Fractional Linear Equations with Discrete Operators of Positive Order

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Abstract. The Caputo- and Riemann–Liouville–type fractional order difference initial value problems for linear and semilinear equations are discussed. We take under our consideration the possible solution via the classical $\mathcal{Z}$-transform method for any positive order. We stress the formulas that used the concept of discrete Mittag–Leffler fractional function.

Keywords: fractional difference operator, linear fractional order system, $\mathcal{Z}$-transform method.

1 Introduction

Recently, in many papers systems with fractional derivatives and differences are widely discussed and their properties are presented usually for fractional orders from the interval $(0,1]$. In the paper the possible solutions of linear and semilinear systems with the Caputo– and Riemann–Liouville–type (difference) operators are studied for any positive order $\alpha > 0$. However we use notation that $\alpha \in (q-1, q]$, where $q \in \mathbb{N}_1$. The possible solution via the classical $\mathcal{Z}$-transform method for any positive order are taken under our consideration. We stress the formulas that used the concept of discrete Mittag–Leffler fractional function.

Basic properties of fractional sums and difference operators were developed firstly in [11] and continued by Atici and Eloe in [5,6], Baleanu and Abdeljawad in [2,3]. Another concept of the fractional sum/difference was introduced in [8,10,7]. In the cited literature there are usually presented the methods of solutions via recurrence or transform methods but not so often via the $\mathcal{Z}$-transform. The problem of stability properties for fractional difference systems with higher orders authors studied in [4,19,20]. In the presented paper we only state formulas for solutions to initial value problems without studying the stability property of the considered systems. As we stressed in paper for example [13] for commensurate case and in [18] for multi–order case, the conversion of the Grünwald–Letnikov–type operator to the Riemann–Liouville–type gives the same result for the first mentioned operator.

Fractional differences used in models of control systems and could be understand as an approximation of continuous operators (see [14]) and a possibility of

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involving some memory to difference systems, i.e. systems in which the current state depends on the full history of systems' states. The main advantage of the use of the $\mathcal{Z}$-transform is to introduce the natural language for discrete systems, it means to work with sequences instead of discrete functions defined on various domains.

The structure of the paper is the following. In Section 2 the preliminary material is presented. Section 3 gives the formula for the $\mathcal{Z}$-transform of the Caputo–type operator, then in Section 4 we investigate Riemann-Liouville–type operator with positive order.

2 Preliminaries

In this section, we make a review of notations, definitions, and some preliminary facts which are useful for the paper. The necessary definitions and technical propositions that are used in the sequel therein the paper are recalled.

Let $h > 0, a \in \mathbb{R}$ and $(h\mathbb{N})_a := \{a, a+h, a+2h, ...\}$.

For a function $x : (h\mathbb{N})_a \to \mathbb{R}$ the forward h-difference operator is defined as (see [10]) $(\Delta_h x)(t) = \frac{x(t+h)-x(t)}{h}$, where $t \in (h\mathbb{N})_a$ and $(\Delta_h^0 x)(t) := x(t)$. Let $q \in \mathbb{N}_0$ and $\Delta_h^q := \Delta_h \circ \cdots \circ \Delta_h$ is q-fold application of operator Δ_h. Then $(\Delta_h^q x)(t) = h^{-q} \sum_{k=0}^{q} (-1)^{q-k} \binom{q}{k} x(t+kh)$.

Let us introduce the family of binomial functions on $\mathbb{Z}$ parameterized by $\mu > 0$ and given by the values: $\tilde{\varphi}_\mu(n) = \binom{n+\mu-1}{n}$ for $n \in \mathbb{N}_0$ and $\tilde{\varphi}_\mu(n) = 0$ for $n < 0$.

Definition 1. *For a function $x : (h\mathbb{N})_a \to \mathbb{R}$ the* fractional h-sum of order *$\alpha > 0$ is given by $\left({}_a\Delta_h^{-\alpha} x\right)(t) := h^\alpha \left(\tilde{\varphi}_\alpha * \overline{x}\right)(n)$, where $t = a + (\alpha+n)h$, $\overline{x}(s) := x(a+sh)$, $n \in \mathbb{N}_0$ and "$*$" denotes a convolution operator, i.e. $\left(\tilde{\varphi}_\alpha * \overline{x}\right)(n) := \sum_{s=0}^{n} \binom{n-s+\alpha-1}{n-s} \overline{x}(s) q$. Additionally, we define $\left({}_a\Delta_h^0 x\right)(t) := x(t)$.*

