Chapter 6
Complex Elections

In this Chapter we discuss cases where several candidates are to be elected simultaneously. Such elections are called *complex* or *multiple* elections.

6.1 The Generalized Hare Method

The Hare system was modified by Andrew Inglis Clark, who was Attorney-General of Tasmania in the late nineteenth century. This *generalized Hare method*, or *quota method*, is used in many countries, including Ireland, Australia and Malta.

The method works as follows. Each voter makes a *preferential* vote, providing a complete list of all the candidates, in order of preference (a *preference list*). The first name on the list is called the voter's *first preference*, and so on. All first preferences are counted; if candidate A receives at least a certain number of votes (called the *quota*) then A is elected, and the unneeded votes, called the *excess*, are distributed proportionally among the second preferences of the electors who chose A. Then the new set of votes is examined again, to see if any more candidates have no achieved the quota after the additional votes are added. If at any stage no one new has been elected, the candidate with the fewest first-place is eliminated, and all those votes go to the second preferences. If, at any stage, the number of candidates not yet elected or eliminated is equal to the number of seats remaining, all those candidates are elected.

As an example, suppose 24,000 people are to elect five representatives from a larger number of candidates, and quota is set at 4,001 votes. (This number is chosen because you cannot have six candidates who each get more than 4,000 first preferences.) Every candidate who gets more than 4,000 votes is elected. If a candidate

W.D. Wallis, *The Mathematics of Elections and Voting*,
DOI 10.1007/978-3-319-09810-4_6

gets more than 4,000 votes, the excess go to voters' second choices, divided proportionally. To illustrate this, suppose A gets 5,000 votes. Of these, 2,500 have B as second choice, 2,000 have C and 500 have D. Since 5,000 is greater than 4,000, A is declared elected.

The 1,000 surplus votes are divided in the proportion 2,500:2,000:500, or $50\,\%$: $40\,\%$: $10\,\%$. That is, B gets 50 % of A's excess, because 2,500 is 50 % of 5,000; C gets 40 %; and D gets 10 %. As A has 1,000 extra votes, we say she has 1,000 preferences to be distributed. B gets 50% of these, so 500 more votes are added to B's total. In the same way C gets 400 added votes (40 %) and D gets 100 votes (10 %).

Now we check whether B, C or D has met the quota. For example, if B previously had 3,700 votes, the new total would be 4,200, exceeding the quota again, so B is declared elected.

In general, suppose there are V voters and N places to be filled. We shall define the *lower bound* for this election to be $\frac{V}{N+1}$, and the quota is the smallest integer greater than the lower bound. A candidate is declared to be elected if his or her number of votes exceeds the lower bound, and the excess is found by subtracting the quota from the candidate's total number of votes. Notice that, if only one candidate were to be elected, the quota requirement is that a candidate receive a majority of the votes.

This quota is called the *Droop quota*, and was suggested by H. R. Droop [13]. The original quota, called the *Hare quota*, is simply $\frac{V}{N}$. (The excess under the Hare quota is found by subtracting the quota from the number of votes) An interesting comparison of the two quota systems can be found in [32].

Some systems use the formula $\frac{V}{N+1} + 1$ for the quota, and say a candidate is elected if the quota is equalled or exceeded. This may require one more vote than the Droop quota in the case where $\frac{V}{N+1}$ is not an integer: not important in most political elections, but it may be significant if a small group is voting.

In a real example, the whole list of preferences is kept. The process may require a great deal of data; moreover it may result in complicated numbers, fractions, and so on. Historically this was a serious problem and caused long delays in announcing the results of elections, but it is no longer an issue now that computers are available and voting machines can be adapted to keep all the data.

For example, given a preference profile of the form

12	9	6	...	...
A	A	A	B	...
B	B	C	A	...
C	D	B	D	...
D	C	D	E	...

and a quota of 19, A's surplus is nine. As B received 21 of the second preferences of A's voters and C received 6, the surplus is divided in the ratio $21 : 6$ between B and C; B receives 7 further votes and C receives 2. Another way of looking at this is to say the votes for A were distributed $12 : 9 : 6$ among the three preference groups, so the 9 surplus votes should be distributed in the same ratio, which comes to 4, 3 and 2 votes respectively. This will make later calculations easier. To look at the second candidate to be elected, delete A from every preference list and replace the votes in those columns where A was originally the first choice with the surplus amounts, so that the profile becomes

4	3	2	...	...
B	B	C	B	...
C	D	B	D	...
D	C	D	E	...

We carefully chose the numbers of votes in the above example so that the surpluses allocated were all integers. This was done for simplicity, but is not essential. Suppose the numbers of votes in the first three columns had been 14, 8 and 5, which still gives a total of 27 and a surplus of 9. In order to find the second candidate elected, the votes allocated to the first three columns would be $\frac{14}{27} \times 9$, $\frac{8}{27} \times 9$ and $\frac{5}{27} \times 9$, or $4\frac{2}{3}$, $2\frac{2}{3}$ and $1\frac{2}{3}$. The arithmetic may become more complicated, but in the real world the calculations will be carried out by computer, so there is no problem.

