On Topological and Hyperbolic Properties of Systems with Homoclinic Tangencies

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Abstract We study dynamical properties of a set Λ of trajectories from a small neighbourhood of a non-transversal Poincaré homoclinic orbit. We show that this problem has no univalent solution, as it takes place in the case of a transversal homoclinic orbit. Here different situations are possible, depending on the character of the homoclinic tangency, when Λ is trivial or contains topological (hyperbolic) horseshoes. In this chapter we find certain conditions for existence of both types of dynamics and give a description (in term of the symbolic dynamics) of the corresponding non-trivial hyperbolic subsets from Λ.

1 Introduction

Homoclinic orbit or *Poincaré homoclinic orbit* is an orbit that is bi-asymptotic to a saddle periodic trajectory. Thus, any such orbit belongs to the intersection of the invariant stable W^s and unstable W^u manifolds of the corresponding periodic orbit. Depending on transversality or non-transversality of the intersection, the homoclinic orbit is called *transversal* or *non-transversal*. In the latter case, one says also about *homoclinic tangency*.

The existence of transversal Poincaré homoclinic orbits is considered as the universal criterium of chaos. The point is that even the set Λ_1 of orbits entirely lying in a small neighbourhood of a transversal homoclinic orbit has a non-trivial structure:

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H. González-Aguilar, E. Ugalde (eds.), *Nonlinear Dynamics New Directions,*
Nonlinear Systems and Complexity 12, DOI 10.1007/978-3-319-09864-7_2

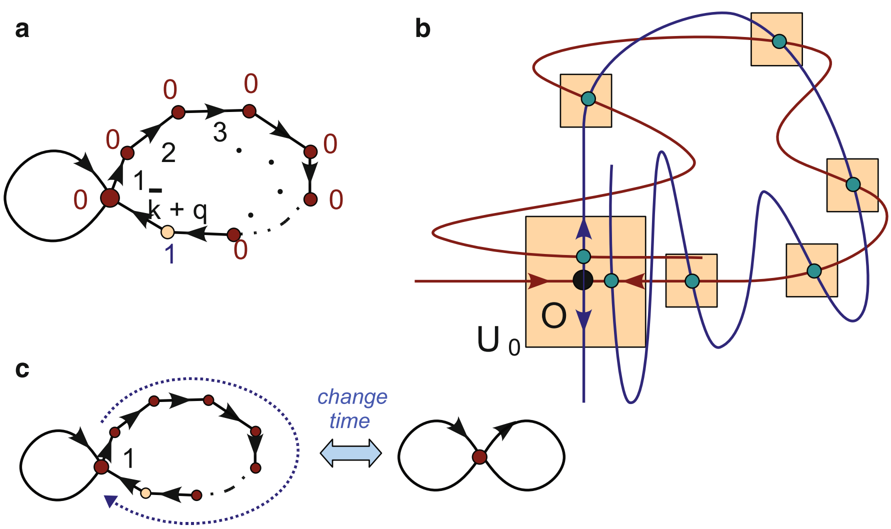


Fig. 1 **a** The graph of the Markov chain $\mathcal{B}_2^{\bar{k}+q}$ related to the symbolic description of Λ_1. **b** A neighbourhood of a transversal homoclinic orbit: $\bar{k}$ can be interpreted as a minimal number of iterations of the diffeomorphism under which orbits of Λ_1 can stay in U_0; and q is a number of neighbourhoods (*small squares*) surrounding those points of Γ_1 which do not belong to U_0. **c** Suspensions over $\mathcal{B}_2^{\bar{k}+q}$ and $\mathcal{B}_2$ are equivalent [3]

it contains infinitely (countable) many periodic and homoclinic orbits, continuum of Poisson stable orbits, etc. Nevertheless, the set Λ_1 can be described completely in terms of the symbolic dynamics, [33]. Namely, let f be a diffeomorphism having a transversal homoclinic orbit Γ_1 to a saddle fixed point O (see Fig. 1). Then the following result, *Shilnikov theorem* [33], takes place:

- *The set Λ_1 is a locally maximal uniformly hyperbolic invariant set on which a diffeomorphism is topologically conjugate to a subsystem $\mathcal{B}_2^{\bar{k}+q}$ of the Bernoulli shift $\mathcal{B}_2$ with two symbols, where $\mathcal{B}_2^{\bar{k}+q}$ is given by the Markov chain of* Fig. 1a *and $\bar{k}, q$ are integers indicated in* Fig. 1b.

Evidently, this result covers also the flow case: then one can consider the set Λ_1 as an invariant set for the Poincaré map of a local section to the corresponding saddle periodic trajectory. However, if we consider Λ_1 as the set of flow orbits, then the result sounds simpler:

- *Λ_1 is topologically equivalent to a suspension over $\mathcal{B}_2$.*

Note that the notion of suspension over a topological Markov chain was introduced in [3]. It generalizes the standard notation of suspension over a map and is convenient for description of flow dynamics (in particular, for classification of critical sets of Morse–Smale systems [2, 3]).

In the case of homoclinic tangency, the corresponding problem (of a description of the orbit structure near a non-transversal homoclinic orbit) becomes much more complicated. Moreover, it cannot be principally solved namely as the problem of "complete description", especially when nearby systems are considered. The point is that arbitrarily small smooth perturbations of any system with a (quadratic) homoclinic tangency can lead to the appearance of homoclinic and periodic orbits of any orders of degeneracy, see [17, 18, 20, 22].

One of the problems of such a type, namely, *the study of topological and hyperbolic properties of systems with homoclinic tangencies*, is the subject of the present chapter. Hyperbolic properties of systems with quadratic homoclinic tangencies were studied first in the paper [5] of N.K. Gavrilov and L.P. Shilnikov and, afterwards, in a series of papers, e.g. [7, 8, 11]. See also the special issue [16] devoted to homoclinic tangencies. Homoclinic tangencies of arbitrary finite orders were studied in papers [6, 11] in which the main attention was given to detecting non-trivial hyperbolic subsets. Note also that certain conditions for the existence of topological horseshoes near homoclinic tangencies were established in [26, 29] for two-dimensional dissipative diffeomorphisms.

In this chapter, see also [24], we extend these results to multidimensional diffeomorphisms (not necessarily dissipative) having homoclinic tangencies to saddle periodic orbits with one-dimensional unstable invariant manifolds.

We assume throughout the paper the following set-up. Let f be an $(m+1)$-dimensional C^r-diffeomorphism, $r \geq 2$, having a hyperbolic fixed point O with multipliers $\lambda_1, \ldots, \lambda_m, \gamma$ such that

$$0 < |\lambda_m| \leq \cdots \leq |\lambda_2| \leq |\lambda_1| < 1 < |\gamma|. \tag{1}$$

We consider the quantity $\sigma \equiv |\lambda_1||\gamma|$ which is called the *saddle value*. Our main assumption is $\sigma \neq 1$ and we consider two different general cases:[1]

(1) the *sectionally dissipative case* when $\sigma < 1$ and
(2) the *sectionally saddle case* when $\sigma > 1$.

We assume also that f has a homoclinic orbit Γ_0 to O for which m-dimensional stable W^s and one-dimensional unstable W^u invariant manifolds of O are tangent and this tangency can be *arbitrary*.

Let U be a small neighbourhood of the contour $O \cup \Gamma_0$. It can be represented as a union of a small neighbourhood U_0 of the point O with a number of small neighbourhood of those points of Γ_0 which do not belong to U_0, as in Fig. 1b. Let Λ be the set of orbits (of f) entirely lying in U.

Our main problem is the study of both topological and hyperbolic properties of Λ. We will keep the following standard terminology.

Definition 1 *We say that*

(i) f possesses a trivial dynamics near Γ_0 if $\Lambda = O \cup \{\Gamma\}$, where $\{\Gamma\}$ is a set of homoclinic orbits to O;

[1] Note that the case $\sigma = 1$ (i.e. O is a neutral saddle) is very specific and we do not consider it here. We only refer the reader to papers [9, 12, 21] in which various cases of neutral saddles ($\sigma = 1$) with homoclinic tangencies were analysed; see also papers [10, 14, 15] in which area-preserving maps were considered.

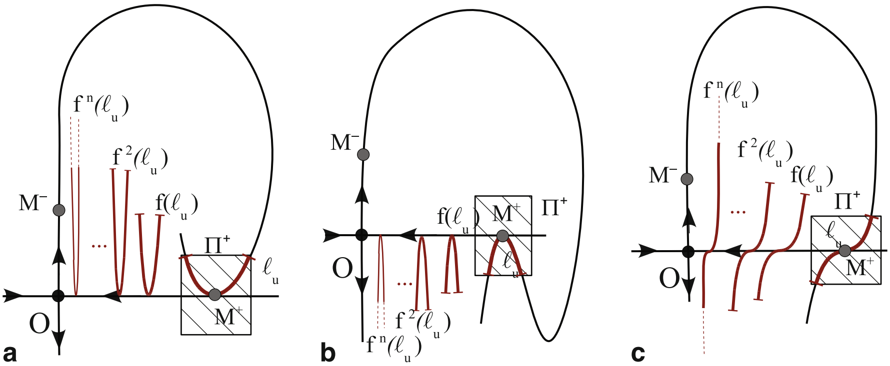


Fig. 2 Homoclinic tangencies. **a** One-sided "from above". **b** One-sided "from below". **c** Topological crossing

(ii) f has a topological horseshoe if Λ contains an f-invariant subset $\tilde{\Lambda}$ such that $f|_{\tilde{\Lambda}}$ is topologically semi-conjugate to a subshift of finite type with positive topological entropy;

(iii) f has a hyperbolic horseshoe if $\tilde{\Lambda}$ from (ii) is uniformly hyperbolic and a topological conjugacy (instead the semi-conjugacy) takes place.

Let M^+ and M^- be a pair of points of Γ_0 such that $M^+ \in W^s_{loc} \cap U_0$, $M^- \in W^u_{loc} \cap U_0$. Let Π^+ and Π^- be sufficiently small neighbourhoods of the points M^+ and M^-, respectively, and let $M^+ = f^q(M^-)$ for some integer positive q. Denote the map $f|U_0$ as T_0 and the map $f^q|\Pi^-$ as T_1 (thus, $T_1(M^-) = M^+$). The map T_0 is called the *local map*, because it is defined in a small neighbourhood of O; while, the map T_1 is called the *global map*, because it acts along a global piece of the orbit Γ_0.

Definition 2 *The homoclinic tangency is* isolated *if, for some Π^+, the point M^+ is the unique intersection point of $l_u = T_1(W^u_{loc} \cap \Pi^-) \cap \Pi^+$ and W^s_{loc}. We say that the (isolated) homoclinic tangency is* one-sided *if W^s_{loc} divides Π^+ onto two half-parts and the curve l_u belongs as whole to the closure of exactly the one half of Π^+, otherwise, the tangency is called* topological crossing. *We say that a one-sided tangency is* from below, *if the point M^- is not an accumulation point of the curves $T_0^i(l_u), i = 0, 1, \ldots,$ i.e.*

$$M^- \notin \overline{\bigcup_{i\geq 0}\{T_0^i(l_u)\}}; \tag{2}$$

and is from above *otherwise. See* Fig. 2 *for an illustration.*[2]

We need to say that the problem under consideration (on a structure of the set Λ of orbits near a non-transversal homoclinic orbit) is sharply different in many aspects

[2] Note that if $\gamma < 0$, then condition (2) can hold only when W^s_{loc} contains l_u. Thus, in this case, any isolated one-sided tangency is, in fact, a tangency "from above". On the other hand, if $\gamma > 0$, condition (2) allows a big variety of non-isolated tangencies.

from the corresponding problem (the so-called Poincaré–Birkhoff problem) for the case of transversal homoclinic orbit.