For $a = 0$ we will write shortly $\Delta_h^{-\alpha}$ instead of ${}_0\Delta_h^{-\alpha}$. Note that ${}_a\Delta_h^{-\alpha} x : (h\mathbb{N})_{a+\alpha h} \to \mathbb{R}$. Let us recall that the $\mathcal{Z}$-transform of a sequence $\{y(n)\}_{n\in\mathbb{N}_0}$ is a complex function given by $Y(z) := \mathcal{Z}[y](z) = \sum_{k=0}^{\infty} \frac{y(k)}{z^k}$, where $z \in \mathbb{C}$ is a complex number for which the series $\sum_{k=0}^{\infty} y(k) z^{-k}$ converges absolutely. Note that since $\binom{k+\alpha-1}{k} = (-1)^k \binom{-\alpha}{k}$, then for $|z| > 1$ we have

$$\mathcal{Z}\left[\tilde{\varphi}_\alpha\right](z) = \sum_{k=0}^{\infty} \frac{1}{z^k} \binom{k+\alpha-1}{k} = \left(\frac{z}{z-1}\right)^\alpha . \tag{1}$$

Proposition 1 ([17]). *For $t = a + \alpha h + nh \in (h\mathbb{Z})_{a+\alpha h}$ let us define $y(n) := \left({}_a\Delta_h^{-\alpha} x\right)(t)$ and $\overline{x}(n) = x(a+nh)$. Then*

$$\mathcal{Z}\left[y\right](z) = \left(\frac{hz}{z-1}\right)^\alpha X(z), \tag{2}$$

where $X(z) := \mathcal{Z}\left[\overline{x}\right](z)$.

In [10] the authors prove the following lemma that gives transition between fractional summation operators for any $h > 0$ and $h = 1$.

Lemma 1. *Let $x : (h\mathbb{N})_a \to \mathbb{R}$ and $\alpha > 0$. Then,* $\left({}_a\Delta_h^{-\alpha} x\right)(t) = h^\alpha \left({}_{\frac{a}{h}}\Delta_1^{-\alpha}\tilde{x}\right)\left(\frac{t}{h}\right)$ *, where $t \in (h\mathbb{N})_{a+\alpha h}$ and $\tilde{x}(s) = x(sh)$.*

We define the discrete version of Mittag-Leffler function and prove that with some values of parameters it is an eigenfunction of difference equation with Caputo– or Riemann–Liouville–type difference operator with order $\alpha \in (q-1, q]$, where $q \in \mathbb{N}_1$. In [17] we use such a function but for orders from $(0, 1]$ or also cite from [3]. Here let us define the *discrete Mittag-Leffler two-parameter function* as follows:

$$E_{(\alpha,\beta)}(\lambda, n) := \sum_{k=0}^{\infty} \lambda^k \widetilde{\varphi}_{k\alpha+\beta}(n - qk) = \sum_{k=0}^{n} \lambda^k \widetilde{\varphi}_{k\alpha+\beta}(n - qk)\,, \tag{3}$$

where the second equation only claims that for $n < qk$ we have values of $\widetilde{\varphi}_{k\alpha+\beta}(n - qk) = 0$. This is not in contradiction with the definition of Mittag–Leffler discrete type functions stated in [1] or used in [17]. Later on we will show that for $\beta = 1$ and $\beta = \alpha$ the formula (3) gives an eigenfunction of difference equation with Caputo– or Riemann–Liouville–type difference operator, respectively. In fact in the paper we will use the following discrete Mittag–Leffler–type functions

$$E_{(\alpha,\alpha)}(\lambda, n) = \sum_{k=0}^{\infty} \lambda^k \widetilde{\varphi}_{k\alpha+\alpha}(n - qk) = \sum_{k=0}^{\infty} \lambda^k \binom{n - qk + (k+1)\alpha - 1}{n - qk}, \tag{4}$$

$$E_{(\alpha)}(\lambda, n) := E_{(\alpha,1)}(\lambda, n) = \sum_{k=0}^{\infty} \lambda^k \widetilde{\varphi}_{k\alpha+1}(n - qk) = \sum_{k=0}^{\infty} \lambda^k \binom{n - qk + k\alpha}{n - qk}, \tag{5}$$

$$E_{(\alpha,0)}(\lambda, n) = \sum_{k=0}^{\infty} \lambda^k \widetilde{\varphi}_{k\alpha}(n - qk) = \sum_{k=0}^{\infty} \lambda^k \binom{n - qk + k\alpha - 1}{n - qk}. \tag{6}$$

Based on (1) for family of functions $\tilde{\varphi}_{k\alpha+\beta}$ we can state the following result for discrete Mittag-Leffler function.