Sample Problem 6.1 *Say there are five candidates for three positions; the preference table is*

6	6	9	6	3	2
A	A	C	C	E	E
B	B	D	D	C	A
E	D	E	E	D	B
D	E	A	B	A	C
C	C	B	A	B	D

Who will be elected?

Solution. There are 32 voters, so the lower bound is 8 and the quota is 9. A gets 12 first place votes and C gets 15, both of which meet the quota. So A and C are elected.

A has a surplus of 4. The second-place candidate in every case is B, but we observe that the votes are divided in proportion 6:6 between A's two lists, so we give two votes to each of A's lists. C is also at the top of two lists, and has a surplus of 7, divided 9:6. This gives surplus allocations of 4.2 and 2.8. After these allocations, the new table is

2	2	4.2	2.8	3	2
B	B	D	D	E	E
E	D	E	E	D	B
D	E	B	B	B	D

B has four votes, D has seven, and E has five. No one meets the quota. (Note that the quota does not change.) So B (who had the fewest votes) is eliminated. We now have

2	2	4.2	2.8	3	2
E	D	D	D	E	E
D	E	E	E	D	D

D now has 9 votes and E has 7, so D is elected. In total, A, C and D are elected.

Practice Exercise. Repeat the above problem for preference profile

5	5	7	8	5	2
A	A	B	C	D	E
B	E	D	D	C	A
E	B	E	E	E	B
D	D	A	B	A	C
C	C	C	A	B	D

The choice of quota can make a significant difference. As an example, consider an election in which five candidates are to be elected, and two political parties are involved. (This was the situation in state elections for the federal senate in Australia, in the middle of the twentieth century.) Typically at least two candidates are elected from each of the two major political parties, with the fifth member being from one of the two parties, from a third party, or an independent. For simplicity we shall assume there are only six candidates, A, B and C from one party (the Republicrats) and D, E and F from the other (the Democans), no third party or independents. We shall assume there are 240 voters, and all electors belong to one or other party. The Hare quota is 48 and the Droop quota is 41. The preference profile is

65	55	4	42	42	32
A	B	C	D	E	F
B	A	A	E	D	D
C	C	B	F	F	E
D	D	D	A	A	A
E	E	E	B	B	B
F	F	F	C	C	C

As you can see, the Republicrat party has 124 votes and the Democans 116, so we would expect all three Republicrats and two Democans to be elected, and this happens when the Droop quota is used: A, B, D and E all achieve the quota, and after the excesses are distributed C has 44 votes to F's 36. However, using the Hare quota, we see that A and B are elected and the new profile is

17	7	4	42	42	32
C	C	C	D	E	F
D	D	D	E	D	D
E	E	E	F	F	E
F	F	F	C	C	C

.

No candidate is elected, so C, with only 28 votes, is eliminated, and D, E and F are elected.

6.2 The Generalized Coombs Rule

The Coombs rule can also be applied to cases where more than one candidate is to be elected. Again, each voter provides a complete preference list. The Droop quota is used. Again, any candidate who receives at least as many votes as the quota is declared elected and removed from the preference profile, and the excess votes are distributed proportionally.

If no candidate has achieved the quota, the candidate with the greatest number of *last place* votes is deleted, and the votes are counted again. If there are two such candidates (with equal numbers of last places), both are eliminated.

This method often provides the same result as the Hare method, but not always. Here is an example with four candidates, where two are elected under one system but the other two are the winners under the other system.

Sample Problem 6.2 *Say there are four candidates for two positions; there are 126 voters, and the preference table is*

36	35	37	18
A	B	C	D
B	C	D	A
C	D	A	C
D	A	B	B

Who will be elected under the Hare system? Who will be elected under the Coombs rule?

Solution. The quota is 43. No candidate gets enough first-place votes to meet the quota.

Under the Hare method, D, with only 18 votes, is eliminated. The new table is

36	35	37	18
A	B	C	A
B	C	A	C
C	A	B	B

A has 54 votes, and is declared elected, with a surplus of 12. Eight of these twelve votes are allocated to the first column and four to the fourth. The result is

8	35	37	4		43	41
B	B	C	C	=	B	C
C	C	B	B		C	B

and B is elected. So A and B are the Hare winners.

Under the Coombs rule, B is the first candidate to be eliminated, with 55 last-place votes. The new table is

36	35	37	18		36	72	18
A	C	C	D		A	C	D
C	D	D	A	=	C	D	A
D	A	A	C		D	A	C

and C, with 72 votes, has achieved the quota. The surplus is 30, and it all goes to the second column, so the new table is

36	30	18		36	48
A	D	D	=	A	D
D	A	A		D	A

and D is the second candidate elected.

Practice Exercise. Who are the winners of the election presented in Practice Exercise 6.1, if the Coombs rule is applied?

