Perfect Observers of Fractional Descriptor Continuous-Time Linear System

Tadeusz Kaczorek

Bialystok University of Technology
Faculty of Electrical Engineering
Wiejska 45D, 15-351 Bialystok
kaczorek@isep.pw.edu.pl

Abstract. Fractional descriptor observers for fractional descriptor continuous-time linear systems are proposed. Necessary and sufficient conditions for the existence of the observers are established. The design procedure of the observers is given and is demonstrated on a numerical example.

Keywords: fractional descriptor linear systems, design, perfect, observer.

1 Introduction

The fractional linear systems have been considered in many papers and books [8, 9, 11, 15, 23]. Positive linear systems consisting of n subsystems with different fractional orders have been proposed in [14, 15]. Descriptor (singular) linear systems have been investigated in [1-6, 12, 13, 18-21, 24, 25]. The eigenvalues and invariants assignment by state and input feedbacks have been addressed in [4, 12, 18]. The computation of Kronecker's canonical form of a singular pencil has been analyzed in [24].

A new concept of perfect observers for linear continuous-time systems has been proposed in [10, 22]. Observers for fractional linear systems have been addressed in [17, 22]. Fractional descriptor full-order observers for fractional descriptor continuous-time linear systems have been proposed in [16].

In this paper perfect fractional descriptor observers for fractional descriptor continuous-time linear systems will be proposed and necessary and sufficient conditions for the existence of the observer will be established.

The paper is organized as follows. In section 2 the basic definitions and theorems of fractional descriptor linear continuous-time systems are recalled and their perfect fractional descriptor observers are defined. In section 3 necessary and sufficient conditions for the existence of the perfect observers are established and design procedure of the perfect observer is proposed. An illustrating example is given in section 4. Concluding remarks are given in section 5.

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2 Fractional Descriptor Systems and Their Perfect Observers

Consider the fractional descriptor continuous-time linear system

$$E\frac{d^{\alpha}x(t)}{dt^{\alpha}} = Ax(t) + Bu(t), \quad x_0 = x(0), \tag{2.1a}$$

$$y = Cx, \tag{2.1b}$$

where $\frac{d^{\alpha}x(t)}{dt^{\alpha}}$ is the fractional α order derivative defined by Caputo [15, 23]

$${}_0D_t^{\alpha}x(t) = \frac{d^{\alpha}x(t)}{dt^{\alpha}} = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\frac{d^n x(\tau)}{d\tau^n}}{(t-\tau)^{\alpha-n+1}}d\tau, \quad n-1<\alpha<n\in N=\{1,2,...\}, \tag{2.2}$$

$\Gamma(x) = \int_0^{\infty} e^{-t}t^{x-1}dt$ is the gamma function, $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors, $E, A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$. It is assumed that $\det E = 0$ and

$$\det[E\lambda - A] \neq 0 \text{ for some } \lambda\in C \text{ (the field of complex number).} \tag{2.3}$$

Let U be the set of admissible inputs $u(t)\in U\subset\Re^m$ and $X_0\subset\Re^n$ be the set of consistent initial conditions $x_0\in X_0$ for which the equation (2.1) has a solution $x(t)$ for $u(t)\in U$.

The solution of the equation (2.1a) for $x_0\in X_0$ has been derived in [16].

Definition 2.1. The fractional descriptor continuous-time linear system

$$E\frac{d^{\alpha}\hat{x}(t)}{dt^{\alpha}} = F\hat{x}(t) + Gu(t) + Hy(t) \tag{2.4}$$

where $\hat{x}(t)\in\Re^n$ is the estimate of $x(t)$, $u(t)\in\Re^m$ and $y(t)\in\Re^p$ are the same input and output vectors as in (2.1), $E, F\in\Re^{n\times n}$, $G\in\Re^{n\times m}$, $H\in\Re^{n\times p}$, $\det E = 0$ is called a (full-order) perfect state observer for the system (2.1) if

$$x(t) = \hat{x}(t) \text{ for } t > 0. \tag{2.5}$$

3 Design of Perfect Fractional Descriptor Observers

The following elementary row (column) operations will be used [13, 15]:

1. Multiplication of the *i*th row (column) by a real number *c*. This operation will be denoted by $L[i\times c]$ ($R[i\times c]$).
2. Addition to the *i*th row (column) of the *j*th row (column) multiplied by a real number *c*. This operation will be denoted by $L[i+j\times c]$ ($R[i+j\times c]$).
3. Interchange of the *i*th and *j*th rows (columns). This operation will be denoted by $L[i,j]$ ($R[i,j]$).

Lemma 3.1. If

$$\operatorname{rank} E = r < n \tag{3.1}$$

then by the use of the elementary row and column operations the matrix *E* can be reduced to the following upper triangular form

$$P_1EQ_1 = \begin{bmatrix} 0 & E_{12} \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} e_{11} & e_{12} & \dots & e_{1r} \\ 0 & e_{22} & \dots & e_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_{rr} \end{bmatrix} \tag{3.2a}$$

or lower triangular form

$$P_2EQ_2 = \begin{bmatrix} 0 & 0 \\ E_{21} & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} e_{11} & 0 & \dots & 0 \\ e_{21} & e_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_{r1} & e_{r2} & \dots & e_{rr} \end{bmatrix} \tag{3.2b}$$

where P_k and Q_k, $k=1,2$ are the matrices of elementary row and column operations.