Proposition 2. *Let $\alpha \in (q-1, q]$ and $\nu = \alpha - q$, $q = \nu h$, $q \in \mathbb{N}_1$. Then*

1. $E_{(\alpha,\beta)}(\lambda, 0) = 1$.
2. *For z such that $|z| > 1$ we have*

$$\mathcal{Z}\left[E_{(\alpha,\beta)}(\lambda, \cdot)\right](z) = \left(\frac{z}{z-1}\right)^{\beta} \left(1 - \frac{\lambda}{z^q}\left(\frac{z}{z-1}\right)^{\alpha}\right)^{-1},$$

where $|z| > 1$ and $|z-1|^\alpha |z|^{q-\alpha} > |\lambda|$.

Proof. The item 1. is obvious, we prove part 2. By basic calculations we have

$$\begin{aligned}
\mathcal{Z}\left[E_{(\alpha,\beta)}(\lambda,\cdot)\right](z) &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\lambda^k\binom{n-qk+k\alpha+\beta-1}{n-qk}z^{-n}\\
&= \sum_{k=0}^{\infty}\lambda^k z^{-qk}\sum_{s=0}^{\infty}\binom{s+k\alpha+\beta-1}{s}z^{-s}\\
&= \sum_{k=0}^{\infty}\left(\frac{\lambda}{z^q}\right)^k\sum_{s=0}^{\infty}(-1)^s\binom{-k\alpha-\beta}{s}z^{-s}\\
&= \left(\frac{z}{z-1}\right)^{\beta}\sum_{k=0}^{\infty}\left(\frac{\lambda}{z^q}\right)^k\left(\frac{z}{z-1}\right)^{k\alpha}\\
&= \left(\frac{z}{z-1}\right)^{\beta}\left(1-\frac{\lambda}{z^q}\left(\frac{z}{z-1}\right)^{\alpha}\right)^{-1},
\end{aligned}$$

where the summation exists for $|z|>1$ and $|z-1|^{\alpha}|z|^{q-\alpha}>|\lambda|$.

3 Caputo-Type Operator with Positive Order

Let us define, like in [15], the family of functions $\varphi_{k,\alpha}:\mathbb{Z}\to\mathbb{R}$ parameterized by $k\in\mathbb{N}_0$ and $\alpha\in(q-1,q]$, $q\in\mathbb{N}_1$, with the following values

$$\varphi^*_{k,\alpha}(n):=\begin{cases}\binom{n-qk+k\alpha}{n-qk}, & \text{for } n\in\mathbb{N}_{qk}\\ 0, & \text{for } n<qk\end{cases}. \tag{7}$$

Proposition 3 ([15]). *Let function $\varphi^*_{k,\alpha}$ be defined by* (7)*. Then*

$$\mathcal{Z}\left[\varphi^*_{k,\alpha}\right](z)=\frac{1}{z^{qk}}\left(\frac{z}{z-1}\right)^{k\alpha+1} \tag{8}$$

for z such that $|z|>1$.

For family of functions $\varphi^*_{k,\alpha}$ we can state the following proposition.

Proposition 4 ([15]). *Let $\alpha\in(q-1,q]$, $q\in\mathbb{N}_1$, and $\nu=\alpha-q$. Then for $n\in\mathbb{N}_0$ one has $\left(\Delta^{-\alpha}\varphi^*_{k,\alpha}\right)(n+\nu)=\varphi^*_{k+1,\alpha}(n)$.*

In this section we recall the definition of Caputo–type operator and give the properties of this operator, in particular the formula for its $\mathcal{Z}$-transform is proved.

The definition of the Caputo–type fractional h-difference operator can be found, for example, in [9] (for $h=1$) or in [12] (for any $h>0$).

Definition 2. *Let $\alpha\in(q-1,q]$, $q\in\mathbb{N}_1$ and $a\in\mathbb{R}$. The* Caputo–type fractional h-difference operator *${}_a\Delta^{\alpha}_h x$ of order α for a function $x:(h\mathbb{N})_a\to\mathbb{R}$ is defined by*

$$\left({}_a\Delta^{\alpha}_{h,*}x\right)(t)=\left({}_a\Delta_h^{-(q-\alpha)}\left(\Delta^q_h x\right)\right)(t), \tag{9}$$

where $t\in(h\mathbb{N})_{a+(q-\alpha)h}$.