First, it does not allow a single (univalent) answer (like the Shilnikov theorem): we select two very different situations when the set Λ has a *trivial structure* and when Λ contains *infinitely many horseshoes*, in a sense of Definition 1.

Second, under the weakest assumptions, related only to geometrical properties of the homoclinic tangency, see Definition 2, we can establish the corresponding classification results for the sectionally dissipative case $\sigma < 1$ (see Theorem 1) or for the two-dimensional case $\sigma > 1$ (which is reduced to the case $\sigma < 1$ for f^{-1}), see Proposition 2. Furthermore, we render concrete the problem by means of additional general assumptions under conditions A, B, C and D in Sect. 3). Conditions A–C define the so-called *simple homoclinic tangency* and D defines an order of this tangency. Importantly, for simple homoclinic tangencies with $\sigma > 1$, we can establish quite readable classification results, see Theorems 2 and 3. However, for non-simple homoclinic tangencies with $\sigma > 1$, even for quadratic ones, we have no hope for a similar classification; see Sect. 4.1. Instead, it was shown in [6, 24] that, in the case of simple homoclinic tangencies of finite orders, a quite detailed description of non-trivial hyperbolic invariant subsets from Λ can be achieved. We collect the corresponding (hyperbolic) results in Sect. 5, see Theorems 4 and 5 and their specifications: Proposition 5 for the case $\sigma < 1$ and Proposition 6 for the case $\sigma > 1$. By "a quite detailed description", we mean that for some dense subset of systems with homoclinic tangency this description (given by Theorem 5) becomes complete; we prove this fact in Sect. 5.1, see also Proposition 7.

2 Topological Horseshoes in the Sectionally Dissipative Case

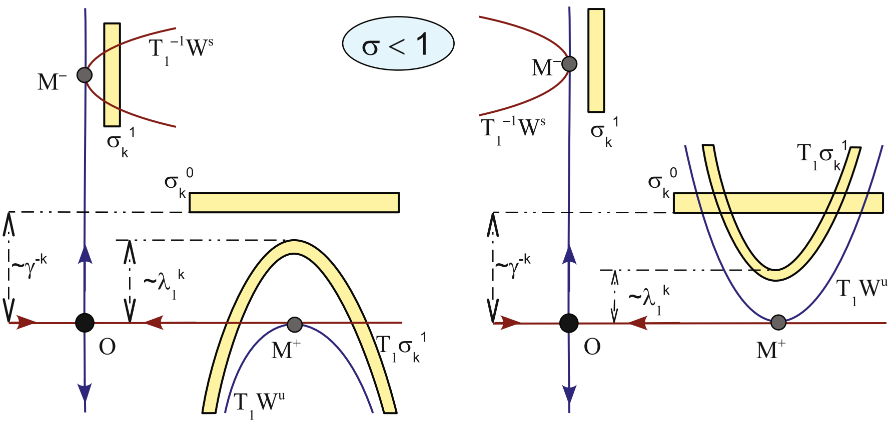
Note, at first, that condition (2) in the sectionally dissipative case ($\sigma < 1$) can be regarded to a certain criterion of trivial dynamics that the following result shows.

Theorem 1 *Let f have a homoclinic tangency to O and $\sigma < 1$. Then*

1. *If the tangency satisfies condition* (2) *(in particular, when the tangency is "from below"), then f possesses the trivial dynamics near Γ_0.*
2. *Otherwise, i.e. when condition* (2) *is not fulfilled (but, again, $\sigma < 1$), f has infinitely many topological horseshoes near Γ_0.*

The proof is given in Sect. 6, see also [24]. Note that the two-dimensional case was analysed in [29, 26]. Nevertheless, it is necessary to note that the main geometric idea of the proof is quite simple that Fig. 3 reflects where the corresponding illustrations (in dimension 2) are shown.

Indeed, the problem under consideration allows a geometric interpretations as follows. In Π^+ and Π^- there exist infinitely many ($m+1$-dimensional) disjoint strips $\sigma_k^0 \subset \Pi^+$ and $\sigma_k^1 \subset \Pi^-$, $k = k_1, k_1+1, \ldots$, such that $\sigma_k^0 = T_0^{-k}(\Pi^-) \cap \Pi^+$ and $\sigma_k^1 = T_0^k(\Pi^+) \cap \Pi^- = T_0^k \sigma_k^0$. Thus, only iterations under f of points from Π^+



a *trivial dynamics* **b** *horseshoes*

Fig. 3 The strips σ_k^0 and σ_k^1 are posed on the distances of order γ^{-k} from W^s_{loc} and λ_1^k from W^u_{loc}, respectively. By a geometry of the tangency, the image of σ_k^1 under the global map T_1 is a horseshoe $T_1(\sigma_k^1)$ whose top is posed on a distance $\sim \lambda_1^k$ from W^s_{loc}. Since $\lambda_1^k \ll \gamma^{-k}$, we have: **a** $T_1(\sigma_k^1) \cap \sigma_k^0 = \emptyset$ in the case of tangency "from below". **b** The return map $T_k \equiv T_1 T_0^k : \sigma_k^0 \to \Pi^+$ is similar to a Smale horseshoe map in the case of tangency "from above"

which belong to the strips σ_k^0 can reach Π^-. In turn, the image of the strip σ_k^1 under the global map T_1 is a horseshoe-shaped figure $T_1(\sigma_k^1) \subset \Pi^+$ (below we will use term "horseshoe $T_1(\sigma_k^1)$" namely in this, geometrical, sense). Therefore, infinitely many first return maps $T_k = T_1 T_0^k : \sigma_k^0 \to \Pi^+$, $k = \bar{k}, \bar{k}+1$, are defined here. If the tangency is "from below" all these maps possess a trivial dynamics, see Fig. 3a. However, in the case of tangency "from above" (or topological crossing), all these maps act, topologically, as Smale horseshoe maps, see Fig. 3b. Therefore, we can say, in the latter case, about existence of infinitely many *geometrical* Smale horseshoes.

Although within the hypothesis of Theorem 1 we can not say directly on hyperbolic properties of these topological horseshoes, one can apply some indirect facts, like the Katok theorem [30], in order to deduce the following.

Proposition 1 *In the case 2 of Theorem 1 the set Λ contains infinitely many hyperbolic horseshoes in sense of Definition 1.*[3]

Indeed, by Theorem 1, we get that the restriction to each topological horseshoe has positive topological entropy. The latter means that there are some orbits with positive first Lyapunov exponent. These orbits have other Lyapunov exponents to

[3] For example in the case of tangency "from above", the topological (geometrical) horseshoe of map T_k (for every value of k from an infinite sequence of integers) contains an T_k-invariant subset Δ_k such that the system $T_k|\Delta_k$ is uniformly hyperbolic and topologically conjugate to a subshift of finite type with positive topological entropy.

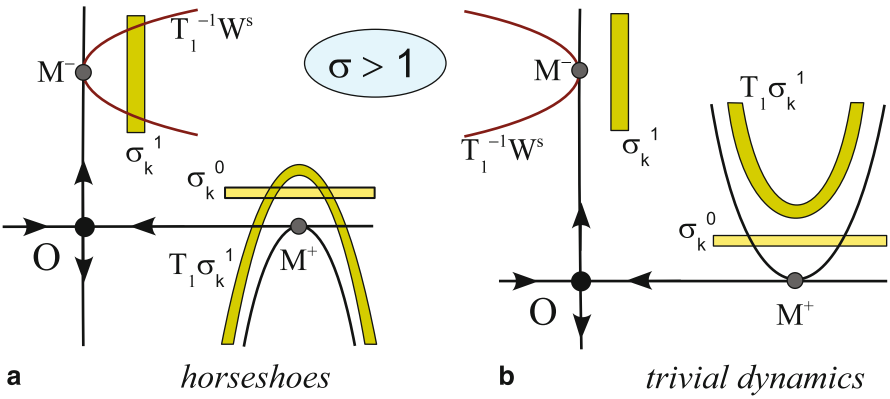


Fig. 4 Geometry of strips and horseshoes in the case $\sigma > 1$

be negative, due to the sectional dissipation that implies the absence of such orbits having zero second Lyapunov exponent.[4]

Thus, in the sectionally dissipative case, relation (2) provides the *necessary condition* for the existence of trivial dynamics near a homoclinic tangency. It is not the case when $\sigma > 1$. In particular, we can see principal differences in geometry between the cases $\sigma < 1$ and $\sigma > 1$ even in dimension two (compare Figs. 3 and 4). However, Theorem 1 can be directly applied to this 2dim case for $\sigma > 1$, since we can here consider f^{-1} instead f. Then the following condition

$$M^+ \notin \overline{\bigcup_{i\geq 0}\{T_0^{-i}(l_s)\}}, \tag{3}$$

where the curve $l_s \subset \Pi^-$ is defined as $l_s = T_1^{-1}(W^s_{loc} \cap \Pi^+) \cap \Pi^-$, plays role of the condition(2). Thus, we obtain

Proposition 2 *Let f be two-dimensional and $\sigma > 1$. Then*

1) If condition (3) *holds, then f possesses a trivial dynamics near Γ_0.*
2) Otherwise, i.e. when (3) *is not valid (but, again, $\sigma > 1$), then f has infinitely many topological horseshoes near Γ_0 every of which contains a hyperbolic horseshoe in sense of Definition* 1.

Unfortunately, this approach is not suitable for multidimensional case with $\sigma > 1$ since dim $W^u(O) > 1$ for f^{-1} and it is not the case under consideration. Moreover, as we will see below, in this case we need, by necessity, an additional specification of the homoclinic tangency, since only geometrical properties are not sufficient even for deriving certain classification results like Theorem 1. However, an analogous specification is required especially (even for the case $\sigma < 1$), if we want to know

[4] We thank D. Turaev who attracted our attention to the interesting fact that the Katok theorem can be directly applied to the sectionally dissipative case. See also [35].

more. question relates to hyperbolic properties. Indeed, the hyperbolic horseshoes from Corollary 1 are not specified. We do not know whether the first return maps $T_1 T_0^i : \sigma_i^0 \to \sigma_i^0$ are hyperbolic. In order to get more information, we need to make more assumptions (see conditions A–D below).

3 The Definition of a Simple Homoclinic Tangency

Let the multipliers $\lambda_1, \dots, \lambda_m, \gamma$ of O are ordered as in (1). We call *leading* (or weak) those multipliers that are equal to $|\lambda_1|$ by the absolute value. Accordingly, the other stable multipliers (less than $|\lambda_1|$ by modulus) are called *non-leading* (or strong stable). We consider the following general condition:

A) the leading stable multipliers of O are simple.
Accordingly, two different types of saddle fixed (periodic) points are defined. Namely,

A1) the point O is a *saddle*, i.e. the multiplier λ_1 is real and $|\lambda_1| > |\lambda_j|$ for $j = 2, \dots, m$;

A2) the point O is a *saddle-focus*, i.e. λ_1 and λ_2 are complex conjugate, $\lambda_1 = \lambda e^{i\psi}, \lambda_2 = \lambda e^{-i\psi}, 0 < \lambda < 1, 0 < \psi < \pi$ and $\lambda > |\lambda_j|$ for $j = 3, \dots, m$.

