constant factorization algebras on the (semi-open) butterfly are equivalent to the ∞-category of triples (A, B, M) where A, B are E_2-algebras and M is a left $A \otimes B^{op}$-module (for the underlying E_1-algebras structures of A and B).

Let $\overline{B} = \{(x, y) \in \mathbb{R}^2 \,|\, |y| \le |x|\}$ be the closure of the butterfly. It has four additional dimension 1 strata, given by the boundary of B_+ and B_-. The above argument yields an equivalence

$$\mathbf{Fac}^{lc}_{\overline{B}} \stackrel{\simeq}{\longrightarrow} \mathbf{Fac}^{lc}_{\mathbb{R}_*} \times^h_{\mathbf{Fac}^{lc}_{\mathbb{R}\setminus\{0\}}} \big(\mathbf{Fac}^{lc}_{\tilde{H}} \times \mathbf{Fac}^{lc}_{\tilde{H}}\big)$$

where $\tilde{H}$ is the pointed half-plane, see Example 37. From Propositions 27, 29 and 32 we see that a locally constant factorization algebra on the closed butterfly $\overline{B}$ is equivalent to the data of two E_2-algebras E_+, E_-, four E_1-algebras A_+, B_+, A_-, B_-, equipped respectively with left or right modules structures over E_+ or E_-, and a left $\big(A_+ \otimes^{\mathbb{L}}_{E_+} B_+\big) \otimes \big(A_- \otimes^{\mathbb{L}}_{E_-} B_-\big)^{op}$-module M.

Example 43 (*Homotopy calculus*) The canonical action of $S^1 = SO(2)$ on $\mathbb{R}^2$ has the origin for fixed point. It follows that S^1 acts canonically on $\mathbf{Fac}_{\mathbb{R}^2_*}$, the category of factorization algebras over the pointed disk. If $\mathcal{F} \in \mathbf{Fac}^{lc}_{\mathbb{R}^2_*}$, its restriction to $\mathbb{R}^2 \setminus \{0\}$ determines an E_2-algebra A with monodromy by Corollary 4, Proposition 18 and Theorem 11. If $\mathcal{F}$ is S^1-equivariant, then its monodromy is trivial and it follows that the global section $\mathcal{F}(\mathbb{R}^2)$ is an E_2-module over A. The S^1-action yields an S^1-action on $\mathcal{F}(\mathbb{R}^2)$ which, algebraically, boils down to an additional differential of homological degree 1 on $\mathcal{F}(\mathbb{R}^2)$. We believe that the techniques in this section suitably extended to the case of compact group actions on factorization algebras allow to prove

Claim The category $\left(\mathbf{Fac}^{lc}_{\mathbb{R}^2_*}\right)^{S^1}$ of S^1-equivariant locally constant factorization algebras on the pointed disk $\mathbb{R}^2_*$ is equivalent to the category of homotopy calculus of Tamarkin-Tsygan [90] describing homotopy Gerstenhaber algebras acting on a BV-module.

There are nice examples of homotopy calculus arising in algebraic geometry [6].

Example 44 Let $K : S^1 \to \mathbb{R}^3$ be a knot, that is a smooth embedding of S^1 inside $\mathbb{R}^3$. Then we can consider the stratified manifold $\mathbb{R}^3_K$ with a 1 dimensional open strata given by the image of K, and another 3-dimensional open strata given by the knot complement $\mathbb{R}^3 \setminus K(S^1)$. Then the category of locally constant factorization algebra on $\mathbb{R}^3_K$ is equivalent to the category of quadruples (A, B, f, ρ) where A is an E_3-algebras, B is an E_1-algebra, $f : B \to B$ is a monodromy and $\rho : A \otimes B \to B$ is an action of A onto B (compatible with all the structures (that is making B an object of E_2-$\mathbf{Mod}_A(E_1$-$\mathbf{Alg})$. We refer to [5] and [91] for details on the invariant of knots produced this way.

7 Applications of Factorization Algebras and Homology

7.1 Enveloping Algebras of E_n-algebras and Hochschild Cohomology of E_n-algebras

In this section we describe the universal enveloping algebra of an E_n-algebra in terms of factorization algebras, and apply it to describe E_n-Hochschild cohomology.

Given an E_n-algebra A, we get a factorization algebra on $\mathbb{R}^n$ and thus on its submanifold $S^{n-1} \times \mathbb{R}$ (equipped with the induced framing); see Example 19. We can use the results of Sect. 6 to study the category of E_n-modules over A.

In particular, from Corollary 8, and Proposition 30, we obtain the following corollary which was first proved by Francis [34].

Corollary 13 *Let A be an E_n-algebra. The functor $N_* : E_n$-$\boldsymbol{Mod}_A \to E_1$-$\boldsymbol{RMod}_{\int_{S^{n-1}\times\mathbb{R}} A}$ is an equivalence.*

Similarly, the functor $(-N)_* : E_n$-$\boldsymbol{Mod}_A \to E_1$-$\boldsymbol{LMod}_{\int_{S^{n-1}\times\mathbb{R}} A}$ is an equivalence.

We call $\int_{S^{n-1}\times\mathbb{R}} A$ *the universal* $(E_1$-)*enveloping algebra of the* E_n *-algebra* A.

A virtue of Corollary 13 is that it reduces the homological algebra aspects in the category of E_n-modules to standard homological algebra in the category of modules over a differential graded algebra (given by any strict model of $\int_{S^{n-1}\times\mathbb{R}} A$).

Remark 29 Corollary 8 remains true for $n = \infty$ (and follows from the above study, see [48]), in which case, since S^∞ is contractible, it boils down to the following result:

Proposition 40 ([71], [66]) *Let A be an E_∞-algebra. There is an natural equivalence of ∞-categories E_∞-$\boldsymbol{Mod}_A \cong E_1$-$\boldsymbol{RMod}_A$ (where in the right hand side, A is identified with its underlying E_1-algebra).*

Example 45 Let A be a smooth commutative algebra (or the sheaf of functions of a smooth scheme or manifold) viewed as an E_n-algebra. Then, by Theorems 3 and 10, we have

Proposition 41 *For $n \geq 2$, there is an equivalence*

$$E_n\text{-}\boldsymbol{Mod}_A \;\cong\; E_1\text{-}\boldsymbol{RMod}_{S^\bullet_A(\Omega^1_A[n-1])}.$$

The right hand side is just a category of modules over a graded commutative algebra. If $A = \mathcal{O}_X$, then one thus has an equivalence between E_n-$\mathbf{Mod}_{\mathcal{O}_X}$ and right graded modules over $\mathcal{O}_{T_X[1-n]}$, the functions on the graded tangent space of X.

Example 46 Let A be an E_n-algebra. It is canonically a E_n-module over itself; thus by Corollary 13 it has a structure of right module over $\int_{S^{n-1}\times\mathbb{R}} A$. The later has an easy geometric description. Indeed, by the dimension axiom, $A \cong \int_{\mathbb{R}^n} A$. The euclidean norm gives the trivialization $S^{n-1} \times (0, +\infty)$ of the end(s) of $\mathbb{R}^n$ so that, by Lemma 5, $\int_{\mathbb{R}^n} A$ has a canonical structure of right module over $\int_{S^{n-1}\times(0,+\infty)} A$.

Let us consider the example of an n-fold loop space. Let Y be an n-connective pointed space ($n \geq 0$) and let $A = C_*(\Omega^n(Y))$ be the associated E_n-algebra. By non-abelian Poincaré duality (Theorem 7) we have an equivalence

$$\int\limits_{S^{n-1}\times\mathbb{R}} A \cong C_*\big(\mathrm{Map}_c(S^{n-1}\times\mathbb{R}, Y)\big) \cong C_*\Big(\Omega\Big(Y^{S^{n-1}}\Big)\Big).$$

By Corollary 13 we get

Corollary 14 *The category of E_n-modules over $C_*(\Omega^n(Y))$ is equivalent to the category of right modules over $C_*\Big(\Omega\Big(Y^{S^{n-1}}\Big)\Big)$.*

The algebra $C_*\Big(\Omega\Big(Y^{S^{n-1}}\Big)\Big)$ in Corollary 14 is computed by the cobar construction of the differential graded coalgebra $C_*\big(\mathrm{Map}(S^{n-1}, Y)\big)$. If Y is of finite type, $n-1$-connected and the ground ring k is a field of characteristic zero, the latter is the linear dual of the commutative differential graded algebra $CH_{S^{n-1}}(\Omega^*_{dR}(Y))$ where $\Omega^*_{dR}(Y)$ is (by Theorem 5) the differential graded algebra of Sullivan polynomial forms on Y. In that case, the structure can be computed using rational homotopy techniques.

Example 47 Assume $Y = S^{2m+1}$, with $2m \geq n$. Then, Y has a Sullivan model given by the CDGA $S(y)$, with $|y| = 2m+1$. By Theorems 3 and 5, $C_*(\mathrm{Map}(S^{n-1}, S^{2m+1}))$ is equivalent to the cofree cocommutative coalgebra $S(u, v)$ with $|u| = -1-2m$ and

$|v| = n - 2m - 2$. By Corollarys 14 and 19 we find that the category of E_n-modules over $C_*(\Omega^n(S^{2m+1})$ is equivalent to the category of right modules over the graded commutative algebra $S(a, b)$ where $|u| = -2m - 2$ and $|v| = n - 2m - 3$.

There is an natural notion of cohomology for E_n-algebras which generalizes Hochschild cohomology of associative algebras. It plays the same role with respect to deformations of E_n-algebras as Hochschild cohomology plays with respect to deformations of associatives algebras.

Definition 23 Let M be an E_n-module over an E_n-algebra A. The E_n-Hochschild cohomology[65] of A with values in M, denoted by $HH_{E_n}(A, M)$, is by definition (see [34]) $\mathbb{R}\mathrm{Hom}_A^{E_n}(A, M)$ (Definition 35).

Corollary 15 *Let A be an E_n-algebra, and M, N be E_n-modules over A.*

1. *There is a canonical equivalence*

$$\mathbb{R}Hom_A^{E_n}(M, N) \cong \mathbb{R}Hom^{left}_{\int_{S^{n-1}\times\mathbb{R}} A}(M, N)$$

where the right hand side are homomorphisms of left modules (Definition 36).

2. *In particular $HH_{E_n}(A, M) \cong \mathbb{R}Hom^{left}_{\int_{S^{n-1}\times\mathbb{R}} A}(A, M)$.*
3. *If A is a CDGA (or E_∞-algebra) and M is a left module over A, then*

$$HH_{E_n}(A, M) \cong \mathbb{R}Hom_A^{E_n}(A, M) \cong CH^{S^n}(A, M).$$

Proof The first two points follows from from Corollary 13. The last one follows from Theorem 10 which yields equivalences

$$\begin{aligned}\mathbb{R}\mathrm{Hom}^{\mathrm{left}}_{\int_{S^{n-1}\times\mathbb{R}} A}(A, M) &\cong \mathbb{R}\mathrm{Hom}^{left}_{CH_{S^{n-1}}(A)}(CH_{\mathbb{R}^n}(A), M)\\ &\cong \mathbb{R}\mathrm{Hom}^{left}_{A}\left(CH_{\mathbb{R}^n}(A) \otimes^{\mathbb{L}}_{CH_{S^{n-1}}(A)} A, M\right)\\ &\cong \mathbb{R}\mathrm{Hom}^{left}_{A}(CH_{S^n}(A), M) \cong CH^{S^n}(A, M) \qquad (42)\end{aligned}$$

when A is an E_∞-algebra. □

Example 48 Let A be a smooth commutative algebra and M a symmetric A-bimodule. By the HKR Theorem (see Theorem 3), one has $CH_{S^d}(A) \cong S_A^\bullet(\Omega_A^1[d])$ which is a projective A-module since A is smooth. Thus, Corollary 15 implies

$$HH_{E_d}(A, M) \cong S_A^\bullet(\mathrm{Der}(A, M)[-d]).$$

We now explain the relationship in between E_n-Hochschild cohomology and deformation of E_n-algebras. Denote E_n**-Alg**$_{|A}$ the ∞-category of E_n-algebras over A.

65 Which is an object of **Chain**(k).

The bifunctor of E_n-derivations Der : $(E_n\text{-}\mathbf{Alg}_{|A})^{op} \times E_n\text{-}\mathbf{Mod}_A \to \mathbf{Chain}(k)$ is defined as

$$\mathrm{Der}(R, N) := \mathrm{Map}_{E_n\text{-}\mathbf{Alg}_{|A}}(R, A \oplus N).$$

The (absolute) *cotangent complex* of A (as an E_n-algebra) is the value on A of the left adjoint of the split square zero extension functor $E_n\text{-}\mathbf{Mod}_A \ni M \mapsto A \oplus M \in E_n\text{-}\mathbf{Alg}_{|A}$. In other words, there is an natural equivalence

$$\mathrm{Map}_{E_n\text{-}\mathbf{Mod}_A}(L_A, -) \xrightarrow{\simeq} \mathrm{Der}(A, -)$$

of functors. The (absolute) *tangent complex of* A (as an E_n-algebra) is the dual of L_A (as an E_n-module):

$$T_A := \mathbb{R}\mathrm{Hom}_A^{E_n}(L_A, A) \cong \mathbb{R}\mathrm{Hom}^{\mathrm{left}}_{\int_{S^{n-1}\times\mathbb{R}} A}(L_A, A).$$

The tangent complex has a structure of an (homotopy) Lie algebra which controls the deformation of A as an E_n-algebra (that is, its deformations are precisely given by the solutions of Maurer-Cartan equations in T_A). Indeed, Francis [34] has proved the following beautiful result which solve (and generalize) a conjecture of Kontsevich [64]. His proof relies heavily on factorization homology and in particular on the excision property to identify $E_n\text{-}\mathbf{Mod}_A$ with the E_{n-1}-Hochschild homology of $E_1\text{-}\mathbf{LMod}_A$ which is a E_{n-1}-monoidal category.

Theorem 13 ([34]) *Let A be an E_n-algebra and T_A be its tangent complex. There is a fiber sequence of non-unital E_{n+1}-algebras*

$$A[-1] \longrightarrow T_A[-n] \longrightarrow HH_{E_n}(A)$$

inducing a fiber sequence of (homotopy) Lie algebras

$$A[n-1] \longrightarrow T_A \longrightarrow HH_{E_n}(A)[n]$$

after suspension.

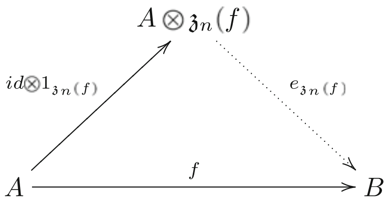
7.2 Centralizers and (Higher) Deligne Conjectures

We will here sketch applications of factorization algebras to study *centralizers* and solve the (relative and higher) Deligne conjecture.

The following definition is due to Lurie [71] (and generalize the notion of center of a category due to Drinfeld).

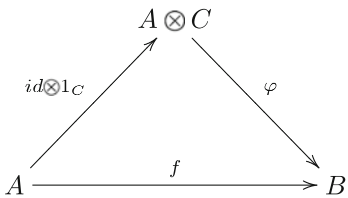
Definition 24 The (derived) centralizer of an E_n-algebra map $f : A \to B$ is the *universal* E_n-algebra $\mathfrak{z}_n(f)$ equipped with a map of E_n-algebras $e_{\mathfrak{z}_n(f)} : A \otimes \mathfrak{z}_n(f) \to B$

making the following diagram



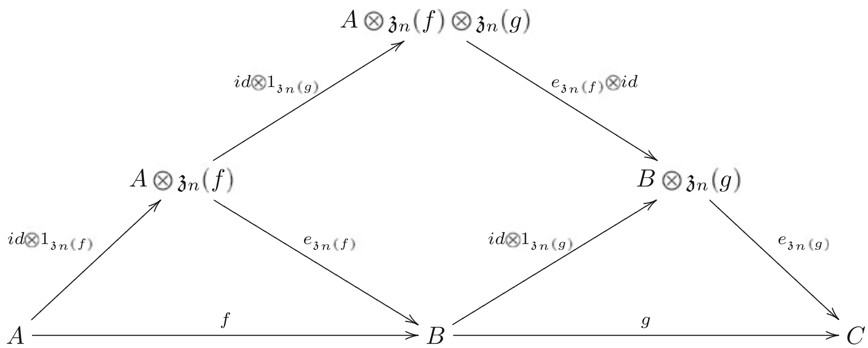
(43)

commutative in E_n-Alg. The *(derived) center* of an E_n-algebra A is the centralizer $\mathfrak{z}_n(A) := \mathfrak{z}_n(A \xrightarrow{id} A)$ of the identity map.
The existence of the derived centralizer $\mathfrak{z}_n(f)$ of an E_n-algebra map $f : A \to B$ is a *non-trivial* result of Lurie [71]. The universal property means that if $C \xrightarrow{\varphi} B$ is an E_n-algebra map fitting inside a commutative diagram



(44)

then there is a unique[66] factorization $\varphi : A \otimes C \overset{id \otimes \kappa}{\dashrightarrow} A \otimes \mathfrak{z}_n(f) \xrightarrow{e_{\mathfrak{z}_n(f)}} B$ of φ by an E_n-algebra map $\kappa : C \to \mathfrak{z}_n(f)$. In particular, the commutative diagram



induces natural maps of E_n-algebras

$$\mathfrak{z}_n(\circ) : \mathfrak{z}_n(f) \otimes \mathfrak{z}_n(g) \longrightarrow \mathfrak{z}_n(g \circ f). \tag{45}$$

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66 Up to a contractible space of choices.

Example 49 Let M be a monoid (for instance a group). That is an E_1-algebra in the (discrete) category of sets with cartesian product for monoidal structure. Then $\mathfrak{z}_1(M) = Z(M)$ is the usual center $\{m \in M, \forall n \in M, n \cdot m = m \cdot n\}$ of M. Let $f : H \hookrightarrow G$ be the inclusion of a subgroup in a group G. Then $\mathfrak{z}_1(f)$ is the usual centralizer of the subgroup H in G. This examples explain the name centralizer.

Similarly, let *k-Mod* be the (discrete) category of k-vector spaces over a field k. Then an E_1-algebra in *k-Mod* is an associative algebra and $\mathfrak{z}_1(A) = Z(A)$ is its usual (non-derived) center. However, if one sees A as an E_1-algebra in the ∞-category **Chain**(k) of chain complexes, then $\mathfrak{z}_1(A) \cong \mathbb{R}\mathrm{Hom}_{A\otimes A^{op}}(A, A)$ is (computed by) the usual Hochschild cochain complex ([67]) in which the usual center embeds naturally, but is different from it even when A is commutative.

Let $A \xrightarrow{f} B$ be an E_n-algebra map. Then B *inherits a* canonical *structure of* E_n*-A -module, denoted* B_f, which is the pullback along f of the tautological E_n-B-module structure on B.

The relative Deligne conjecture claims that the centralizer is computed by E_n-Hochschild cohomology.

Theorem 14 (Relative Deligne conjecture) *Let* $A \xrightarrow{f} B$ *and* $B \xrightarrow{g} C$ *be maps of* E_n*-algebras.*

1. *There is an* E_n*-algebra structure on* $HH_{E_n}(A, B_f) \cong \mathbb{R}Hom_A^{E_n}(A, B_f)$ *which makes* $HH_{E_n}(A, B_f)$ *the centralizer* $\mathfrak{z}_n(f)$ *of* f *(in particular,* $\mathfrak{z}(f)$ *exists);*
2. *the diagram*

$$\begin{array}{ccc} \mathfrak{z}_n(f) \otimes \mathfrak{z}_n(g) & \xrightarrow{\mathfrak{z}_n(\circ)} & \mathfrak{z}_n(g\circ f) \\ \uparrow \cong & & \uparrow \cong \\ \mathbb{R}Hom_A^{E_n}(A, B_f) \otimes \mathbb{R}Hom_B^{E_n}(B, C_g) & \xrightarrow{\circ} & \mathbb{R}Hom_A^{E_n}(A, C_{g\circ f}) \end{array}$$

   is commutative in E_n**-Alg** *(where the lower arrow is induced by composition of maps in* E_n**-Mod***).*
3. *If* $A \xrightarrow{f} B$ *is a map of* E_∞*-algebras, then there is an equivalence of* E_n*-algebras* $\mathfrak{z}_n \cong CH^{S^n}(A, B_f)$ *where* $CH^{S^n}(A, B)$ *is endowed with the structure given by Proposition 7.*

Sketch of proof This result is proved in [48] (the techniques of [34] shall also give an independent proof) and we only briefly sketch the main point of the argument. We first define an E_n-algebra structure on $\mathbb{R}\mathrm{Hom}_A^{E_n}(A, B_f)$. By Corollary 8 and Theorem 9, we can assume that A, B are factorization algebras on $\mathbb{R}^n$ and that a morphism of modules is a map of the underlying (stratified) factorization algebras. We are left to prove that there is a locally constant factorization algebra structure on $\mathbb{R}^n$ whose global sections are $HH_{E_n}(A, B_f) \cong \mathbb{R}\mathrm{Hom}_A^{E_n}(A, B_f)$. It is enough to define it on the basis of convex open subsets $\mathcal{CV}$ of $\mathbb{R}^n$ (by Proposition 17). To any convex open set U (with central point x_U), we associate the chain complex $\mathbb{R}\mathrm{Hom}_{A_{|U}}^{\mathbf{Fac}_U}(A_{|U}, B_{f\,|U})$ of factorization algebras morphisms from the restrictions $A_{|U}$ to $B_{|U}$ which, on the restriction to $U \setminus \{x_U\}$ are given by f. Note that since U is convex, there is a quasi-isomorphism

$$\mathbb{R}\mathrm{Hom}_{A_{|U}}^{\mathbf{Fac}_U}(A_{|U}, B_{f\,|U}) \cong \mathbb{R}\mathrm{Hom}_A^{E_n}(A, B_f) = \mathbb{R}\mathrm{Hom}_{A_{|\mathbb{R}^n}}^{\mathbf{Fac}_{\mathbb{R}^n}}(A_{|\mathbb{R}^n}, B_{f\,|\mathbb{R}^n}). \quad (46)$$

Now, given convex sets $U_1, \ldots, U_r$ which are pairwise disjoint inside a bigger convex V, we define a map

$$\rho_{U_1,\ldots,U_r,V} : \bigotimes_{i=1\ldots r} \mathbb{R}\mathrm{Hom}_{A_{|U_i}}^{\mathbf{Fac}_{U_i}}(A_{|U_i}, B_{f\,|U_i}) \longrightarrow \mathbb{R}\mathrm{Hom}_{A_{|V}}^{\mathbf{Fac}_V}(A_{|V}, B_{f\,|V})$$

as follows. To define $\rho_{U_1,\ldots,U_r,V}(g_1, \ldots, g_r)$, we need to define a factorization algebra map on V and for this, it is enough to do it on the open set consisting of convex subsets of V which either are included in one of the U_i and contains x_{U_i} or else does not contains any x_{U_i}. To each open set $x_{U_i} \in D_i \subset U_i$ of the first kind, we define

$$\rho_{U_1,\ldots,U_r,V}(g_1, \ldots, g_r)(D_i) : A(D_i) \xrightarrow{g_i} B_f(D_i)$$

to be given by g_i, while for any open set $D \subset V \setminus \{x_{U_1}, \ldots, x_{|U_r}\}$, we define

$$\rho_{U_1,\ldots,U_r,V}(g_1, \ldots, g_r)(D) : A(D) \xrightarrow{f} B_f(D)$$

to be given by f. The conditions that $g_1, \ldots, g_r$ are maps of E_n-modules over $A(U_i)$ ensures that $\rho_{U_1,\ldots,U_r,V}$ define the structure maps of a factorization algebra which is further locally constant since $\rho_{U,\mathbb{R}^n}$ is the equivalence (46).