Moreover, for $\alpha = q \in \mathbb{N}_1$ we have $\left({}_a\Delta^q_{h,*}x\right)(t) = \left(\Delta^q_h x\right)(t)$.

There exists the transition formula for the Caputo–type operator between the cases for any $h > 0$ and $h = 1$. Let $x : (h\mathbb{N})_a \to \mathbb{R}$ and $\alpha \in (q-1, q]$, $q \in \mathbb{N}_1$. Then, $\left({}_a\Delta^\alpha_{h,*}x\right)(t) = h^{-\alpha}\left({}_{\frac{a}{h}}\Delta^\alpha_{1,*}\tilde{x}\right)\left(\frac{t}{h}\right)$, where $t \in (h\mathbb{N})_{a+(q-\alpha)h}$ and $\tilde{x}(s) = x(sh)$. For the case $h = 1$ we will write: ${}_{\frac{a}{h}}\Delta^\alpha_* :={}_{\frac{a}{h}}\Delta^\alpha_{1,*}$ and $\Delta^q := \Delta^q_1$.

Proposition 5 ([15]). *For $a \in \mathbb{R}$, $\alpha \in (q-1, q]$, $q \in \mathbb{N}_1$ let us define $y(n) := \left({}_a\Delta^\alpha_{h,*}x\right)(t)$, where $t \in (h\mathbb{N})_{a+(q-\alpha)h}$ and $t = a + (q-\alpha)h + nh$. Then*

$$\mathcal{Z}\left[y\right](z) = h^{-\alpha} z^q \left(\frac{z}{z-1}\right)^{-\alpha}\left(X(z) - \frac{z}{z-1}\sum_{k=0}^{q-1}(z-1)^{-k}\left(\Delta^k_h x\right)(a)\right), \quad (10)$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ and $\overline{x}(n) := x(a+nh)$.

The $\mathcal{Z}$-transform can be used to show some properties of the Caputo–type operator of functions related with the solutions of initial value problems. In [15] we proved that for $k \in \mathbb{N}_1$

$$\left({}_0\Delta^\alpha \varphi^*_{k,\alpha}\right)(n + q - \alpha) = \varphi^*_{k-1,\alpha}(n) . \quad (11)$$

Using the notation of the family of functions $\varphi^*_{k,\alpha}$ we can write the formula for Mittag–Leffler function defined by (5) as

$$E_{(\alpha)}(\lambda, n) := E_{(\alpha,1)}(\lambda, n) = \sum_{k=0}^{\infty}\lambda^k \varphi^*_{k,\alpha}(n) .$$

Moreover, by equations: (5) and (11) we easily see the following property.

Proposition 6. *Let $\alpha \in (q-1, q]$ and $\nu = q - \alpha$, $q = \nu h$, $q \in \mathbb{N}_1$.*

1. $\left(\Delta^{-\alpha}E_{(\alpha)}(\lambda, \cdot)\right)(n - \nu) = \frac{1}{\lambda}E_{(\alpha)}(\lambda, n-1)$,
2. $\left(\Delta^\alpha_* E_{(\alpha)}(\lambda, \cdot)\right)(n + \nu) = \lambda E_{(\alpha)}(\lambda, n)$.

The next proposition states that the function $E_{(\alpha)}(\lambda h^\alpha, \cdot)$ is an eigenfunction of fractional difference equation with the Caputo–type operator.

Proposition 7. *Let $\alpha \in (q-1, q]$ and $a = (\alpha - q)h$. The initial value problem*

$$\left({}_a\Delta^\alpha_{h,*}x\right)(nh) = \lambda x(nh + a) ,\ n \in \mathbb{N}_q \quad (12)$$

$$\left(\Delta^i_h x\right)(a) = b_i ,\quad i = 0, \dots, q-1 \quad (13)$$

has the unique solution given by the formula

$$x(a + nh) = \overline{x}(n) = \sum_{i=0}^{q-1}\left(\Delta^{q-i-1}E_{(\alpha)}(\lambda h^\alpha, \cdot)\right)(n - (q-1))b_i . \quad (14)$$