When the point O has also non-leading stable multipliers we need more assumptions related to the homoclinic tangency. Recall some necessary facts.

First, if Condition A holds, the manifold $W^s_{loc}(O)$ contains a C^r-smooth *strong stable manifold* W^{ss}_{loc} which touches at O the eigenspace of Df corresponding to the non-leading multipliers λ_i (thus, the W^{ss}_{loc} has dimension $(m-1)$ or $(m-2)$ when, respectively, A1 or A2 holds). Moreover, it is well-known (see e.g. [25, 34]) that W^s_{loc} is foliated by the C^r *strong stable foliation* F^{ss} containing W^{ss}_{loc} as the leaf.

Note also that the manifold $W^u(O)$ is a part of the so-called *extended unstable manifold* $W^{ue}(O)$ (see, for example, [25, 34]). It is a smooth (at least $C^{1+\epsilon}$) invariant manifold which is tangent, at O, to the eigenspace of Df corresponding to the unstable and the leading stable multipliers, thus, W^{ue} is two- or three-dimensional if O is a saddle or a saddle focus, respectively. Although, the manifold W^{ue} is not defined uniquely, any two such manifolds contain W^u_{loc} and are tangent to each other at the points of W^u_{loc}. Thus, at the homoclinic point $M^- \in W^u_{loc}$ the tangent space to W^{ue}, denoted as $\mathcal{T}_{M^-} W^{ue}$ is defined uniquely. Since $M^+ = T_1(M^-)$, we can extend W^{ue} up to the homoclinic point M^+. Denote the tangent space to W^{ue} at M^+ as $\mathcal{T}_{M^+} W^{ue}$. Evidently, $\mathcal{T}_{M^+} W^{ue} = DT_1\, (\mathcal{T}_{M^-} W^{ue}))$, where DT_1 denotes the differential of the global map $T_1 \equiv f^q : \Pi^- \to \Pi^+$ at the point M^-.

We introduce the following general conditions: *the off strong stable manifold condition*

B) $M^+ \notin W^{ss}_{loc}$
and *subtransversality condition*

C) $\mathcal{T}_{M^+} W^{ue}$ is transversal to $F^{ss}(M^+)$ at M^+, where $F^{ss}(M^+)$ is the leaf of the foliation F^{ss} containing the point M^+.

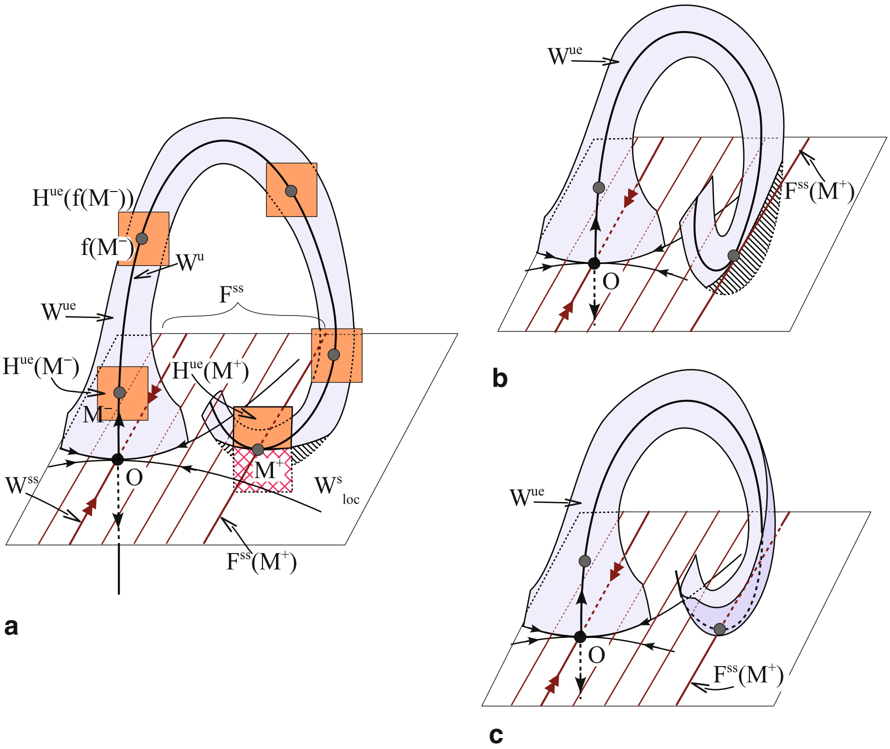


Fig. 5 Examples of three-dimensional diffeomorphisms with simple (**a**), and non-simple (**b**)–(**c**) homoclinic tangencies

An example of three-dimensional diffeomorphism satisfying the conditions A1, B and C is shown in Fig. 5a. Main cases where C is violated (while A1 and B are kept) are illustrated in Fig. 5b and c. Here, either $\mathcal{T}_{M^+} W^{ue}$ is transversal to W^s_{loc} but touches $F^{ss}(M^+)$ at M^+, as in case (b), or $\mathcal{T}_{M^+} W^{ue}$ belongs to W^s_{loc} (i.e., any surface $T_1(W^{ue})$ touches W^s_{loc} at M^+), as in case (c).

Definition 3 *A homoclinic tangency satisfying conditions A–C is called* simple.

We can also adapt this definition to homoclinic tangencies of finite orders as follows.

Definition 4 Let f be a C^r-diffeomorphism under consideration and n be an integer such that $2 \le n \le r$. Then we say that the homoclinic tangency at M^+ is of order n if there exists local (near M^+) C^r-coordinates $(\xi_1, \dots, \xi_m, \eta)$ in which W^s_{loc} has the equation $\eta = 0$ and a piece of W^u containing M^+ can be written (in the parameter form) as follows

$$\xi_i = b_i \alpha + O(\alpha^2),\ (i = 1, \dots, m),\ \eta = g(\alpha), \tag{4}$$

where $g(\alpha)$ is C^r and

$$g(0) = \frac{dg(0)}{d\alpha} = \cdots = \frac{d^{n-1}g(0)}{d\alpha^{n-1}} = 0, \frac{d^n g(0)}{d\alpha^n} = n!, \quad d \neq 0, \tag{5}$$

where α is a parameter varying near zero, b_i and d are constants and $\sum |b_i| \neq 0$. If all derivatives $d^i g(0)/d\alpha^i$ vanish for $i = 0, \ldots, s$, where $s \geq r$, then we say that the tangency is of indefinite order.

By definition, tangencies of even orders are one-sided, while, tangencies of odd orders correspond to the topological crossings. Tangencies of some small orders have special notations: *quadratic* for $n = 2$, *cubic* for $n = 3$, *quadric* for $n = 4$. Note that a type of the tangency can depend on coordinate changes. For instance, even a quadratic tangency can be transformed into a tangency of indefinite order under C^1-change of coordinates.[5]

We introduce the following condition:

D) the manifolds $W^s(O)$ and $W^u(O)$ have the tangency of a finite order $n \geq 2$ at the homoclinic point M^+.

Since f is diffeomorphism, condition D implies that $W^s(O)$ and $W^u(O)$ have tangency of order n at any point of Γ_0. Note that, in the real analytical case, the condition D holds always in that sense that any possible homoclinic tangency can be only of finite order here (except for infinitely degenerate cases when $W^s(O)$ and $W^u(O)$ coincide).

By Definition 3, a homoclinic tangency satisfying conditions A–D should be labeled as *simple homoclinic tangency of order n*. Note that notation of simple quadratic homoclinic tangency was introduced in [19] which is, in fact, a certain variant of the so-called quasi-transversal homoclinic intersection, [31].

Note that conditions A–C have very important dynamical sense. Namely, when these conditions hold the corresponding diffeomorphism f has, see [34, 36],

- *A global smooth invariant center manifold W^c which contains the orbits O and Γ_0 as well as all orbits entirely lying in U.*

This manifold is normally-hyperbolic (in sense of [25]), since f is exponentially contracting along transversal to W^c directions which correspond, at O, to the u-directions. Therefore, $\dim W^c = 2$ or $\dim W^c = 3$ depending on A1 or A2 holds and, thus, the problem under consideration allows the so-called *dimension reduction* to $\dim = 2$ or $\dim = 3$, respectively. Therefore, it has a certain sense to single out this type of homoclinic tangencies, i.e. *the simple homoclinic tangencies*.

However, an insufficient smoothness of W^c (only $C^{1+\epsilon}$, the same as for W^{ue}) "destroys" the condition D in the restricted system $f|W^c$. Therefore, when this condition is principally important (see Sect. 5) we should work with the initial multidimensional system (even in the sectionally dissipative case $\sigma < 1$). On the other hand,

[5] Therefore, in problems of such type, it is not reasonable to use a C^1-linearization (which, by the way, does not always exist in the multidimensional case). This can lead to non-repairable mistakes in the proofs or to absurd results, and, in the best case, only very rough topological properties can be established [32, 4].

the condition D is hardly controlled one (for specific systems) and, therefore, it is reasonable to assume that only its topological variant takes place, i.e. the homoclinic tangency is *simple one-sided* or *simple topological crossing* (here, "simple" means again that conditions A–C are fulfilled). In this case one can obtain certain meaningful results related to the existence of horseshoes (topological or even hyperbolic) in the case $\sigma > 1$. See Sect. 4.

3.1 On a Coordinate Expression of the Simple Homoclinic Tangency

We will use in U_0 local coordinates in which the saddle map T_0 takes the so-called *main normal form* or *normal form of the first order*. This form is very convenient for calculations and, in contrast to the linear form, exists always. Thus, the following result holds.

Lemma 1 [13, 8, 34]. *Let f be C^r ($r \geq 2$) and O have multipliers $\lambda_1, \dots, \lambda_m, \gamma$ satisfying* (1). *Then the map $T_0 = f|U_0$ can be written, in some C^r-coordinates (x, u, y) on U, as follows:*

$$(\bar{x}, \bar{u}, \bar{y}) = \left(\hat{A}x + h_1(x, u, y), \hat{B}u + h_2(x, u, y), \gamma y + h_3(x, u, y)\right), \tag{6}$$

where eigenvalues of the matrix $\hat{A}$ are equal to $|\lambda_1|$ by absolute values, whereas, eigenvalues of $\hat{B}$ are all smaller. Besides, the functions h_1, h_2, h_3 satisfy conditions

$$\begin{aligned} & h_1(0,0,y) \equiv 0,\ h_2(0,0,y) \equiv 0,\ h_3(x,y,0) \equiv 0, \\ & h_1(x,u,0) \equiv 0,\ h_3(0,0,y) \equiv 0, \\ & \frac{\partial h_1}{\partial x}|_{x=0,u=0} \equiv 0,\quad \frac{\partial h_2}{\partial x}|_{x=0,u=0} \equiv 0\quad \frac{\partial h_3}{\partial y}|_{y=0} \equiv 0. \end{aligned} \tag{7}$$

Remark 1 The proof of Lemma 1 is based on the so-called "Afraimovich changes of variables" [1]. In turn, these changes generalize the method by E.A. Leontovich for construction of finitely-smooth normal forms of two-dimensional flows near saddle equilibria [27, 28]. See also [21, 34] for a modern treatment of this theory.