The construction is roughly described in Fig. 4.

One can check that the natural evaluation map $eval : A \otimes \mathbb{R}\mathrm{Hom}_A^{E_n}(A, B_f) \to B$ is a map of E_n-algebras.

Now let C be an E_n-algebra fitting in the commutative diagram (44), which we again identify with a factorization algebra map (over $\mathbb{R}^n$). By adjunction (in **Chain**(k)), the map $\varphi : A \otimes C \to B$ has a (derived) adjoint $\theta_\varphi : C \to RHom(A, B)$. Since φ is a map of factorization algebras and diagram (43) is commutative, one check that θ_φ factors through a map

$$\tilde{\theta_\varphi} : C \longrightarrow RHom_A^{E_n}(A, B) \cong CH^{S^n}(A, B). \quad (47)$$

which can be proved to be a map of factorization algebras. Further, by definition of θ_φ, the identity

$$eval \circ \big(id_A \otimes \theta_\varphi\big) = \varphi$$

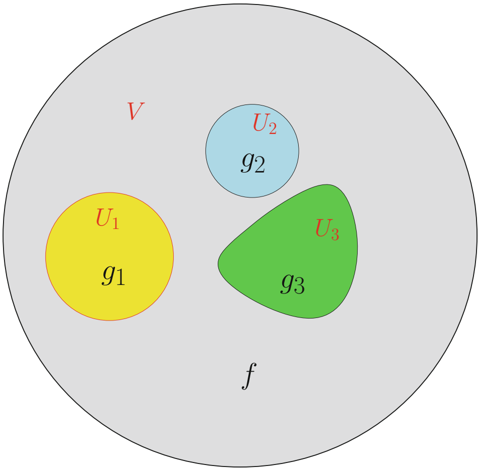

Fig. 4 The factorization algebra map $A_{|V} \to B_{|V}$ obtained by applying the relevant maps of modules g_1, g_2, g_3 (viewed as maps of factorizations algebras) and the E_n-algebra map $f : A \to B$ on the respective regions

holds. The uniqueness of the map $\tilde{\theta_\varphi}$ follows from the fact that the composition

$$RHom_A^{E_n}(A, B) \xrightarrow{1_{RHom_A^{E_n}(A,A)}\otimes id} RHom_A^{E_n}\Big(A, A \otimes RHom_A^{E_n}(A, B)\Big) \xrightarrow{ev_*} RHom_A^{E_n}(A, B)$$

is the identity map.

Finally the equivalence between $\mathfrak{z}_n(f)$ and $CH^{S^n}(A, B_f)$ in the commutative case follows from the string of equivalences (42) which can be checked to be an equivalence of E_n-algebras using diagram (30) connecting algebras over the operads of little rectangles of dimension n and factorization algebras. □

Example 50 Let $A \xrightarrow{f} B$ be a map of CDGAs. then by Proposition 15 we have an equivalence $HH_{E_n}(A, B) \cong \mathbb{R}\mathrm{Hom}^{left}_{CH_{\partial I^n}(A)}(CH_{I^n}(A), CH_{I^n}(B))$ where $I = [0, 1]$ and $\partial I^n \cong S^{n-1}$ is the boundary of the unit cube I^n. We have a simplicial model $I^n_\bullet$ of I^n where $I_\bullet$ is the standard model of Example 1; its boundary $\partial I^n_\bullet$ is a simplicial model for S^{n-1}. Then Theorem 14 identifies the derived composition (45) as the usual composition (of left dg-modules)

$$Hom^{left}_{CH_{\partial I^n_\bullet}(A)}\Big(CH_{I^n_\bullet}(A), CH_{I^n_\bullet}(B)\Big) \otimes Hom^{left}_{CH_{\partial I^n_\bullet}(B)}\Big(CH_{I^n_\bullet}(B), CH_{I^n_\bullet}(C)\Big) \xrightarrow{\circ} Hom^{left}_{CH_{\partial I^n_\bullet}(A)}\Big(CH_{I^n_\bullet}(A), CH_{I^n_\bullet}(C)\Big).$$

The relative Deligne conjecture implies easily the standard one and also the Swiss cheese conjecture. Indeed, Theorem 14 implies that the multiplication $\mathfrak{z}_n(A) \otimes \mathfrak{z}_n(A) \xrightarrow{\mathfrak{z}_n(\circ)} \mathfrak{z}_n(A)$ makes $\mathfrak{z}_n(A)$ into an E_1-algebra in the ∞-category E_n**-Alg**. The ∞-category version of Dunn Theorem (Theorem 11) gives an equivalence

E_1**-Alg**$\big(E_n$**-Alg**$\big) \cong E_{n+1}$**-Alg**. This yields the following solution to the higher Deligne conjecture, see [48, 71].

Corollary 16 (Higher Deligne Conjecture) *Let A be an E_n-algebra. The E_n-Hochschild cohomology $HH_{E_n}(A, A)$ has an E_{n+1}-algebra structure lifting the Yoneda product which further lifts the E_n-algebra structure of the centralizer $\mathfrak{z}(A \stackrel{id}{\to} A)$.*

In particular, if A is commutative, there is a natural E_{n+1}-algebra structure on $CH^{S^n}(A, A) \cong HH_{E_n}(A, A)$ whose underlying E_n-algebra structure is the one given by Theorem 7. Hence the underlying E_1-algebra structure is given by the cup-product (Example 9).

Example 51 Let $\mathcal{C}$ be a monoidal (ordinary) category. Then the center $\mathfrak{z}_1(\mathcal{C})$ is in E_2-**Alg**(Cat), that is a braided monoidal category. One can prove that $\mathfrak{z}_1(\mathcal{C})$ is actually the Drinfeld center of $\mathcal{C}$, see [71].

Remark 30 Presumably, the E_{n+1}-structure on $HH_{E_n}(A)$ given by Corollary 16 shall be closely related to the one given by Theorem 13.

Example 52 Let $1_A : k \to A$ be the unit of an E_n-algebra A. Then $\mathfrak{z}_n(1_A) \cong A$ as an E_n-algebra. The derived composition (45) yields canonical map[67] of E_n-algebras $\mathfrak{z}_n(A) \otimes \mathfrak{z}_n(1_A) \longrightarrow \mathfrak{z}_n(1_A)$ which exhibits $A \cong \mathfrak{z}_n(1_A)$ as a right E_1-module over $\mathfrak{z}_n(A) \cong HH_{E_n}(A, A)$ in the category of E_n-algebras (by Theorem 16). Hence, in view of Example 37, we obtain, as an immediate corollary (see [14]), a proof of the Swiss-Cheese version of Deligne conjecture[68]:

Corollary 17 (Deligne conjecture with action) *Let A be an E_n-algebra. Then the pair $(HH_{E_n}(A, A), A)$ is canonically an object of E_1-**RMod**$(E_n$-**Alg**$)$, that is A has an natural action of the E_{n+1}-algebra $HH_{E_n}(A, A)$.*

Example 53 (*(Higher homotopy) calculus again* [14]) Let A be an E_n-algebra. Assume $n = 0, 1, 3, 7$, so that A defines canonically a (locally constant) factorization algebra A_{S^n} on the framed manifold S^n (see Example 19). Similarly the E_{n+1}-algebra given by the higher Hochschild cohomology $HH_{E_n}(A, A)$ defines canonically a (locally constant) factorization algebra on the manifold $S^n \times (0, \infty)$ endowed with the product framing.

The Deligne conjecture with action (Corollary 17) shows that A is also a left module over $HH_{E_n}(A, A)$. Thus, according to Proposition 30, the pair $(HH_{E_n}(A, A), A)$ yields a stratified locally constant factorization algebra $\mathcal{H}$ on $D^{n+1} \setminus \{0\}$, the closed disk in which we have removed the origin.

By Theorem 8, we have that $(A_{S^n})(\partial D^{n+1}) \cong \int_{S^n} A$. Collapsing the boundary ∂D^{n+1} to a point yields an adequatly stratified map $\tau : D^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1}_*$ so that $\mathcal{A} := \tau_*(\mathcal{H})$ is stratified locally constant on $\mathbb{R}^{n+1}_*$. It is further $SO(n+1)$-equivariant. Together with Example 43, the above paragraph thus sketches a proof of the following fact:

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[67] Which is equivalent to $e_{\mathfrak{z}(1_A)}$.

[68] Originally proved, for a slightly different variant of the swiss cheese operad, in [25, 100].

Corollary 18 *Let A be an E_n-algebra and $n = 0, 1, 3, 7$. Then A gives rise to an $SO(n+1)$-equivariant stratified locally constant factorization algebra $\mathcal{A}$ on the pointed disk $\mathbb{R}^{n+1}_*$ such that $\mathcal{A}(\mathbb{R}^{n+1}) \cong \int_{S^n} A$ and for any sub-disk $D \subset \mathbb{R}^{n+1} \setminus \{0\}$, there is an natural (with respect to disk inclusions) equivalence of E_{n+1}-algebras $\mathcal{A}(D) \stackrel{\simeq}{\to} HH_{E_n}(A, A)$.*

In particular, for $n = 1$, we recover that the pair $(HH_{E_1}(A, A), CH_{S^1}(A))$, given by Hochschild cohomology and Hochschild homology of an associative or A_∞-algebra A, defines an homotopy calculus (see [65, 90] or Example 43).

This corollary is proved in details (using indeed factorization homology techniques) in the interesting paper [59] along with many other examples in which D^n is replaced by other framed manifold.

7.3 Higher String Topology

The formalism of factorization homology for CDGAs and higher Deligne conjecture was applied in [45, 48] to higher string topology which we now explain briefly. We also refer to the work [57, 58] for a related approach.

Let M be a closed oriented manifold, equipped with a Riemannian metric. String topology is about the algebraic structure of the chains and homology of the free loop space $LM := \mathrm{Map}(S^1, M)$ and its higher free sphere spaces $M^{S^n} := \mathrm{Map}(S^n, M)$. These spaces have Fréchet manifold structures and there is a submersion $ev : M^{S^n} \to M$ given by evaluating at a chosen base point in S^n. The canonical embedding $Map(S^n \vee S^n, M) \xrightarrow{\rho_{in}} Map(S^n, M) \times Map(S^n, M)$ has an oriented normal bundle.[69] It follows that there is a Gysin map $(\rho_{in})_! : H_*\left(M^{S^n \coprod S^n}\right) \longrightarrow H_{*-\dim(M)}\left(M^{S^n \vee S^n}\right)$. The pinching map $\delta_{S^n} : S^n \to S^n \vee S^n$ (obtained by collapsing the equator to a point) yields the map $\delta^*_{S^n} : \mathrm{Map}(S^n \vee S^n, M) \longrightarrow M^{S^n}$. The *sphere product* is the composition

$$\begin{aligned}\star_{S^n} : H_{*+\dim(M)}\left(M^{S^n}\right)^{\otimes 2} &\to H_{*+2\dim(M)}\left(M^{S^n \coprod S^n}\right)\\ &\xrightarrow{(\rho_{in})_!} H_{*+\dim(M)}\left(M^{S^n \vee S^n}\right) \xrightarrow{(\delta^*_{S^n})_*} H_{*+\dim(M)}\left(M^{S^n}\right).\end{aligned} \tag{48}$$

The circle action on itself induces an action $\gamma : LM \times S^1 \to LM$.

[69] Which can be obtained as a pullback along ev of the normal bundle of the diagonal $M \to M \times M$.

Theorem 15

1. (***Chas-Sullivan*** [17]) *Let* $\Delta : H_*(LM) \stackrel{\times[S^1]}{\longrightarrow} H_{*+1}(LM \times S^1) \stackrel{\gamma_*}{\longrightarrow} H_{*+1}(LM)$ *be induced by the* S^1*-action. Then* $(H_{*+\dim(M)}(LM), \star_{S^1}, \Delta)$ *is a Batalin Vilkoviski-algebra and in particular a* P_2*-algebra.*
2. (***Sullivan-Voronov*** [20]) $(H_{*+\dim(M)}(LM), \star_{S^n})$ *is a graded commutative algebra.*
3. (***Costello*** [21], ***Lurie*** [70]) *If* M *is simply connected, the chains* $C_*(LM)$ $[\dim(M)]$ *have a structure of* E_2*-algebra (and actually of* Disk_2^{or}*-algebra) which, in characteristic* 0*, induces Chas-Sullivan* P_2*-structure in homology (by* [32]*).*

In [20], Sullivan-Voronov also sketched a proof of the fact that $H_{\bullet+\dim(M)}(M^{S^n})$ is an algebra over the homology $H_*(\mathrm{Disk}_{n+1}^{or})$ of the operad Disk_{n+1}^{or} and in particular has a P_{n+1}-algebra structure (see Example 64). Their work and the aforementioned work for $n = 1$ (Theorem 15(3)) rise the following

Question 1 Is there a natural E_{n+1}-algebra (or even Disk_{n+1}^{or}-algebra) on the *chains* $C_*\left(M^{S^n}\right)[\dim(M)]$ which induces Sullivan-Voronov product in homology?

Using the solution to the higher Deligne conjecture and the relationship between factorization homology and mapping spaces, one obtains a positive solution to the above question for sufficiently connected manifolds.

Theorem 16 ([48]) *Let* M *be an* n*-connected*[70] *Poincaré duality space. The shifted chain complex* $C_{\bullet+\dim(X)}(X^{S^n})$ *has a natural*[71] E_{n+1}*-algebra structure which induces the sphere product* $\star_{S^n}$ *(given by the map* (48)*) of Sullivan-Voronov* [20]

$$H_p\left(X^{S^n}\right) \otimes H_q\left(X^{S^n}\right) \to H_{p+q-\dim(X)}\left(X^{S^n}\right)$$

in homology when X *is an oriented closed manifold.*

Sketch of proof We only gives the key steps of the proof following [45, 48].

- Let $[M]$ be the fundamental class of M. The Poincaré duality map $\chi_M : x \mapsto x \cap [M]$ is a map of left modules and thus, by Proposition 40 (and since, by assumption, the biduality homomorphism $C_*(X) \to (C^*(X))^\vee$ is a quasi-isomorphism),

$$\chi_M : C^*(X) \to C_*(X)[\dim(X)] \;\cong\; \left(C^*(X)\right)^\vee[\dim(X)]$$

  has an natural lift as an E_∞-module. And thus as an E_n-module as well.
- From the previous point we deduce that there is an equivalence

[70] It is actually sufficient to assume that M is nilpotent, connected and has finite homotopy groups $\pi_i(M, m_0)$ for $1 \le i \le n$.

[71] With respect to maps of Poincaré duality spaces.

$$
\begin{aligned}
HH_{E_n}(C^*(X), C^*(X)) &\cong \mathbb{R}\mathrm{Hom}^{E_n}_{C^*(X)}\big(C^*(X), C^*(X)\big)\\
&\xrightarrow{(\chi_M)^*} \mathbb{R}\mathrm{Hom}^{E_n}_{C^*(X)}\big(C^*(X), \big(C^*(X)\big)^{\vee}\big)[\dim(M)]\\
&\cong \mathbb{R}\mathrm{Hom}^{left}_{\int_{S^{n-1}\times\mathbb{R}} C^*(X)}\big(C^*(X), \big(C^*(X)\big)^{\vee}\big)[\dim(M)]\\
&\cong \mathbb{R}\mathrm{Hom}^{left}_{C^*(X)}\big(C^*(X) \otimes^{\mathbb{L}}_{\int_{S^{n-1}\times\mathbb{R}} C^*(X)} C^*(X), k\big)[\dim(M)]\\
&\cong \mathbb{R}\mathrm{Hom}^{left}_{C^*(X)}\big(CH_{S^n}(C^*(X)), k\big)[\dim(M)]. \qquad (49)
\end{aligned}
$$

where the last equivalence follows from Theorem 10.

- By Theorem 5 relating Factorization homology of singular cochains with mapping spaces, the above equivalence (49) induces an equivalence

$$C_*\Big(M^{S^n}\Big)[\dim(M)] \xrightarrow{\simeq} HH_{E_n}(C^*(X), C^*(X)). \qquad (50)$$

- Now, one uses the higher Deligne conjecture (Corollary 16) and the latter equivalence (50) to get an E_{n+1}-algebra structure on $C_*\Big(M^{S^n}\Big)[\dim(M)]$. The explicit definition of the cup-product given by Proposition 7 (and Theorem 14) allows to describe explicitly the E_1-algebra structure at the level of the cochains $C^*\Big(M^{S^n}\Big)$ through the equivalence (50), which, in turn allows to check it induces the product $\star_{S^n}$.

□

Example 54 Let $M = G$ be a Lie group and $A = S(V) \xrightarrow{\simeq} \Omega_{dR}(G)$ be its minimal model. The graded space V is concentrated in positive odd degrees. If G is n-connected, by the generalized HKR Theorem 3, there is an equivalence

$$S(V \oplus V^*[-n]) \;\cong\; CH^{S^n}(A, A) \;\cong\; C_*\Big(G^{S^n}\Big)[\dim(G)] \qquad (51)$$

in **Chain**(k). The *higher formality conjecture* shows that the equivalence (51) is an equivalence of E_{n+1}-algebras. Here the left hand side is viewed as an E_{n+1}-algebra obtained by the formality of the E_{n+1}-operad from the P_{n+1}-structure on $S(V \oplus V^*[-n])$ whose multiplicative structure is the one given by the symmetric algebra and the bracket is given by the pairing between V and V^*.

7.4 Iterated Loop Spaces and Bar Constructions

In this section we apply the formalism of factorization homology to describe iterated Bar constructions *equipped with* their algebraic structure and relate them to the E_n-algebra structure of nth-iterated loop spaces. We follow the approach of [4, 34, 48].

Bar constructions have been introduced in topology as a model for the coalgebra structure of the cochains on $\Omega(X)$, the based loop space of a pointed space X. Similarly the cobar construction of coaugmented coalgebra has been studied originally as a model for the (E_1-)algebra structure of the chains on $\Omega(X)$. The Bar and coBar constructions also induce equivalences between algebras and coalgebras under sufficient nilpotence and degree assumptions [31, 33, 61].

Let (A, d) be a differential graded unital associative algebra which is *augmented*, that is equipped with an algebra homomorphism $\varepsilon : A \to k$.

Definition 25 The standard *Bar functor* of the augmented algebra (A, d, ε) is

$$Bar(A) := k \overset{\mathbb{L}}{\underset{A}{\otimes}} k.$$

If A is flat over k, it is computed by the standard chain complex $Bar^{std}(A) = \bigoplus_{n\geq 0} \overline{A}^{\otimes n}$ (where $\overline{A} = ker(A \overset{\epsilon}{\to} k)$ is the augmentation ideal of A) endowed with the differential

$$b(a_1 \otimes \cdots a_n) = \sum_{i=1}^{n} \pm a_1 \otimes \cdots \otimes d(a_i) \otimes \cdots \otimes a_n$$
$$+ \sum_{i=1}^{n-1} \pm a_1 \otimes \cdots \otimes (a_i \cdot a_{i+1}) \otimes \cdots \otimes a_n$$

see [31, 37] for details (and signs).

The Bar construction has a standard coalgebra structure. It is well-known that if A is a commutative differential graded algebra, then the shuffle product makes the Bar construction $Bar^{std}(A)$ a CDGA and a bialgebra as well. It was proved by Fresse [37] that Bar constructions of E_∞-algebras have an (augmented) E_∞-structure, allowing to consider iterated Bar constructions and further that there is a canonical n^{th}-iterated Bar construction functor for augmented E_n-algebras as well [38].

Let us now describe the factorization homology/algebra point of view on Bar constructions.

An *augmented E_n-algebra* is an E_n-algebra A equipped with an E_n-algebra map $\varepsilon : A \to k$, called the augmentation. We denote E_n**-Alg**^{aug} *the ∞-category of augmented E_n-algebras* (see Definition 31). The augmentation makes k an E_n-module over A.

By Proposition 35, an augmented E_n-algebra defines naturally a locally constant factorization algebra on the closed unit disk (with its stratification given by its boundary) of dimension less than n. Indeed, we obtain functors

$$\omega_{D^i} : E_n\textbf{-Alg}^{aug} \longrightarrow E_i\textbf{-Alg}^{aug}(E_{n-i}\textbf{-Alg}^{aug}) \longrightarrow \textbf{Fac}^{lc}_{D^i}(E_{n-i}\textbf{-Alg}^{aug}). \quad (52)$$

Definition 26 Let A be an augmented E_n-algebra. Its Bar construction is

$$Bar(A) := \int_{I\times\mathbb{R}^{n-1}} A \overset{\mathbb{L}}{\underset{\int_{S^0\times\mathbb{R}^{n-1}} A}{\otimes}} k.$$

This definition agrees with Definition 25 for differential graded associative algebras and further we have equivalences

$$Bar(A) := \int_{I\times\mathbb{R}^{n-1}} A \overset{\mathbb{L}}{\underset{\int_{S^0\times\mathbb{R}^{n-1}} A}{\otimes}} k \;\cong\; k \overset{\mathbb{L}}{\underset{A}{\otimes}} k \;\cong p_*(\omega_{D^1}(A)) \tag{53}$$

where I is the closed interval $[0, 1]$ and $p : I \to pt$ is the unique map; in particular the right hand side of (53) is just the factorization homology of the associated factorization algebra on D^1.

The functor (52) shows that $Bar(A) \cong p_*(\omega_{D^1}(A))$ has an natural structure of augmented E_{n-1}-algebra (which can also be deduced from Lemma 5(2)).

We can thus iterate (up to n-times) the Bar constructions of an augmented E_n-algebra.

Definition 27 Let $0 \le i \le n$. The ith-iterated Bar construction of an augmented E_n-algebra A is the augmented E_{n-i}-algebra

$$Bar^{(i)}(A) := Bar(\cdots(Bar(A))\cdots).$$

Using the excision axiom of factorization homology, one finds

Lemma 8 (Francis [34, 48]) *Let A be an E_n-algebra and $0 \le i \le n$. There is a natural equivalence of E_{n-i}-algebras*

$$Bar^{(i)}(A) \cong \int_{D^i\times\mathbb{R}^{n-i}} A \overset{\mathbb{L}}{\underset{\int_{S^{i-1}\times\mathbb{R}^{n-i+1}} A}{\otimes}} k \;\cong\; p_*(\omega_{D^i}(A))$$

In particular, taking $n = \infty$, we recover an E_∞-structure on the iterated Bar construction $Bar^{(i)}(A)$ of an augmented E_∞-algebra.