Proof. Let $y(n) = \left({}_a\Delta^{\alpha}_{h,*}x\right)(nh)$ and $\overline{x}(n) = x(nh+a)$. Then using the $\mathcal{Z}$-transform of both sides of (12) we get the algebraic equation

$$\left(z^q\left(\frac{z}{z-1}\right)^{-\alpha} - h^{\alpha}\lambda\right)X(z) = z^q\left(\frac{z}{z-1}\right)^{-\alpha}\frac{z}{z-1}\sum_{i=0}^{q-1}(z-1)^{-i}b_i\,.$$

Then we calculate that

$$X(z) = \frac{z}{z-1}\left(1-\frac{1}{z^q}\left(\frac{z}{z-1}\right)^{\alpha}\lambda h^{\alpha}\right)^{-1}\sum_{i=0}^{q-1}(z-1)^{-i}b_i\,.$$

Taking

$$W(z) = \frac{z}{z-1}\left(1-\frac{1}{z^q}\left(\frac{z}{z-1}\right)^{\alpha}\lambda h^{\alpha}\right)^{-1}$$

and

$$\begin{aligned}G(z) &= \frac{1}{z^{q-1}}\sum_{i=0}^{q-1}(z-1)^{q-i-1}b_i\\ &= \frac{1}{z^{q-1}}\sum_{i=0}^{q-1}\sum_{s=0}^{q-i-1}\binom{q-i-1}{s}(-1)^{q-i-s-1}z^s b_i\\ &= \sum_{i=0}^{q-1}\sum_{s=0}^{q-i-1}\binom{q-i-1}{s}(-1)^{q-i-s-1}\frac{1}{z^{q-1-s}}b_i\end{aligned}$$

we get

$$X(z) = W(z)G(z)\,.$$

For $W(z)$ from Proposition 2 we see that

$$\mathcal{Z}^{-1}[W](n) = E_{(\alpha)}(\lambda h^{\alpha}, n)\,.$$

Then

$$\begin{aligned}x(a+nh) =&\overline{x}(n) = \mathcal{Z}^{-1}[X](n) = \mathcal{Z}^{-1}[W(z)G(z)](n)\\ =&\sum_{i=0}^{q-1}\sum_{s=0}^{q-i-1}\binom{q-i-1}{s}(-1)^{q-i-s-1}E_{(\alpha)}(\lambda h^{\alpha}, n-(q-s-1))b_i\\ =&\sum_{i=0}^{q-1}\left(\Delta^{q-i-1}E_{(\alpha)}(\lambda h^{\alpha},\cdot)\right)(n-(q-1))b_i\,.\end{aligned}$$

The immediate consequence of Proposition 7 is the formula for solution of semilinear equation.

Proposition 8. *Let $\alpha \in (q-1, q]$ and $a = (\alpha - q)h$. The initial value problem*

$$\left({}_a\Delta^{\alpha}_{h,*}x\right)(nh) = \lambda x(nh+a) + f(nh)\,, \tag{15}$$

$$\left(\Delta^i_h(x)\right)(a) = b_i\,,\quad i = 0,\ldots,q-1 \tag{16}$$

where $n \in \mathbb{N}_q$ has the unique solution given by the formula

$$x(a+nh) = \sum_{i=0}^{q-1} \left(\Delta^{q-i-1} E_{(\alpha)}(\lambda h^\alpha, \cdot)\right)(n-(q-1))b_i + E_{(\alpha)}(\lambda h^\alpha, n-q) * \overline{f}(n)$$

and $\overline{f}(n) = f(nh)$.

Example 1. Let us consider the following initial value problem

$$\left({}_a\Delta^\alpha_{h,*} x\right)(nh) = 0.1 \cdot x(nh+a)\,, \tag{17a}$$
$$x(a) = 1 \tag{17b}$$
$$\left(\Delta_h x\right)(a) = 0.1 \tag{17c}$$

Then Figure 1 presents the graphs of solutions to initial value problems (17) with the Caputo–type operator with orders: 1.1, 1.5, 1.9.