Proof. If the condition (3.1) is satisfied then by elementary row and column operations the matrix *E* can be reduced to the form

$$\begin{bmatrix} 0 & E'_{12} \\ 0 & 0 \end{bmatrix}, \quad E'_{12} \in \Re^{r\times r}. \tag{3.3}$$

Next applying the elementary column operations we can reduced the matrix E'_{12} to the upper triangular form E_{12}. The proof for (3.2b) is similar. □

Definition 3.1. The smallest nonnegative integer q is called the nilpotent index of the nilpotent matrix N if $N^q = 0$ and $N^{q-1} \neq 0$.

Lemma 3.2. If

$$\text{rank } E = r < \frac{n}{2} \tag{3.4}$$

then the nilpotent index q of the matrix E is

$$q = 2 \text{ for } r = 1,2,\ldots,\frac{n}{2}-1. \tag{3.5}$$

Proof. If $r < \frac{n}{2}$ then by Definition 3.1 and (3.2a) we have

$$(P_1 E Q_1)^2 = \begin{bmatrix} 0 & E_{12} \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ for } r = 1,2,\ldots,\frac{n}{2}-1. \tag{3.6}$$

Proof for (3.2b) is similar. □

Lemma 3.3. If the nilpotent matrix $N \in \Re^{n\times n}$ has the index $q = 2$ i.e. $N^2 = 0$ and

$$D = \det[d_1,\ldots,d_n],\ d_k \neq 0,\ k = 1,2,\ldots,n \tag{3.7}$$

then the solution $x(t)$ of the fractional differential equation

$$N\frac{d^\alpha x(t)}{dt^\alpha} = Dx,\ 0 < \alpha < 1 \tag{3.8}$$

satisfy the condition

$$x(t) = \begin{cases} -\sum_{k=0}^{q-2} \overline{N}^{(k+1)} \delta^{(k+1)\alpha-1}(t) & \text{for } t = 0 \\ 0 & \text{for } t > 0 \end{cases} \tag{3.9}$$

where $\delta^{(k)}(t)$ is the k-order derivative of the Dirac function $\delta(t)$.

Proof. Applying the Laplace transform to (3.8) and taking into account that

$$\mathcal{L}\left[\frac{d^\alpha x(t)}{dt^\alpha}\right] = \int_0^\infty \frac{d^\alpha x(t)}{dt^\alpha} e^{-st} dt = s^\alpha X(s) - s^{\alpha-1} x_0 \tag{3.10}$$

we obtain

$$Ns^{\alpha}X(s) - Ns^{\alpha-1}x_0 = DX(s), \tag{3.11}$$

where $X(s) = \mathcal{L}[x(t)] = \int_0^{\infty} x(t)e^{-st}dt$ and $x_0 = x(0)$.

Premultiplying (3.11) by the inverse matrix D^{-1} we obtain

$$X(s) = -[I_n - \overline{N}s^{\alpha}]^{-1}\overline{N}s^{\alpha-1}x_0, \tag{3.12}$$

where $\overline{N} = D^{-1}N$ and $\overline{N}^q = D^{-q}N^q = 0$.

Taking into account that

$$[I_n - \overline{N}s^{\alpha}]^{-1} = \sum_{k=0}^{q-1}\overline{N}^k s^{k\alpha}, \tag{3.13}$$

from (3.12) w obtain

$$X(s) = -\sum_{k=0}^{q-2}\overline{N}^{(k+1)}s^{(k+1)\alpha-1}x_0. \tag{3.14}$$

Applying the inverse Laplace transform to (3.14) we obtain (3.9) since $\mathcal{L}^{-1}[s^{k\alpha}] = \delta^{(k\alpha)}(t)$. □

Let

$$e(t) = x(t) - \hat{x}(t). \tag{3.15}$$

Then using (2.1) and (2.4) we obtain

$$\begin{aligned} E\frac{d^{\alpha}e(t)}{dt^{\alpha}} &= E\frac{d^{\alpha}x(t)}{dt^{\alpha}} - E\frac{d^{\alpha}\hat{x}(t)}{dt^{\alpha}} \\ &= Ax(t) + Bu(t) - (F\hat{x}(t) + Gu(t) + HCx(t)) \\ &= (A - HC)x(t) - F\hat{x}(t) + (B - G)u(t) \end{aligned} \tag{3.16}$$

and

$$E\frac{d^{\alpha}e(t)}{dt^{\alpha}} = Fe(t) \tag{3.17}$$

if

$$F = A - HC, \tag{3.18}$$

$$H = B. \tag{3.19}$$

By Lemma 3.1 using the elementary row and column operations the singular matrix E can be reduced to a suitable nilpotent matrix N and from (3.17) we obtain

$$N\frac{d^{\alpha}\bar{e}(t)}{dt^{\alpha}} = \bar{F}\bar{e}(t) \tag{3.20}$$

where

$$N = PEQ, \quad \bar{F} = PFQ, \quad \bar{e}(t) = Q^{-1}e(t) \tag{3.21}$$

and P and Q are matrices of elementary row and column operations.

