Minimum Energy Control of Fractional Positive Continuous-Time Linear Systems with Two Different Fractional Orders and Bounded Inputs

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Abstract. The minimum energy control problems for fractional positive continuous-time linear systems with two different fractional orders and with bounded input is formulated. Solution to the minimum energy control problem with bounded input is derived. Procedure is proposed and demonstrated on example of electrical circuit.

Keywords: fractional calculus, different order, positive systems, continuous-time, bounded input.

1 Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. Positive linear systems are defined on cones and not on linear spaces. An overview of state of art in positive systems theory is given in the monographs [4, 12].

The first definition of the fractional derivative was introduced by Liouville and Riemann in the middle of the 19^{th} century [24, 25] and another on was proposed in 20^{th} century by Caputo [26]. This idea has been used by engineers for modeling different processes [3, 5]. Mathematical fundamentals of fractional calculus are given in the monographs [17, 23-25]. The positive fractional linear systems have been investigated in [7, 8, 13, 14, 17]. Stability of fractional linear 1D discrete-time and continuous-time systems has been investigated in [1, 20] of 2D fractional positive linear systems in [6] and of continuous-time linear systems consisting of n subsystem with different fractional orders [2]. The minimum energy control problem for standard linear systems has been formulated and solved in [18-22], for fractional positive continuous-time linear systems in [10, 11] and for systems with two different fractional orders in [28]. Reachability and observability of fractional positive continuous-time linear systems has been investigated in [16] and reachability of systems with two different fractional orders in [27].

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In this paper minimum energy problems for fractional positive continuous-time linear systems with two different fractional orders and bounded input will be formulated and solved.

The paper is organized as follows. In section 2 the basic definitions and theorems of the fractional continuous-time linear systems with two different fractional orders are recalled and the necessary and sufficient conditions for positivity and reachability of the systems are given. Section 3 gives the formulation and solution to the minimum energy control problem. Illustrating example of electrical circuit is given in section 4. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix, A^T - the transpose matrix *A*. A real square matrix is called monomial if each its row and each its column contains only one positive entry and the remaining entries are zero.

2 System with Two Different Fractional Orders

In this paper the following Caputo definition of the fractional derivative will be used [17, 26]

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)}\int_0^\infty \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, \tag{2.1}$$

where $n-1<\alpha<n$, $n\in W=\{1,2,...\}$,

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt \tag{2.2}$$

is the Euler gamma function and

$$f^{(n)}(\tau)=\frac{d^n f(\tau)}{d\tau^n}. \tag{2.3}$$

It is well known [17] that the Laplace transform ($\mathcal{L}$) of (2.1) is given by the formula

$$\mathcal{L}\left[\frac{d^\alpha f(t)}{dt^\alpha}\right]=\int_0^\infty \frac{d^\alpha f(t)}{dt^\alpha}e^{-st}dt = s^\alpha F(s)-\sum_{k=1}^n s^{\alpha-k}f^{(k-1)}(0+), \tag{2.4}$$

$n-1<\alpha<n$, $n\in W$, where $F(s)=\mathcal{L}[f(t)]$ and $n-1<\alpha<n$, $n\in W$.

Consider a fractional linear system described by the equation [14]

$$\begin{bmatrix} \frac{d^{\alpha} x_1(t)}{dt^{\alpha}} \\ \frac{d^{\beta} x_2(t)}{dt^{\beta}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t), \tag{2.5a}$$

$$y(t) = [C_1 \quad C_2] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{2.5b}$$

and $p-1<\alpha,\beta<p;\quad p,q\in W$ where $x_1(t)\in\Re^{n_1}$, $x_2(t)\in\Re^{n_2}$, $u(t)\in\Re^{m}$ and $y(t)\in\Re^{p}$ are the state, input and output vectors respectively, $A_{ij}\in\Re^{n_i\times n_j}$, $B_i\in\Re^{n_i\times m}$; $i,j=1,2$.