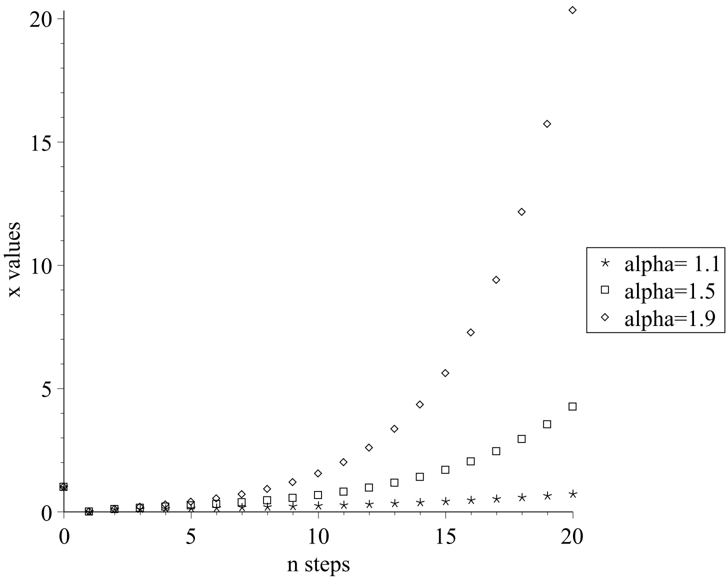

Fig. 1. Trajectories of equation $\left({}_a\Delta^\alpha_{h,*} x\right)(nh) = \lambda \cdot x(nh+a)$ with the Caputo–type operator for $\lambda = 0.1$ and the initial conditions $x(a) = 1$, $\left(\Delta_h x\right)(a) = \lambda$, see Example 1

4 Riemann–Liouville–Type Operator with Positive Order

Let us define, like in [16], the family of functions $\varphi_{k,\alpha} : \mathbb{Z} \to \mathbb{R}$ parameterized by $k \in \mathbb{N}_0$ and $\alpha \in (q-1, q]$, $q \in \mathbb{N}_1$, with the following values

$$\varphi_{k,\alpha}(n) := \begin{cases} \binom{n-qk+k\alpha+\alpha-1}{n-qk}, & \text{for } n \in \mathbb{N}_{qk} \\ 0, & \text{for } n < qk \end{cases}. \tag{18}$$

Observe that $\varphi_{k,\alpha}(n) = \widetilde{\varphi}_{k\alpha+\alpha}(n-qk)$.

Proposition 9 ([16]). *Let function $\varphi_{k,\alpha}$ be defined by* (18). *Then*

$$\mathcal{Z}\left[\varphi_{k,\alpha}\right](z) = \frac{1}{z^{qk}}\left(\frac{z}{z-1}\right)^{(k+1)\alpha} \tag{19}$$

for z such that $|z| > 1$.

For family of functions $\varphi_{k,\alpha}$, $k \in \mathbb{N}_0$, $\alpha > 0$, we can state the following proposition.

Proposition 10 ([16]). *Let $\alpha \in (q-1, q]$, $q \in \mathbb{N}_1$, and $\nu = \alpha - q$. Then for $n \in \mathbb{N}_0$ one has*

$$\left(\Delta^{-\alpha}\varphi_{k,\alpha}\right)(n+\nu) = \varphi_{k+1,\alpha}(n)\,. \tag{20}$$

The definition of the fractional h-difference Riemann-Liouville–type operator can be found, for example, in [5] (for $h = 1$) or in [8] (for any $h > 0$).

Definition 3. *Let $\alpha \in (q-1, q]$, $q \in \mathbb{N}_1$ and $a \in \mathbb{R}$. The* Riemann-Liouville–type fractional h-difference operator ${}_a\Delta_h^\alpha x$ *of order α for a function $x : (h\mathbb{N})_a \to \mathbb{R}$ is defined by*

$$\left({}_a\Delta_h^\alpha x\right)(t) = \left(\Delta_h^q\left({}_a\Delta_h^{-(q-\alpha)}x\right)\right)(t)\,, \tag{21}$$

where $t \in (h\mathbb{N})_{a+(q-\alpha)h}$.

Moreover, for $\alpha = q \in \mathbb{N}_1$ we have: $\left({}_a\Delta_h^q x\right)(t) = \left(\Delta_h^q x\right)(t)$.

There exists the transition formula for the Riemann-Liouville–type operator between the cases for any $h > 0$ and $h = 1$. Let $x : (h\mathbb{N})_a \to \mathbb{R}$ and $\alpha \in (q-1, q]$, $q \in \mathbb{N}_1$. Then, $\left({}_a\Delta_h^\alpha x\right)(t) = h^{-\alpha}\left({}_{\frac{a}{h}}\Delta_1^\alpha \tilde{x}\right)\left(\frac{t}{h}\right)$, where $t \in (h\mathbb{N})_{a+(q-\alpha)h}$ and $\tilde{x}(s) = x(sh)$. For the case $h = 1$ we will write: ${}_{\frac{a}{h}}\Delta^\alpha := {}_{\frac{a}{h}}\Delta_1^\alpha$ and $\Delta^q := \Delta_1^q$.