6.3 The Single Transferable Vote

The *single transferable vote* (or STV) system is a variation of the generalized Hare method that contains some aspects of approval voting, a system that we shall examine in the next chapter. Every elector decides whether or not a candidate is approved, and supplies a preference profile of the approved candidates. In the preference table, some columns will have blank spaces at the bottom. In some countries the name "single transferable vote" is also applied to the Hare method and generalized Hare method, but we shall restrict it to the case where a voter is allowed to omit a candidate.

There are various ways in which an STV system can be implemented. Typically, any vote that lists *no* approved candidates is eliminated before the quota is calculated. Then elections proceed similarly to elections under the generalized Hare method. However, it is possible for a vote to disappear because all the candidates named on it have been eliminated. In the basic system this can mean that no candidate reaches the quota, even though not all vacancies have been filled; the standard procedure is to hold another election for the remaining seats. However, another possibility is to recalculate the quotas at each stage. We shall refer to this as *dynamic* STV.

Sample Problem 6.3 *An election for two positions is held under the single transferable vote system; the preference table is*

5	5	10	3	1
A	A	A	B	C
B	C		C	B
C	B		A	A

What is the result under the basic STV system? What is the result under dynamic STV?

Solution. There are 24 electors, so the quota is 9. A is elected and has 12 surplus votes. The new preference table is

3	3	6	3	1
B	C		B	C
C	B		C	B

, or equivalently

6	4	6
B	C	
C	B	

B has six votes and C has four; neither is elected and a new election is held for the remaining position.
Under the dynamic system, the new preference table is treated as if it were the preference table for a new election; the "empty" ballots are discarded, and we proceed as though there were ten votes with preference table

6	4
B	C
C	B

B is elected.

Practice Exercise. In the above problem, suppose the voters all decided to vote for at least two candidates, and the preference table was

5	5	5	5	3	1
A	A	A	A	B	C
B	C	B	C	C	B
C	B			A	A

What is the result under the basic STV system? What is the result under dynamic STV?

Exercises 6

In Exercises 1–4, a preference table is shown for an election where two candidates are to be elected using the generalized Hare method. In each case, what is the quota? What is the outcome of the election? What would be the outcome using the Coombs rule?

1.

7	8	7	5
A	B	C	D
B	D	B	C
C	C	A	B
D	A	D	A

2.

4	6	6	6	2	6
A	D	B	E	C	B
D	A	D	A	E	A
E	E	C	D	D	E
B	B	E	C	A	C
C	C	A	B	B	D

3.

3	5	8	8	6
A	A	B	C	D
B	C	D	B	C
C	D	C	A	B
D	B	A	D	A

4.

6	7	7	7	7	5
A	B	C	D	C	E
E	D	B	E	D	B
D	A	D	A	E	A
B	E	E	C	A	C
C	C	A	B	B	D

5. In Exercises 1 and 3, what would be the quota if the Hare quota were used instead of the Droop quota? Would the result be changed?

6. In Exercises 2 and 4, what would be the quota if the Hare quota were used instead of the Droop quota? Would the result be changed?

In Exercises 7–12, a preference table is shown for an election where three candidates are to be elected using the generalized Hare method. In each case, what is the quota? What is the outcome of the election? What would be the outcome using the Coombs rule?

7.

4	7	9	8	7	5
A	A	B	C	D	D
B	B	D	D	C	A
C	D	C	A	B	B
D	C	A	B	A	C

8.

8	5	6	7	4	3
A	B	C	D	E	E
B	A	D	C	C	A
E	D	E	E	D	B
D	E	B	B	B	D
C	C	A	A	A	C

9.

6	6	6	6	5	3
A	A	C	C	E	E
B	D	D	E	C	A
E	E	E	D	D	B
D	B	A	B	A	C
C	C	B	A	B	D

10.

9	8	7	6	4	2
A	B	D	D	C	A
E	D	E	E	D	B
D	A	C	A	E	E
B	E	B	C	A	C
C	C	A	B	B	D

11.

6	6	7	5	6	5	5
A	B	B	C	D	E	E
B	E	D	E	E	B	B
D	A	A	B	A	C	A
C	C	E	A	B	D	D
E	D	C	D	C	A	C

12.

8	6	7	7	9	3	8
A	A	B	C	C	D	E
D	B	C	D	E	C	A
E	D	A	E	D	B	B
B	E	D	A	A	E	D
C	C	E	B	B	A	C

13. Suppose a generalized Hare election were held using a quota smaller than the Droop quota. Construct an example to show that there could be no result.

14. An election is to be held under the STV system, and three candidates are to be elected. The preference table is

7	7	4	2	4	6
A	B	C	D	B	F
B	A	B	B	A	C
		D	A	E	E
		A	C	D	
		E	F		

(i) What is the quota?

(ii) What is the result under the basic STV system?

(iii) What is the result under dynamic STV?

15. In an STV election, 39 voters must elect two candidates and three candidates are to be elected. The preference table is

6	6	4	4	7	3	6	0
A	A	B	B	C	C	D	
B		A	D	D	B	A	
C		C	A	A	D	B	
		D				C	

(i) What is the quota?

(ii) What is the result under the basic STV system?

(iii) What is the result under dynamic STV?