We now describes the expected coalgebras structures. We start with the E_∞-case, for which we can use the derived Hochschild chains from Sect. 2. Then, Lemma 8, Theorem 10 and the excision axiom give natural equivalences of E_∞-algebras ([48]):

$$CH_{S^i}(A, k) \;\cong\; Bar^{(i)}(A). \tag{54}$$

Recall the continuous map (15) $pinch : \mathrm{Cube}_d(r) \times S^d \longrightarrow \bigvee_{i=1\ldots r} S^d$. Similarly to the definition of the map (17), applying the singular set functor to the map (15) we get a morphism

$$pinch_*^{S^n,r} : C_*\big(\mathrm{Cube}_d(r)\big) \otimes CH_{S^d}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k \xrightarrow{pinch_* \otimes^{\mathbb{L}}_A id} CH_{\bigvee_{i=1}^r S^d}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k \cong \Big(CH_{\coprod_{i=1}^r S^d}(A) \overset{\mathbb{L}}{\underset{A^{\otimes r}}{\otimes}} A\Big) \overset{\mathbb{L}}{\underset{A}{\otimes}} k$$
$$\cong \Big(CH_{\coprod_{i=1}^r S^d}(A)\Big) \overset{\mathbb{L}}{\underset{A^{\otimes r}}{\otimes}} k \cong \Big(CH_{S^d}(A,k)\Big)^{\otimes r} \tag{55}$$

Proposition 42 ([48]) *Let A be an augmented E_∞-algebra. The maps* (55) $pinch_*^{S^d,r}$ *makes the iterated Bar construction $Bar^{(d)}(A) \cong CH_{S^d}(A,k)$ a natural E_n-coalgebra in the ∞-category of E_∞-algebras.*

If Y is a pointed space, its E_∞-algebra of cochains $C^*(Y)$ has a canonical augmentation $C^*(Y) \to C^*(pt) \cong k$ induced by the base point $pt \to Y$. By Theorem 4, we have an E_∞-algebra morphism

$$\mathcal{I}t^{\Omega^n} : Bar^{(n)}(C^*(Y)) \cong CH_{S^n}(C^*(Y),k) \xrightarrow{\mathcal{I}t \otimes^{\mathbb{L}}_{C^*(Y)} k} C^*\big(Y^{S^n}\big) \otimes^{\mathbb{L}}_{C^*(Y)} k \longrightarrow C^*\big(\Omega^n(Y)\big). \tag{56}$$

We now have a nice application of factorization homology in algebraic topology.

Corollary 19 *1. The map* (56) $\mathcal{I}t^{\Omega^n} : Bar^{(n)}(C^*(Y)) \to C^*\big(\Omega^n(Y)\big)$ *is an E_n-coalgebra morphism in the category of E_∞-algebras. It is further an equivalence if Y is connected, nilpotent and has finite homotopy groups $\pi_i(Y)$ in degree $i \le n$.*

2. *The dual of* (56) $C_*\big(\Omega^n(Y)\big) \longrightarrow \big(C_*\big(\Omega^n(Y)\big)\big)^{\vee\vee} \xrightarrow{\mathcal{I}t_{\Omega^n}} \Big(Bar^{(n)}(C^*(Y))\Big)^{\vee}$ *is a morphism in E_n-**Alg**. If Y is n-connected, it is an equivalence.*

We now sketch the construction of the E_i-coalgebra structure of the ith-iterated Bar construction of an E_n-algebra. To do so, we only need to define a locally constant *cofactorization algebra* structure[72] whose global section is $Bar^{(i)}(A)$. By (the dual of) Proposition 17, it is enough to define such a structure on the basis of convex open disks of $\mathbb{R}^i$. Let $\mathcal{A} \in \mathbf{Fac}^{lc}_{\mathbb{R}^i}(E_{n-i}\text{-}\mathbf{Alg}^{aug})$ be the factorization algebra associated to A (by Theorems 9 and 11).

Let V be a convex open subset. By Corollary 10 (and Theorem 9), the augmentation gives us a stratified locally constant factorization algebra $\widehat{\omega}(\mathcal{A}_{|V})$ on $\widehat{V} = V \cup \{\infty\}$ (with values in E_{n-i}-$\mathbf{Alg}^{aug}$).

If $U \subset V$ is a convex open subset, we have a continuous map $\pi_U : \widehat{V} \to \widehat{U}$ which maps the complement of U to a single point. Further, the augmentation defines maps of factorization algebras (on $\widehat{U}$)

[72] That is a locally constant coalgebra over the ∞-operad Disk($\mathbb{R}^i$) see Remark 24.

$$\epsilon_U : \pi_{U*}(\widehat{\omega}(\mathcal{A}_{|V})) \longrightarrow \widehat{\omega}(\mathcal{A}_{|U})$$

which, on every open convex subset of U is the identity, and, on every open convex neighborhood of ∞ is given by the augmentation.

Define

$$Bar^{(i)}(A)(U) := \int_V \widehat{\omega}(\mathcal{A}_{|V}) \cong \int_U \pi_{U*}(\widehat{\omega}(\mathcal{A}_{|V}))$$

to be the factorization homology of $\widehat{\omega}(\mathcal{A}_{|V})$. We finally get, for $U_1, \dots, U_s$ pairwise disjoint convex subsets of a convex open subset V, a structure map

$$\nabla_{U_1,\dots,U_s,V} : Bar^{(i)}(A)(V) = \int_V (\widehat{\omega}(\mathcal{A}_{|V}) \xrightarrow{\otimes \int_{U_i} \varepsilon_{U_i}} \bigotimes_{i=1\cdots s} \int_{U_i} \widehat{\omega}(\mathcal{A}_{|U_i}) \xrightarrow{\simeq} Bar^{(i)}(A)(U_1) \otimes \cdots \otimes Bar^{(i)}(A)(U_s). \tag{57}$$

The maps $\nabla_{U_1,\dots,U_s,V}$ are the structure maps of a locally constant factorization coalgebras (see [48]) hence they make $Bar^{(i)}(A)$ into an E_i-coalgebra (with values in the category of E_{n-i}-algebras), naturally in A:

Theorem 17 ([4, 34, 48]) *The iterated Bar construction lifts into an ∞-functor*

$$Bar^{(i)} : E_n\text{-}Alg^{aug} \longrightarrow E_i\text{-}coAlg\Big(E_{n-i}\text{-}Alg^{aug}\Big).$$

One has an natural equivalence[73] $Bar^{(i)}(A) \cong k \overset{\mathbb{L}}{\underset{A}{\otimes}} k$ *in* E_{n-i}-***Alg***.

Further this functor is equivalent to the one given by Proposition 42 when restricted to augmented E_∞-algebras.

Example 55 Let Free_n be the free E_n-algebra on k as in Example 16. By Definition 26, we have equivalences of E_{n-1}-algebras.

$$\begin{aligned} Bar(\mathrm{Free}_n) &= \int_{D^1\times\mathbb{R}^{n-1}} \mathrm{Free}_n \overset{\mathbb{L}}{\underset{\int_{S^0\times\mathbb{R}^{n-1}} \mathrm{Free}_n}{\otimes}} k \\ &\cong \int_{S^1\times\mathbb{R}^{n-1}} \mathrm{Free}_n \overset{\mathbb{L}}{\underset{\int_{\mathbb{R}^{n-1}} \mathrm{Free}_n}{\otimes}} k \\ &\cong \Big(\mathrm{Free}_n \otimes \mathrm{Free}_{n-1}(k[1])\Big) \overset{\mathbb{L}}{\underset{\int_{\mathbb{R}^{n-1}} \mathrm{Free}_n}{\otimes}} k \qquad \text{(by Proposition 8)} \\ &\cong \mathrm{Free}_{n-1}(k[1]). \end{aligned}$$

[73] The relative tensor product being the tensor product of E_i-modules over A.

The result also holds for $\mathrm{Free}_n(V)$ instead of $Free_n$, see [35]. Iterating, one finds

Proposition 43 ([35]) *There is a natural equivalence of E_{n-i}-algebras*

$$Bar^{(i)}(Free_n(V)) \cong Free_{n-i}(V[i]).$$

If one works in $\mathbf{Top}_*$ instead of $\mathbf{Chain}(k)$, then $\mathrm{Free}_n(X) = \Omega^n \Sigma^n X$ and the above proposition boils down to $Bar^{(i)}(\Omega^n \Sigma^n X) \cong B^i \Omega^i (\Omega^{n-i} \Sigma^n X) \overset{\simeq}{\leftarrow} \Omega^{n-i} \Sigma^{n-i}(\Sigma^i X)$.

7.5 E_n-Koszul Duality and Lie Algebras Homology

Let $A \overset{\varepsilon}{\to} k$ be an augmented E_n-algebra. The linear dual $RHom(Bar^{(n)}(A), k)$ of the nth-iterated Bar construction inherits an E_n-algebra structure (Theorem 17).

Definition 28 ([34, 71]) The E_n-algebra $A^{(n)!} := RHom(Bar^{(n)}(A), k)$ dual to the iterated Bar construction is called the (derived) E_n *-Koszul dual* of A.

The terminology is chosen because it agrees with the usual notion of Koszul duality for quadratic associative algebras but it is really more like a E_n-*Bar*-duality.

Direct inspection of the E_n-algebra structures show that the dual of the iterated Bar construction is equivalent to the centralizer of the augmentation. (see Sect. 7.2):

Lemma 9 ([48, 71]) *Let $A \overset{\varepsilon}{\to} k$ be an augmented E_n-algebra. There is an natural equivalence of E_n-algebras $A^{(n)!} \cong \mathfrak{z}_n(A \overset{\varepsilon}{\to} k)$.*

Let M be a dimension m manifold endowed with a framing of $M \times \mathbb{R}^n$. By Proposition 35, Theorem 11 and Proposition 18, we have the functor

$$\omega_{M\times D^n} : E_{n+m}\text{-}\mathbf{Alg}^{aug} \to E_n\text{-}\mathbf{Alg}^{aug}(E_m\text{-}\mathbf{Alg}) \overset{\omega_{D^n}}{\longrightarrow} \mathbf{Fac}^{lc}_{D^n}(E_m\text{-}\mathbf{Alg}) \cong \mathbf{Fac}^{lc}_{\mathbb{R}^m\times D^n} \longrightarrow \mathbf{Fac}^{lc}_{M\times D^n}$$

where the last map is induced by the framing of $M \times \mathbb{R}^n$ as in Example 19. Let p be the map $p : M \times D^n \to pt$. We can compute the factorization homology $p_*(\omega_{M\times D^n})$ by first pushing forward along the projection on D^n and then applying p_* or first pushing forward on M and then pushing forward to the point. By Theorem 17, we thus obtain an equivalence

$$p_*(\omega_{M\times D^n}) \cong Bar^{(n)}\Big(\int_{M\times\mathbb{R}^n} A\Big) \cong \int_M \big(Bar^{(n)}(A)\big) \tag{58}$$

where the right equivalence is an equivalence of E_n-coalgebras. When M is further *closed*, this result can be extended to obtain :

Proposition 44 *(*Francis [4, 35]*) Let A be an augmented E_{n+m}-algebra, $M \times \mathbb{R}^n$ be a framed closed manifold. There is an natural equivalence of E_n-algebras*

$$\int_{M\times\mathbb{R}^n} A^{(n+m)!} \cong \Big(\int_{M\times\mathbb{R}^n} A\Big)^{(n)!}$$

if $Bar^{(n)}\Big(\int_{M\times\mathbb{R}^n} A\Big)$ has projective finite type homology groups in each degree. In particular, $\int_M A^{(m)!} \cong \big(\int_M A\big)^{\vee}$ when M is framed (and the above condition is satisfied).

In plain english, we can say that the factorization homology over a closed framed manifold of an algebra and its E_n-Koszul dual are the same (up to finiteness issues).

Example 56 (Lie algebras and their E_n-enveloping algebras) Let **Lie-Alg** be the ∞-category of Lie algebras.[74] The forgetful functor E_n**-Alg** $\to$ **Lie-Alg**, induced by $A \mapsto A[n-1]$, has a left adjoint $U^{(n)}$: **Lie-Alg** $\to$ E_n**-Alg**, the E_n-enveloping algebra functor (see [33, 39] for a construction). For $n = 1$, this functor agrees with the standard universal enveloping algebra.

Proposition 45 *(*Francis [35]*) Let $\mathfrak{g}$ be a (differential graded) Lie algebra. There is an natural equivalence of E_n-coalgebras*

$$(U^{(n)}(\mathfrak{g}))^{(n)!} \;\cong\; C^{\bullet}_{Lie}(\mathfrak{g})$$

where $C^{\bullet}_{Lie}(\mathfrak{g})$ is the usual Chevalley-Eilenberg cochain complex (endowed with its differential graded commutative algebra structure).

Then using Propositions 44 and 45, we obtain for $n = 1, 3, 7$ that

$$\int_{S^n} U^{(n)}(\mathfrak{g}) \cong \left(\int_{S^n} C^{\bullet}_{Lie}(\mathfrak{g})\right)^{\vee}$$

which for $n = 1$ gives the following standard result computing the Hochschild homology groups of an universal enveloping algebra: $HH_*(U(\mathfrak{g})) \cong HH_*(C^{\bullet}_{Lie}(\mathfrak{g}))^{\vee}$. Applying the Fubini formula, we also find

$$\int_{S^1\times S^1} U^{(2)}(\mathfrak{g}) \cong CH_*\big(CH_*(C^{\bullet}_{Lie}(\mathfrak{g}))^{(1)!}\big).$$

[74] Which is equivalent to the category of L_∞-algebras.

7.6 Extended Topological Quantum Field Theories

In [70], Lurie introduced factorization homology as a (generalization of) an invariant of an extended topological field theory and offshoot of the cobordism hypothesis. We wish now to reverse this construction and explain very roughly how factorization homology can be used to produce an extended topological field theory.

Following [70], there is an ∞-category[75] of *extended topological field theories* with values in a symmetric monoidal (∞, n)-category with duals $(\mathcal{C}, \otimes)$. It is the category of symmetric monoidal functors $\mathbf{Fun}^{\otimes}\big((\mathrm{Bord}_n^{fr}, \coprod), (\mathcal{C}, \otimes)\big)$ where Bord_n^{fr} is the (∞, n)-category of bordisms of framed manifolds with monoidal structure given by disjoint union. In [70], Bord_n^{fr} is defined as an n-fold Segal space which precisely models the following *intuitive* notion of an (∞, n)-category whose objects are framed compact 0-dimensional manifolds. The morphisms between objects are framed 1-bordism, that is $\mathrm{Hom}_{\mathrm{Bord}_n^{fr}}(X, Y)$ consists of 1-dimensional framed manifolds T with boundary $\partial T = Y \coprod X^{op}$ (where X^{op} has the opposite framing to the one of the object X). The 2-morphisms in Bord_n^{fr} are framed 2-bordisms between 1-dimensional framed manifolds (with corners) and so on. The n-morphisms are n-framed bordisms between $n-1$-dimensional framed manifolds with corners, its $n+1$-morphisms diffeomorphisms and the higher morphisms are isotopies. Note that in the precise model of Bord_n^{fr}, the boundary component $N_1, \dots, N_r$ of a manifold M are represented by an open manifold with boundary components $N_1 \times \mathbb{R}, \dots, N_r \times \mathbb{R}$ (in other words are replaced by open collars).

There is an $(\infty, n+1)$-category $E_{\leq n}$**-Alg** whose construction is only sketched in [70] and detailled in [16] using a model based on factorization algebras. The category $E_{\leq n}\text{-}Alg$ can be described informally as the ∞-category with objects the E_n-algebras, 1-morphisms $\mathrm{Hom}_{E_{\leq n}\textbf{-Alg}}(A, B)$ is the space of all (A, B)-bimodules in E_{n-1}**-Alg** and so on. The (∞-)category $n\text{-}\mathrm{Hom}_{E_{\leq n}\textbf{-Alg}}(P, Q))$ of n-morphisms is the ∞-category of (P, Q)-bimodules where P, Q are E_1-algebras (with additional structure). In other words we have

$$n\text{-}\mathrm{Hom}_{E_{\leq n}\textbf{-Alg}}(P, Q)) \cong \{P\} \underset{\mathbf{Fac}^{lc}_{(-\infty,0)}}{\times} \mathbf{Fac}^{lc}_{\mathbb{R}_*} \underset{\mathbf{Fac}^{lc}_{(0,+\infty)}}{\times} \{R\}$$
$$\cong \{P\} \underset{E_1\textbf{-Alg}}{\times} \mathbf{BiMod} \underset{E_1\textbf{-Alg}}{\times} \{R\}$$

see Example 34. The composition

$$n\text{-}\mathrm{Hom}_{E_{\leq n}\textbf{-Alg}}(P, Q)) \times n\text{-}\mathrm{Hom}_{E_{\leq n}\textbf{-Alg}}(Q, R)) \longrightarrow n\text{-}\mathrm{Hom}_{E_{\leq n}\textbf{-Alg}}(P, R))$$

is given by tensor products of bimodules: ${}_P M_Q \otimes^{\mathbb{L}}_Q {}_Q N_R$ which in terms of factorization algebras is induced by the pushforward along the map $q : \mathbb{R}_* \times_{\mathbb{R}} \mathbb{R}_* \to \mathbb{R}_*$

[75] The cobordism hypothesis actually ensures that it is an ∞-groupoid.

where $\mathbb{R}_* \times_{\mathbb{R}} \mathbb{R}_*$ is identified with $\mathbb{R}$ stratified in two points -1 and 1 and q is the quotient map identifying the interval $[-1, 1]$ with the stratified point $0 \in \mathbb{R}_*$.

Let $E_{\leq n}$**-Alg**$_{(0)}$ be the (∞, n)-category obtained from $E_{\leq n}$**-Alg** by discarding non-invertible $n+1$-morphisms, that is $E_{\leq n}\textbf{-Alg}_{(0)} = Gr^{(n)}(E_{\leq n}\textbf{-Alg})$ where $Gr^{(n)}$ is the right adjoint of the forgetful functor $(\infty, n+1)$**-Cat** $\to (\infty, n)$**-Cat**.

The (∞, n)-category $E_{\leq n}\textbf{-Alg}_{(0)}$ is fully dualizable (the dual of an algebra is its opposite algebra) hence every E_n-algebra determines in an unique way a fully extended topological field theory by the cobordism hypothesis.

In fact, this extended field theory can be constructed by factorization homology. Let M be a m-dimensional manifold. We say that M is *stably n-framed* if $M \times \mathbb{R}^{n-m}$ is framed. Assume that M has two ends which are trivialized as $L \times \mathbb{R}^{op} \subset M$ and $R \times \mathbb{R} \subset M$, where L, R are stably framed codimension 1 closed sub-manifolds; here $\mathbb{R}^{op}$ means $\mathbb{R}$ endowed with the opposite framing to the standard one. For instance, M can be the interior of compact manifold $\overline{M}$ with two boundary component L, R and trivializations $L \times [0, \infty) \hookrightarrow \overline{M}$ and $R \times (-\infty, 0] \hookrightarrow \overline{M}$ where the trivialization on $L \times [0, \infty)$ has the opposite orientation as the one induced by M.

In that case, Lemma 5 (and Proposition 29) imply that the factorization homology $\int_M A$ is an E_{n-m}-algebra which is also a bimodule over the E_{n-m+1}-algebras $\big(\int_{L\times\mathbb{R}^{n-m+1}} A, \int_{R\times\mathbb{R}^{n-m+1}} A\big)$:

$$\int_{M\times\mathbb{R}^{n-m}} A \in \Big\{ \int_{L\times\mathbb{R}^{n-m+1}} A\Big\} \underset{E_1\textbf{-Alg}}{\times} \textbf{BiMod}\big(E_{n-m}\textbf{-Alg}\big) \underset{E_1\textbf{-Alg}}{\times} \Big\{ \int_{R\times\mathbb{R}^{n-m+1}} A\Big\}.$$

Thus $\int_M A$ is a m-morphism in $E_{\leq n}\textbf{-Alg}_{(0)}$ from R to L. In fact, one can prove

Theorem 18 ([16]) *Let A be an E_n-algebra. The rule which, to a stably n-framed manifold M of dimension m, associates*

$$Z_A(M) := \int_{M\times\mathbb{R}^{n-m}} A$$

extends as an extended field theory $Z_A \in \textbf{Fun}^{\otimes}\big(Bord_n^{fr}, E_{\leq n}\textbf{-Alg}_{(0)}\big)$.

8 Commutative Factorization Algebras

In this Section, we explain in details the relationship in between classical homology theory *à la* Eilenberg-Steenrod with factorization homology and more generally between (co)sheaves and factorization algebras. The main point is that when $\mathcal{C}$ is endowed with the monoidal structure given by the coproduct, then Factorization algebras boils down to the usual theory of cosheaves. This is in particular the case of factorization algebras with values in a category of commutative algebras E_∞-**Alg**$(\mathcal{C}, \otimes)$ in $(\mathcal{C}, \otimes)$.

8.1 Classical Homology as Factorization Homology

In this section we explain the relationship between factorization homology and singular homology (as well as generalized (co)homology theories for spaces).

Let **Chain**$(\mathbb{Z})$ be the (∞-)category of differential graded abelian groups (i.e. chain complexes of $\mathbb{Z}$-modules). It has a symmetric monoidal structure given by the direct sum of chain complexes, which is the coproduct in **Chain**$(\mathbb{Z})$. We can thus define homology theory for manifolds with values in (**Chain**$(\mathbb{Z}), \oplus)$. These are precisely (restrictions of) the (generalized) cohomology theories for spaces and nothing more. Recall that Mfld_n^{or} is the (∞-)category of oriented manifolds, Example 11.

Corollary 20 *Let G be an abelian group.*[76] *There is an unique homology theory for oriented manifolds with coefficient in G (Definition 11), that is (continuous) functor* $\mathcal{H}_G : \mathit{Mfld}_n^{or} \to \mathbf{Chain}(\mathbb{Z})$ *satisfying the axioms*

- ***(dimension)*** $\mathcal{H}_G(\mathbb{R}^n) \cong G$ *;*
- ***(monoidal)*** *the canonical map* $\bigoplus_{i\in I} \mathcal{H}_G(M_i) \xrightarrow{\simeq} \mathcal{H}_G(\coprod_{i\in I} M_i)$ *is a quasi-isomorphism;*
- ***(excision)*** *If M is an oriented manifold obtained as the gluing* $M = R \cup_{N\times\mathbb{R}} L$ *of two submanifolds along a a codimension* 1 *submanifold N of M with a trivialization* $N \times \mathbb{R}$ *of its tubular neighborhood in M, there is an natural equivalence*

$$\mathcal{H}_G(M) \xleftarrow{\simeq} \mathit{cone}\Big(\mathcal{H}_G(N\times\mathbb{R}) \xrightarrow{i_L - i_N} \mathcal{H}_G(L) \oplus \mathcal{H}_G(R)\Big).$$

  Here $\mathcal{H}_G(N\times\mathbb{R}) \xrightarrow{i_L} \mathcal{H}_G(L)$ *and* $\mathcal{H}_G(N\times\mathbb{R}) \xrightarrow{i_R} \mathcal{H}_G(R)$ *are the maps induced by functoriality by the inclusions of* $N \times \mathbb{R}$ *in L and R.*

Then, this homology theory is singular homology[77] *with coefficient in G. In particular, it extends as an homology theory for spaces.*

The uniqueness means of course up to a contractible choice, meaning that any two homology theory with coefficient in G will be naturally equivalent and any two choices of equivalences will also be naturally equivalent and so on.