Proposition 11 ([16]). *For $a \in \mathbb{R}$, $\alpha \in (q-1, q]$, $q \in \mathbb{N}_1$ let us define $y(n) := \left({}_a\Delta_h^\alpha x\right)(t)$, where $t \in (h\mathbb{N})_{a+(q-\alpha)h}$ and $t = a + (q-\alpha)h + nh$, $t_0 = a + (q-\alpha)h$. Then*

$$\begin{aligned}\mathcal{Z}[y](z) =& h^{-\alpha}\left(z^q\left(\frac{z}{z-1}\right)^{-\alpha} X(z)\right.\\ &\left. - z\sum_{k=0}^{q-1}(z-1)^{q-k-1}\left(\Delta_h^k\left({}_a\Delta^{-(q-\alpha)}x\right)\right)(t_0)\right),\end{aligned} \tag{22}$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ and $\overline{x}(n) := x(a+nh)$.

Using the $\mathcal{Z}$-transform of the Riemann-Liouville–type operator we get

$$\left({}_0\Delta^\alpha\varphi_{k,\alpha}\right)(n+q-\alpha) = \varphi_{k-1,\alpha}(n)\,, \tag{23}$$

for $k \in \mathbb{N}_q$.

Using the notation of the family of functions $\varphi_{k,\alpha}$ we can write the formula for Mittag–Leffler function defined by (4) as $E_{(\alpha,\alpha)}(\lambda, n) = \sum_{k=0}^{\infty}\lambda^k\varphi_{k,\alpha}(n)\,.$

Moreover, by equations: (4) and (23) we easily see the property of $E_{(\alpha,\alpha)}(\lambda,\cdot)$ analogous to that one for $E_{(\alpha)}(\lambda,\cdot)$ stated in Proposition 6

The next proposition states that the function $E_{(\alpha,\alpha)}(\lambda h^\alpha,\cdot)$ is an eigenfunction of fractional difference equation with the Riemann–Liouville–type operator.

Proposition 12. *Let $\alpha \in (q-1,q]$ and $a=(\alpha-q)h$. The initial value problem*

$$\left({}_a\Delta_h^\alpha x\right)(nh) = \lambda x(nh+a)\,,\ n\in\mathbb{N}_q \tag{24}$$

$$\left(\Delta_h^i\left({}_a\Delta_h^{-(q-\alpha)}x\right)\right)(a) = b_i\,,\quad i=0,\ldots,q-1 \tag{25}$$

has the unique solution given by the formula

$$x(a+nh) = \overline{x}(n) = \sum_{i=0}^{q-1}\left(\Delta^{q-i-1}E_{(\alpha,\alpha)}(\lambda h^\alpha,\cdot)\right)(n-(q-1))b_i\,. \tag{26}$$

Proof. Let $y(n) = \left({}_a\Delta_h^\alpha x\right)(nh)$ and $\overline{x}(n) = x(nh+a)$. Then using the $\mathcal{Z}$-transform of both sides of (24) we get the algebraic equation

$$\left(z^q\left(\frac{z}{z-1}\right)^{-\alpha} - h^\alpha\lambda\right)X(z) = z\sum_{i=0}^{q-1}(z-1)^{q-i-1}b_i\,.$$

Then we calculate that

$$X(z) = W(z)G(z)\,,$$

where

$$W(z) = \left(\frac{z}{z-1}\right)^\alpha\left(1-\frac{1}{z^q}\left(\frac{z}{z-1}\right)^\alpha\lambda h^\alpha\right)^{-1}$$

and

$$\begin{aligned}G(z) &= \frac{1}{z^{q-1}}\sum_{i=0}^{q-1}(z-1)^{q-i-1}b_i = \frac{1}{z^{q-1}}\sum_{i=0}^{q-1}\sum_{s=0}^{q-i-1}\binom{q-i-1}{s}(-1)^{q-i-s-1}z^sb_i\\ &= \sum_{i=0}^{q-1}\sum_{s=0}^{q-i-1}\binom{q-i-1}{s}(-1)^{q-i-s-1}\frac{1}{z^{q-1-s}}b_i\,.\end{aligned}$$