Initial conditions for (2.5) have the form

$$x_1(0)=x_{10},\ x_2(0)=x_{20} \text{ and } x_0=\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}. \tag{2.6}$$

Theorem 2.1. The solution of the equation (2.5) for $0<\alpha<1;\quad 0<\beta<1$ with initial conditions (2.6) has the form

$$x(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \Phi_0(t)x_0 + \int_0^t M(t-\tau)u(\tau)d\tau, \tag{2.7a}$$

where

$$M(t)=\Phi_1(t)B_{10}+\Phi_2(t)B_{01}=\begin{bmatrix} \Phi_{11}^1(t) & \Phi_{12}^1(t) \\ \Phi_{21}^1(t) & \Phi_{22}^1(t) \end{bmatrix}\begin{bmatrix} B_1 \\ 0 \end{bmatrix}+\begin{bmatrix} \Phi_{11}^2(t) & \Phi_{12}^2(t) \\ \Phi_{21}^2(t) & \Phi_{22}^2(t) \end{bmatrix}\begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$
$$=\begin{bmatrix} \Phi_{11}^1(t)B_1+\Phi_{12}^2(t)B_2 \\ \Phi_{21}^1(t)B_1+\Phi_{22}^2(t)B_2 \end{bmatrix}=\begin{bmatrix} \Phi_{11}^1(t) & \Phi_{12}^2(t) \\ \Phi_{21}^1(t) & \Phi_{22}^2(t) \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{2.7b}$$

and

$$\Phi_0(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}T_{kl}\frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)}, \tag{2.8a}$$

$$\Phi_1(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}T_{kl}\frac{t^{(k+1)\alpha+l\beta-1}}{\Gamma[(k+1)\alpha+l\beta]},\quad \Phi_2(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}T_{kl}\frac{t^{k\alpha+(l+1)\beta-1}}{\Gamma[k\alpha+(l+1)\beta]}, \tag{2.8b}$$

$$T_{kl} = \begin{cases} I_n & \text{for} \quad k = l = 0 \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} & \text{for} \quad k = 1, l = 0 \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} & \text{for} \quad k = 0, l = 1 \\ T_{10}T_{k-1,l} + T_{01}T_{k,l-1} & \text{for} \quad k + l > 1 \end{cases} \tag{2.8c}$$

Proof is given in [14].

Definition 2.1. The fractional system (2.5) is called positive if $x_1(t) \in \Re_+^{n_1}$ and $x_2(t) \in \Re_+^{n_2}$, $t \geq 0$ for any initial conditions $x_{10} \in \Re_+^{n_1}$, $x_{20} \in \Re_+^{n_2}$ and all input vectors $u \in \Re_+^m$, $t \geq 0$.

Theorem 2.2. The fractional system (2.5) for $0 < \alpha < 1$; $0 < \beta < 1$ is positive if and only if

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in M_N, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \Re_+^{N \times m}, \quad C = [C_1 \quad C_2] \in \Re_+^{p \times N}, \quad N = n_1 + n_2. \tag{2.9}$$

Proof is given in [12, 15].

Theorem 2.3. The positive fractional continuous-time linear system with two different fractional orders (2.5a) is reachable if and only if $A \in M_N$ is diagonal and $B \in \Re_+^{N \times N}$ is a monomial matrix.

Proof is given in [28].

3 Minimum Energy Control with Bounded Input

Consider the fractional positive system with two different fractional orders (2.5) with $A \in M_N$, $B \in \Re_+^{N \times N}$ and $C \in \Re_+^{N \times N}$ monomial. If the system is reachable in time $t \in [0, t_f]$, then usually there exists many different inputs $u(t) \in \Re_+^N$ that steers the state of the system from $x_0 = [x_{10} \quad x_{20}]^T = 0$ to $x_f = [x_{1f} \quad x_{2f}]^T \in \Re_+^N$. Among these inputs we are looking for input $u(t) \in \Re_+^N$, $t \in [0, t_f]$ satisfying the condition

$$u(t) \le U \in \Re_+^n,\ t \in [0, t_f], \tag{3.1}$$

that minimizes the performance index [9-11]

$$I(u) = \int_0^{t_f} u^T(\tau) Q u(\tau) d\tau, \tag{3.2}$$

where $Q \in \Re_+^{N\times N}$ is a symmetric, positive defined matrix and $Q^{-1} \in \Re_+^{N\times N}$.