Proof This is a consequence of Proposition 46 below applied to $\mathcal{C} = \mathbf{Chain}(\mathbb{Z})$ and the fact that the homotopy colimit is precisely computed by the cone[78] of the map $i_L - i_R$. □

[76] Or even a graded abelian group or chain complex of abelian groups.

[77] Or generalized exceptional homology when G is graded or a chain complex.

[78] Note that in this case, we know *a posteriori* that we can choose $\mathcal{H}_G$ to be singular chains, so that $\mathcal{H}_G(i_L) \oplus \mathcal{H}_G(i_R)$ is injective and the cone is equivalent to a quotient of chain complexes.

Remark 31 Let H be a topological group, $f : H \to \text{Homeo}(\mathbb{R}^n)$ a map of topological groups and $Bf : BH \to B\text{Homeo}(\mathbb{R}^n)$ the induced map, so that we have the category $\text{Mfld}_n^{(BH,Bf)}$ of manifolds with H-structure, see Example 11. As shown by its proof, Corollary 20 also holds with oriented manifolds replaced by manifold with a H-structure; in particular for *all manifolds or* a contrario for *framed manifolds*.

Remark 32 Corollary 20 and Theorem 10 implies that $\mathcal{H}_G(M)$ is computed by (derived) Hochschild homology $CH_M(G)$ (in $(\mathbf{Chain}(\mathbb{Z}), \oplus)$). If $M_\bullet$ is a simplicial set model of M, then $\mathcal{H}_G(M) \cong CH_{M_\bullet}(G)$ which is exactly (by Sect. 2.3) the chain complex of the simplicial abelian group $G[M_\bullet]$. In particular, for $M_\bullet = \Delta_\bullet(M) = \text{Hom}(\Delta^\bullet, M)$, one recovers *exactly* the singular chain complex $C_*(M)$ of M.

Corollary 20 is a particular case of a more general result which we now describe. Let $\mathcal{C}$ be a category with coproducts. Then $(\mathcal{C}, \coprod)$ is symmetric monoidal, with unit given by its initial object $\emptyset$. As we have seen in Sect. 2.1, any object X of $\mathcal{C}$ carries a canonical (thus natural in X) structure of commutative algebra in $(\mathcal{C}, \coprod)$ which is given by the "multiplication" $X \coprod X \stackrel{\coprod id_X}{\longrightarrow} X$ induced by the identity map $id_X : X \to X$ on each component. This algebra structure is further unital, with unit given by the unique map $\emptyset \to X$. This defines a functor *triv* $: \mathcal{C} \to E_\infty\textbf{-Alg}(\mathcal{C})$ which to an object associates its trivial commutative algebra structure. In fact, the latter algebras are the only only possible commutative and even associative ones in $(\mathcal{C}, \coprod)$.

Lemma 10 (Eckman-Hilton principle) *Let $\mathcal{C}$ be a category with coproducts and $(\mathcal{C}, \coprod)$ the associated symmetric monoidal category. Let $H \stackrel{f}{\to} \text{Homeo}(\mathbb{R}^n)$ be a topological group morphism and $\iota : \text{Disk}_n^{(BH,Bf)}\textbf{-Alg}(\mathcal{C}) \to \mathcal{C}$ be the underlying object functor* (19) *(Definition 9). We have a commutative diagram of equivalences*

$$\begin{array}{ccc} E_\infty\textbf{-Alg}(\mathcal{C}) & \xrightarrow{\simeq} & \text{Disk}_n^{(BH,Bf)}\textbf{-Alg}(\mathcal{C}) \\ & {\scriptstyle triv}\nwarrow^{\simeq} \quad \swarrow^{\simeq}{\scriptstyle \iota} & \\ & \mathcal{C} & \end{array}$$

where the horizontal arrow is the canonical functor of Example 13.

In particular, any E_n-algebra ($n \geq 1$) in $(\mathcal{C}, \coprod)$ is a (trivial) commutative algebra.

Proof Let $I = \{1, \ldots, n\}$ be a finite set and $m_I : \coprod_{i \in I} C \to C$ be any map. The universal property of the coproduct yields a commutative diagram

$$\begin{array}{ccc}
\left(\coprod_{i\in I} C\right)\coprod\left(\coprod_{i\in I} C\right) & \xrightarrow{m_I \coprod m_I} & C\coprod C \\
\Big\downarrow {\scriptstyle \left(\coprod_{i\in I}(\coprod id_C)\right)\circ\sigma_I} & & \Big\downarrow {\scriptstyle \coprod id_C} \\
\coprod_{i\in I} C & \xrightarrow[m_I]{} & C
\end{array}$$

where σ_I is the permutation induced by the bijection $(1, n+1)(2, n+2)\cdots(n, 2n)$ of $I\coprod I = \{1, \ldots, 2n\}$ on itself. Hence, if C is an E_n-algebra, it is naturally an object of E_n**-Alg**$(E_\infty$**-Alg**$)$ where the commutative algebra structure is the trivial one $C\coprod C \stackrel{\coprod id_C}{\rightarrow} C$. By Dunn Theorem 11 (or Eckman-Hilton principle), the forgetful map (induced by the pushforward of factorization algebras) E_n**-Alg**$((E_\infty$**-Alg**$)) \longrightarrow E_0$**-Alg**$(E_\infty$**-Alg**$)$ is an equivalence. It follows that the E_n-algebra C is an E_∞-algebra whose structure is equivalent to *triv*(C).

The group map $\{1\} \to H$ induces a canonical functor $\text{Disk}_n^{(BH,Bf)}$**-Alg** $\to E_n$**-Alg** so that the above result implies that such a $\text{Disk}^{(BH,Bf)}$-algebra with underlying object C is necessarily of the form *triv*(C). □

Proposition 46 *Let* $(\mathcal{C}, \coprod)$ *be a* ∞*-category whose monoidal structure is given by the coproduct and* $f : H \to \text{Homeo}(\mathbb{R}^n)$ *be a topological group morphism .*

- *Any homology theory for* (BH, Bf)*-structured manifolds (Definition 10)* extends uniquely *into an homology theory for spaces (Definition 1).*
- *Any object* $C \in \mathcal{C}$ *determines a unique homology theory for* (BH, Bf)*-manifolds with values in C (Definition 11); further the evaluation map* $\mathcal{H} \mapsto \mathcal{H}(\mathbb{R}^n)$ *is an equivalence between the category of homology theories for* (BH, Bf)*-manifolds in* $(\mathcal{C}, \coprod)$ *and* $\mathcal{C}$*.*

Proof By Theorem 6, homology theories for (BH, Bf)-structured manifolds are equivalent to $\text{Disk}_n^{(BH,Bf)}$-algebras which, by Lemma 10, are equivalent to $\mathcal{C}$. In particular, any $\text{Disk}_n^{(BH,Bf)}$-algebra is given by the commutative algebra associated to an object of $\mathcal{C}$ so that by Theorem 10, it extends to an homology theory for spaces.

8.2 Cosheaves as Factorization Algebras

In this Section we identify (pre-)cosheaves and (pre-)factorization algebras when the monoidal structure is given by the coproduct.

Let $(\mathcal{C}, \otimes)$ be symmetric monoidal and let $\mathcal{F}$ be in $\mathbf{PFac}_X(\mathcal{C})$. Then the structure maps $\mathcal{F}(U) \to \mathcal{F}(V)$ for any open U inside an open V induces a functor $\gamma_{\mathcal{C}} : \mathbf{PFac}_X(\mathcal{C}) \longrightarrow \mathbf{PcoShv}_X(\mathcal{C})$ where $\mathbf{PcoShv}_X(\mathcal{C}) := \mathbf{Fun}(\text{Open}(X), \mathcal{C})$ is the ∞-category of *precosheaves on* X *with values in* $\mathcal{C}$.

Lemma 11 *Let $(\mathcal{C}, \coprod)$ be an ∞-category with coproducts whose monoidal structure is given by the coproduct and X be a topological space.*

1. *The functor $\gamma_{\mathcal{C}} : \boldsymbol{PFac}_X(\mathcal{C}) \longrightarrow \boldsymbol{PcoShv}_X(\mathcal{C})$ is an equivalence.*
2. *If X has a factorizing basis of opens,*[79] *then the functor $\gamma_{\mathcal{C}} : \boldsymbol{PFac}_X(\mathcal{C}) \longrightarrow \boldsymbol{PcoShv}_X(\mathcal{C})$ restricts to an equivalence*

$$\boldsymbol{Fac}_X(\mathcal{C}) \xrightarrow{\simeq} \boldsymbol{coShv}_X(\mathcal{C})$$

   between factorization algebras on X and the ∞-category $\boldsymbol{coShv}_X(\mathcal{C})$ of (homotopy) cosheaves on X with values in $\mathcal{C}$.
3. *If X is a manifold, then the above equivalence also induces an equivalence $\boldsymbol{Fac}^{lc}_X(\mathcal{C}) \xrightarrow{\simeq} \boldsymbol{coShv}^{lc}_X(\mathcal{C})$ between locally constant factorization algebras and locally constant cosheaves.*

Proof Let $\mathcal{F}$ be in $\mathbf{PFac}_X$ and $U_1, \ldots, U_r$ be open subsets of $V \in Open(X)$, which are pairwise disjoint. Let $\rho_{U_1,\ldots,U_i,V} : \mathcal{F}(U_1) \coprod \cdots \coprod \mathcal{F}(U_i) \to \mathcal{F}(V)$ be the structure map of $\mathcal{F}$. The associativity of the structure maps (diagram (22)) shows that the structure map $\rho_{U_j,V} : \mathcal{F}(U_j) \to \mathcal{F}(V)$ factors as

$$\begin{array}{ccc}
\mathcal{F}(U_j) & \xrightarrow{\quad\rho_{U_j,V}\quad} & \mathcal{F}(V) \\
\cong\big\downarrow & & \big\uparrow \rho_{U_1,\ldots,U_i,V} \\
\varnothing \coprod \ldots \varnothing \coprod \mathcal{F}(U_j) \coprod \varnothing \ldots \coprod \varnothing & \xrightarrow{\left(\coprod_{k=1}^{j-1} \rho_{\varnothing,U_k}\right) \coprod id \coprod \left(\coprod_{k=j+1}^{i} \rho_{\varnothing,U_k}\right)} & \mathcal{F}(U_1) \coprod \ldots \coprod \mathcal{F}(U_i).
\end{array}$$

The universal property of the coproduct implies that the following diagram

$$\begin{array}{ccc}
\mathcal{F}(U_1) \coprod \ldots \coprod \mathcal{F}(U_i) & \xrightarrow{\quad\rho_{U_1,\ldots,U_i,V}\quad} & \mathcal{F}(V) \\
\rho_{U_1,V} \coprod \cdots \coprod \rho_{U_i,V} \big\downarrow & \nearrow_{\coprod id_{\mathcal{F}(V)}} & \\
\mathcal{F}(V) \coprod \ldots \coprod \mathcal{F}(V) & &
\end{array} \tag{59}$$

is commutative. It follows that the structure maps are completely determined by the precosheaf structure. Conversely, any precosheaf on $\mathcal{C}$ gives rise functorially to a prefactorization algebra with structure maps given by the composition

$$\mathcal{F}(U_1) \coprod \cdots \coprod \mathcal{F}(U_i) \longrightarrow \mathcal{F}(V) \coprod \cdots \coprod \mathcal{F}(V) \xrightarrow{\coprod id_{\mathcal{F}(V)}} \mathcal{F}(V)$$

[79] For instance when X is Hausdorff.

yielding a functor $\theta_{\mathcal{C}} : \mathbf{PcoShv}_X(\mathcal{C}) \to \mathbf{PFac}_X(\mathcal{C})$. The commutativity of diagram (59) implies that this functor $\theta_{\mathcal{C}}$ is inverse to $\gamma_{\mathcal{C}} : \mathbf{PFac}_X(\mathcal{C}) \to \mathbf{PcoShv}_X(\mathcal{C})$.

Note that both the cosheaf condition and the factorization algebra conditions implies that the canonical map $\mathcal{F}(U_1)\coprod\cdots\coprod\mathcal{F}(U_i) \to \mathcal{F}(U_1\coprod\cdots\coprod U_i)$ is an equivalence (of constant simplicial objects). Now, we can identify the two gluing conditions. Since $\mathcal{U}$ is a (factorizing) cover of V, then

$$\mathcal{P}(U) := \{\{U_1, \dots, U_k\}, \text{ which are pairwise disjoint}\}$$

is a cover of V. Further, if $\mathcal{F} \in \mathbf{Fac}_X$, the Čech complex $\check{C}(\mathcal{U}, \mathcal{F})$ is precisely the (standard) Čech complex $\check{C}^{\text{cosheaf}}(P\mathcal{U}, \gamma_{\mathcal{C}}(\mathcal{F}))$ of the cosheaf $\gamma_{\mathcal{C}}(\mathcal{F})$ computed on the cover $P\mathcal{U}$ so that the map $\check{C}(\mathcal{U}, \mathcal{F}) \to \mathcal{F}(V)$ is an equivalence if and only if $\check{C}^{\text{cosheaf}}(P\mathcal{U}, \gamma_{\mathcal{C}}(\mathcal{F})) \to \mathcal{F}(V)$ is an equivalence.

Assume X has a factorizing basis of opens. Both factorization algebras and cosheaf are determined by their restriction on a basis of opens. It follows that $\gamma_{\mathcal{C}}$ sends factorization algebras to cosheaves and $\theta_{\mathcal{C}}$ sends cosheaves to factorization algebras.

It remains to consider the locally constant condition when X is a manifold, thus has a basis of euclidean neighborhood. On each euclidean neighborhood D, by Theorem 9, the restriction of $\mathcal{F} \in \mathbf{Fac}_X^{lc}$ to D is the factorization algebra given by an E_n-algebra $A \in E_n\text{-}\mathbf{Alg}((\mathcal{C}, \coprod))$ in $\mathcal{C}$. Lemma 10 implies that A is given by the trivial commutative algebra $triv(C)$ associated to an object $C \in \mathcal{C}$. It follows from the identification of Čech complexes above, that $\mathcal{F}_{|D}$ is thus equivalent to the constant cosheaf on D associated to the object C. The converse follows from the fact if $\mathcal{G} \in \mathbf{coShv}_X^{lc}(\mathcal{C})$, then for any point $x \in X$, there is an euclidean neighborhood $D_x \cong \mathbb{R}^n$ on which $\mathcal{G}_{|D_x}$ is constant. The identification of the Čech complexes above implies that $\theta_{\mathcal{C}}(\mathcal{G}_{|D_x})$ is locally constant on D_x. Proposition 13 implies that the factorization algebra $\theta_{\mathcal{C}}(\mathcal{G})$ is locally constant on X, hence finishes the proof. □

From the identification between cosheaves and factorization algebras, we deduce that factorization homology in $(\mathcal{C}, \coprod)$ agrees with homology with local coefficient:

Proposition 47 *Let $(\mathcal{C}, \coprod)$ be a ∞-category whose monoidal structure is given by the coproduct and X a manifold.*

- *There is an equivalence between homology theories for (X, TX)-structured manifolds (Definition 10) and $\mathbf{coShv}_X^{lc}(\mathcal{C})$, the ($\infty$-)category of locally constant cosheaves on X with values in $\mathcal{C}$.*
- *The above equivalence is given, for any (X, TX)-structured manifold M and (homotopy) cosheaf $\mathcal{G} \in \mathbf{coShv}_X^{lc}(\mathcal{C})$, by*

$$\int_M \mathcal{G} := \mathbb{R}\Gamma(M, \mathcal{G})$$

the cosheaf homology of M with values in the cosheaf $p^*(\mathcal{G})$ where $p : M \to X$ is the map defining the (X, TX)-structure.

Proof The first claim is an immediate application of Lemma 11(3) and Theorem 8. The latter result implies that factorization homology is computed by the Čech complex of the locally constant factorization algebra associated to a $\mathrm{Disk}_n^{(M,TM)}$-algebra given by the pullback along $p: M \to X$ of some $A \in \mathrm{Disk}_n^{(X,TX)}$**-Alg**. Now the second claims follows from Lemma 11(2). □

Remark 33 Factorization homology on a (X, e)-structured manifold M depends only on its value on open sub sets of M. Thus Proposition 47 implies that, for any manifold M and $A \in \mathrm{Disk}_n^{(X,e)}\text{-}\mathbf{Alg}(\mathcal{C})$, factorization homology $\int_M A$ is given by cosheaf homology of the locally constant cosheaf $\mathcal{G}$ given by the image of A under the functor $\mathrm{Disk}_n^{(X,e)}\text{-}\mathbf{Alg} \to \mathrm{Disk}_n^{(M,TM)}\text{-}\mathbf{Alg}$ (of Example 12) and the equivalence $\mathbf{Fac}_X^{lc}(\mathcal{C}) \xrightarrow{\simeq} \mathbf{coShv}_X^{lc}(\mathcal{C})$ of Lemma 11.

Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal (∞-)category. We say that a factorization algebra $\mathcal{F}$ on X is *commutative* if each $\mathcal{F}(U)$ is given a structure of differential graded commutative (or E_∞-) algebra and the structure maps are maps of algebras. In other words, the category of *commutative factorization algebras* is $\mathbf{Fac}_X(E_\infty\text{-}\mathbf{Alg})$.

A peculiar property of (differential graded) commutative algebras is that their coproduct is given by their tensor product (that is the underlying tensor product in $\mathcal{C}$ endowed with its canonical algebra structure). From Lemma 11, we obtain the following:

Proposition 48 *Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal (∞-)category. The functor*

$$\gamma_{E_\infty\text{-}\mathbf{Alg}(\mathcal{C})} : \mathbf{Fac}_X(E_\infty\text{-}\mathbf{Alg}(\mathcal{C})) \longrightarrow \mathbf{coShv}_X(E_\infty\text{-}\mathbf{Alg}(\mathcal{C}))$$

is an equivalence.

In other words, commutative factorization algebras are cosheaves (in E_∞-**Alg**).

Remark 34 In view of Proposition 48, one can think general factorization algebras as non-commutative cosheaves.

Combining Propositions 47 and 48 and Theorem 10, we obtain:

Corollary 21 *Let $\mathcal{F} \in \mathbf{Fac}_X^{lc}(E_\infty\text{-}\mathbf{Alg})$ be a locally constant commutative factorization algebra on X. Then*

$$\int_X A \cong \mathcal{CH}_X(\mathcal{F})$$

where $\mathcal{CH}_X(\mathcal{F})$ is the (derived) global section of the cosheaf which to any open U included in an euclidean Disk D associates the derived Hochschild homology $CH_U(\mathcal{F}(D))$.

9 Complements on Factorization Algebras

In this Section, we give several proofs of results, some of them probably known by the experts, about factorization algebras that we have postponed and for which we do not know any reference in the literature.

9.1 Some Proofs Related to the Locally Constant Condition and the Pushforward

9.1.1 Proof of Propositions 13.13 and 13.24

Let $U \subset D \subset M$ be an inclusion of open disks; we need to prove that $\mathcal{F}(U) \to \mathcal{F}(D)$ is a quasi-isomorphism. We can assume $D = \mathbb{R}^n$ (by composing with a homeomorphism); the proof in the stratified case is similar to the non-stratified one by replacing $\mathbb{R}^n$ with $\mathbb{R}^i \times [0, +\infty)^{n-i}$. We first consider the case where $\mathcal{F}_{|U}$ is locally constant and further, that U is an euclidean disk (with center x and radius r_0). Denote $D(y, r)$ an euclidean open disk of center y and radius $r > 0$ and let

$$T_+ := \sup(t \in \mathbb{R},\ \text{such that} \forall\ \frac{r_0}{2} \leq s < t, \mathcal{F}(D(x, \frac{r_0}{2}) \to \mathcal{F}(D(x, s)) \text{is an equivalence}).$$

By assumption $T_+ \geq r_0$. We claim that $T = +\infty$. Indeed, let T be finite and such that , $\mathcal{F}(D(x, \frac{r_0}{2}) \to \mathcal{F}(D(x, s))$ is an equivalence for all $s < T$. We will prove that T can not be equal to T_+. Every point y on the sphere of center x and radius T has a neighborhood in which $\mathcal{F}$ is locally constant. In particular, there is a number $\epsilon_y > 0$, an open angular sector $S_{[0,T+\epsilon_y)}$ of length $T + \epsilon_y$ and angle θ_y containing y such that $\mathcal{F}_{|S_{(T-\epsilon_y,T+\epsilon_y)}}$ is locally constant. Here, $S_{(\tau,\gamma)}$ denotes the restriction of the angular sector to the band containing numbers of radius lying in (τ, γ).

We first note that $S_{[0,T+\epsilon_y)}$ has a factorizing cover $\mathcal{A}_y$ consisting of open angular sectors of the form $S_{(T-\tau,T+\epsilon_y)}$ $(0 < \tau \leq \epsilon_y)$ and $S_{[0,\kappa)}$ $(0 < \kappa < T)$; there is an induced similar cover $\mathcal{A}_y \cap U$ of $S_{[0,T)}$ given by the angular sectors of the form $S_{(T-\tau,T)}$ $(0 < \tau \leq \epsilon_y)$ and $S_{[0,\kappa)}$ $(0 < \kappa < T)$. The structure maps $\mathcal{F}(S_{(T-\tau,T)}) \to \mathcal{F}(S_{(T-\tau,T+\epsilon_y)})$ induce a map of Čech complex $\psi_y : \check{C}(\mathcal{A}_y \cap U, \mathcal{F}) \to \check{C}(\mathcal{A}_y, \mathcal{F})$ so that the following diagram is commutative:

$$\begin{array}{ccc} \check{C}(\mathcal{A}_y \cap U, \mathcal{F}) & \xrightarrow{\psi_y} & \check{C}(\mathcal{A}_y, \mathcal{F}) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathcal{F}(S_{[0,T)}) & \longrightarrow & \mathcal{F}(S_{[0,T+\epsilon_y)}) \end{array} \quad . \tag{60}$$

Since the map $S_{(T-\tau,T+\epsilon_y)} \to S_{(T-\tau,T+\epsilon_y)}$ is the inclusion of a sub-disk inside a disk in $S_{(T-\epsilon_y,T+\epsilon_y)}$, it is a quasi-isomorphism and, thus, so is the map ψ_y : $\check{C}(\mathcal{A}_y \cap U, \mathcal{F}) \to \check{C}(\mathcal{A}_y, \mathcal{F})$. It follows from diagram (60) that the structure map $\mathcal{F}(S_{[0,T)}) \to \mathcal{F}(S_{[0,T+\epsilon_y)})$ is a quasi-isomorphism. In the above proof, we could also have taken any angle $\theta \leq \theta_y$ or replaced ϵ_y by any $\epsilon \leq \epsilon_y$ without changing the result.