If condition A holds, we have in Lemma 1 that either $x \in \mathbb{R}^1$ and $\hat{A} = \lambda_1$ in case A1; or $x \in \mathbb{R}^2$ and $\hat{A}_2 = \lambda \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$ in case A2.

Using the main normal form one can easily calculate any iterations T_0^k, especially, when k is large that the following result shows.

Lemma 2 [13, 8, 34] *Let $(x_k, u_k, y_k) = T_0^k(x_0, u_0, y_0)$. When the local map T_0 is written in form (6) and identities (7) hold, the following relations take place for all large k:*

$$
\begin{aligned}
x_k \; - \; \hat{A}^k x_0 \; &= \; \hat{\lambda}^k \xi_k(x_0, u_0, y_k), \\
u_k \; &= \; \hat{\lambda}^k \hat{\xi}_k(x_0, u_0, y_k), \\
y_0 \; - \; \gamma^{-k} y_k \; &= \; \hat{\gamma}^{-k} \eta_k(x_0, u_0, y_k)
\end{aligned}
\tag{8}
$$

where $\hat{\lambda}$ and $\hat{\gamma}$ are some constants such that $0 < \hat{\lambda} < |\lambda_1|$, $\hat{\gamma} > |\gamma|$ and functions $\xi_k, \eta_k, \hat{\xi}_k, \hat{\eta}_k$ are uniformly bounded for all k, along with the derivatives up to order $(r-2)$. The derivatives of orders $(r-1)$ and r are estimated as follows $\|x_k, u_k\|_{C^{r-1}} = o(|\lambda_1|^k)$, $\|y_0\|_{C^{r-1}} = o(|\gamma|^{-k})$ and $\|x_k, u_k, y_0\|_{C^r} = o(1)_{k\to\infty}$.

In the coordinates of Lemma 1, the manifolds $W^s_{loc}(O)$, $W^u_{loc}(O)$ as well as W^{ss}_{loc} are straightened, i.e. they have the following equations:

$$
W^s_{loc}(O) \; - \; \{y = 0\}, \;\; W^u_{loc}(O) \; - \; \{(x, u) = 0\}, \;\; W^{ss}_{loc} \; - \; \{x = 0, y = 0\}.
$$

Hence, we can write that $M^+ = (x^+, u^+, 0)$ and $M^-(0, 0, y^-)$, where $y^- > 0$. If condition B holds, then $\|x^+\| \neq 0$. In case A1, since $x \in \mathbb{R}^1$ (and $u \in \mathbb{R}^{m-1}$), it means that $x^+ \neq 0$ and we assume that $x^+ > 0$ here.

Define the neighbourhoods Π^+ and Π^- of M^+ and M^-, respectively, as follows

$$
\Pi^+ = \{\|(x - x^+, u - u^+)\| \le \varepsilon_0, |y| \le \varepsilon_0\}, \;\; \Pi^- = \{\|(x, u)\| \le \varepsilon_1, |y - y^-| \le \varepsilon_1\},
\tag{9}
$$

where $\varepsilon_0 > 0, \varepsilon_1 > 0$ are sufficiently small and $T_0\Pi^+ \cap \Pi^+ = \emptyset$, $T_0^{-1}\Pi^- \cap \Pi^- = \emptyset$.

The global map $T_1 \equiv f^q : \Pi^- \to \Pi^+$ can be written as follows

$$
(\bar{x} - x^+, \bar{u} - u^+) \; = \; F(x_1, u_1, y_1 - y^-), \;\; \bar{y} \; = \; G(x_1, u_1, y_1 - y^-),
\tag{10}
$$

where C^r-functions F and G are defined on Π^- and $F(0) = 0, G(0) = 0$ as well as $G_y(0) = 0$. Then we can write the map T_1 in the following form

$$
\begin{aligned}
&(\bar{x} - x^+, \bar{u} - u^+) = ax + \hat{a}u + b(y - y^-) + O\left(\|(x, u)\|^2 + (y - y^-)^2\right), \\
&\bar{y} = cx + \hat{c}u + \varphi(y - y^-) + O\left(\|(x, u)\|^2 + \|(x, u)\||y - y^-|\right),
\end{aligned}
\tag{11}
$$

where $\varphi(0) = 0$ and $\varphi'(0) = 0$, since the curve $T_1(W^u_{loc})$ touches W^s_{loc} at the point M^+, and

$$
\det \begin{pmatrix} a & \hat{a} & b \\ c & \hat{c} & 0 \end{pmatrix} \neq 0.
\tag{12}
$$

If condition D holds, then

$$
\varphi(y - y^-) \; \equiv \; d(y - y^-)^n + o\left((y - y^-)^n\right) \;\; \text{and} \;\; d \neq 0.
\tag{13}
$$

Note also that, in the coordinates of Lemma 1, the foliation F^{ss} has a form $\{x = const, y = 0\}$ and the tangent space $\mathcal{T}_M W^{ue}$ to W^{ue} at any point $M \in W^u_{loc}$ is the plane $u = 0$. Then condition C means, by (11), that the planes $\mathcal{T}_{M^+} W^{ue}$: $\{(\bar{x} - x^+, \bar{u} - u^+) = ax + b(y - y^-),\ \ \bar{y} = cx\}$ and $F^{ss}(M^+) : \ \left\{\bar{x} = x^+, \bar{y} = 0\right\}$ are transversal (here, $\mathcal{T}_{M^+} W^{ue}$ is given in a parameter form where x and $(y - y^-)$ are parameters). It means that the system $(0, u - u^+) = ax + b(y - y^-), 0 = cx$ has a unique solution. Thus, condition C reads as

$$b_1 \neq 0, \quad c \neq 0 \ \text{ in case A1} \tag{14}$$

or as

$$b_1^2 + b_2^2 \neq 0, \quad c_1^2 + c_2^2 \neq 0 \ \text{ in case A2.} \tag{15}$$

Note that Fig. 5b and c correspond to the case A1 with $b_1 = 0$ and $c = 0$, respectively.

4 On Simple Homoclinic Tangencies in the Sectionally Saddle Case

In this section we consider, essentially, the multidimensional sectionally saddle case $\sigma > 1$. Concerning a type of the homoclinic tangencies, we assume in this section that they are *isolated and one-sided.*

Then f can possess both trivial and non-trivial dynamics near Γ_0 depending on type of the tangency.

Remark 2 In the case of topological intersection, infinitely many topological horseshoes near the homoclinic tangency always exist. It follows from the fact that the system $f|\Lambda$ is *semi-conjugate* to $\mathcal{B}_2^{\bar{k}+q}$ (compare with the Shilnikov theorem from Introduction). However, even in this case, if conditions A1, B and C hold (i.e. O is a saddle and the tangency is simple), infinitely many *hyperbolic horseshoes* (in the sense of Definition 1) exist. This fact follows directly from the Katok theory, since the problem allows reduction to dim $= 2$ in this situation. Note that in [11] certain classes of systems with simple homoclinic tangencies of odd order are described for which $f|\Lambda$ is *topologically conjugate* to $\mathcal{B}^2_{\bar{k}+q}$ and all orbits of Λ, except Γ_0, are of saddle type.

We introduce the so-called "index of one-sided tangency" ν_0 that can take values $+1$ or -1 and is defined as follows. Consider the piece $T_1(W^u_{loc}) \cap \Pi^+$ of $W^u(O)$ which, by (11), has the equation

$$(\bar{x} - x^+, \bar{u} - u^+) = b\alpha + O\left(\alpha^2\right), \quad \bar{y} = \varphi(\alpha), \tag{16}$$

written in the parametric form, where $\alpha = y - y^-$ is a parameter. Since $\varphi(0) = 0, \varphi'(0) = 0$, the curve (16) touches the plane $\bar{y} = 0$ at $\alpha = 0$. Let this tangency be

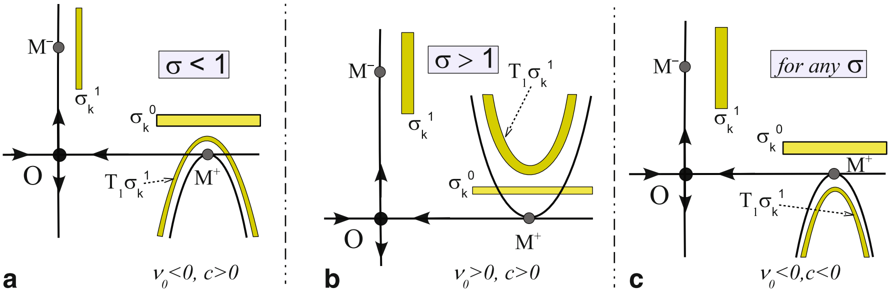


Fig. 6 Examples of simple homoclinic tangencies with trivial dynamics for $\lambda_1 > 0, \gamma > 0$

one-sided, then we define

$$\nu_0 \;=\; \text{sign}\;\varphi(\xi) \;\; \text{at}\; \xi \neq 0.$$

Thus, if $\gamma > 0$ the homoclinic tangency (one-sided and isolated) is "from below" if $\nu_0 = -1$ and it is "from above" if $\nu_0 = +1$. If $\gamma < 0$ the value of index ν_0 depends on a choice of homoclinic points in such a way that, for example, $\nu_0(M^+) = -\nu_0(f(M^+))$; it means that we can always take such pairs of the homoclinic points that $\nu_0 = +1$. Note that if the tangency is of even order, then $\nu_0 \;=\;$ sign d that follows directly from (13).

Theorem 2 [Simple homoclinic tangencies with trivial dynamics]
Let f have a one-sided homoclinic tangency satisfying A1, B and C and one of the following conditions: (1) $\sigma < 1$ and $\gamma > 0, \nu_0 < 0$; (2) $\sigma > 1$ and $\lambda_1 > 0, c\nu_0 > 0$ or 3) $\lambda_1 > 0, \gamma > 0, c < 0, \nu_0 < 0$ (independently on σ).

Then f possesses trivial dynamics near Γ_0, i.e. $\Lambda = O \cup \Gamma_0$.

Proof Item 1 of the theorem is a partial case of Theorem 1, see Fig. 6a.

Consider item 2 of the theorem.