The minimum energy control problem for the fractional positive continuous-time linear systems with two different fractional orders (2.5) can be stated as follows [9-11]: Given the matrices $A \in M_N$, $B \in \Re_+^{N\times N}$, α, β and $Q \in \Re_+^{N\times N}$ of the performance matrix (3.2), $x_f \in \Re_+^N$ and $t_f > 0$, find an input $u(t) \in \Re_+^N$ for $t \in [0, t_f]$ satisfying (3.1), that steers the state vector of the system from $x_0 = 0$ to $x_f \in \Re_+^N$ and minimizes the performance index (3.2).

To solve the problem, following [9-11], we define the matrix

$$W(t_f) = \int_0^{t_f} M(t_f - \tau) Q^{-1} M^T(t_f - \tau) d\tau, \tag{3.3}$$

where $M(t)$ is defined by (2.7b). From (3.3) and Theorem 2.3 it follows that the matrix (3.3) is monomial if and only if the fractional positive system (2.5) is reachable in time $[0, t_f]$. In this case we may define the input

$$\hat{u}(t) = Q^{-1} M^T(t_f - t) W^{-1}(t_f) x_f \ \text{ for } t \in [0, t_f]. \tag{3.4}$$

Note that the input (3.4) satisfies the condition $u(t) \in \Re_+^N$ for $t \in [0, t_f]$ if

$$Q^{-1} \in \Re_+^{N\times N} \ \text{ and } W^{-1}(t_f) \in \Re_+^{N\times N}. \tag{3.5}$$

Theorem 3.1. Let $\overline{u}(t) \in \Re_+^N$ for $t \in [0, t_f]$ be an input satisfying (3.1), that steers the state of the fractional positive system (2.5) from $x_0 = 0$ to $x_f \in \Re_+^N$. Then the input (3.4) satisfying (3.1) also steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^N$ and minimizes the performance index (3.2), i.e. $I(\hat{u}) \le I(\overline{u})$.

The minimal value of the performance index (3.2) is equal to

$$I(\hat{u}) = x_f^T W^{-1}(t_f) x_f. \tag{3.6}$$

Proof is given in [28].

Now following [11], to find $t \in [0, t_f]$ for which $\hat{u}(t) \in \Re_+^n$ reaches its minimal value, using (3.4) we compute the derivative

$$\frac{d\hat{u}(t)}{dt} = Q^{-1} B^T \Psi(t) W^{-1}(t_f) x_f \,,\ t \in [0, t_f] \,, \tag{3.7}$$

where

$$\Psi(t) = \frac{d}{dt}[M^T(t_f - t)]. \tag{3.8}$$

Knowing $\Psi(t)$ and using the equality

$$\Psi(t) W^{-1}(t_f) x_f = 0 \tag{3.9}$$

we can find $t \in [0, t_f]$ for which $\hat{u}(t)$ reaches its maximal value.

Note that if the system is asymptotically stable $\lim_{t\to\infty} M(t) = 0$ then $\hat{u}(t)$ reaches its maximal value for $t = t_f$ and if it is unstable then for $t = 0$.

From the above considerations we have the following procedure for computation of the optimal inputs:

Procedure 4.1

Step 1. Using (2.7b) and (2.8) compute the matrix $M(t)$.

Step 2. Knowing the matrices A, B, Q and α, β, t_f using (3.3) compute the matrix W.

Step 3. Using (3.4) and (3.9) find t_f for which $\hat{u}(t)$ satisfying (3.1) reaches its maximal value and the desired $\hat{u}(t)$ for given $U \in \Re_+^n$ and $x_f \in \Re_+^n$.

Step 4. Using (3.6) compute the maximal value of the performance index $I(\hat{u})$.

4 Example

Consider the fractional electrical circuit shown on Figure 4.1 [15, 27, 28] with given source voltages e_1, e_2, ultracapacitor C_1 of the fractional order $\alpha = 0.7$, ultracapacitor C_2 of the fractional order $\beta = 0.6$, conductances G_1, G_2, G'_1, G'_2 and $G_{12} = 0$.

