Chapter 4
Fair Elections; Polls; Amendments

We have seen several different electoral systems. Even in a small election, the same preferences can give rise to a different results depending on the system used. In this section we shall present an extreme case of how the method chosen might affect the outcome of an election in a realistic situation. This means that an electoral system could be chosen in order to favor one or another candidate.

There are also many instances in which voters could misrepresent their preferences, manipulating an election to give a certain result. This can be unintentional; when polls are held before an election, the results may convince some voters that their favorite candidate has no chance, and they may decide to vote for their second-favorite rather than risk their least preferred candidate gaining election. Or it can be intentional, as we shall see in our discussion of amendments.

4.1 Five Candidates, Six Methods, Six Results

Consider a political party convention at which six different voting schemes are adopted. Assume that there are 110 delegates to this national convention, at which five of the party members, denoted by A, B, C, D, and E, have been nominated as the party's presidential candidate. Each delegate must rank all five candidates according to his or her choice. Although there are $5! =: 5 \times 4 \times 3 \times 2 \times 1 = 120$ possible rankings, many fewer will appear in practice because electors typically split into blocs with similar rankings. Let's assume that our 110 delegates submit only six different preference lists, as indicated in the following preference profile:

W.D. Wallis, *The Mathematics of Elections and Voting*,
DOI 10.1007/978-3-319-09810-4_4

36	24	20	18	8	4
A	B	C	D	E	E
D	E	B	C	B	C
E	D	E	E	D	D
C	C	D	B	C	B
B	A	A	A	A	A

The 36 delegates who most favor nominee A rank D second, E third, C fourth, and B fifth. Although A has the most first-place votes, he is actually ranked last by the other 74 delegates. The 12 electors who most favor nominee E split into two subgroups of 8 and 4 because they differ between B and C on their second and fourth rankings.

We shall assume that our delegates must stick to these preference schedules throughout the following six voting agendas. That is, we will not allow any delegate to switch preference ordering in order to vote in a more strategic manner or because of new campaigning.

We report the results when six popular voting methods are used. There are six different results.

1. **Majority.** As one might expect with five candidates, there is no majority winner.
2. **Plurality.** If the party were to elect its candidate by a simple plurality, nominee A would win with 36 first-place votes, in spite of the fact that A was favored by less than one-third of the electorate and was ranked dead last by all other delegates.
3. **Runoff.** On the other hand, if the party decided that a runoff election should be held between the top two contenders (A and B), who together received a majority of the first-place votes in the initial plurality ballot, then candidate B outranks A on 74 of the 110 preference schedules and is declared the winner in the runoff.
4. **Hare Method.** Suppose the Hare method is used: a sequence of ballots is held, and at each stage the nominee with the fewest first-place votes is eliminated. The last to survive this process becomes the winning candidate. In our example E, with only 12 first-place votes, is eliminated in the first round. E can then be deleted from the preference profile, and all 110 delegates will vote again on successive votes. On the second ballot, the 12 delegates who most favored E earlier now vote for their second choices, that is, 8 for B and 4 for C; the number of first-place votes for the 4 remaining nominees is

$$\begin{array}{cccc} A & B & C & D \\ 36 & 32 & 24 & 18 \end{array}$$

Thus, D is eliminated. On the third ballot the 18 first-place votes for D are reassigned to C, their second choice, giving

A	B	C
36	32	42

Now B is eliminated. On the final round, 74 of the 110 delegates favor C over A, and therefore C wins.