Since A1 holds, the map T_0, by Lemma 1, takes the form

$$(\bar{x}, \bar{u}, \bar{y}) \;=\; \Big(\lambda_1 x + h_1(x,u,y), \hat{B}u + h_2(x,u,y), \gamma y + h_3(x,u,y)\Big), \tag{17}$$

where $x, y \in \mathbb{R}^1$, $u \in \mathbb{R}^{m-1}$ and the matrix $\hat{B}$ has eigenvalues $\lambda_2, \ldots, \lambda_m$. Then, by Lemma 2, map T_0^k can be written in the following cross-form (compare with (8))

$$x_k \;=\; \lambda_1^k x_0 \left(1 + O([\hat{\lambda}/\lambda_1]^k)\right), u_k = O\left(\hat{\lambda}^k\right), \;\; y_0 \;=\; \gamma^{-k} y_k (1 + O\left([\hat{\gamma}/\gamma]^{-k}\right)). \tag{18}$$

Using (11) we can write the first return map $T_k = T_1 T_0^k : \sigma_k^0 \mapsto \Pi^+$ as follows

$$\begin{gathered}(\bar{x}_0 - x^+, \bar{u}_0 - u^+) = b(y_1 - y^-) + O\left(|\lambda_1|^k \|(x_0,u_0)\| + (y_1 - y^-)^2\right), \\ \bar{y}_0 = c_1 \lambda_1^k x_0 + \varphi(y_1 - y^-) + O\left(\tilde{\lambda}^k \|(x_0,u_0)\| + |\lambda_1^k| \|(x_0,u_0)\| |y_1 - y^-|\right),\end{gathered} \tag{19}$$

where $x_0, y_0, y_1 \in \mathbb{R}^1, u_0 \in \mathbb{R}^{m-1}$. Let $\bar{y}_0$ be the y-coordinate of a point inside some strip $\sigma_j^0 \subset \Pi^+$. Then we can write, by (18), that $\bar{y}_0 = \gamma^{-j}\bar{y}_1 + O(\tilde{\gamma}^{-j})$. Introduce the coordinates $\xi = x_0 - x^+, \hat{\xi} = u_0 - u^+$ and $\eta = y_1 - y^-$. Then we can write the second equation of (19) in the form

$$\gamma^{-j}\left(y^- + \bar{\eta} + O([\hat{\gamma}/\gamma]^{-j})\right) = c_1\lambda_1^k\left(x^+ + O(|\xi| + |\eta|) + O([\hat{\lambda}/\lambda]^j)\right) + \varphi(\eta). \tag{20}$$

Since $\xi, \hat{\xi}, \eta$ are small, $x^+ > 0, y^- > 0$ as well as $\lambda_1 > 0$ and $c_1\nu_0 > 0$, Eq. (20) can have solutions only in the case $|\gamma|^{-j} \geq \lambda_1^k$. Since $|\lambda_1\gamma| > 1$, this inequality can be fulfilled if only $k \gg j$. Thus, any horseshoe $T_1(\sigma_k^1)$ can intersect only those strips σ_j^0 whose numbers are strictly less than k, see Fig. 6b. It implies that some forward iteration of any point from Π^+ must leave U. Thus, only two orbits, O and Γ_0, will always stay in U which implies that the dynamics is trivial.

In the case $\lambda_1 > 0, \gamma > 0, c_1 < 0, \nu_0 < 0$, Eq. (20) has no solutions at all, independently on σ. It means that the horseshoes $T_1(\sigma_k^1)$ do not intersect any strip σ_i^0, see Fig. 6c, i.e. the dynamics is trivial. This completes item (3) of the theorem.

Theorem 3 [Simple one-sided tangencies with non-trivial dynamics at $\sigma > 1$] *Let f have a one-sided homoclinic tangency satisfying conditions A–C and let $\sigma > 1$. Then the set $\Lambda(f)$ contains infinitely many topological horseshoes in the following cases:*

(1) *the point O is a saddle-focus, i.e. conditions A2 holds;* (2) *the point O is a saddle, i.e. A1 holds, and the combination $\lambda_1 > 0, \nu_0 > 0$ of the signs takes no place.*

Proof In the case of *item 1* of the theorem, we have, by Lemma 1, that the local map T_0 has the form

$$(\bar{x}, \bar{u}, \bar{y}) = \left(\lambda R_\psi x + h_1(x,u,y), \hat{B}u + h_2(x,u,y), \gamma y + h_3(x,u,y)\right),$$

where $x = (x_1, x_2)$ and R_ψ is the rotation matrix (on the angle ψ). The global map T_1 has now form (11), where $c = (c_1, c_2)$ and $c_1^2 + c_2^2 \neq 0$ by the condition C and $\varphi(s)$ is a function of fixed sign at $s \neq 0$ because the homoclinic tangency is one-sided.

Consider the first return map $T_k = T_1 T_0^k : \sigma_k^0 \mapsto \Pi^+$ which can be written now as

$$\begin{aligned} &(\bar{x} - x^+, \bar{u} - u^+) = (b_1, b_2)^\mathsf{T}(y - y^-) + O\left(\lambda^k\|(x,u)\| + (y - y^-)^2\right), \\ &\bar{y} = \lambda^k\left((c_1\cos k\psi + c_2\sin k\psi)x_1 + (c_2\cos k\psi - c_1\sin k\psi)x_2\right) \\ &\qquad + \varphi(y - y^-) + + O\left(\tilde{\lambda}^k\|(x,u)\| + \lambda^k\|(x,u)\||y - y^-|\right). \end{aligned} \tag{21}$$

Let us show that, for infinitely many values of k, these maps T_k are, geometrically, Smale horseshoe maps. Introduce new x-coordinates as $\xi_1 = x_1 - x_1^+, \xi_2 = x_2 - x_2^+$. Then the second equation from (21) can be written as

$$\begin{aligned} \bar{y} = \lambda^k&\left(\hat{C}\cos(k\psi + \theta) + O(\|\xi\|)\right) + \varphi(y - y^-) \\ &+ O\left(\tilde{\lambda}^k\|(\xi,u)\| + \lambda^k\|(\xi,u)\||y - y^-|\right), \end{aligned} \tag{22}$$

where $\hat{C} = \sqrt{(c_1^2+c_2^2)((x_1^+)^2+(x_2^+)^2)}$ and $\theta \in [0,2\pi)$ is an angle such that $\cos\theta = (c_1x_1^+ + c_2x_2^+)\hat{C}^{-1},\ \cos\theta = (c_2x_1^+ - c_1x_2^+)\hat{C}^{-1}$.

Note that $\hat{C} > 0$, since the conditions B and C imply, respectively, that $(x_1^+)^2 + (x_2^+)^2 \neq 0$ and $c_1^2 + c_2^2 \neq 0$.

Let $\varphi(s) \geq 0$ and $\gamma > 0$, for the sake of definiteness. Then $\bar{y}$ from (22) can run values from $\bar{y}_{min} = \lambda^k(\hat{C}\cos(k\psi+\theta) + O(\|\xi\|))$ till $\bar{y}_{max} = \max\varphi(s)_{|s|\leq\varepsilon_1} + O(\lambda^k)$. However, values of the coordinate y on the strip σ_k^0 satisfy the inequality

$$\gamma^{-k}(y^- - \varepsilon_1) < y < \gamma^{-k}(y^- + \varepsilon_1).$$

Evidently, there are such $\delta_0 > 0$ and $\delta_1 > 0$ that (i) $\bar{y}_{max} > \delta_0$ for all sufficiently large k and (ii) for any ψ, since $\hat{C} > 0$ and $\|\xi\|$ is small, there are infinitely many such k that $(\hat{C}\cos(k\psi+\theta) + O(\|\xi\|)) < -\delta_1$. Thus, the first return map T_1T^k for such values of k transforms the strip σ_k^0 into the horseshoe $T_k(\sigma_k^0)$ such that its top is posed below σ_k^0 (and even below W^s_{loc}) and the horseshoe intersects σ_k^0 forming (at least) two connected components. Thus, f possesses, in this case, infinitely many geometrical Smale horseshoes. Other cases are considered analogously.

Consider *item 2* of the theorem. Since $A1$ holds, the map $T_k = T_1T_0^k : \sigma_k^0 \mapsto \Pi^+$ can be written now as

$$\begin{aligned}(\bar{x}-x^+, \bar{u}-u^+) &= (b_1,b_2)^{\mathsf{T}}(y-y^-) + O\left(|\lambda_1|^k\|(x,u)\| + (y-y^-)^2\right),\\ \bar{y} &= c\lambda_1^k x + \varphi(y-y^-) + O\left(\tilde{\lambda}^k\|(x,u)\| + |\lambda_1|^k\|(x,u)\||y-y^-|\right).\end{aligned} \tag{23}$$

Denote $\xi = x - x^+$. Then the second equation from (23) is rewritten as

$$\bar{y} = c\lambda_1^k\left(x^+ + O(|\xi|) + O([\tilde{\lambda}/\lambda_1]^k)\right) + \varphi(y-y^-). \tag{24}$$

Consider the model equation $\gamma^{-k}y^- = c\lambda_1^k x^+ + \varphi(s)$ where $s \in [-\varepsilon_1, \varepsilon_1]$ and for some $\hat{\delta} > 0$, $\varphi(s) \in [0,\hat{\delta}]$ or $\varphi(s) \in [-\hat{\delta}, 0]$ and $\nu_0 = \text{sign}\varphi(s)_{s\neq 0}$. Since $|\lambda_1\gamma| > 1$, this equation has no solution (with respect to s) only in the case where $\lambda_1 > 0$ and $c\nu_0 > 0$. In other cases, at least two solutions exist.

It gives us the desired result.

4.1 On the Necessity of Conditions A and C for the Existence/Absence of Topological Horseshoes

Note that, in the sectionally dissipative case $\sigma < 1$, as we can see from Theorem 1, conditions A, B and C play only a role of auxiliary conditions which help to establish certain hyperbolic properties. However, in the sectionally saddle case $\sigma > 1$, these conditions become necessary even for the topological horseshoe property. If they are violated, then the corresponding system can possess either trivial dynamics or horseshoes depending on new characteristics of the corresponding (non-simple) homoclinic tangency. We illustrate this fact by means of considering a three-dimensional model below.

Let g_0 be a three-dimensional diffeomorphism having a saddle fixed point $\hat{O}$ with multipliers $\lambda_1, \lambda_2, \gamma$, where $0 < \lambda_2 < \lambda_1 < 1 < \gamma$ and $\lambda_1\gamma > 1$, and a homoclinic orbit $\hat{\Gamma}_0$ at whose points the manifolds $W^u(\hat{O})$ and $W^s(\hat{O})$ have a quadratic tangency. We assume also that this homoclinic tangency is not simple, the local map T_0 is linear and the global map T_1 is of model form. We take T_0 in the linear form $(\bar{x}, \bar{u}, \bar{y}) = (\lambda_1 x, \lambda_2 u, \gamma y)$ and T_1 in the following form

$$(\bar{x}_0 - x^+, \bar{u}_0 - u^+, \bar{y}_0) = \big(b_1(y - y^-), a_{21}x, \hat{c}u + d(y - y^-)^2\big), \tag{25}$$

where $b_1 a_{21} \hat{c} \neq 0$, since the map T_1 should be diffeomorphism. We see that condition C is violated in this case: the model map T_1 corresponds to the case where $c = 0$ in (11), i.e. (14) is not valid here. We assume, for more definiteness, that $d < 0$, i.e. the quadratic homoclinic tangency is "from below". We assume also that $u^+ > 0$ in the given example.