By compactness of the sphere of radius T, we can thus find an $\epsilon > 0$ and a $\theta > 0$ such that the structure map $\mathcal{F}(S \cap U) \to \mathcal{F}(S)$ is a quasi-isomorphism for any angular sector S around x of radius $r = T + \epsilon' < T + \epsilon$ and arc length $\phi_S < \theta$. The collection of such angular sectors S is a (stable by intersection) factorizing basis of the disk $D(x, T + \epsilon')$ while the collection of sectors $S \cap U$ is a (stable by intersection) factorizing basis of the disk $D(x, T)$. Further, we have proved that the structure maps $\mathcal{F}(S \cap U) \to \mathcal{F}(S)$ is a quasi-isomorphism for any such S. It follows that the map $\mathcal{F}(D(x, T)) \to \mathcal{F}(D(x, T + \epsilon')$ is a quasi-isomorphism (since again the induced map in between the Čech complexes associated to this two covers is a quasi-isomorphism). It follows that $T_+ > T$ for any finite T hence is infinite as claimed above. In particular, the canonical map $\mathcal{F}(D(x, T)) \to \mathcal{F}(D(x, T + r))$ is a quasi-isomorphism for any $r \geq 0$.

Now, since the collection of disks of radius $T > 0$ centered at x is a factorizing cover of $\mathbb{R}^n$, we deduce that $\mathcal{F}(D(x, T)) \to \mathcal{F}(\mathbb{R}^n)$ is a quasi-isomorphism. Indeed, fix some $R > 0$ and let $j_R : x + y \mapsto x + R/(R - |y|)y$ be the homothety centered at x mapping $D(x, R)$ homeomorphically onto $\mathbb{R}^n$. The map j_R is a bijection between the set $\mathcal{D}_R$ of (euclidean) sub-disks of $D(x, R)$ centered at x and the set $\mathcal{D}$ of all (euclidean) disks of $\mathbb{R}^n$ centered at x. For any disk centered at x, the inclusion $D(x, T) \hookrightarrow D(x, j_R(T))$ yields a quasi-isomorphism $\mathcal{F}(D(x, T)) \to \mathcal{F}(D(x, j_R(T))$. If $\alpha = \{D(x, r_0), \ldots, D(x, r_i)\} \in (P\mathcal{D}_R)^{i+1}$, we thus get a quasi-isomorphism

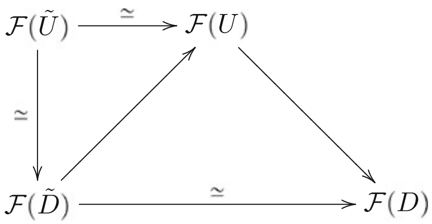
$$\mathcal{F}(\alpha) \cong \mathcal{F}\Big(D\big(x, \min(r_j, j = 0 \ldots i)\big)\Big) \xrightarrow{\simeq} \mathcal{F}\Big(D\big(x, \min(j_R(r_j), j = 0 \ldots i)\big)\Big) \cong \mathcal{F}(j_R(\alpha)).$$

Assembling those for all α's yields a quasi-isomorphism $\check{C}(\mathcal{D}_R, \mathcal{F}) \xrightarrow{\simeq} \check{C}(\mathcal{D}, \mathcal{F})$ which fit into a commutative diagram

$$\begin{array}{ccc}
\check{C}(\mathcal{D}_R, \mathcal{F}) & \xrightarrow{\simeq} & \check{C}(\mathcal{D}, \mathcal{F}) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathcal{F}(D(x, R)) & \longrightarrow & \mathcal{F}(\mathbb{R}^n)
\end{array}$$

whose vertical arrows are quasi-isomorphisms since $\mathcal{F}$ is a factorization algebra. It follows that the lower horizontal arrow is a quasi-isomorphism as claimed.

We are left to prove the result for $U \hookrightarrow D = \mathbb{R}^n$ when U is not necessarily an euclidean disk. Choose an euclidean open disk $\tilde{D}$ inside U small enough so that $\mathcal{F}_{|\tilde{D}}$ is locally constant. Let $h : \tilde{D} \cong \mathbb{R}^n$ be an homothety (with the same center as $\tilde{D}$) identifying $\tilde{D}$ and $\mathbb{R}^n$. Then $\tilde{U} := h^{-1}(U) \subset D \subset U$ is an open disk homothetic to U. So that by the above reasoning (after using an homeomorphism between U and an euclidean disk $\mathbb{R}^n$) we have that the structure map $\mathcal{F}(\tilde{U}) \to \mathcal{F}(U)$ is a quasi-isomorphism as well. Since $\mathcal{F}_{|\tilde{D}}$ is locally constant, the structure map $\mathcal{F}(\tilde{U}) \to \mathcal{F}(\tilde{D})$ is a quasi-isomorphism. Now, Proposition 13 follows from the commutative diagram



which implies that the structure map $\mathcal{F}(U) \to \mathcal{F}(D)$ is a quasi-isomorphism.

9.1.2 Proof of Proposition 18

First we check that if $\mathcal{F}$ is a factorization algebra on $X \times Y$ and $U \subset X$ is open, then $\underline{\pi_1}_*(\mathcal{F}(U))$ is a factorization algebra over Y. If $\mathcal{V}$ is a factorizing cover of an open set $V \subset Y$, then $\{U\} \times \mathcal{V}$ is a factorizing cover of $U \times V$ and the Čech complex $\check{C}\big(\mathcal{V}, \underline{\pi_1}_*\mathcal{F}(U)\big)$ is equal to $\check{C}(\{U\} \times \mathcal{V}, \mathcal{F})$. Hence the natural map $\check{C}\big(\mathcal{V}, \underline{\pi_1}_*\mathcal{F}(U)\big) \to \underline{\pi_1}_*(\mathcal{F})(U, V)$ factors as

$$\check{C}\big(\mathcal{V}, \underline{\pi_1}_*\mathcal{F}(U)\big) = \check{C}(\{U\} \times \mathcal{V}, \mathcal{F}) \to \mathcal{F}(U \times V) = \underline{\pi_1}_*(\mathcal{F})(U, V).$$

It is a quasi-isomorphism since $\mathcal{F}$ is a factorization algebra. We have proved that $\underline{\pi_1}_*(\mathcal{F}) \in \mathbf{PFac}_X(\mathbf{Fac}_Y)$. To show that $\underline{\pi_1}_*(\mathcal{F}) \in \mathbf{Fac}_X(\mathbf{Fac}_Y)$, we only need to check that for every open $V \subset Y$, and any factorizing cover $\mathcal{U}$ of U, the natural map $\check{C}\big(\mathcal{U}, \underline{\pi_1}_*(\mathcal{F})(-, V)\big) \to \underline{\pi_1}_*(\mathcal{F})(U, V)$ is a quasi-isomorphism, which follows by the same argument. Hence $\underline{\pi_1}_*$ factors as a functor $\underline{\pi_1}_* : \mathbf{Fac}_{X\times Y} \longrightarrow \mathbf{Fac}_X(\mathbf{Fac}_Y)$.

When $\mathcal{F}$ is locally constant, Proposition 15 applied to the first and second projection implies that $\underline{\pi_1}_*(\mathcal{F}) \in \mathbf{Fac}_X^{lc}(\mathbf{Fac}_Y^{lc})$.

Now we build an inverse of $\underline{\pi_1}_*$ in the locally constant case. Let $\mathcal{B}$ be in $\mathbf{Fac}_X(\mathbf{Fac}_Y)$. A (stable by finite intersection) basis of neighborhood of $X \times Y$ is given by the products $U \times V$, with $(U, V) \in \mathcal{CV}(X) \times \mathcal{CV}(Y)$ where $\mathcal{CV}(X)$, $\mathcal{CV}(Y)$ are bounded geodesically convex neighborhoods (for some choice of Riemannian metric on X and Y). Thus by Sect. 5.2, in order to extend $\mathcal{B}$ to a factorization algebra on $X \times Y$, it is enough to prove that the rule $(U \times V) \mapsto \mathcal{B}(U)(V)$ (where $U \subset X$, $V \subset Y$) defines an $\mathcal{CV}(X) \times \mathcal{CV}(Y)$-factorization algebra. If $U \times V \in \mathcal{CV}(X) \times \mathcal{CV}(Y)$, then U

and V are canonically homeomorphic to $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Now, the construction of the structure maps for opens in $\mathcal{CV}(U)\times\mathcal{CV}(V)$ restricts to proving the result for $\mathcal{B}_{|U\times V} \in \mathbf{Fac}^{lc}_{\mathbb{R}^n}(\mathbf{Fac}^{lc}_{\mathbb{R}^m})$. By Theorem 9, $\mathbf{Fac}_{\mathbb{R}^d} \cong E_d\textbf{-Alg}$, hence by Dunn Theorem 11 below, we have that $\mathbf{Fac}^{lc}_{\mathbb{R}^{n+m}} \xrightarrow{\underline{\pi_1}_*} \mathbf{Fac}^{lc}_{\mathbb{R}^n}(\mathbf{Fac}^{lc}_{\mathbb{R}^m})$ is an equivalence which allows to define a $\mathcal{CV}(X)\times\mathcal{CV}(Y)$-factorization algebra structure associated to $\mathcal{B}$. We denote $j(\mathcal{B}) \in \mathbf{Fac}_{X\times Y}$ the induced factorization algebra on $X\times Y$. Note that $j(\mathcal{B})$ is locally constant, since, again, the question reduces to Dunn Theorem.

It remains to prove that $j : \mathbf{Fac}^{lc}_X(\mathbf{Fac}^{lc}_Y) \to \mathbf{Fac}_{X\times Y}$ is a natural inverse to $\underline{\pi_1}_*$. This follows by uniqueness of the factorization algebra extending a factorization algebra on a factorizing basis, that is, Proposition 17.

9.2 *Complements on Subsection 6.2*

Here we collect the proofs of statements relating factorization algebras and intervals.

9.2.1 Proof of Proposition 27

The (sketch of) proof is extracted from the excision property for factorization algebras in [47]. By definition of a Disk_1^{fr}-algebra, a factorization algebra $\mathcal{G}$ on $\mathbb{R}$ carries a structure of Disk_1^{fr}-algebra (by simply restricting the value of $\mathcal{G}$ to open sub-intervals, just as in Remark 23).

Similarly, if $\mathcal{G}$ is a factorization algebra on $[0,+\infty)$, it carries a structure of a Disk_1^{fr}-algebra and a (pointed) right module over it, while a factorization algebra on $(-\infty,0]$ carries the structure of a left (pointed) module over a Disk_1^{fr}-algebra (see Definition 36). It follows that a factorization algebra over the closed interval $[0,1]$ determines an E_1-algebra $\mathcal{A}$ and pointed left module $\mathcal{M}^\ell$ and pointed right module $\mathcal{M}^r$ over $\mathcal{A}$.

By strictification we can replace the E_1-algebra and modules by differential graded associative ones so that we are left to the case of a factorization algebra $\mathcal{F}$ on $[-1,1]$ which, on the factorizing basis $\mathcal{I}$ of $[0,1]$, is precisely the $\mathcal{I}$-prefactorization algebra $\mathcal{F}$ defined before the Proposition 27.

Now, we are left to prove that, for any $A, M^\ell, M^r, m^r, m^\ell, \mathcal{F}$ is a $\mathcal{I}$-factorization algebra, and then to compute its global section $\mathcal{F}([0,1])$. Theorem 9 implies that the restriction $\mathcal{F}_A$ of $\mathcal{F}$ to $(0,1)$ is a factorization algebra. In order to conclude we only need to prove that the canonical maps

$$\check{C}\big(\mathcal{U}_{[0,1)},\mathcal{F}\big) \longrightarrow \mathcal{F}([0,1)) = M^r, \quad \check{C}\big(\mathcal{U}_{(0,1]},\mathcal{F}\big) \longrightarrow \mathcal{F}((0,1]) = M^\ell$$

$$\text{and} \quad \check{C}\big(\mathcal{U}_{[0,1]},\mathcal{F}\big) \longrightarrow \mathcal{F}([0,1]) \cong M^r \overset{\mathbb{L}}{\underset{A}{\otimes}} M^\ell$$

are quasi-isomorphisms. Here, $\mathcal{U}_{[0,1]}$ is the factorizing covers given by all opens $U_t := [0,1]\setminus\{t\}$ where $t \in [0,1]$ (in other words by the complement of a singleton). Similarly, $\mathcal{U}_{[0,1)}, \mathcal{U}_{(0,1]}$ are respectively covers given by all opens $U_t^{\ell} := ([0,1)\setminus\{t\}$ where $t \in [0,1)$ and all opens $U_t^r := (0,1]\setminus\{t\}$ where $t\in(0,1]$.

The proof in the 3 cases are essentially the same so we only consider the case of the opens U_t. Since $M^r \overset{\mathbb{L}}{\underset{A}{\otimes}} M^{\ell} \cong M^r \underset{A}{\otimes} B(A,A,A)\otimes_A M^{\ell}$ where $B(A,A,A)$ is the two-sided Bar construction of A, it is enough to prove the result for $M^r = M^{\ell} = A$ in which case we are left to prove that the canonical map

$$\check{C}\big(\mathcal{U}_{[0,1]},\mathcal{F})\big) \longrightarrow A \overset{\mathbb{L}}{\underset{A}{\otimes}} A \cong B(A,A,A) \xrightarrow{\simeq} A$$

is an equivalence.

Any two open sets in $\mathcal{U}_{[0,1]}$ intersect non-trivially so that the set $P\mathcal{U}$ are singletons. We have $\mathcal{F}(U_t)\cong \mathcal{F}([-1,t))\otimes\mathcal{F}((t,1])$ which is $A\otimes A$ if $t\neq\pm1$ and is $A\otimes k$ or $k\otimes A$ if $t=1$ or $t=-1$. More generally,

$$\mathcal{F}(U_{t_0},\dots,U_{t_n},U_{\pm1})\cong\mathcal{F}(U_{t_0},\dots,U_{t_n})\otimes k.$$

Further, if $0<t_0<\cdots<t_n<1$, then $\mathcal{F}(U_{t_0},\dots,U_{t_n})\cong A\otimes A^{\otimes n}\otimes A$ and the structure map $\mathcal{F}(U_{t_0},\dots,U_{t_n})\to\mathcal{F}(U_{t_0},\dots,\widehat{U_{t_i}},\dots,U_{t_n})$ is given by the multiplication

$$a_0\otimes\cdots\otimes a_{n+1}\mapsto a_0\otimes\cdots(a_ia_{i+1})\otimes\cdots\otimes a_{n+1}.$$

This identifies the Čech complex $\check{C}\big(\mathcal{U},\mathcal{F}\big)$ with a kind of parametrized analogue of the standard two sided Bar construction with coefficients in A. We have canonical maps

$$\phi_t:\mathcal{F}(U_t)\cong\mathcal{F}([-1,t))\otimes\mathcal{F}((t,1])\to\mathcal{F}([-1,1))\otimes\mathcal{F}((-1,1])\cong A\otimes A\to A$$

induced by the multiplication in A. The composition $\bigoplus_{U_r,U_s\in P\mathcal{U}}\mathcal{F}(U_r,U_s)[1]\to\bigoplus_{U_t\in P\mathcal{U}}\mathcal{F}(U_t)[0]\to A$ is the zero map so that we have a map of (total) chain complexes: $\eta:\check{C}\big(\mathcal{U},\mathcal{F}\big)\to A$. In order to prove that η is an equivalence, we consider the retract $\kappa: A\cong\mathcal{F}(U_1)\hookrightarrow\bigoplus_{U_t\in P\mathcal{U}}\mathcal{F}(U_t)[0]\hookrightarrow\check{C}\big(\mathcal{U},\mathcal{F}\big)$ which satisfies $\eta\circ\kappa=id_A$. Let h be the homotopy operator on $\check{C}\big(\mathcal{U},\mathcal{F}\big)$ defined, on $\mathcal{F}(U_{t_0},\dots,U_{t_n})[n]$, by

$$\sum_{i=0}^{n}(-1)^i s_i^{t_0,\dots,t_n}:\mathcal{F}(U_{t_0},\dots,U_{t_n})[n]\longrightarrow\bigoplus_{U_{r_0},\dots U_{r_{n+1}}\in P\mathcal{U}}\mathcal{F}(U_{r_0},\dots,U_{r_{n+1}})[n+1]$$

where, for $0\le i\le n-1$, $s_i^{t_0,\dots,t_n}$ is defined as the suspension of the identity map

$$\mathcal{F}(U_{t_0},\dots,U_{t_n})[n] \to \mathcal{F}(U_{t_0},\dots,U_{t_n})[n+1] \cong \mathcal{F}(U_{t_0},\dots,U_{t_i},U_{t_i},\dots,U_{t_n})[n+1]$$

followed by the inclusion in the Čech complex.

Similarly, the map $s_n^{t_0,\dots,t_n}$ is defined as the suspension of the identity map $\mathcal{F}(U_{t_0},\dots,U_{t_n})[n] \to \mathcal{F}(U_{t_0},\dots,U_{t_n})[n+1] \cong \mathcal{F}(U_{t_0},\dots,U_{t_n},U_1)[n+1]$ (followed by the inclusion in the Čech complex). Note that $dh + hd = id - \kappa \circ \eta$ where d is the total differential on $\check{C}(\mathcal{U},\mathcal{F})$. It follows that $\eta : \check{C}(\mathcal{U},\mathcal{F}) \to A$ is an equivalence.

9.2.2 Proof of Corollary 7

The functor $\underline{\pi_1}_* : \mathbf{Fac}^{lc}_{X\times[0,+\infty)} \to E_1\textbf{-RMod}(\mathbf{Fac}^{lc}_X)$ is well-defined by Corollary 6 and Proposition 28. In order to check it is an equivalence, as in the proof of Proposition 18, we only need to prove it when $X = \mathbb{R}^n$, that is that, if $\mathcal{F} \in \mathbf{Fac}^{lc}_{[0,+\infty)}(\mathbf{Fac}^{lc}_X)$, then it is in the essential image of $\underline{\pi_1}_*$. By Proposition 18, we can also assume that the restriction $\mathcal{F}_{|(0,+\infty)}$ is in $\underline{\pi_1}_*(\mathbf{Fac}^{lc}_{\mathbb{R}^n\times(0,+\infty)})$.

Let $\mathcal{I}_\varepsilon$ be the factorizing cover of $[0,+\infty)$ consisting of all intervals with the restriction that intervals containing 0 are included in $[0,\varepsilon)$ note that $I_\tau \subset I_\varepsilon$ if $\epsilon > \tau$. We can replace $\mathcal{F}$ by its Čech complex on $\mathcal{I}_\varepsilon$ (for any ε) and thus by its limit over all $\varepsilon > 0$, which we still denote $\mathcal{F}$. As in the proof of Proposition 18, we only need to prove that $(U,V) \mapsto \big(\mathcal{F}(V)\big)(U)$ extends as a factorization algebra relative to the factorizing basis of $\mathbb{R}^n \times [0,+\infty)$ consisting of cubes (with sides parallel to the axes). The only difficulty is to define the prefactorization algebra structure on this basis (since we already know it is locally constant, and thus will extend into a factorization algebra). As noticed above, we already have such structure when no cubes intersect $\mathbb{R}^n \times \{0\}$. Given a finite family of pairwise disjoint cubes lying in a bigger cube $K \times [0,R)$ intersecting $\{0\}$, we can find $\varepsilon > 0$ such that no cubes of the family lying in $\mathbb{R}^n \times (0,+\infty)$ lies in the band $\mathbb{R}^n \times [0,\varepsilon)$. The value of $\mathcal{F}$ on each square containing $\mathbb{R}^n \times \{0\}$ can be computed using the Čech complex associated to I_ε. This left us, in every such cube, with one term containing a summand $[0,\tau)$ $(\tau \leq \epsilon)$ and cubes in the complement. Now choosing the maximum of the possible τ allows to first maps $(\mathcal{F}(c))(d)$ to $\big(\mathcal{F}(\tau,R)\big)(K)$ for every cube $c \times d$ in $\mathbb{R}^n \times (\tau,R)$ (since we already have a factorization algebra on $\mathbb{R}^n \times (0,+\infty)$). Then to maps all other summands to terms of the form $\mathcal{F}([0,\tau))(d)$, then all of them in $\mathcal{F}([0,\tau))(K)$ and finally to evaluate the last two remaining summand in $\mathcal{F}([0,R))(K)$ using the prefactorization algebra structure of $\mathcal{F}$ with respect to intervals in $[0,\infty)$. This is essentially the same argument as in the proof of Corollary 8.

9.3 Complements on Subsection 6.3

Here we collect proofs of statements relating factorization algebras and E_n-modules.

9.3.1 Proof of Theorem 12

The functoriality is immediate from the construction. Let $\mathbf{Fin}_*$ be the ∞-category associated to the category Fin_* of pointed finite sets. If $\mathcal{O}$ is an operad, the ∞-category $\mathcal{O}\text{-}\mathbf{Mod}_A$ of $\mathcal{O}$-modules[80] over an $\mathcal{O}$-algebra A is the category of $\mathcal{O}$-linear functors $\mathcal{O}\text{-}\mathbf{Mod}_A := \mathrm{Map}_{\mathbf{O}}(\mathbf{O}_*, \mathbf{Chain}(k))$. Here, following the notations of Appendix 10.2, $\mathbf{O}$ is the ∞-categorical enveloppe of $\mathcal{O}$ as in [71] and $\mathbf{O}_* := \mathbf{O} \times_{\mathbf{Fin}} \mathbf{Fin}_*$. There is an natural fibration $\pi_{\mathcal{O}} : \mathcal{O}\text{-}\mathbf{Mod} \longrightarrow \mathcal{O}\text{-}\mathbf{Alg}$ whose fiber at $A \in \mathcal{O}\text{-}\mathbf{Alg}$ is $\mathcal{O}\text{-}\mathbf{Mod}_A$.