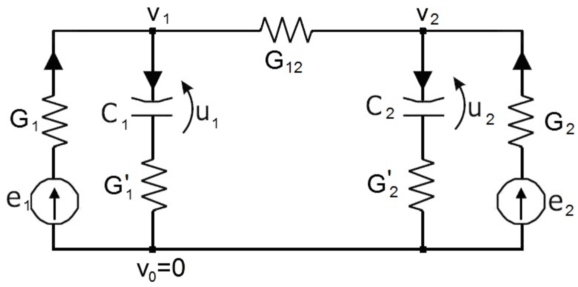

Fig. 1. Fractional electrical circuit

Using the Kirchhoff's laws we can write the equations

$$C_1 \frac{d^\alpha u_1}{dt^\alpha} = G'_1 (v_1 - u_1), \quad C_2 \frac{d^\beta u_2}{dt^\beta} = G'_2 (v_2 - u_2) \tag{4.1}$$

and

$$\begin{bmatrix} (G_1 + G'_1) & 0 \\ 0 & (G_2 + G'_2) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} G'_1 & 0 \\ 0 & G'_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{4.2}$$

From (4.2) we obtain

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} (G_1 + G'_1) & 0 \\ 0 & (G_2 + G'_2) \end{bmatrix}^{-1} \begin{bmatrix} G'_1 & 0 \\ 0 & G'_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} (G_1 + G'_1) & 0 \\ 0 & (G_2 + G'_2) \end{bmatrix}^{-1} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{4.3}$$

Substitution of (4.3) into

$$\begin{bmatrix} \frac{d^\alpha u_1}{dt^\alpha} \\ \frac{d^\beta u_2}{dt^\beta} \end{bmatrix} = \begin{bmatrix} -\frac{G'_1}{C_1} & 0 \\ 0 & -\frac{G'_2}{C_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \frac{G'_1}{C_1} & 0 \\ 0 & \frac{G'_2}{C_2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{4.4}$$

yields

$$\begin{bmatrix} \frac{d^\alpha u_1}{dt^\alpha} \\ \frac{d^\beta u_2}{dt^\beta} \end{bmatrix} = A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{4.5}$$

where

$$A = \begin{bmatrix} \frac{G'_1}{C_1}\left(\frac{G'_1}{(G_1+G'_1)}-1\right) & 0 \\ 0 & \frac{G'_2}{C_2}\left(\frac{G'_2}{(G_2+G'_2)}-1\right) \end{bmatrix},$$
$$B = \begin{bmatrix} \frac{G'_1 G_1}{C_1(G_1+G'_1)} & 0 \\ 0 & \frac{G'_2 G_2}{C_2(G_2+G'_2)} \end{bmatrix}. \tag{4.6}$$

From (4.6) it follows that A is a diagonal Metzler matrix and the matrix B is monomial matrix with positive diagonal entries. Therefore, the fractional electrical circuit is positive for all values of the conductances and capacitances.

Without lost of generality, to simplify the notation, wet take for example values of the electrical circuit: $C_1 = 1$, $\alpha = 0.7$, $C_2 = 2$, $\beta = 0.6$, $G_1 = 4$, $G'_1 = 4$, $G_2 = 3$, $G'_2 = 6$, $G_{12} = 0$ and $N = n_1 + n_2 = 2$.

The matrices (4.6) takes the form

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{4.7}$$

Find the optimal input (source voltage) $\hat{e}(t) \in \Re^2_+$, $t \in [0, t_f]$ satisfying the condition

$$\hat{e}(t) = \begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for } t \in [0, t_f] \tag{4.8}$$

for the performance index (3.2) with $Q = \mathrm{diag}[2,2]$, which steers the system from initial state (voltage drop on capacitances) $u_0 = [0 \quad 0]^T$ to the finite state $u_f = [2 \quad 3]^T$ and minimize the performance index (3.2) with (4.7).