Proposition 3 *The following dynamical properties of g_0 hold.*

(i) *If $\hat{c} < 0$, then $\Lambda(g_0)$ is trivial, i.e. $\Lambda(g_0) = \hat{O} \cup \hat{\Gamma}_0$.*
(ii) *If $\hat{c} > 0$ and $\lambda_2\gamma < 1$, then $\Lambda(g_0)$ is trivial, i.e. $\Lambda(g_0) = \hat{O} \cup \hat{G}_0$.*
(iii) *If $\hat{c} > 0$ and $\lambda_2\gamma > 1$, then $\Lambda(g_0)$ contains infinitely many geometrical Smale horseshoes.*

Proof Since the corresponding results have an independent interest and the proof consists mainly in direct calculations, we analyse the problem under consideration in more detail than before. Since T_0 is linear, the equations of $W^s_{loc}(O)$ and $W^u_{loc}(O)$ are $y = 0$ and $(x = 0, u = 0)$, respectively. We choose a pair of homoclinic points: $M^+(x^+, u^+, 0) \in W^s_{loc}$ and $M^-(0, 0, y^-) \in W^u_{loc}$ assuming that $u^+ > 0, y^- > 0$. We consider sufficiently small rectangle neighbourhoods $\Pi^+ = \{|x - x^+| \leq \varepsilon_0, |u - u^+| \leq \varepsilon_0, |y| \leq \varepsilon_0)\}$ and $\Pi^- = \{|x| \leq \varepsilon_1, |u| \leq \varepsilon_1, |y - y^-| \leq \varepsilon_1)\}$ of the points M^+ and M^-, respectively, such that $T_0(\Pi^+) \cap \Pi^+ = \emptyset$ and $T_0^{-1}(\Pi^-) \cap \Pi^- = \emptyset$. Note that (for any $\varepsilon_{0,1} > 0$) there exist points on Π^+ whose iterations under T_0 reach Π^-. The set of such initial points on Π^+ consists from countable many disjoint strips $\sigma_k^0, k = \bar{k}, \bar{k}+1, \ldots$ Accordingly, a countable many disjoint strips $\sigma_k^1, k = \bar{k}, \bar{k}+1, \ldots$ exists on Π^- such that $\sigma_k^1 = T_0^k(\sigma_k^0)$. Note also that the strips σ_k^0 and σ_k^1 are defined as $\sigma_k^0 = \Pi^+ \cap T_0^{-k}(\Pi^-)$ and $\sigma_k^1 = \Pi^- \cap T_0^k(\Pi^+)$. In the case under consideration, since the map T_0^k can be written in the form

$$x_k = \lambda_1^k x_0, \ u_k = \lambda_2^k u_0, \ y_0 = \gamma^{-k} y_k \ , \tag{26}$$

we can write exact formulas for the strips:

$$\sigma_k^0 = \{(x, u, y) | |x - x^+| \leq \varepsilon_0, |u - u^+| \leq \varepsilon_0, |y - \gamma^{-k} y^-| \leq \gamma^{-k}\varepsilon_1\}, \tag{27}$$

$$\sigma_k^1 = \{(x, u, y) | |x - \lambda_1^k x^+| \leq \lambda_1^k \varepsilon_0, |u - \lambda_2^k u^+| \leq \lambda_2^k \varepsilon_0, |y - y^-| \leq \varepsilon_1\}. \tag{28}$$

For the sake of definiteness, we will denote coordinates x, u, y of points in Π^+ as x_0, u_0, y_0 and in Π^- as x_1, u_1, y_1. Now we take the strip σ_k^1 and consider its image, horseshoe $T_1(\sigma_k^1)$, under the global map T_1.

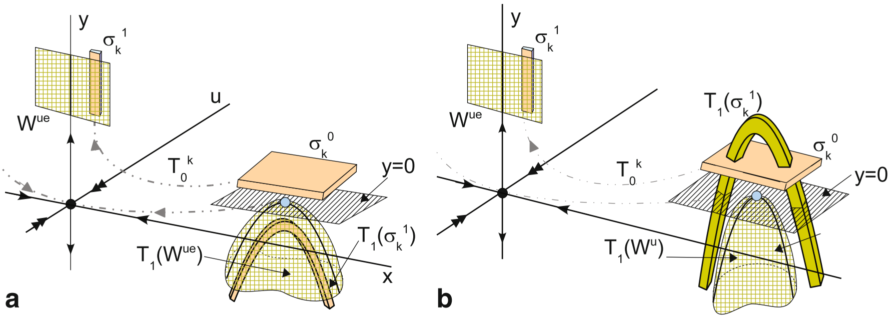


Fig. 7 Examples of homoclinic tangencies with $c = 0$ in the sectionally saddle case $\sigma > 1$: **a** The dynamics is trivial; **b** Smale horseshoes exist

By (25), we obtain that $T_1(\sigma_k^1)$ is a 3D-figure in Π^+ which can be given in the coordinate form as follows

$$(x_0 - x^+, u_0 - u^+, y_0) = \big(b_1(y_1 - y^-), a_{21}x_1,\ \hat{c}u_1 + d(y - y^-)^2\big), \tag{29}$$

where the coordinates (x_1, u_1, y_1) run values along σ_k^1, see formula (28).

Consider now *item (i)*, $\hat{c} < 0$, of Proposition 3 and show that here $T_1(\sigma_k^1) \cap \sigma_j^0 = \emptyset$ for all sufficiently large k and j. Suppose, however, that $T_1(\sigma_k^1) \cap \sigma_j^0 \neq \emptyset$ for some k and j. Then, evidently, by virtue of (27), (28) and (29), the equation

$$\gamma^{-j}\eta = \hat{c}\lambda_2^k\xi + dz^2 \tag{30}$$

has solutions with respect to $\eta \in [y^- - \varepsilon_1, y^- + \varepsilon_1]$ when (ξ, z) run values from the domain $(|\xi - u^+| \leq \varepsilon_0, |z| \leq \varepsilon_1)$. Note that since $u^+ > 0, y^- > 0$ such η and ξ take only positive values for sufficiently small $\varepsilon_{0,1}$. Then, since $d < 0, \lambda_2 > 0, \gamma > 0$, Eq. (30) can have solutions only in that case where

$$\gamma^{-j}\eta - \hat{c}\lambda_2^k\xi \leq 0. \tag{31}$$

However, since $\hat{c} < 0$, the latter inequality is not valid for any $j, k \geq \bar{k}$. It implies that the horseshoes $T_1(\sigma_k^1)$ and strips σ_j^0 do not intersect each other in this case. Geometrically, it follows from the fact that in the case $\hat{c} < 0$ the strips σ_j^0 and the horseshoes $T_1(\sigma_k^1)$ are posed in Π^+ on different sides from a plane $W^s_{loc} : y = 0$, see Fig. 7a. Hence, the diffeomorphism g_0 has a trivial dynamics here: $\Lambda(g_0) = \hat{O} \cup \hat{\Gamma}_0$.
Consider now <col1>item (ii), $\hat{c} > 0$ and $\lambda_2\gamma < 1$, of Proposition 3. Again we obtain that if $T_1(\sigma_k^1) \cap \sigma_j^0 \neq \emptyset$, then the inequality (31) has solutions. Since $\gamma^{-1} > \lambda_2$ in this case, the Eq. (30) can have solutions only if $j > k$. It follows that $T_1(\sigma_k^1) \cap \sigma_k^0 = \emptyset$, i.e. g_0 has no (topological) horseshoes in U, and, moreover, $\Lambda(g_0) = \hat{O} \cup \hat{\Gamma}_0$ here.
Consider now *item (iii)*, $\hat{c} > 0$ and $\lambda_2\gamma > 1$, of Proposition 3. We obtain from (29) and (28) that the horseshoe $T_1(\sigma_k^1)$ has a top with coordinate $y_0^{top} \sim \hat{c}\lambda_2^k u^+$ and its bottom (i.e. T_1-image of the top and bottom of the strip σ_k^1) has coordinate $y_0^{bot} \sim d\varepsilon_1^2 + \hat{c}\lambda_2^k u^+$. Since $\lambda_2\gamma > 1$ we have that $y_0^{top} > \gamma^{-k}(y^- + \varepsilon_1)$, i.e. the

top of the horseshoe $T_1(\sigma_k^1)$ is posed above the strip σ_k^0. Since $d < 0$ we have that $y_0^{bot} < 0$ for sufficiently large k, i.e. the bottom of $T_1(\sigma_k^1)$ is posed below the strip σ_k^0. Thus, the first return maps $T_k = T_1 T_0^k$ for all sufficiently large k are in this case, topologically, Smale horseshoe maps. This completes the proof.

5 Hyperbolic Properties of Diffeomorphisms with Simple Homoclinic Tangencies of Finite Order

We assume now that f satisfies conditions A)–D). Then reformulating Theorem 2 we select class of simple homoclinic tangencies of finite order. Namely, Theorem 2 implies the following result

Proposition 4 *Let a diffeomorphism f satisfying conditions A–D with even n be such that 1) either $\gamma > 0, d < 0$ in the case $\sigma < 1$; 2) or $\lambda_1 > 0, cd > 0$ in the case $\sigma > 1$; 3) or $\lambda_1 > 0, \gamma > 0, c < 0, d < 0$ independently on σ. Then f possesses trivial dynamics near Γ_0, i.e. $\Lambda = O \cup \Gamma_0$.*

Other diffeomorphisms under consideration will contain non-trivial hyperbolic subsets inside Λ. The corresponding results were proved in [5, 6, 11] and [24]. Therefore, we give here only some review of results.

Among the diffeomorphisms with hyperbolic subsets we select first those which admits a complete description of Λ. Namely, let f satisfy conditions A1, B, C and D with $\gamma > 0, \lambda_1 > 0, c < 0$ and also $d > 0$ when n is even. We bring such diffeomorphisms to the *complete class*.

In this case the geometry of the strips σ_k^0 and horseshoes $T_1(\sigma_i^1)$ for all possible $i, k \geq \bar{k}$ is of such type as in Fig. 8: all strips and horseshoes intersect "regularly". As it was shown still in [5, 6], such a geometry implies a non-uniform hyperbolicity: all orbits of Λ, except for Γ_0, are of saddle type and the set Λ can be described completely in terms of the symbolic dynamics.

If n is odd (the tangency if like cubic), then Λ can be identified with $\mathcal{B}_2^{\bar{k}+q}$ as in the case of transversal homoclinic orbit. If n is even, the set Λ is described now by means of the topological Bernoulli scheme (shift) $\mathcal{B}_3$ with three symbols. Namely, let $\mathcal{B}_3^{\bar{k}+q}$ be a subsystem of $\mathcal{B}_3$ which contains all bi-infinite sequences of form

$$(\ldots, 0, \alpha_i, \overbrace{0, 0, \ldots 0, 1,}^{j_i+q} \overbrace{0, 0, \ldots, 0, \alpha_{i+1},}^{j_{i+1}+q}, 0, \ldots), \tag{32}$$

where $\alpha_i \in \{1, 2\}$ and $j_i \geq \bar{k}$ for all i.[6] We identify in $\mathcal{B}_3^{\bar{k}+q}$ two homoclinic orbits $\omega_1 = (\ldots, 0, \ldots, 0, 1, 0, \ldots, 0, \ldots)$ and $\omega_2 = (\ldots, 0, \ldots, 0, 2, 0, \ldots, 0, \ldots)$ and let $\hat{\omega}$ be the glued orbit. We denote the resulting factor-system as $\hat{\mathcal{B}}_3^{\bar{k}+q}$.