Let $\mathcal{D}_{isk}$ be the set of all open disks in $\mathbb{R}^n$. Recall from Remark 24 that $\mathcal{D}_{isk}$-prefactorization algebras are exactly algebras over the operad $\mathrm{Disk}(\mathbb{R}^n)$ and that locally constant $\mathcal{D}_{isk}$-prefactorization algebras are the same as locally constant factorization algebras on $\mathbb{R}^n$ (Proposition 12). The map of operad $\mathrm{Disk}(\mathbb{R}^n) \to \mathbb{E}_{\mathbb{R}^n}$ of [71, Sect. 5.2.4] induces a fully faithful functor $E_n\text{-}\mathbf{Alg} \to \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}$ and thus a functor

$$\tilde{\psi} : E_n\text{-}\mathbf{Mod} \longrightarrow \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Mod} \longrightarrow \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}.$$

The map $\tilde{\psi}$ satisfies that, for every convex subset $C \subset \mathbb{R}^n$, one has

$$\tilde{\psi}(M)(C) = M(C) = \mathcal{F}_M(C).$$

By definition, $\tilde{\psi} \circ can : E_n\text{-}\mathbf{Alg} \to \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}$ is the composition

$$E_n\text{-}\mathbf{Alg} \longrightarrow \mathbf{Fac}^{lc}_{\mathbb{R}^n} \longrightarrow \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}.$$

Hence, the commutativity of the diagram in the Theorem will follow automatically once we have proved that $\tilde{\psi}$ factors as a composition of functors

$$\tilde{\psi} : E_n\text{-}\mathbf{Mod} \xrightarrow{\psi} \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \longrightarrow \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}. \tag{61}$$

Assuming for the moment that we have proved that $\tilde{\psi}$ factors through $\mathbf{Fac}^{lc}_{\mathbb{R}^n_*}$, let us show that ψ is fully faithful. By definition of categories of modules, we have a commutative diagram

$$\begin{array}{ccc} E_n\text{-}\mathbf{Mod} & \longrightarrow & \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Mod} \\ \Big\downarrow{\scriptstyle \pi_{E_n}} & & \Big\downarrow{\scriptstyle \pi_{\mathrm{Disk}(\mathbb{R}^n)}} \\ E_n\text{-}\mathbf{Alg} & \longrightarrow & \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg} \end{array}$$

80 In $\mathbf{Chain}(k)$.

whose bottom arrow is a fully faithful embedding by [71, Sect. 5.2.4]. Since the mapping spaces of $\mathcal{F} \in \mathbf{Fac}_X^{lc}$ are the mapping spaces of the underlying prefactorization algebra, the map $\mathbf{Fac}_{\mathbb{R}^n_*}^{lc} \to \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}$ is fully faithful, and we are left to prove that

$$\tilde{\psi} : \mathrm{Map}_{E_n\text{-}\mathbf{Mod}}(M, N) \to \mathrm{Map}_{\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}}(\tilde{\psi}(M), \tilde{\psi}(N))$$

is an equivalence for all $M \in E_n\text{-}\mathbf{Mod}_A$ and $N \in E_n\text{-}\mathbf{Mod}_B$. The fiber at (the image of) an E_n-algebra A of $\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Mod} \to \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}$ is the (homotopy) pullback

$$\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Mod}_A := \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}^{/A} \times^h_{\mathrm{Disk}(\mathbb{R}^n\setminus\{0\})\text{-}\mathbf{Alg}^{/A}} \mathrm{Iso}_{\mathrm{Disk}(\mathbb{R}^n\setminus\{0\})\text{-}\mathbf{Alg}}(A).$$

Here we write $\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}^{/A}$ for the ∞-category of $\mathrm{Disk}(\mathbb{R}^n)$-algebras under A and $\mathrm{Iso}_{\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}}(A)$ its subcategory of objects $A \xrightarrow{f} B$ such that f is an equivalence. In plain english, $\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Mod}_A$ is the ∞-category of maps $A \xrightarrow{f} B$ (where B runs through $\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}$) whose restriction to $\mathbb{R}^n \setminus \{0\}$ is an equivalence.

It is now sufficient to prove, given E_n-algebras A and B (identified with objects of $\mathbf{Fac}_{\mathbb{R}^n}^{lc}$) and two locally constant factorization algebras $\tilde{\psi}(M)$, $\tilde{\psi}(N)$ on $\mathbb{R}^n_*$ together with two maps of factorizations algebras $f : A \to \tilde{\psi}(M)$, $g : B \to \tilde{\psi}(N)$ whose restrictions to $\mathbb{R}^n \setminus \{0\}$ are quasi-isomorphisms, that the canonical map

$$\begin{aligned}\mathrm{Map}_{\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}}(A, B) \times^h_{\mathrm{Map}_{\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}}(A, \tilde{\psi}(N))} &\mathrm{Map}_{\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}}(\tilde{\psi}(M), \tilde{\psi}(N)) \\ &\longrightarrow \mathrm{Map}_{\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}}(\tilde{\psi}(M), \tilde{\psi}(N)) \end{aligned} \tag{62}$$

is an equivalence. This pullback is the mapping space $\mathrm{Map}_{\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Mod}}(\tilde{\psi}(M), \tilde{\psi}(N))$ and the maps to $\mathrm{Map}_{\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}}(A, \tilde{\psi}(N))$ are induced by post-composition by g and precomposition by f.

The fiber of the map (62) at $\Theta : \tilde{\psi}(M) \to \tilde{\psi}(N))$ is the mapping space of $\mathrm{Disk}(\mathbb{R}^n)$-algebras $A \xrightarrow{\tau} B$ such that, for any disk $U \subset \mathbb{R}^n \setminus \{0\}$, which is a sub-disk of a disk D containing 0, the following diagram is commutative:

$$\begin{array}{ccccc} A(D) & \xleftarrow{\rho^A_{U,D}} & A(U) & \xrightarrow[f]{\simeq} & \tilde{\psi}(M)(U) \\ \downarrow{\scriptstyle \tau(D)} & & \downarrow{\scriptstyle \tau(U)} & & \downarrow{\scriptstyle \Theta(U)} \\ B(D) & \xleftarrow{\rho^B_{U,D}} & B(U) & \xrightarrow[g]{\simeq} & \tilde{\psi}(N)(U). \end{array} \tag{63}$$

Here $\rho^A_{U,D}$ and $\rho^B_{U,D}$ are the structure maps of the factorization algebras associated to A and B. The right hand square of Diagram (63) shows that τ is uniquely determined by Θ on every open disk in $\mathbb{R}^n \setminus \{0\}$.

Since A and B are locally constant factorization algebras on $\mathbb{R}^n$, the maps $\rho^A_{U,D}$ and $\rho^B_{U,D}$ are natural quasi-isomorphisms. It follows from the left hand square in Diagram (63) that the restriction of τ to $\mathbb{R}^n \setminus \{0\}$ also determines the map τ on $\mathbb{R}^n$. Hence the map (62) is an equivalence which concludes the proof that ψ is fully faithful.

It remains to prove that $\tilde{\psi}$ factors through a functor ψ, that is that we have a composition as written in (61). This amounts to prove that for any $M \in E_n\textbf{-Mod}_A$ (that is M is an E_n-module over A), $\tilde{\psi}(M)$ is a locally constant factorization algebra on the stratified manifold given by the pointed disk $\mathbb{R}^n_*$. Since the convex subsets are a factorizing basis stable by finite intersection, we only have to prove this result on the cover $\mathcal{CV}(\mathbb{R}^n)$ (by Propositions 13 and 17).

Note that if $V \in \mathcal{CV}(\mathbb{R}^n)$ is a subset of $\mathbb{R}^n \setminus \{0\}$, then $\psi(M)_{|V}$ lies in the essential image of $\psi \circ can(M)_{|V}$ where $\psi \circ can(M)$ is the functor inducing the equivalence between E_n-algebras and locally constant factorization algebras on $\mathbb{R}^n$ (Theorem 9). We denote $\mathcal{F}_A := \psi \circ can(A)$ the locally constant factorization algebra on $\mathbb{R}^n$ induced by A. In particular, the canonical map

$$\check{C}(\mathcal{CV}(V), \mathcal{F}_M) = \check{C}(\mathcal{CV}(V), \mathcal{F}_A) \to \mathcal{F}_A(V) \cong \mathcal{F}_M(V)$$

is a quasi-isomorphism and further, if $U \subset V$ is a sub-disk, then $\mathcal{F}_M(U) \to \mathcal{F}_M(V)$ is a quasi-isomorphism.

We are left to consider the case where V is a convex set containing 0. Let $\mathcal{U}_V$ be the cover of V consisting of all open sets which contains 0 and are a finite union of disjoint convex subsets of V. It is a factorizing cover, and, by construction, two open sets in $\mathcal{U}_V$ intersects non-trivially since they contain 0. Hence $P\mathcal{U}_V = \mathcal{U}_V$. Since $\mathcal{U}_V \subset P\mathcal{CV}(V)$, we have a diagram of short exact sequences of chain complexes

$$\begin{array}{ccccccccc}
0 & \to & \check{C}(\mathcal{U}_V, \mathcal{F}_M) & \xrightarrow{i_M} & \check{C}(\mathcal{CV}(V), \mathcal{F}_M) & \to & \check{C}(\mathcal{CV}(V), \mathcal{F}_M)/\check{C}(\mathcal{U}_V, \mathcal{F}_M) & \to & 0 \\
 & & & & & & \downarrow \simeq & & \\
0 & \to & \check{C}(\mathcal{U}_V, \mathcal{F}_A) & \xrightarrow{i_A} & \check{C}(\mathcal{CV}(V), \mathcal{F}_A) & \to & \check{C}(\mathcal{CV}(V), \mathcal{F}_A)/\check{C}(\mathcal{U}_V, \mathcal{F}_A) & \to & 0
\end{array}$$

where the vertical equivalence follows from the fact that $\mathcal{F}_M(U) \cong \mathcal{F}_A(U)$ if U is a convex set not containing 0. Moreover, since $\mathcal{U}_V$ is a factorizing cover of V and $\mathcal{F}_A$ a factorization algebra, i_A is a quasi-isomorphism, hence $\check{C}(\mathcal{CV}(V), \mathcal{F}_A)/\check{C}(\mathcal{U}_V, \mathcal{F}_A)$ is acyclic. It follows that $i_M : \check{C}(\mathcal{U}_V, \mathcal{F}_M) \to \check{C}(\mathcal{CV}(V), \mathcal{F}_M)$ is a quasi-isomorphism as well.

We are left to prove that the canonical map $\check{C}(\mathcal{U}_V, \mathcal{F}_M) \to \mathcal{F}(M) \cong M$ is a quasi-isomorphism. Note that for any $U \in \mathcal{U}_V$, we have $\mathcal{F}_M(U) \cong M \otimes^{\mathbb{L}}_A \mathcal{F}_A(U)$.

We deduce that $\check{C}(\mathcal{U}_V, \mathcal{F}_M) \cong M \otimes^{\mathbb{L}}_A \check{C}(\mathcal{U}_V, \mathcal{F}_A)$ as well. The chain map

$$M \otimes^{\mathbb{L}}_A \check{C}(\mathcal{U}_V, \mathcal{F}_A) \cong \check{C}(\mathcal{U}_V, \mathcal{F}_M) \longrightarrow \mathcal{F}(M) \cong M \otimes^{\mathbb{L}}_A A$$

is an equivalence since it is obtained by tensoring (by M over A) the quasi-isomorphism $\check{C}(\mathcal{U}_V, \mathcal{F}_A) \to \mathcal{F}_A(V) \cong A$ (which follows from the fact that $\mathcal{F}_A$ is a factorization algebra).

It remains to prove that $\mathcal{F}_M(U) \to \mathcal{F}_M(V)$ is a quasi-isomorphism if both U, V are convex subsets containing 0. This is immediate since $M(U) \to M(V)$ is an equivalence by definition of an E_n-module over A.

9.3.2 Proof of Corollary 8

By Theorem 12, we have a commutative diagram

$$\begin{array}{ccc}
E_n\textbf{-Mod} & \longrightarrow & \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times^h_{\mathbf{Fac}^{lc}_{\mathbb{R}^n \setminus \{0\}}} \mathbf{Fac}^{lc}_{\mathbb{R}^n} \\
\downarrow{\scriptstyle \pi_{E_n}} & & \downarrow \\
E_n\textbf{-Alg} & \longrightarrow & \mathbf{Fac}^{lc}_{\mathbb{R}^n}
\end{array}$$

with fully faithful horizontal arrows. Since $E_n\textbf{-Alg} \to \mathbf{Fac}^{lc}_{\mathbb{R}^n}$ is an equivalence, we only need to prove that, for any E_n-algebra A, the induced fully faithful functor $E_n\textbf{-Mod}_A \longrightarrow \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times_{\mathbf{Fac}^{lc}_{\mathbb{R}^n \setminus \{0\}}} \{A\}$ between the fibers is essentially surjective.[81]

Let $\mathcal{M}$ be a locally constant factorization algebra on $\mathbb{R}^n_*$ such that $\mathcal{M}_{|\mathbb{R}\setminus\{0\}}$ is equal to $\mathcal{A}_{|\mathbb{R}^n\setminus\{0\}}$ where $\mathcal{A}$ is the factorization algebra associated to A (by Theorem 9). Recall that $N : \mathbb{R}^n \to [0, +\infty)$ is the euclidean norm map. Lemma 7 implies that $N_*(\mathcal{M})$ is is locally constant on the stratified half-line $[0, +\infty)$ and thus equivalent to a right module over the E_1-algebra $N_*(\mathcal{M})(\mathbb{R}^n \setminus \{0\}) \cong \mathcal{A}(\mathbb{R}^n \setminus \{0\}) \cong \int_{S^{n-1}\times\mathbb{R}} A$.

By homeomorphism invariance of (locally constant) factorization algebras, we can replace $\mathbb{R}^n$ by the unit open disk D^n of $\mathbb{R}^n$ in the above analysis. We also denote D^n_* the disk D^n viewed as a pointed space with base point 0. We now use this observation to define a structure of E_n-module over A on $M := \mathcal{M}(D^n) = N_*(\mathcal{M})\big([0, 1)\big)$. It amount to define, for any finite set I, continuous maps (compatible with the structure of the operad of little disks of dimension n)

[81] That is we are left to prove Corollary 9.

$$\mathrm{Rect}_*\Big(D^n_* \coprod \big(\coprod_{i\in I} D^n\big), D^n_*\Big) \longrightarrow \mathrm{Map}_{\mathbf{Chain}(k)}\big(M \otimes A^{\otimes I}, M\big)$$
$$\xrightarrow{\simeq} \mathrm{Map}_{\mathbf{Chain}(k)}\Big(M \otimes \Big(\int_{D^n} A\Big)^{\otimes I}, M\Big)$$

where Rect_* is the space of rectilinear embeddings which maps the center of the first copy D^n_* to the center of D^n_* (i.e. preserves the base point of D^n). Let I_N be the map that sends an element $f \in \mathrm{Rect}_*\big(D^n \coprod \big(\coprod_{i\in I} D^n\big), D^n\big)$ to the smallest open subinterval $I_N(f) \subset (0, 1)$ which contains $N\big(f\big(\coprod_{i\in I} D^n\big)\big)$, that is the smallest interval that contains the image of the non-pointed disks. By definition I_N is continuous (meaning the lower and the upper bound of $I_N(f)$ depends continuously of f) and its image is disjoint from the image $N(D^n_*)$ of the pointed copy of D^n. Similarly we define $r(f)$ to be the radius of $f(D^n_*)$. We have a continuous map

$$\tilde{N} : \mathrm{Rect}_*\Big(D^n_* \coprod \big(\coprod_{i\in I} D^n\big), D^n_*\Big) \longrightarrow \mathrm{Rect}\Big([0, 1) \coprod (0, 1), [0, 1)\Big)$$

given by $\tilde{N}(f)\big((0, 1)\big) = I_N(f)$ and $\tilde{N}(f)\big([0, 1)\big) = [0, r(f))$. Since $f(\coprod_{i\in I} D^n) \subset S^{n-1} \times (0, 1)$, we have the composition

$$\Upsilon : \mathrm{Rect}_*\Big(D^n_* \coprod \big(\coprod_{i\in I} D^n\big), D^n_*\Big) \longrightarrow \mathrm{Rect}\Big(\coprod_{i\in I} D^n, S^{n-1} \times (0, 1)\Big)$$
$$\longrightarrow \mathrm{Map}_{\mathbf{Chain}(k)}\Big(\Big(\int_{D^n} A\Big)^{\otimes I}, \int_{S^{n-1}\times(0,1)} A\Big)$$

where the first map is induced by the restriction to $\coprod_{i\in I} D^n$ and the last one by functoriality of factorization homology with respect to embeddings. We finally define

$$\mu : \mathrm{Rect}_*\Big(D^n_* \coprod \big(\coprod_{i\in I} D^n\big), D^n_*\Big) \xrightarrow{\tilde{N}\times\Upsilon}$$
$$\mathrm{Rect}\Big([0, 1) \coprod (0, 1), [0, 1)\Big) \times \mathrm{Map}_{\mathbf{Chain}(k)}\Big(\Big(\int_{D^n} A\Big)^{\otimes I}, \int_{S^{n-1}\times(0,1)} A\Big) \longrightarrow$$
$$\mathrm{Map}_{\mathbf{Chain}(k)}\Big(M \otimes \int_{S^{n-1}\times(0,1)} A, M\Big) \times \mathrm{Map}_{\mathbf{Chain}(k)}\Big(\Big(\int_{D^n} A\Big)^{\otimes I}, \int_{S^{n-1}\times(0,1)} A\Big)$$
$$\xrightarrow{id_M \otimes \circ} \mathrm{Map}_{\mathbf{Chain}(k)}\Big(M \otimes \Big(\int_{D^n} A\Big)^{\otimes I}, M\Big)$$

where the second map is induced by the E_1-module structure of $M = N_*(\mathcal{M})([0, 1))$ over $\int_{S^{n-1}\times(0,1)} A$ and the last one by composition. That μ is compatible with the action of the little disks operad follows from the fact that Υ is induced by the E_n-algebra structure of A and M is an E_1-module over $\int_{S^{n-1}\times(0,1)} A$. Hence, M is in E_n-$\mathbf{Mod}_A$.

We now prove that the factorization algebra $\psi(M)$ is $\mathcal{M}$. For all euclidean disks D centered at 0, one has $\psi(M)(D) = \mu(\{D \hookrightarrow D^n\})(M) = \mathcal{M}(D)$ and further $\psi(M)(U) = \mathcal{A}(U)$ if U is a disk that does not contain 0. The $\mathcal{D}$-prefactorization algebra structure of $\psi(M)$ (where $\mathcal{D}$ is the basis of opens consisting of all euclidean disks centered at 0 and all those who do not contain 0) is precisely given by μ according to the construction of ψ (see Theorem 12). Hence, by Proposition 17, $\psi(M) \cong \mathcal{M}$ and the essential surjectivity follows.

9.3.3 Proof of Proposition 29

By [71, Sect. 4.3], we have two functors $i_\pm : \mathbf{BiMod} \to E_1\text{-}\mathbf{Alg}$ and the (homotopy) fiber of $\mathbf{BiMod} \overset{(i_-,i_+)}{\longrightarrow} E_1\text{-}\mathbf{Alg} \times E_1\text{-}\mathbf{Alg}$ at a point (L, R) is the category of (L, R)-bimodules which is equivalent to the category $E_1\text{-}\mathbf{LMod}_{L\otimes R^{op}}$. We have a factorization

$$\begin{array}{ccccc} \mathbf{Fac}^{lc}_{\mathbb{R}_*} & \xrightarrow{(j_\pm^*,(-N)_*)} & \mathbf{BiMod} & \xrightarrow{(i_-,i_+)} & E_1\text{-}\mathbf{Alg} \times E_1\text{-}\mathbf{Alg} \\ & \searrow^{(j_-^*,j_+^*)} & & & \uparrow \simeq \\ & & & & \mathbf{Fac}^{lc}_{(-\infty,0)} \times \mathbf{Fac}^{lc}_{(0,+\infty)}. \end{array} \tag{64}$$

We can assume that L, R are strict and consider the fiber

$$\left(\mathbf{Fac}^{lc}_{\mathbb{R}_*}\right)_{L,R} := \{\mathcal{F}_L, \mathcal{F}_R\} \times_{\mathbf{Fac}^{lc}_{(-\infty,0)} \times \mathbf{Fac}^{lc}_{(0,+\infty)}} \mathbf{Fac}^{lc}_{\mathbb{R}_*}$$

of (j_-^*, j_+^*) at the pair of factorization algebras $(\mathcal{F}_L, \mathcal{F}_R)$ on $(-\infty, 0)$, $(0, +\infty)$ corresponding to L, R respectively (using Proposition 27). The pushforward along the opposite of the euclidean norm map gives the functor $(-N)_* : \left(\mathbf{Fac}^{lc}_{\mathbb{R}_*}\right)_{L,R} \to E_1\text{-}\mathbf{LMod}_{L\otimes R^{op}}$.

We further have a locally constant factorization algebra $\mathcal{G}_M^{L,R}$ on $\mathbb{R}_*$ which is defined on the basis of disks by the same rule as for the open interval (for disks included in a component $\mathcal{R} \setminus \{0\}$) together with $\mathcal{G}_M^{L,R}(\alpha, \beta) = M$ for $\alpha < 0 < \beta$. For $r < t_1 < u_1 \cdots < t_n < u_n < \alpha < 0 < \beta < x_1 < y_1 < \cdots < x_m < y_m < s$, the structure maps

$$\Big(\bigotimes_{i=1\dots n} \mathcal{G}_M^{L,R}\big((u_i, t_i)\big)\Big) \otimes \mathcal{G}_M^{L,R}\big((\alpha, \beta)\big) \otimes \Big(\bigotimes_{i=1\dots n} \mathcal{G}_M^{L,R}\big((u_i, t_i)\big)\Big)$$
$$\cong\ L^{\otimes n} \otimes M \otimes R^{\otimes m} \longrightarrow M \cong \mathcal{G}_M^{L,R}\big((r, s)\big)$$

are given by $\ell_1 \otimes \cdots \otimes \ell_n \otimes a \otimes r_1 \otimes \cdots \otimes r_n \mapsto (\ell_1 \cdots \ell_n) \cdot a \cdot (r_1 \cdots r_n)$.

One checks as in Proposition 27 that $\mathcal{G}_M^{L,R}$ is a locally constant factorization algebra on $\mathbb{R}_*$. The induced functor $E_1\text{-}\mathbf{LMod}_{L\otimes R^{op}} \to \mathbf{Fac}^{lc}_{\mathbb{R}_*}$ is an inverse of $(-N)_*$. Thus the fiber $\left(\mathbf{Fac}^{lc}_{\mathbb{R}_*}\right)_{L,R}$ of (j_-^*, j_+^*) is equivalent to $E_1\text{-}\mathbf{LMod}_{L\otimes R^{op}}$. It now follows from diagram (64) that the functor $(j_{\pm}^{*}, (-N)_{*}) : \mathbf{Fac}^{lc}_{\mathbb{R}_*} \cong \mathbf{BiMod}$ is an equivalence.

9.3.4 Proof of Proposition 30

We define a functor $G : E_1\text{-}\mathbf{RMod}_{\mathcal{A}(S^{n-1}\times\mathbb{R})} \to \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \{\mathcal{A}\}$ (which will be an inverse of N_*) as follows. By Proposition 28 we have an equivalence

$$E_1\text{-}\mathbf{RMod}_{\mathcal{A}(S^{n-1}\times\mathbb{R})} \cong \mathbf{Fac}^{lc}_{[0,+\infty)} \times_{\mathbf{Fac}^{lc}_{(0,+\infty)}} \{N_*(\mathcal{A})\}.$$

It is enough to define G as a functor from $\mathbf{Fac}^{lc}_{[0,+\infty)} \times_{\mathbf{Fac}^{lc}_{(0,+\infty)}} \{N_*(\mathcal{A})\}$ to locally constant $\mathcal{U}$-factorization algebras, where $\mathcal{U}$ is a (stable by finite intersections) factorizing basis of $\mathbb{R}^n$ (by Proposition 17). We choose $\mathcal{U}$ to be the basis consisting of all euclidean disks centered at 0 and all convex open subsets not containing 0. Let $\mathcal{R} \in \mathbf{Fac}^{lc}_{[0,+\infty)} \times_{\mathbf{Fac}^{lc}_{(0,+\infty)}} \{N_*(\mathcal{A})\}$. If $U \in \mathcal{U}$ does not contains 0, then we set $G(\mathcal{R})(U) = \mathcal{A}(U)$ and structure maps on open sets in $\mathcal{U}$ not containing 0 to be the one of $\mathcal{A}$; this defines a locally constant factorization algebra on $\mathbb{R}^n \setminus \{0\}$ since $\mathcal{A}$ does.