Using (2.8) and (4.7) we obtain

$$M(t) = \begin{bmatrix} \Phi_{11}^1(t) & 0 \\ 0 & \Phi_{22}^2(t) \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{4.9}$$

where

$$\begin{aligned} \Phi_{11}^1(t) &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^{(k+1)0.7+l0.6-1}}{\Gamma[(k+1)0.7+l0.6]}(-2)^k, \\ \Phi_{22}^2(t) &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^{k0.7+(l+1)0.6-1}}{\Gamma[k0.7+(l+1)0.6]}(-1)^l. \end{aligned} \tag{4.10}$$

From (2.7b), (4.7) and (3.3) we have

$$\begin{aligned} W(t_f) &= \int_0^{t_f} M(t_f-\tau)Q^{-1}M^T(t_f-\tau)d\tau \\ &= \int_0^{t_f} \begin{bmatrix} \Phi_{11}^1(\tau) & 0 \\ 0 & \Phi_{22}^2(\tau) \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \Phi_{11}^1(\tau) & 0 \\ 0 & \Phi_{22}^2(\tau) \end{bmatrix}^T d\tau. \end{aligned} \tag{4.11}$$

Note that the electrical circuit is stable. Therefore, $\hat{e}(t)$ reach its maximal value for $t = t_f$.

Now using (3.4) and (4.11) we obtain

$$\hat{e}(t) = Q^{-1}M^T(t_f-t)W^{-1}(t_f)u_f = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} \Phi_{11}^1(t_f-t) & 0 \\ 0 & \Phi_{22}^2(t_f-t) \end{bmatrix}^T W^{-1}(t_f) \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \Phi_{11}^1(t_f-t) & 0 \\ 0 & 0.5\Phi_{22}^2(t_f-t) \end{bmatrix}^T \left[\int_0^{t_f} \begin{bmatrix} 8\Phi_{11}^1(\tau)[\Phi_{11}^1(\tau)]^T & 0 \\ 0 & 2\Phi_{22}^2(\tau)[\Phi_{22}^2(\tau)]^T \end{bmatrix} d\tau \right]^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.25[\Phi_{11}^1(t_f-t)]^T \int_0^{t_f} [\Phi_{11}^1(\tau)[\Phi_{11}^1(\tau)]^T]^{-1} d\tau \\ 1.5[\Phi_{22}^2(t_f-t)]^T \int_0^{t_f} [\Phi_{22}^2(\tau)[\Phi_{22}^2(\tau)]^T]^{-1} d\tau \end{bmatrix}, \tag{4.12}$$

The minimal value of t_f satisfying the condition (3.1) can be found from the inequality

$$\begin{bmatrix} 0.25[\Phi_{11}^1(t_f - t)]^T \int_0^{t_f} [\Phi_{11}^1(\tau)[\Phi_{11}^1(\tau)]^T]^{-1} d\tau \\ 1.5[\Phi_{22}^2(t_f - t)]^T \int_0^{t_f} [\Phi_{22}^2(\tau)[\Phi_{22}^2(\tau)]^T]^{-1} d\tau \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for } t \in [0, t_f] \quad (4.13)$$

and the minimal value of the performance index (3.2) is equal to

$$I(\hat{e}) = u_f^T W^{-1} u_f = [2 \quad 3]\left[\int_0^{t_f} \begin{bmatrix} 8\Phi_{11}^1(\tau)[\Phi_{11}^1(\tau)]^T & 0 \\ 0 & 2\Phi_{22}^2(\tau)[\Phi_{22}^2(\tau)]^T \end{bmatrix} d\tau\right]^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= 0.5\int_0^{t_f} [\Phi_{11}^1(\tau)[\Phi_{11}^1(\tau)]^T]^{-1} d\tau + 3\int_0^{t_f} [\Phi_{22}^2(\tau)[\Phi_{22}^2(\tau)]^T]^{-1} d\tau. \quad (4.14)$$

5 Concluding Remarks

The minimum energy control problems for the fractional positive continuous-time linear systems with two different fractional orders and bounded input have been formulated. Solution to the minimum energy control problem has been given. Effectiveness of the proposed considerations has been demonstrated on example of electrical circuit. Extension of this considerations on systems consisting of n subsystems with different fractional orders is also possible.

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