[6] We include also sequences with $j_i = \infty$ or $j_{i+1} = \infty$. Then such sequences contain infinite strings from zeros either on the left or, respectively, right end and correspond either α- or ω-asymptotic orbit to the fixed point $(\ldots, 0, \ldots, 0, \ldots)$.

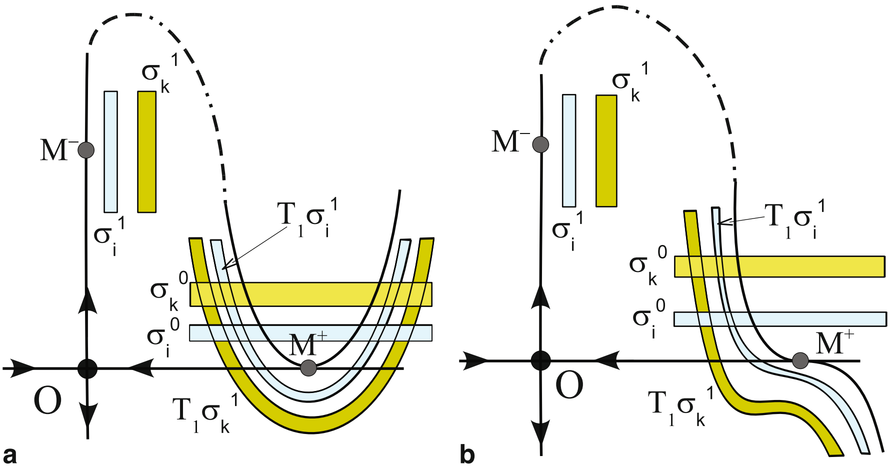


Fig. 8 Homoclinic tangencies for diffeomorphisms from the complete class. **a** The order of tangency n is even and the corresponding homoclinic tangency is "from above", i.e. $d > 0$ and $\gamma > 0$, and the horseshoes $T_1\sigma_k^1$ are posed under $T_1(W^u_{loc})$, i.e. $c < 0$ and $\lambda_1 > 0$. **b** n is odd and $\gamma > 0, \lambda_1 > 0$ and $c < 0$

Theorem 4 *Let f belong to the complete class. Then there exists such $\bar{k}$ that all orbits from the set Λ, except for Γ_0, are of saddle type and the system $f|_\Lambda$ is topologically conjugate either to $\mathcal{B}_2^{\bar{k}+q}$ for odd n or $\hat{\mathcal{B}}_3^{\bar{k}+q}$ for even n.*

Note that we do not require here that the saddle value σ is less or greater than 1. Therefore, the structure of the set $\Lambda(f)$ for systems from the complete class is the same for all cases of the sectionally dissipative, saddle or neutral ($\sigma = 1$) ones.

In the remaining cases (except for diffeomorphisms of trivial and complete classes), the set Λ does not allow, in general, the complete description. Moreover, as it is shown in [17–20], Λ can contain periodic and homoclinic orbits of any orders of degeneracy (including homoclinic and heteroclinic tangencies of indefinite orders). Nevertheless, we can observe here certain elements of hyperbolicity and, moreover, we are able to give a description of hyperbolic subsets by means of methods of the symbolic dynamics like it was done for diffeomorphisms of complete class.

However, while for diffeomorphisms of the complete class, all strips σ_i^0 and horseshoes $T_1(\sigma_j^1)$ have regular intersections, in other cases we have both to detect regular intersections and remove, from a description, all irregular and empty ones. As result, we obtain some sufficient conditions (in form of inequalities, see below) which provide the existence of certain (non-uniformly) hyperbolic subsets.

Formally speaking, we consider in this section such diffeomorphisms which satisfy conditions A–D but are not diffeomorphisms with trivial and complete description. We will call them as diffeomorphisms *with partial description*.[7]

[7] Thus, the diffeomorphisms with partial description in the main case $\sigma \neq 1$ are such that conditions A–D are valid and the following combinations of signs of the parameters λ_1, γ, c and d are excluded:

Let $\Omega_2^{\bar{k}+q}$ be such a subsystem of $\mathcal{B}_2^{\bar{k}+q}$ which contains the fixed point $\hat{O}$, the homoclinic orbit $\hat{\omega} = (\ldots, 0, \ldots, 0, 1, 0, \ldots, 0, \ldots)$ and all such orbits of form (32), where $\alpha_i = 1$ for any i and the following estimates hold for any successive integers $\bar{k} \le j_i \le \infty$ and $\bar{k} \le j_{i+1} \le \infty$

(H1) $|\gamma^{-j_{i+1}} y^- - c\lambda_1^{j_i} x_1^+| > S_{\bar{k}}(j_i, j_{i+1})$ in the case A1;

(H2) $|\gamma^{-j_{i+1}} y^- - Q\cos(j_i\psi + \omega)\cdot\lambda^{j_i}| > S_{\bar{k}}(j_i, j_{i+1})$ in the case A2;

where

$$S_{\bar{k}}(j_i, j_{i+1}) = \nu\left(|\lambda_1|^{j_i} + |\gamma|^{-j_{i+1}}\right)\cdot(|\lambda_1|^{\bar{k}/n} + |\gamma|^{-\bar{k}/n}),$$
$$Q = \left[(c_1x_1^+ + c_2x_2^+)^2 + (c_1x_2^+ - c_2x_1^+)^2\right]^{1/2}, \quad \omega = \arccos\frac{c_1x_1^+ + c_2x_2^+}{Q} \qquad (33)$$

and ν is some positive constant independent of $\bar{k}$, j_i and j_{i+1}.

Let $\hat{\Omega}_3^{\bar{k}+q}$ be such a subsystem of $\hat{\mathcal{B}}_3^{\bar{k}+q}$ which contains the fixed point $\hat{O}$ the homoclinic orbit $\hat{\omega}$ and all such orbits of form (32) for which the following estimates hold

(H3) $d\left[\gamma^{-j_{i+1}} y^- - c_1\lambda_1^{j_i} x_1^+\right] > S_{\bar{k}}(j_i, j_{i+1})$ in the case A1;

(H4) $d\left[\gamma^{-j_{i+1}} y^- - Q\cos(j_i\psi + \omega)\cdot\lambda^{j_i}\right] > S_{\bar{k}}(j_i, j_{i+1})$ in the case A2;

Theorem 5 [6, 24] *Let f be a diffeomorphism with partial description. Then there is a closed invariant subset $\tilde{\Lambda}_{\bar{k}+q} \subset \Lambda$ such that* (i) *$\tilde{\Lambda}_{\bar{k}+q}$ contains the orbits O and Γ_0;* (ii) *all orbits of $\tilde{\Lambda}_{\bar{k}+q}$, except for Γ_0, are of saddle type;* (iii) *the system $f|\tilde{\Lambda}_{\bar{k}+q}$ is conjugate either to $\Omega_2^{\bar{k}+q}$ for odd n or to $\tilde{\Omega}_3^{\bar{k}+q}$ for even n.*

One can deduce certain simple consequences from this theorem on the existence of Smale horseshoes in the first return maps. For the sake of definiteness, we consider case of even order n (like quadratic) of the tangency. Then it follows directly from Theorem 5 that

- If the estimates H3 or H4 hold for $j_i = j_{i+1} = k$, then the first return map $T_k \equiv T_1T_0^k : \sigma_k^0 \to \sigma_k^0$ has in σ_k^0 a Smale horseshoe, i.e. a closed invariant hyperbolic set Ω_{k+q} such that the system $T^k|\Omega_{k+q}$ is topologically conjugate to $\mathcal{B}_2$.

In the sectionally dissipative case $\sigma < 1$, we have that $|\lambda_1|^k \ll |\gamma|^{-k}$. Then both the inequalities (H3) and (H4) with sufficiently large $j_i = j_{i+1} = k$ can be rewritten as follows $d\gamma^{-k}y^- > 0$. Since $y^- > 0$, it implies the following result.

(1) those ones which correspond to the trivial class, i.e. n is even and (i) $\gamma > 0, d < 0$ if $\sigma < 1$, (ii) $\lambda_1 > 0, dc > 0$ if $\sigma > 1$; and (2) those ones which correspond to the complete class, i.e (iii) $\gamma > 0, \lambda_1 > 0, c < 0, d > 0$ with even n and (iv) $\gamma > 0, \lambda_1 > 0, c < 0$ with odd n.

Proposition 5 *Let f satisfy conditions A–D with even n and $\sigma < 1$, and let the case $\gamma > 0, d < 0$ be excluded. Then, f has infinitely many Smale horseshoes Ω_{k+q}, where $k \geq \bar{k}$ and k run all integers if $\gamma > 0$ or all odd (even) integers if $\gamma < 0, d < 0$ ($\gamma < 0, d > 0$).*

In the sectionally saddle case $\sigma > 1$, we have, conversely, that $|\gamma|^{-k} \ll |\lambda_1|^k$. Then, for sufficiently large $\bar{k}$, the inequalities (H3) and (H4), for $j_i = j_{i+1} = k \geq \bar{k}$, take, respectively, the following forms

$$dc_1 x_1^+ \lambda_1^k \ < \ 0 \tag{34}$$

and

$$-dQ\cos(k\psi + \omega) \ > \ \nu(\bar{k}) \ > \ 0, \tag{35}$$

where $\nu(\bar{k}) \to 0$ as $\bar{k} \to \infty$. (Note that we can not write in (35) simply "> 0" since values of $\cos(k\psi + \omega)$ are not uniformly bounded from zero when ψ/π is irrational).

The inequality (34) has no solutions only in the case $\lambda_1 > 0, dc_1 > 0$, i.e. for diffeomorphisms with trivial dynamics. Since $Q > 0$ and $\psi \neq 0, \pi$, the inequality (35) has always infinitely many integer solutions. Thus, we obtain the following.

Proposition 6 *Let f satisfy conditions A–D with even n and $\sigma > 1$, and let the case $\lambda_1 > 0, dc_1 > 0$ be excluded. Then, f has infinitely many Smale horseshoes Ω_{k+q}, where $k \geq \bar{k}$ and k run all integers such that the inequalities* (34) *in case A1 or* (35) *in case A2 hold.*

5.1 *Invariants θ and τ and a Complete Description of $\tilde{\Lambda}_{\bar{k}}$*

Note that the inequalities H1–H4 generally define an "infinite net" of the strips and horseshoes which have regular (hyperbolic) intersections. Naturally, conditions H1–H4 are only sufficient. However, using them we describe quite large (non-uniformly) hyperbolic subsets $\tilde{\Lambda}_{\bar{k}+q} \subset \Lambda$. Moreover, for some dense subset of systems with homoclinic tangency, $\tilde{\Lambda}_{\bar{k}+q}$ can provide a complete description for Λ, i.e. $\Lambda = \tilde{\Lambda}_{\bar{k}+q}$.

In this section we will prove the corresponding result for the case where O is a saddle, i.e. condition A1 holds.