For $W(z)$ from Proposition 2 we see that

$$\mathcal{Z}^{-1}[W](n) = E_{(\alpha,\alpha)}(\lambda h^\alpha,n)\,.$$

Then

$$\begin{aligned}x(a+nh) =& \overline{x}(n) = \mathcal{Z}^{-1}[X](n) = \mathcal{Z}^{-1}[W(z)G(z)](n)\\ =&\sum_{i=0}^{q-1}\sum_{s=0}^{q-i-1}\binom{q-i-1}{s}(-1)^{q-i-s-1}E_{(\alpha,\alpha)}(\lambda h^\alpha,n-(q-s-1))b_i\\ =&\sum_{i=0}^{q-1}\left(\Delta^{q-i-1}E_{(\alpha,\alpha)}(\lambda h^\alpha,\cdot)\right)(n-(q-1))b_i\,.\end{aligned}$$

The immediate consequence of Proposition 12 is the formula for solution of semilinear equation.

Proposition 13. *Let $\alpha \in (q-1, q]$ and $a = (\alpha - q)h$. The initial value problem*

$$\left({}_a\Delta_h^\alpha x\right)(nh) = \lambda x(nh+a) + f(nh)\,, \tag{27}$$

$$\left(\Delta_h^i \left({}_a\Delta_h^{-(q-\alpha)} x\right)\right)(a) = b_i\,, \quad i = 0, \ldots, q-1 \tag{28}$$

where $n \in \mathbb{N}_q$ has the unique solution given by the formula

$$x(a+nh) = \sum_{i=0}^{q-1} \left(\Delta^{q-i-1} E_{(\alpha,\alpha)}(\lambda h^\alpha, \cdot)\right)(n-(q-1))b_i + E_{(\alpha,\alpha)}(\lambda h^\alpha, n-q) * \overline{f}(n)$$

and $\overline{f}(n) = f(nh)$.

Example 2. Let us consider the following equations

$$\left({}_a\Delta_{h,*}^{1.5} x\right)(nh) = 0.1 \cdot x(nh+a)\,, \tag{29}$$

$$\left({}_a\Delta_h^{1.5} x\right)(nh) = 0.1 \cdot x(nh+a) \tag{30}$$

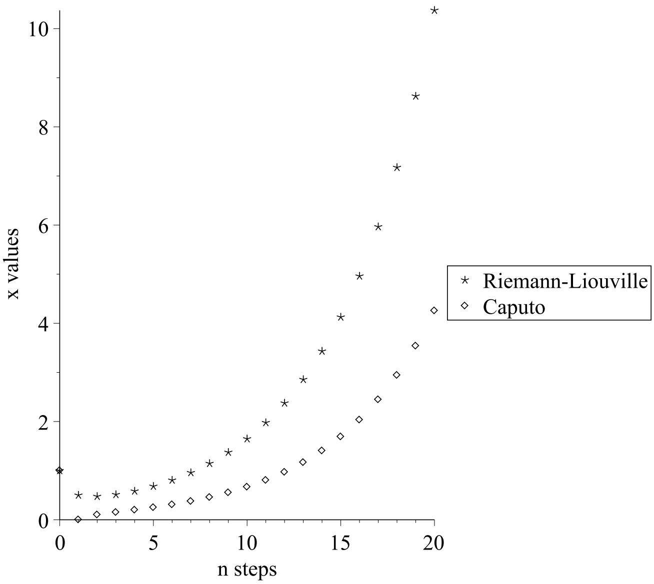

Fig. 2. Trajectories of equations $\left({}_a\Delta_{h,*}^\alpha x\right)(nh) = \lambda \cdot x(nh+a)$ and $\left({}_a\Delta_h^\alpha x\right)(nh) = \lambda \cdot x(nh+a)$ with the Caputo–type and Riemann-Liouville–type operators, respectively, for $\lambda = 0.1$ and order $\alpha = 1.5$, see Example 2

and initial conditions

$$b_0 = 1 \tag{31a}$$

$$b_1 = 0.1 \tag{31b}$$

Then Figure 2 presents the graphs of solutions to equations (29) and (30) with order 1.5 and the initial conditions (31) with the Caputo–type and Riemann-Liouville–type operators, respectively.

5 Conclusions

The Caputo– and Riemann-Liouville–type fractional order difference initial value problems for linear and semilinear systems with any positive order are discussed. The next step is to consider the semilinear control systems and develop conditions on controllability or stability properties via the complex domain.

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