We denote $D(0,r)$ the euclidean disk of radius $r > 0$ and set $G(\mathcal{R})(D(0,r)) = \mathcal{R}([0,\epsilon))$. Let $D(0,r), U_1, \dots, U_i$ be pairwise subsets of $\mathcal{U}$ which are sub-sets of an euclidean disk $D(0,s)$. Then, $U_1, \dots, U_i$ lies in $S^{n-1}\times(r,s)$. Denoting respectively $\rho^{\mathcal{A}}$, $\rho^{\mathcal{R}}$ the structure maps of the factorization algebras $\mathcal{A} \in \mathbf{Fac}^{lc}_{S^{n-1}\times(0,+\infty)}$ and $\mathcal{R} \in \mathbf{Fac}^{lc}_{[0,+\infty)}$, we have the following composition

$$\begin{aligned} G(\mathcal{R})\big(D(0,r)\big) \otimes G(\mathcal{R})(U_1) \otimes \cdots \otimes G(\mathcal{R})(U_i) &\cong \mathcal{R}\big([0,r)\big) \otimes \mathcal{A}(U_1) \otimes \cdots \otimes \mathcal{A}(U_i) \\ \xrightarrow{id\otimes\rho^{\mathcal{A}}_{U_1,\dots,U_i,S^{n-1}\times(r,s)}} \mathcal{R}\big([0,r)\big) \otimes \mathcal{A}\big(S^{n-1}\times(r,s)\big) &\cong \mathcal{R}\big([0,r)\big) \otimes N_*(\mathcal{A})\big((r,s)\big) \\ \xrightarrow{\rho^{\mathcal{R}}_{[0,r),(r,s),[0,s)}} \mathcal{R}\big([0,s)\big) &= G(\mathcal{R})\big(D(0,s)\big). \end{aligned} \tag{65}$$

The maps (65) together with the structure maps of $\mathcal{A}_{|\mathbb{R}^n\setminus\{0\}} \cong \mathcal{R}_{|\mathbb{R}^n\setminus\{0\}}$ define the structure of a $\mathcal{U}$-factorization algebra since $\mathcal{R}$ and $\mathcal{A}$ are factorization algebras.

The maps $G(\mathcal{R})\big(D(0,r)\big) \to G(\mathcal{R})\big(D(0,s)\big)$ are quasi-isomorphisms since $\mathcal{R}$ is locally constant. Since the maps (65) only depend on the structure maps of $\mathcal{R}$ and $\mathcal{A}$, the rule $\mathcal{R} \mapsto G(\mathcal{R})$ extends into a functor

$$G : E_1\text{-}\mathbf{RMod}_{\mathcal{A}(S^{n-1}\times\mathbb{R})} \cong \mathbf{Fac}^{lc}_{[0,+\infty)} \times_{\mathbf{Fac}^{lc}_{(0,+\infty)}} \{N_*(\mathcal{A})\} \to \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \{\mathcal{A}\}.$$

In order to check that $N_* \circ G$ is equivalent to the identity functor of E_1 $\text{-}\mathbf{RMod}_{\mathcal{A}(S^{n-1}\times\mathbb{R})}$ it is sufficient to check it on the basis of opens of $[0,+\infty)$ given by the open intervals and the half-closed intervals $[0,s)$ for which the result follows from the definition of the maps (65). Similarly, one can check that $G \circ N_*$ is equivalent to the identity of $\mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \{\mathcal{A}\}$ by checking it on the open cover $\mathcal{U}$.

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10 Appendix

In this appendix, we briefly collect several notions and results about ∞-categories and (∞-)operads and in particular the E_n-operad and its algebras and their modules.

10.1 A ∞-category Overview

There are several equivalent (see [9]) notions of (symmetric monoidal) ∞-categories and the reader shall feel free to use its favorite ones in these notes though we choose

Definition 29 In this paper, an ∞-category means a complete Segal spaces [70, 81].

Other appropriate models[82] are given by Segal category [56, 94] or Joyal quasi-categories [69]. Almost all ∞-categories in these notes arise as some (derived) topological (or simplicial or dg) categories or localization of a category with weak equivalences. They carry along derived functors (such as derived homomorphisms) lifting the usual derived functors of usual derived categories. We recall below (Examples 59 and 58) how to go from a model or topological category to an ∞-category.

Following [70, 81], a ***Segal space*** is a functor $X_\bullet : \Delta^{op} \to \mathrm{Top}$, that is a simplicial space,[83] which is Reedy fibrant (see [60]) and satisfies the condition that for every integers $n \geq 0$, the natural map (induced by the face maps)

$$X_n \longrightarrow X_1 \times_{X_0} \times X_1 \times_{X_0} \cdots \times_{X_0} X_1 \tag{66}$$

(where there is n copies of X_1) is a weak homotopy equivalence. [84]

[82] Depending on the context some models are more natural to use than others.

[83] Here a space can also mean a simplicial set and it is often technically easier to work in this setting.

[84] Alternatively, one can work out an equivalent notion fo Segal spaces which forget about the Reedy fibrancy condition and replace condition (66) by the condition that the following natural map is a weak homotopy equivalence:

Associated to a Segal space $X_\bullet$ is a (discrete) category $ho(X_\bullet)$ with objects the points of X_0 and morphisms $ho(X_\bullet)(a, b) = \pi_0\big(\{a\} \times_{X_0} X_1 \times_{X_0} \{b\}\big)$. We *call* $ho(X_\bullet)$ *the homotopy category of* $X_\bullet$.

A Segal space $X_\bullet$ is *complete* if the canonical map $X_0 \to \mathrm{Iso}(X_1)$ is a weak equivalence, where $\mathrm{Iso}(X_1)$ is the subspace of X_1 consisting of maps f whose class $[f] \in ho(X_\bullet)$ is invertible.

There is a simplicial closed model category structure, denoted $\mathcal{S}e\mathcal{S}p$ on the category of simplicial spaces such that a fibrant object in $\mathcal{S}e\mathcal{S}p$ is precisely a Segal space. The category of simplicial spaces has another simplicial closed model structure, denoted $\mathcal{CS}e\mathcal{S}p$, whose fibrant objects are precisely complete Segal spaces [81, Theorem 7.2]. Let $\mathbb{R} : \mathcal{S}e\mathcal{S}p \to \mathcal{S}e\mathcal{S}p$ be a fibrant replacement functor and $\widehat{\cdot} : SeSp \to \mathcal{CS}e\mathcal{S}p$ be the completion functor that assigns to a Segal space $X_\bullet$ an equivalent complete Segal space $\widehat{X_\bullet}$. The composition $X_\bullet \mapsto \widehat{\mathbb{R}(X_\bullet)}$ gives a fibrant replacement functor $L_{\mathcal{CS}e\mathcal{S}p}$ from simplicial spaces to complete Segal spaces.

Example 57 (*Discrete categories*) Let $\mathcal{C}$ be an ordinary category (which we also referred to as a discrete category since its Hom-spaces are discrete). Its nerve is a Segal space which is not complete in general. However, one can form its *classifying diagram*, abusively denoted $N(\mathcal{C})$ which is a complete Segal space [81]. This is the ∞-category associated to $\mathcal{C}$.

By definition, the classifying diagram is the simplicial space $n \mapsto \big(N(\mathcal{C})\big)_n := N_\bullet(\mathrm{Iso}(\mathcal{C}^{[n]}))$ given by the ordinary nerves (or classifying spaces) of $\mathrm{Iso}(\mathcal{C}^{[n]})$ the subcategories of isomorphisms of the categories of n-composables arrows in $\mathcal{C}$.

Example 58 (*Topological category*) Let T be a topological (or simplicial) category. Its nerve $N_\bullet(T)$ is a simplicial space. Applying the complete Segal Space replacement functor we get the ∞*-category* $T_\infty := L_{\mathcal{CS}e\mathcal{S}p}(N_\bullet(T))$ *associated to* T.

Note that there is a model category structure on topological category which is Quillen equivalent[85] to $\mathcal{CS}e\mathcal{S}p$, ([9]). The functor $T \mapsto T_\infty$ realizes this equivalence. If T is a discrete topological category (in other words an usual category viewed as a topological category), then T_∞ is equivalent to the ∞-category $N(T)$ associated to T in Example 57 ([10]). It is worth mentioning that the functor $T \mapsto T_\infty$ is the analogue for complete Segal spaces of the *homotopy coherent nerve* ([69]) for quasi-categories, see [11] for a comparison.

Example 59 (*The* ∞*-category of a model category*) Let $\mathcal{M}$ be a model category and $\mathcal{W}$ be its subcategory of weak-equivalences. We denote $L^H(\mathcal{M}, \mathcal{W})$ its *hammock localization*, see [28]. One of the main property of $L^H(\mathcal{M}, \mathcal{W})$ is that it is a simplicial category and that the (usual) category $\pi_0(L^H(\mathcal{M}, \mathcal{W}))$ is the homotopy category of

(Footnote 84 continued)

$$X_n \longrightarrow holim\big(X_1 \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1 \xrightarrow{d_0} \cdots \xleftarrow{d_1} X_1 \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1\big).$$

[85] More precisely there is a zigzag of Quillen equivalences in between them; zigzag which goes through the model category structure of Segal categories.

$\mathcal{M}$. Further, every weak equivalence has a (weak) inverse in $L^H(\mathcal{M},\mathcal{W})$. If $\mathcal{M}$ is a simplicial model category, then for every pair (x, y) of objects the simplicial set of morphisms $Hom_{L^H(\mathcal{M},\mathcal{W})}(x, y)$ is naturally homotopy equivalent to the function complex $\mathrm{Map}_{\mathcal{M}}(x, y)$.

By construction, the nerve $N_\bullet(L^H(\mathcal{M},\mathcal{W}))$ is a simplicial space. Applying the complete Segal Space replacement functor we get

Proposition 49 ([9]) *The simplicial space* $\boldsymbol{L}_\infty(\mathcal{M}) := L_{\mathcal{CS}e\mathcal{S}p}(N_\bullet(L^H(\mathcal{M},\mathcal{W})))$ *is a complete Segal space, which is the* ∞*-category associated to* $\mathcal{M}$.

Note that the above construction extends to any category with weak equivalences.

Also, the *limit and colimit* in the ∞-category $\boldsymbol{L}_\infty(\mathcal{M})$ associated to a closed model category $\mathcal{M}$ can be computed by the *homotopy limit and homotopy colimit* in $\mathcal{M}$, that is by using fibrant and cofibrant resolutions. The same is true for derived functors. For instance a right Quillen functor $f : \mathcal{M} \to \mathcal{N}$ has a lift $\mathbb{L}f : \boldsymbol{L}_\infty(\mathcal{M}) \to \boldsymbol{L}_\infty(\mathcal{N})$.

Remark 35 There are other functors that yields a complete Segal space out of a model category. For instance, one can generalize the construction of Example 57. For $\mathcal{M}$ a model category and any integer n, let $\mathcal{M}^{[n]}$ be the (model) category of n-composables morphisms, that is the category of functors from the poset $[n]$ to $\mathcal{M}$. The *classification diagram* of $\mathcal{M}$ is the simplicial space $n \mapsto N_\bullet(\mathcal{W}e(\mathcal{M}^{[n]}))$ where $\mathcal{W}e(\mathcal{M}^{[n]})$ is the subcategory of weak equivalences of $\mathcal{M}^{[n]}$. Then taking a *Reedy* fibrant replacement yields another complete Segal space $N_\bullet(\mathcal{W}e(\mathcal{M}^{[n]}))^f$ ([10, Theorem 6.2], [81, Theorem 8.3]). It is known that the Segal space $N_\bullet(\mathcal{W}e(\mathcal{M}^{[n]}))^f$ is equivalent to $\boldsymbol{L}_\infty(\mathcal{M}) = L_{\mathcal{CS}e\mathcal{S}p}(N_\bullet(L^H(\mathcal{M},\mathcal{W})))$ ([10]).

Definition 30 The *objects* of an ∞-category $\boldsymbol{C}$ are the points of $\boldsymbol{C}_0$. By definition, an ∞-category has a *space* (and not just a set) of morphisms

$$\mathrm{Map}_{\boldsymbol{C}}(x, y) := \{x\} \times^h_{\boldsymbol{C}_0} \boldsymbol{C}_1 \times^h_{\boldsymbol{C}_0} \{y\}$$

between two objects x and y. A morphism $f \in \mathrm{Map}_{\boldsymbol{C}}(x, y)$ *is called an equivalence* if its image $[f] \in \mathrm{Map}_{ho}(\boldsymbol{C})(x, y)$ is an isomorphism.

From Example 59, we get an ∞*-category of* ∞*-categories*, denoted ∞**-Cat**, whose morphisms are called ∞*-functors* (or just functors for short). An equivalence of ∞-categories is an equivalence in ∞**-Cat** in the sense of Definition 30.

The model category of complete Segal spaces is cartesian closed [81] hence so is the ∞-category ∞**-Cat**. In particular, given two ∞-categories $\mathcal{C}, \mathcal{D}$ we have an ∞-category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ of functors[86] from $\mathcal{C}$ to $\mathcal{D}$. There is an natural weak equivalence of spaces:

$$\mathrm{Map}_{\infty\text{-}\mathbf{Cat}}\big(\mathcal{B}, \mathbf{Fun}(\mathcal{C}, \mathcal{D})\big) \stackrel{\simeq}{\to} \mathrm{Map}_{\infty\text{-}\mathbf{Cat}}\big(\mathcal{B} \times \mathcal{C}, \mathcal{D}\big). \tag{67}$$

[86] Computed from the Hom-space in the category of simplicial spaces using the fibrant replacement functor $L_{\mathcal{CS}e\mathcal{S}p}$.

Remark 36 **(The case of simplicial model categories)** When $\mathcal{M}$ is a simplicial closed model category, there are natural equivalences ([81]) of spaces

$$\mathrm{Map}_{\boldsymbol{L}_\infty(\mathcal{M})}(x, y) \cong \mathrm{Map}_{\left(L^H(\mathcal{M},\mathcal{W})\right)_\infty}(x, y) \cong \mathrm{Map}_{L^H(\mathcal{M},\mathcal{W})}(x, y) \cong \mathrm{Map}_{\mathcal{M}}(x, y)$$

where the right hand side is the function complex of $\mathcal{M}$ and x, y two objects. The first two equivalences also hold for general model categories ([9]). In particular, the two constructions of an ∞-category associated to a simplicial model category, either viewed as topological category as in Example 58, or as a model category as in Example 59, are equivalent:

Proposition 50 *Let $\mathcal{M}$ be a simplicial model category. Then $\mathcal{M}_\infty \cong \boldsymbol{L}_\infty(\mathcal{M})$.*

Let $\boldsymbol{I}$ be the ∞-category associated to the trivial category $\Delta^1 = \{0\} \to \{1\}$ which has two objects and only one non-trivial morphism. We have two maps $i_0, i_1 : \{pt\} \to \boldsymbol{I}$ from the trivial category to $\boldsymbol{I}$ which respectively maps the object pt to 0 and 1.

Definition 31 Let A be an object of an ∞-category $\mathcal{C}$. The ∞-*category* $\mathcal{C}_A$ of objects over A is the pullback

$$\begin{array}{ccc} \mathcal{C}_A & \longrightarrow & \mathbf{Fun}(\boldsymbol{I}, \mathcal{C}) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle i_1^*} \\ \{A\} & \longrightarrow & \mathcal{C}. \end{array}$$

The ∞-*category* ${}_A\mathcal{C}$ of objects under A is the pullback

$$\begin{array}{ccc} \mathcal{C}_A & \longrightarrow & \mathbf{Fun}(\boldsymbol{I}, \mathcal{C}) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle i_0^*} \\ \{A\} & \longrightarrow & \mathcal{C}. \end{array}$$

Informally, the ∞-category $\mathcal{C}_A$ is just the category of objects $B \in \mathcal{C}$ equipped with a map $f : B \to A$ in $\mathcal{C}$.

There is a notion of symmetric monoidal ∞-category generalizing the classical notion for discrete categories. There are several equivalent way to define this notion, see [69, 71, 98] for details. Let Γ be the skeleton of the category Fin_* of finite pointed sets, that is the subcategory spanned by the objects $n_+ := \{0, \dots, n\}$, $n \in \mathbb{N}$. For $i = 1 \dots n$, let $s_i : n_+ \to 1_+$ be the map sending i to 1 and everything else to 0.

Definition 32 A symmetric monoidal ∞-category is a functor $T \in \mathbf{Fun}(\Gamma, \infty\text{-}\mathbf{Cat})$ such that the canonical map $T(n_+) \xrightarrow{\prod_{i=0}^n s_i} \big(T(1_+)\big)^n$ is an equivalence. The full subcategory of $\mathbf{Fun}(\Gamma, \infty\text{-}\mathbf{Cat})$ spanned by the symmetric monoidal categories is denoted $\infty\text{-}\mathbf{Cat}^{\otimes}$. Its morphisms are called *symmetric monoidal functors*. Further $\infty\text{-}\mathbf{Cat}^{\otimes}$ is enriched over $\infty\text{-}\mathbf{Cat}$.

A symmetric monoidal category $T : \Gamma \to \infty\text{-}\mathbf{Cat}$ will usually be denoted as $(\boldsymbol{T}, \otimes)$ where $\boldsymbol{T} := T(1)$. If $C : \Gamma \to \infty\text{-}\mathbf{Cat}$ and $D : \Gamma \to \infty\text{-}\mathbf{Cat}$ are symmetric monoidal categories, we will *denote* $\mathbf{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ *the* ∞*-categories of symmetric monoidal functors*.

Equivalently, a symmetric monoidal category is an E_∞-algebra object in the ∞-category $\infty\text{-}\mathbf{Cat}$. An ($\infty$-)category with finite coproducts has a canonical structure of symmetric monoidal ∞-category and so does a category with finite products.

Example 60 (*The* ∞*-category* **Top**) Applying the above procedure (Example 59) to the model category of simplicial sets, we obtain the ∞-category **sSet**. Similarly, the model category of topological spaces yields the ∞-category **Top** of topological spaces. By Remark 36, we can also apply Example 58 to the standard enrichment of these categories into topological (or simplicial) categories to construct (equivalent) models of **sSet** and **Top**.

Since the model categories $sSet$ and Top are Quillen equivalent [49, 60], their associated ∞-categories are equivalent. The left and right equivalences $|-| : sSet \underset{\sim}{\overset{\sim}{\rightleftarrows}}$ Top : $\Delta_\bullet(-)$ are respectively induced by the singular set and geometric realization functors. The *disjoint union of simplicial sets and topological spaces make* **sSet** *and* **Top** *into symmetric monoidal* ∞*-categories*.

The above analysis also holds for the pointed versions $\mathbf{sSet}_*$ and $\mathbf{Top}_*$ of the above ∞-categories (using the model categories of these pointed versions [60]).

Example 61 (*Chain complexes*) The model category of (unbounded) chain complexes over k (say with the projective model structure) [60] yields the ∞-category of chain complexes $\mathbf{Chain}(k)$ (Example 59). The mapping space between two chain complex P_*, Q_* is equivalent to the geometric realization of the simplicial set $n \mapsto \mathrm{Hom}_{\mathbf{Chain}(k)}(P_* \otimes C_*(\Delta^n), Q_*)$ where $\mathrm{Hom}_{\mathbf{Chain}(k)}$ stands for morphisms of chain complexes. It follows from Proposition 50, that one can also use Example 58 applied to the category of chain complexes endowed with the above topological space of morphisms to define $\mathbf{Chain}(k)$. In particular a chain homotopy between two chain maps $f, g \in \mathrm{Map}_{\mathbf{Chain}(k)}(P_*, Q_*)$ is a path in $\mathrm{Map}_{\mathbf{Chain}(k)}(P_*, Q_*)$.

In fact, $ho(\mathbf{Chain}(k)) \cong \mathcal{D}(k)$ is the usual *derived category of* k*-modules*. The (derived) tensor product over k yields a symmetric monoidal structure to $\mathbf{Chain}(k)$ which will usually simply denote by $\otimes$. Note that $\mathbf{Chain}(k)$ *is enriched over itself*, that is, for any $P_*, Q_* \in \mathbf{Chain}(k)$, there is an object $\mathbb{R}\mathrm{Hom}_k(P_*, Q_*) \in \mathbf{Chain}(k)$ together with an adjunction

$$\mathrm{Map}_{\mathbf{Chain}(k)}\big(P_* \otimes Q_*, R_*\big) \cong \mathrm{Map}_{\mathbf{Chain}(k)}\big(P_*, \mathbb{R}\mathrm{Hom}_k(Q_*, R_*)\big).$$

The interested reader can refer to [43, 71] for details of ∞-categories enriched over ∞-categories and to [26, 53] for model categories enriched over symmetric monoidal closed model categories (which is the case of the category of chain complexes).

Example 62 In characteristic zero, there is a standard closed model category structure on the category of commutative differential graded algebras (CDGA for short), see [55, Theorem 4.1.1]. Its fibrations are epimorphisms and (weak) equivalences are quasi-isomorphisms (of CDGAs). We thus get the ∞-category **CDGA** of CDGAs. The category **CDGA** also has a monoidal structure given by the (derived) tensor product (over k) of differential graded commutative algebras, which makes $CDGA$ a symmetric monoidal model category. Given $A, B \in CDGA$, the mapping space $\mathrm{Map}_{\mathbf{CDGA}}(A, B)$ is the (geometric realization of the) simplicial set of maps $[n] \mapsto Hom_{\text{dg-Algebras}}(A, B \otimes \Omega^*(\Delta^n))$ (where $\Omega^*(\Delta^n)$ is the CDGA of forms on the n-dimensional standard simplex and $\mathrm{Hom}_{\text{dg-Algebras}}$ is the module of differential graded algebras maps). It has thus a canonical enrichment over chain complexes.

The model categories of left modules and commutative algebras over a CDGA A yield the ∞-categories E_1**-LMod**$_A$ and **CDGA**$_A$. The base change functor lifts to a functor of ∞-categories. Further, if $f : A \to B$ is a weak equivalence, the natural functor $f_* : E_1\text{-}\mathbf{LMod}_B \to: E_1\text{-}\mathbf{LMod}_A$ induces an equivalence $E_1\text{-}\mathbf{LMod}_B \overset{\sim}{\to} E_1\text{-}\mathbf{LMod}_A$ of ∞-categories since it is induced by a Quillen equivalence.

Moreover, if $f : A \to B$ is a morphism of CDGAs, we get a natural functor $f^* : E_1\text{-}\mathbf{LMod}_A \to E_1\text{-}\mathbf{LMod}_B$, $M \mapsto M \otimes^{\mathbb{L}}_A B$, which is an equivalence of ∞-categories when f is a quasi-isomorphism, and is a (weak) inverse of f_* (see [95] or [66]). The same results applies to monoids in $E_1\text{-}\mathbf{LMod}_A$ that is to the categories of commutative differential graded A-algebras.