If we choose the matrix H so that

$$\bar{F} = D \tag{3.22}$$

where D is given by (3.7) then by Lemma 3.3 $\bar{e}(t) = 0$ for $t > 0$ and the fractional descriptor observer (2.4) will be a perfect observer for the system (2.1).

Theorem 3.1. There exists the perfect fractional descriptor observer (2.4) of the fractional descriptor system (2.1) if and only if

$$\operatorname{rank}\begin{bmatrix} \bar{A} - D \\ \bar{C} \end{bmatrix} = \operatorname{rank}[\bar{C}] \tag{3.23}$$

where

$$\bar{A} = PAQ, \quad \bar{C} = CQ \tag{3.24}$$

and the matrices P, Q satisfy (3.21).

Proof. To design the perfect observer (2.4) for the system (2.1) with given matrices A, B, C we have to choose the matrices F, G, H of the observer so that the conditions (3.18), (3.19) and (3.22) are met. From (3.19) we have $H = B$ and the conditions (3.18) and (3.22) are met if and only if

$$\bar{A} - \overline{HC} = D \tag{3.25}$$

where $\bar{H} = PH$.

The equation (3.25) has a solution $\bar{H}$ (and $H = P^{-1}\bar{H}$) for given $\bar{C}$ and D if and only if the condition (3.23) is satisfied. Therefore, there exists the perfect observer (2.4) for the system (2.1) if and only if the condition (3.23) is satisfied. □

From the above considerations we have the following procedure for designing of the perfect observer (2.4) for the system (2.1).

Procedure 3.1

Step 1. Find the matrices P and Q of the elementary row and column operations reducing the matrix E to its nilpotent form $N = PEQ$.

Step 2. Knowing the matrices P, Q compute $\overline{A}$ and $\overline{C}$ defined by (3.24).

Step 3. Choose a diagonal matrix (3.7) and check the condition (3.23). If the condition is satisfied then there exists the perfect observer (2.4) for the system (2.1).

Step 4. Knowing the matrices $\overline{A}$ and $\overline{C}$ find the solution $\overline{H}$ of the equation (3.25).

Step 5. Compute the matrices of the perfect observer (2.4)

$$F = A - HC\,,\ G = B\,,\ H = P^{-1}\overline{H}\,. \tag{3.26}$$

4 Example

Consider the fractional descriptor system (2.1) with the matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix},\quad B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},\quad C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{4.1}$$

The descriptor system is regular since

$$\det[Es - A] = \begin{vmatrix} s-2 & 0 & -1 \\ -3 & s & -2 \\ 0 & -2 & 0 \end{vmatrix} = 2(1-2s) \neq 0\,. \tag{4.2}$$

To design the perfect fractional descriptor observer for the system we use Procedure 3.1 and we obtain the following:

Step 1. In this case we have

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{4.3}$$

and

$$N = PEQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.4}$$

Step 2. Using (3.24) and (4.1) we obtain

$$\bar{A} = PAQ = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \bar{C} = CQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{4.5}$$

Step 3. In this case we choose

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{4.6}$$

and the condition (3.23) is satisfied since

$$\operatorname{rank}\begin{bmatrix} \bar{A} - D \\ \bar{C} \end{bmatrix} = \begin{bmatrix} -2 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = 2 = \operatorname{rank}[\bar{C}] = \operatorname{rank}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{4.7}$$

Therefore, there exists the perfect observer (2.4) for the system (2.1) with (4.1).

Step 4. The equation

$$\overline{HC} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ h_{31} & h_{32} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \bar{A} - D = \begin{bmatrix} -2 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.8}$$

has the solution

$$\bar{H} = \begin{bmatrix} -2 & 2 \\ 2 & 1 \\ 0 & 0 \end{bmatrix} = H \tag{4.9}$$

since $P = I_3$.

Step 5. Using (3.26), (4.1) and (4.9) we obtain

$$F = A - HC = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} - \begin{bmatrix} -2 & 2 \\ 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix},$$
$$G = B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}. \tag{4.10}$$

The perfect observer is described by the equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d^\alpha \hat{x}(t)}{dt^\alpha} = \begin{bmatrix} 0 & 0 & 3 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} u(t) + \begin{bmatrix} -2 & 2 \\ 2 & 1 \\ 0 & 0 \end{bmatrix} y(t). \tag{4.11}$$

5 Concluding Remarks

Perfect fractional descriptor observers for fractional descriptor continuous-time linear systems have been proposed. Necessary and sufficient conditions for the existence of perfect observers for the fractional descriptor linear systems have been established. Designing procedure of the fractional descriptor observers has been proposed and illustrated on a numerical example. The considerations can be easily extended to fractional descriptor discrete-time linear systems. An open problem is an extension for fractional descriptor 2D continuous-discrete linear systems.

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