Consider the following inequality

$$|d| \left| \gamma^{-j_{i+1}} y^- - c_1 \lambda_1^{j_i} x_1^+ \right| \leq S_{\bar{k}}(j_i, j_{i+1}). \tag{36}$$

By geometric constructions (see [6, 24]), integer solutions (j_i, j_{i+1}) of this inequality include *all* numbers of those strips and horseshoes which can intersect non-hyperbolically. By (33), the inequality (36) is equivalent to the following system of inequalities

$$\gamma^{-j_{i+1}}(y^- - \rho_{\bar{k}}) \leq \lambda_1^{j_i}(cx^+ + \rho_{\bar{k}}), \quad \gamma^{-j_{i+1}}(y^- + \rho_{\bar{k}}) \geq \lambda_1^{j_i}(cx^+ - \rho_{\bar{k}}), \tag{37}$$

where

$$\rho_{\bar{k}} = \frac{\nu}{|d|}\left(|\lambda_1|^{\bar{k}/n} + |\gamma|^{-\bar{k}/n}\right).$$

If both sides of the inequalities (37) are of the same sign (that can be always fulfilled for diffeomorphisms with partial description), we take the logarithm of them.

As result, we obtain the following double inequalities

$$\epsilon^2_{\bar{k}} \;\leq\; j_{i+1} - j_i\theta + \tau \;\leq\; \epsilon^1_{\bar{k}}, \tag{38}$$

where

$$\theta = -\frac{\ln|\lambda_1|}{\ln|\gamma|}, \quad \tau = \frac{1}{\ln|\gamma|}\ln\frac{|cx^+|}{y^-}$$

and

$$\epsilon^1_{\bar{k}} = \frac{1}{\ln|\gamma|}\ln\left(\frac{1+\rho_{\bar{k}}|cx^+|^{-1}}{1-\rho_{\bar{k}}(y^-)^{-1}}\right), \quad \epsilon^2_{\bar{k}} = \frac{1}{\ln|\gamma|}\ln\left(\frac{1-\rho_{\bar{k}}|cx^+|^{-1}}{1+\rho_{\bar{k}}(y^-)^{-1}}\right),$$

Now we assume that the following condition holds:

S1) θ is rational, i.e. $\theta = r/s$ for some relatively prime natural r and s, and the number $s\tau$ is not integer.

Then the straight line $j - i\theta + \tau = 0$ is posed, in the (i, j)-plane, on a finite distance (depending on θ and τ) from points of the integer lattice. This means that the inequality (38) has no integer solutions for sufficiently large $\bar{k}$. In turn, it implies that all orbits of $\Lambda_{\bar{k}+q}$, except for Γ_0, are of saddle type.

In the case of odd n we have that if there is some non-saddle orbit in $\Lambda_{\bar{k}+q}$, then the inequality opposite to H1, i.e. again inequality (36), has to be fulfilled for at least one pair j_i, j_{i+1}.

Thus, the following result takes place.[8]

Proposition 7 *Let f be a diffeomorphism with partial description satisfying conditions A1, B, C, D, $\sigma \neq 1$ and S1. Then there is such $\bar{k}_1 = \bar{k}_1(\theta, \tau)$, where $\bar{k}_1 \to \infty$ as $s \to \infty$ or $s\tau$ tends to an integer, that all orbits of $\Lambda_{\bar{k}_1+q}$, except for Γ_0, are of saddle type and $f|\Lambda_{\bar{k}_1+q}$ is conjugate either to $\Omega_2^{\bar{k}+q}$ for odd n or to $\tilde{\Omega}_3^{\bar{k}+q}$ for even n.*

6 Proof of Theorem 1

Here we assume only that a diffeomorphism f has a homoclinic tangency of the invariant manifolds of a saddle fixed point O with multipliers $\lambda_1, \ldots, \lambda_m, \gamma$ ordered by the rule (1) and such that $\sigma \equiv |\lambda_1||\gamma| < 1$. In the case under consideration, by

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[8] This result was proved also in [6] for the sectionally dissipative case $\sigma < 1$.

Lemma 1, we can write the local map T_0 in the following form

$$\bar{x} = \mathcal{A}, x + \hat{h}_1(x, y), \quad \bar{y} = \gamma y + h_3(x, y),$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^1$, $\mathcal{A}$–$(m \times m)$-matrix with eigenvalues $\lambda_1, \ldots, \lambda_m$. This form of T_0 can be considered as form (6) with identities (7), where $\mathcal{A} = \mathrm{diag}\ (\hat{A}, \hat{B})$, $\hat{h}_1 = (h_1, h_2)$ and $x_{new} = (x, u)$. Then, by (10), the global map T_1 takes the form

$$\begin{aligned} &\bar{x} - x^+ = F(x, y - y^-), \\ &\bar{y} = G(x, y - y^-) \equiv cx + \varphi(y - y^-) + O(\|x\|^2 + \|x\||y - y^-|) \end{aligned} \tag{39}$$

where $(x, y) \in \Pi^-, (\bar{x}, \bar{y}) \in \Pi^+$, $F(0, 0) = 0, G(0, 0) = 0$ and $\varphi(0) = \varphi'(0) = 0$. If the homoclinic tangency is isolated, then $\varphi \neq 0$ at $y \neq y^-$. Besides, in the case of one-sided tangency, we have that either $\varphi \geq 0$ (the tangency "from above") or $\varphi \leq 0$ (the tangency "from below" if also $\gamma > 0$); in the case of topological intersection, we have that either $s\varphi(s) \geq 0$ (the tangency like "$y = x^3$") or $s\varphi(s) \leq 0$ (the tangency like "$y = -x^3$"), where $s = (y - y^-)$.

Again, by Lemma 2, the map T_0^k can be written in the following cross-form (compare with (8))

$$x_k - \mathcal{A}^k x_0 = \hat{\lambda}^k \tilde{\xi}_k(x_0, y_k), \quad y_0 - \gamma^{-k} y_k = \hat{\gamma}^{-k} \tilde{\eta}_k(x_0, y_k) \tag{40}$$

Proof [of item 1 of Theorem 1] Take some strip $\sigma_k^1 \subset \Pi^-$. Suppose that the corresponding horseshoe $T_1(\sigma_k^1)$ has non-empty intersection with a strip σ_i^0. Then, by (40) and (39), the equation

$$\gamma^{-i} \bar{y} \left(1 + O([\hat{\gamma}/\gamma]^{-i})\right) = \alpha_k(x, y) + \varphi(y - y^-), \tag{41}$$

has a solution with respect to $\bar{y}$ (for some (x, y)). Note that the following estimate $\|\alpha_k\| < \tilde{\lambda}^k$ holds for sufficiently large k, where $\tilde{\lambda} = |\lambda_1| + \delta$ and $\delta \geq 0$ is a sufficiently small constant such that, in any case, $|\tilde{\lambda}\gamma| < 1$.

Let the homoclinic tangency satisfy the condition (2). If $\gamma > 0$ (the main case), it means that $\varphi(y_1 - y^-) \leq 0$. Then the Eq. (41) can have a solution if only

$$\gamma^{-i} \bar{y} \left(1 + O([\hat{\gamma}/\gamma]^{-i})\right) - \alpha_k(x, y) \leq 0.$$

Since $\gamma > 0$, $\|\alpha_k\| < \tilde{\lambda}^k$ and $|\tilde{\lambda}\gamma| < 1$, the inequality above can hold only in the case where $i \gg k$. Thus, any horseshoe $T_1(\sigma_k^1)$ can intersect only those strips σ_i^0 whose numbers are much bigger than k, see Fig. 9a. In turn, the horseshoe $T_1(\sigma_i^1)$, again, can intersect only some strip σ_j^0 with $j > i$, etc. It implies that some backward iteration of any point from Π^+ must leave U. Thus, only two orbits, O and Γ_0, will stay always in U.

If $\gamma < 0$, the condition (2) imply the identity $\varphi(y_1 - y^-) \equiv 0$. Evidently, Eq. (41) can have a solution in this case again only for $i \gg k$.

Proof [of item 2 of Theorem 1] Consider first the case of one-sided tangency "from above". If $\gamma < 0$ we can always choose such homoclinic points M^- and M^+ that

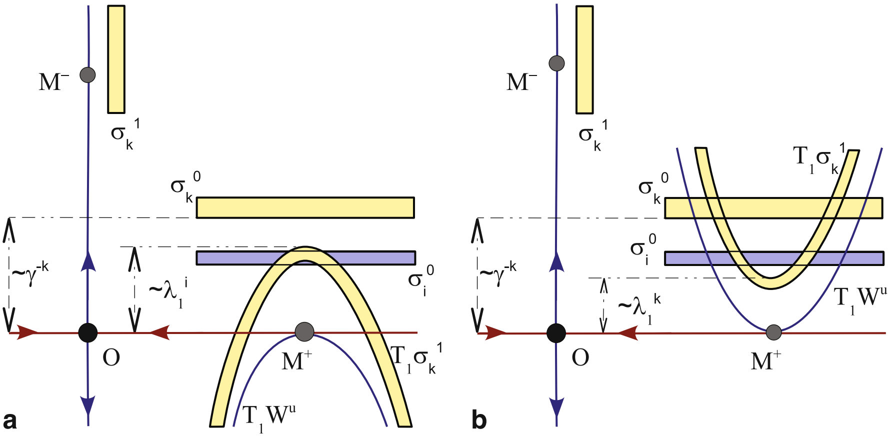

Fig. 9 A horseshoe geometry in the case $\sigma < 1$ for homoclinic tangencies **a** from below and **b** from above

$T_1(W^u)$ touches W^u_{loc} from above (if it not the case for a given point M^+, we take the point $T_0(M^+)$). Then, by condition $\varphi \geq 0$ and $\varphi > 0$ if $y_1 \neq y^-$, we have that the curve $T_1(W^u): x_0 - x^+ = F(0, y_1 - y^-), y_0 = \varphi(y_1 - y^-)$ will intersect all the strips σ_k^0 with sufficiently large k (even k if $\gamma < 0$) at least two connected components. (Note that the vector $\hat{l} = F_y(0,0)$ is non-zero, since T_1 is diffeomorphism). The strips σ_k^0 are posed on distance $\rho_k \geq |\gamma|^{-k}(y^- - \varepsilon_1)$ from the plane $y_0 = 0$. While, the strips σ_k^0 are posed from the line $x = 0$ on the distance of order $\tilde{\lambda}^k$. The latter means that the horseshoes $T_1(\sigma_k^1)$ will be posed from $T_1 W^u_{loc}$ on the distance of order $\tilde{\lambda}^k$. Since $\tilde{\lambda}^k \ll |\gamma|^{-k}$, it follows that the strips σ_k^0 with sufficiently large k will intersect own horseshoes $T_1(\sigma_k^1)$ along at least two connected components (the same as they intersect the curve $T_1 W^u_{loc}$). See Fig. 9b.

Concerning the case of topological intersection, we note that the curve $T_1 W^u_{loc}$ intersect infinitely many strips σ_k^0. In turn, strips σ_j^1 accumulate to W^u_{loc}. It implies that infinitely many strips σ_k^0 and horseshoes $T_1(\sigma_j^1)$ for all sufficiently large k and j are mutually intersect.

Acknowledgements The authors thank M. Malkin and D. Turaev for very fruitful discussions. AG and SG have been partially supported by the Russian Scientific Foundation Grant 14-41-00044 and the grants of RFBR No.13-01-00589, 13-01-97028-povolzhye and 14-01-00344. ML was partially supported by a NSC research grant.

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