10.2 E_n-Algebras and E_n-Modules

The classical definition of an E_n-algebra (in chain complexes) is an algebra over any E_n-operad in chain complexes, that is an operad weakly homotopy equivalent to the chains on the little (n-dimensional) cubes operad $(\mathrm{Cube}_n(r))_{r\geq 0}$ [76]. Here

$$\mathrm{Cube}_n(r) := \mathrm{Rect}\Big(\coprod_{i=1}^{r}(0,1)^n, (0,1)^n\Big)$$

is the space of rectilinear embeddings of r-many disjoint copies of the unit open cube in itself. It is topologized as the subspace of the space of all continuous maps. By a *rectilinear embedding*, we mean a composition of a translation and dilatations in the direction given by a vector of the canonical basis of $\mathbb{R}^n$. In other words, $\mathrm{Cube}_n(r)$ is the configuration space of r-many disjoint open rectangles[87] parallel to the axes lying in the unit open cube. The operad structure $\mathrm{Cube}_n(r) \times \mathrm{Cube}_n(k_1) \times \cdots \times \mathrm{Cube}_n(k_r) \to \mathrm{Cube}_n(k_1 + \cdots + k_r)$ is simply given by composition of embeddings.

[87] More precisely rectangular parallelepiped in dimension bigger than 2.

An E_n*-algebra in chain complexes* is thus a chain complex A together with chain maps $\gamma_r : C_*(\mathrm{Cube}_n(r)) \otimes A^{\otimes r} \longrightarrow A$ compatible with the composition of operads [12, 40, 76]. By definition of the operad Cube_n, we are only considering (weakly) unital versions of E_n-algebras.

The model category of E_n-algebras gives rises to *the* ∞*-category* E_n**-Alg** of E_n-algebras in the symmetric monoidal ∞-category of differential graded k-modules. The symmetric structure of **Chain**(k) lifts to a a symmetric monoidal structure on $(E_n\textbf{-Alg}, \otimes)$ given by the tensor product of the underlying chain complexes.[88]

One can extend the above notion to define E_n-algebras with coefficient in any symmetric monoidal ∞-category following [71]. One way is to rewrite it in terms of symmetric monoidal functor as follows. Any topological (resp. simplicial) operad $\mathcal{O}$ defines a symmetric monoidal category, denoted **O**, fibered over the category of pointed finite sets Fin_*. This category **O** has the finite sets for objects. For any sets $n_+ := \{0, \dots n\}$, $m_+ := \{0, \dots, m\}$ (with base point 0), its morphism space $\mathbf{O}(n_+, m_+)$ (from n_+ to m_+) is the disjoint union $\coprod_{f:n_+\to m_+} \prod_{i\in m_+} \mathcal{O}((f^{-1}(i))_+)$ and the composition is induced by the operadic structure. The rule $n_+ \otimes m_+ = (n+m)_+$ makes canonically **O** into a symmetric monoidal topological (resp. simplicial) category. We abusively denote **O** its associated ∞-category. Note that this construction extends to colored operad and is a special case of an ∞-operad. [89]

Then, if $(\mathcal{C}, \otimes)$ is a symmetric monoidal ∞-category, *a* $\mathcal{O}$*-algebra in* $\mathcal{C}$ *is a symmetric monoidal functor* $A \in \mathbf{Fun}^{\otimes}(\mathbf{O}, \mathcal{C})$. We call $A(1_+)$ the underlying algebra object of A and we usually denote it simply by A.

Definition 33 [34, 71] Let $(\mathcal{C}, \otimes)$ be symmetric monoidal (∞-)category. The ∞-category of E_n*-algebras with values in* $\mathcal{C}$ is

$$E_n\textbf{-Alg}(\mathcal{C}) := \mathbf{Fun}^{\otimes}(\mathbf{Cube}_n, \mathcal{C}).$$

Similarly $E_n\textbf{-coAlg}(\mathcal{C}) := \mathbf{Fun}^{\otimes}(\mathbf{Cube}_n, \mathcal{C}^{op})$ is the category of E_n*-coalgebras in* $\mathcal{C}$. We denote $Map_{E_n\textbf{-Alg}}(A, B)$ *the mapping space of* E_n*-algebras maps from A to B*.

Note that Definition 33 is a definition of categories of (weakly) *(co)unital* E_n-coalgebra objects.

One has an equivalence $E_n\textbf{-Alg} \cong E_n\textbf{-Alg}(\mathbf{Chain}(k))$ of symmetric monoidal ∞-categories (see [35, 71]) where E_n**-Alg** is the ∞-category associated to algebras over the operad Cube_n considered above. It is clear from the above definition that any (∞-)operad $\mathbb{E}_n$ weakly homotopy equivalent (as an operad) to Cube_n gives rise to an equivalent ∞-category of algebra. In particular, the inclusion of rectilinear embeddings into all framed embeddings gives us an alternative definition for E_n-algebras:

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[88] Other possible models for the symmetric monoidal ∞-category $(E_n\textbf{-Alg}, \otimes)$ are given by algebraic Hopf operads such as those arising from the filtration of the Barratt-Eccles operad in [8].

[89] *An* ∞*-operad* $\mathcal{O}^{\otimes}$ is a ∞-category together with a functor $\mathcal{O}^{\otimes} \to N(Fin_*)$ satisfying a list of axioms, see [71]. It is to colored topological operads what ∞-categories are to topological categories.

Proposition 51 ([71]) *Let $Disk_n^{fr}$ be the category with objects the integers and morphism the spaces $Disk_n^{fr}(k,\ell) := Emb^{fr}(\coprod_k \mathbb{R}^n, \coprod_\ell \mathbb{R}^n)$ of framed embeddings of k disjoint copies of a disk $\mathbb{R}^n$ into ℓ such copies (see Example 12). The natural map $\mathbf{Fun}^{\otimes}(Disk_n^{fr}, \mathbf{Chain}(k)) \xrightarrow{\simeq} E_n\text{-}\mathbf{Alg}$ is an equivalence.*

Example 63 (*Iterated loop spaces*) The standard examples[90] of E_n-algebras are given by iterated loop spaces. If X is a pointed space, we denote $\Omega^n(X) := \mathrm{Map}_*(S^n, X)$ the set of all pointed maps from $S^n \cong I^n/\partial I^n$ to X, equipped with the compact-open topology. The pinching map (15) $pinch : \mathrm{Cube}_n(r) \times S^n \longrightarrow \bigvee_{i=1\dots r} S^n$ induces an E_n-algebra structure (in (Top, $\times$)) given by

$$\mathrm{Cube}_n(r) \times \left(\Omega^n(X)\right)^r \cong \mathrm{Cube}_n(r) \times \mathrm{Map}_*(\bigvee_{i=1\dots r} S^n, X) \stackrel{pinch^*}{\longrightarrow} \Omega^n(X).$$

Since the construction is functorial in X, the singular chain complex $C_*(\Omega^n(X))$ is also an E_n-algebra in chain complexes, and further this structure is compatible with the E_∞-coalgebra structure of $C_*(\Omega^n(X))$ (from Example 65). Similarly, the singular cochain complex $C^*(\Omega^n(X))$ is an E_n-coalgebra in a way compatible with its E_∞-algebra structure; that is an object of $E_n\text{-}\mathbf{coAlg}(E_\infty\text{-}\mathbf{Alg})$.

Example 64 (*P_n-algebras*) A standard result of Cohen [19] shows that, for $n \geq 2$, the homology of an E_n-algebra is a P_n-algebra (also see [40, 88]). A *P_n-algebra* is a graded vector space A endowed with a degree 0 multiplication with unit which makes A a graded commutative algebra, and a (cohomological) degree 1-n operation $[-,-]$ which makes $A[1-n]$ a graded Lie algebra. These operations are also required to satisfy the Leibniz rule $[a\cdot b, c] = a[b,c] + (-1)^{|b|(|c|+1-n)}[a,c]\cdot b$.

For $n=1$, P_n-algebras are just usual Poisson algebras while for $n=2$, they are Gerstenhaber algebras.

In characteristic 0, the operad Cube_n is formal, thus equivalent as an operad to the operad governing P_n-algebras (for $n \geq 2$). It follows that P_n-algebras gives rise to E_n-algebras in that case, that is there is a functor[91] $P_n\text{-}\mathbf{Alg} \to E_n\text{-}\mathbf{Alg}$.

There are natural maps (sometimes called the stabilization functors)

$$\mathrm{Cube}_0 \longrightarrow \mathrm{Cube}_1 \longrightarrow \mathrm{Cube}_2 \longrightarrow \cdots \tag{68}$$

(induced by taking products of cubes with the interval (0, 1)). It is a fact ([71, 76]) that the colimit of this diagram, denoted by $\mathbb{E}_\infty$ is equivalent to the commutative operad *Com* (whose associated symmetric monoidal ∞-category is $\mathbf{Fin}_*$).

Definition 34 The(∞-) category of E_∞-algebras with value in $\mathcal{C}$ is $E_\infty\text{-}\mathbf{Alg}(\mathcal{C}) := \mathbf{Fun}^{\otimes}(\mathbf{Cube}_\infty, \mathcal{C})$. It is simply denoted $E_\infty\text{-}\mathbf{Alg}$ if $(\mathcal{C},\otimes) = (\mathbf{Chain}(k),\otimes)$. Similarly, the category of E_∞-coalgebras. is $E_\infty\text{-coAlg} := \mathbf{Fun}^{\otimes}(\mathbf{Cube}_\infty, \mathcal{C}^{op})$.

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90 May's recognition principle [76] actually asserts that any E_n-algebra in (Top, $\times$) which is group-like is homotopy equivalent to such an iterated loop space.

91 Which is *not* canonical, see [89].

Note that Definition 34 is a definition of (weakly) unital E_∞-algebras.

The category E_∞**-Alg** is (equivalent to) the ∞-category associated to the model category of $\mathbb{E}_\infty$-algebras for any E_∞-operad $\mathbb{E}_\infty$.

The natural map $\mathbf{Fun}^{\otimes}(\mathbf{Fin}_*, \mathcal{C}) \longrightarrow \mathbf{Fun}^{\otimes}(\mathbf{Cube}_\infty, \mathcal{C}) = E_\infty\textbf{-Alg}$ is also an equivalence.

For any $n \in \mathbb{N} - \{0\} \cup \{+\infty\}$, the map $\mathrm{Cube}_1 \to \mathrm{Cube}_n$ (from the nested sequence (68)) induces a *functor* E_n**-Alg** $\longrightarrow E_1$**-Alg** which associates to an E_n-algebra its underlying E_1-algebra structure.

Example 65 (*Singular (co)chains*) Let X be a topological space. Its *singular cochain complex $C^*(X)$ has a natural structure of E_∞-algebra*, whose underlying E_1-structure is given by the usual (strictly associative) cup-product (for instance see [75]). The *singular chains $C_*(X)$ have a natural structure of E_∞-coalgebra* which is the predual of $(C^*(X), \cup)$. There are similar constructions for simplicial sets $X_{\cdot}$ instead of spaces, see [8]. We recall that $C^*(X)$ is the linear dual of the singular chain complex $C_*(X)$ with coefficient in k which is the geometric realization (in the ordinary category of chain complexes) of the simplicial k-module $k[\Delta_\bullet(X)]$ spanned by the singular set $\Delta_\bullet(X) := \{\Delta^\bullet \stackrel{f}{\to} X, f \text{ continuous}\}$. Here Δ^n is the standard n-dimensional simplex.

Remark 37 The mapping space $Map_{E_\infty\textbf{-Alg}}(A, B)$ of two E_∞-algebras A, B (in the model category of E_∞-algebras) is the (geometric realization of the) simplicial set $[n] \mapsto Hom_{E_\infty\textbf{-Alg}}\big(A, B \otimes C^*(\Delta^n)\big)$.

The ∞-category E_∞-*Alg* is enriched over **sSet** (hence **Top** as well by Example 60) and has all (∞-)colimits. In particular, it is *tensored* over **sSet**, see [69, 71] for details on tensored ∞-categories or [30, 73] in the context of topologically enriched model categories. We recall that it means that there is a functor $E_\infty\textbf{-Alg} \times \textbf{sSet} \to E_\infty\textbf{-Alg}$, denoted $(A, X_\bullet) \mapsto A \boxtimes X_\bullet$, together with natural equivalences

$$Map_{E_\infty\textbf{-Alg}}\big(A \boxtimes X_\bullet, B\big) \;\cong\; Map_{\textbf{sSet}}\big(X_{\cdot}, Map_{E_\infty\textbf{-Alg}}(A, B)\big).$$

To compute explicitly this tensor, it is useful to know the following proposition.

Proposition 52 *Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal ∞-category. In the symmetric monoidal ∞-category E_∞**-Alg**$(\mathcal{C})$, the tensor product is a coproduct.*

For a proof see Proposition 3.2.4.7 of [71] (or [66, Corollary 3.4]); for $\mathcal{C} = \mathbf{Chain}(k)$, this essentially follows from the observation that an E_∞-algebra is a commutative monoid in $(\mathbf{Chain}(k), \otimes)$, see [71] or [66, Section 5.3]. In particular, Proposition 52 implies that, for any finite set I, $A^{\otimes I}$ has a natural structure of E_∞-algebra.

10.2.1 Modules over E_n-algebras

In this paragraph, we give a brief account of various categories of modules over E_n-algebras. Note that by definition (see below), the categories we considered are categories of *pointed* modules. Roughly, an A-modules M being pointed means it is equipped with a map $A \to M$.

Let Fin, (resp. Fin_*) be the category of (resp. pointed) finite sets. There is a forgetful functor $Fin_* \to Fin$ forgetting which point is the base point. There is also a functor $Fin \to Fin_*$ which adds an extra point called the base point. We write **Fin**, $\mathbf{Fin}_*$ for the associated ∞-categories (see Example 57). Following [34, 71], if $\mathcal{O}$ is a (coherent) operad, *the* ∞*-category* $\mathcal{O}$**-Mod**$_A$ of $\mathcal{O}$*-modules*[92] *over an* $\mathcal{O}$*-algebra* A is the category of $\mathcal{O}$-linear functors $\mathcal{O}\text{-}\mathbf{Mod}_A := \mathrm{Map}_{\mathbf{O}}(\mathbf{O}_*, \mathbf{Chain}(k))$ where $\mathbf{O}$ is the (∞-)category associated[93] to the operad $\mathcal{O}$ and $\mathbf{O}_* := \mathbf{O} \times_{\mathbf{Fin}} \mathbf{Fin}_*$ (also see [36] for similar constructions in the model category setting of topological operads).

The categories $\mathcal{O}\text{-}\mathbf{Mod}_A$ for $A \in \mathcal{O}\text{-}\mathbf{Alg}$ assemble to form an ∞-category $\mathcal{O}\text{-}\mathbf{Mod}$ describing pairs consisting of an $\mathcal{O}$-algebra and a module over it. More precisely, there is an natural fibration $\pi_{\mathcal{O}} : \mathcal{O}\text{-}\mathbf{Mod} \longrightarrow \mathcal{O}\text{-}\mathbf{Alg}$ whose fiber at $A \in \mathcal{O}\text{-}\mathbf{Alg}$ is $\mathcal{O}\text{-}\mathbf{Mod}_A$. When $\mathcal{O}$ is an $\mathbb{E}_n$-operad (that is an operad equivalent to Cube_n), we simply write E_n instead of $\mathcal{O}$:

Definition 35 Let A be an E_n-algebra (in $\mathbf{Chain}(k)$). We denote $E_n\text{-}\mathbf{Mod}_A$ the ∞-category of (pointed) E_n-modules over A. Since $\mathbf{Chain}(k)$ is bicomplete and enriched over itself, $E_n\text{-}\mathbf{Mod}_A$ is naturally enriched over $\mathbf{Chain}(k)$ as well.

We *denote*[94] $\mathbb{R}Hom_A^{E_n}(M, N) \in \mathbf{Chain}(k)$ *the enriched mapping space of morphisms of* E_n*-modules over* A. Note that if $\mathbb{E}_n$ is a cofibrant E_n-operad and further M, N are modules over an $\mathbb{E}_n$-algebra A, then $\mathbb{R}Hom_A^{E_n}(M, N)$ is computed by $Hom_{\mathrm{Mod}_A^{\mathbb{E}_n}}(Q(M), R(N))$. Here $Q(M)$ is a cofibrant replacement of M and $R(N)$ a fibrant replacement of N in the model category $\mathrm{Mod}_A^{\mathbb{E}_n}$ of modules over the $\mathbb{E}_n$-algebra A. In particular, $\mathbb{R}Hom_A^{E_n}(M, N) \cong Hom_{E_n\text{-}\mathbf{Mod}_A}(M, N)$. This follows from the fact that $E_n\text{-}\mathbf{Mod}_A$ is equivalent to the ∞-category associated to the model category $\mathrm{Mod}_A^{\mathbb{E}_n}$.

If $(\mathcal{C}, \otimes)$ is a symmetric monoidal (∞-)category and $A \in E_n\text{-}\mathbf{Alg}$, then we *denote* $E_n\text{-}\mathbf{Mod}_A(\mathcal{C})$ *the* ∞*-category of* E_n*-modules over* A *(in* $\mathcal{C}$*)*.

We *denote respectively* $E_n\text{-}\mathbf{Mod}$ the ∞-category of all E_n-modules in $\mathbf{Chain}(k)$ *and* $E_n\text{-}\mathbf{Mod}(\mathcal{C})$ the ∞-category of all E_n-modules in $(\mathcal{C}, \otimes)$.

By definition, the canonical functor[95] $\pi_{E_n} : E_n\text{-}\mathbf{Mod}(\mathcal{C}) \to E_n\text{-}\mathbf{Alg}(\mathcal{C})$ gives rise, for any E_n-algebra A, to a (homotopy) pullback square:

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92 In $\mathbf{Chain}(k)$. Of course, similar construction hold with $\mathbf{Chain}(k)$ replaced by a symmetric monoidal ∞-category

93 In the paragraph above Definition 33

94 The $\mathbb{R}$ in the notation is here to recall that this corresponds to a functor that can be computed as a derived functor associated to ordinary model categories using standard techniques of homological/homotopical algebras

95 Which essentially forget the module in the pair (A, M)

$$\begin{array}{ccc} E_n\text{-}\mathbf{Mod}_A(\mathcal{C}) & \longrightarrow & E_n\text{-}\mathbf{Mod}(\mathcal{C}) \\ \downarrow & & \downarrow{\scriptstyle \pi_{E_n}} \\ \{A\} & \longrightarrow & E_n\text{-}\mathbf{Alg}(\mathcal{C}) \end{array} \tag{69}$$

Note that the functor π_{E_n} is monoidal.

We also have a canonical functor $can : E_n\text{-}\mathbf{Alg} \to E_n\text{-}\mathbf{Mod}$ induced by the tautological module structure that any algebra has over itself.

Example 66 If A is a differential graded algebra, $E_1\text{-}\mathbf{Mod}_A$ is equivalent to the ∞-category of (pointed) A-bimodules. If A is a CDGA, $E_\infty\text{-}\mathbf{Mod}_A$ is equivalent to the ∞-category of (pointed) left A-modules.

Example 67 (*Left and right modules*) If $n = 1$, we also have naturally defined ∞-categories of left and right modules over an E_1-algebra A (as well as ∞-categories of all right modules and left modules). They are the immediate generalization of the (∞-categories associated to the model) categories of pointed left and right differential graded modules over a differential graded associative unital algebra. We refer to [71] for details.

Definition 36 We write respectively $E_1\text{-}\mathbf{LMod}_A(\mathcal{C})$, $E_1\text{-}\mathbf{RMod}_A(\mathcal{C})$, $E_1\text{-}\mathbf{LMod}(\mathcal{C})$ and $E_1\text{-}\mathbf{RMod}(\mathcal{C})$ for the ∞-categories of left modules over a fixed A, right modules over A, and all left modules and all right modules (with values in $(\mathcal{C}, \otimes)$).

If $\mathcal{C} = \mathbf{Chain}(k)$, we simply write $E_1\text{-}\mathbf{LMod}_A$, $E_1\text{-}\mathbf{RMod}_A$, $E_1\text{-}\mathbf{LMod}$, $E_1\text{-}\mathbf{RMod}$. Further, we will *denote* $\mathbb{R}Hom_A^{left}(M, N) \in \mathbf{Chain}(k)$ *the enriched mapping space of morphisms of left modules over* A (induced by the enrichment of $\mathbf{Chain}(k)$). In particular $\mathbb{R}Hom_A^{left}(M, N) \cong Hom_{E_1\text{-}\mathbf{LMod}_A}(M, N)$.

There are standard models for these categories. For instance, the category of right modules over an E_1-algebra can be obtained by considering a colored operad Cube_1^{right} obtained from the little interval operad Cube_1 as follows. Denote c, i the two colors. We define $\mathrm{Cube}_1^{right}(\{X_j\}_{j=1}^r, i) := \mathrm{Cube}_1(r)$ if all $X_j = i$. If $X_1 = c$ and all others $X_j = i$, we set $\mathrm{Cube}_1^{right}(\{X_j\}_{j=1}^r, c) := \mathrm{Rect}\big([0,1) \coprod \big(\coprod_{i=1}^r (0,1)\big), [0,1)\big)$ where Rect is the space of rectilinear embeddings (mapping 0 to itself). All other spaces of maps are empty. Then the ∞-category associated to the category of Cube_1^{right}-algebras is equivalent to $E_1\text{-}\mathbf{RMod}$.

Let A be an E_1-algebra, then the usual tensor product of right and left A-modules has a canonical lift

$$-\underset{A}{\overset{\mathbb{L}}{\otimes}}- : E_1\text{-}\mathbf{RMod}_A \times E_1\text{-}\mathbf{LMod}_A \longrightarrow \mathbf{Chain}(k)$$

which, for a differential graded associative algebra over a field k is computed by the two-sided Bar construction. There is a similar derived functor $E_1\text{-}\mathbf{RMod}_A(\mathcal{C}) \times$

E_1**-LMod**$_A(\mathcal{C}) \longrightarrow \mathcal{C}$, still denoted $(R, L) \mapsto R \otimes_A^{\mathbb{L}} L$, whenever $(\mathcal{C}, \otimes)$ is a symmetric monoidal ∞-category with geometric realization and such that $\otimes$ preserves geometric realization in both variables, see [71]. There are (derived) adjunction

$$\mathrm{Map}_{E_1\text{-}\mathbf{LMod}_A}\big(P_* \otimes L, N\big) \;\cong\; \mathrm{Map}_{\mathbf{Chain}(k)}\big(P_*, \mathbb{R}\mathrm{Hom}_A^{left}(L, N)\big),$$

$$\mathrm{Map}_{\mathbf{Chain}(k)}\big(R \overset{\mathbb{L}}{\underset{A}{\otimes}} L, N\big) \;\cong\; \mathrm{Map}_{E_1\text{-}\mathbf{LMod}_A}\big(L, \mathbb{R}\mathrm{Hom}_k(R, N)\big)$$

which relates the tensor product with the enriched mapping spaces of modules.

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