5. **Borda count.** Given that they have the complete preference schedule for each delegate, the party might choose to use a straight Borda count to pick the winner. They assign five points to each first-place vote, four points for each second, three points for a third, two points for a fourth, and one point for a fifth. The scores are:

$$A\colon\ 254 = (5)(36) + (4)(0) + (3)(0) + (2)(0) + (1)(24 + 20 + 18 + 8 + 4)$$
$$B\colon\ 312 = (5)(24) + (4)(20 + 8) + (3)(0) + (2)(18 + 4) + (1)(36)$$
$$C\colon\ 324 = (5)(20) + (4)(18 + 4) + (3)(0) + (2)(36 + 24 + 8) + (1)(0)$$
$$D\colon\ 382 = (5)(18) + (4)(36) + (3)(24 + 8 + 4) + (2)(20) + (1)(0)$$
$$E\colon\ 378 = (5)(8 + 4) + (4)(24) + (3)(36 + 20 + 18) + (2)(0) + (1)(0)$$

The highest total score of 382 is achieved by D, who then wins. A has the lowest score (254) and B the second lowest (312).

6. **Condorcet.** With five candidates, there is often no Condorcet winner. However, when we make the head-to-head comparisons, we see that E wins out over:

- A by a vote of 74–36
- B by a vote of 66–44
- C by a vote of 72–38
- D by a vote of 56–54

So there is a Condorcet winner, namely E.

In summary, our political party has employed six different common voting procedures and has come up with five different winning candidates. We see from this illustration that those with the power to select the voting method may well determine the outcome.

4.2 Manipulating the Vote

The term *strategic voting* means voting in a way that does not represent your actual preferences, in order to change the result of the election. We would call the resulting ballot *insincere.*

Suppose your favorite is candidate X. (We will call you an X *supporter.*) Then X would normally appear at the top of your preference list. But sometimes you can achieve X's election by voting for another candidate in first place! This is most common in runoff situations; you can ensure that your candidate does not have to face a difficult opponent. The following example illustrates this.

Sample Problem 4.1 *A runoff election has preference profile*

6	2	7	5	4
A	C	C	B	D
D	B	A	A	A
C	D	D	C	B
B	A	B	D	C

Show that the supporters of C can change the result so that their candidate wins, by the two voters in the second column changing their ballots by demoting *their candidate.*

Solution. Initially the first-place votes are A–6, B–5, C–9, D–4, so runoff election will be between A and C, and A wins 15–9. The revised profile is

6	2	7	5	4
A	B	C	B	D
D	C	A	A	A
C	D	D	C	B
B	A	B	D	C

,

the first-place votes are A–6, B–7, C–7, D–4, so the runoff election is between A and C, and C wins 13–11.

Even when you cannot ensure victory for your opponent, you may still be able to obtain a preferable result. For example, suppose you support candidate X; you think candidate Y is acceptable, but hate candidate Z. Even if insincere voting cannot ensure victory for candidate X, you may be able to swing the election to Y rather than Z.

Sample Problem 4.2 *An election with four candidates and seven voters is to be decided by the Hare system. The preference profile is*

2	1	2	1	2
B	B	D	C	C
A	D	C	A	D
D	A	B	B	A
C	C	A	D	B

Show that one of the two voters with preference list B, A, D, C can change the outcome to a more favorable one by insincere voting.

Solution. First consider the result of sincere voting. Initially A is eliminated, having no first-place votes:

2	1	2	1	2
B	B	D	C	C
D	D	C	B	D
C	C	B	D	B

Next D is eliminated, leaving

2	1	2	1	2
B	B	C	C	C
C	C	B	B	B

The winner is C.

Now suppose one voter changes his ballot from B, A, D, C to D, A, B, C. The profile is

1	1	1	2	1	2
B	D	B	D	C	C
A	A	D	C	A	D
D	B	A	B	B	A
C	C	C	A	D	B

Again A is first eliminated, leaving:

1	1	1	2	1	2
B	D	B	D	C	C
D	B	D	C	B	D
C	C	C	B	D	B

In the next round, B is eliminated.

1	1	1	2	1	2
D	D	D	D	C	C
C	C	C	C	D	D

The winner is D. This is a preferable outcome for the voter who switched.

Practice Exercise. Consider a Hare system election with preference profile

8	6	5
P	Q	R
Q	R	S
R	P	P
S	S	Q

Show that P would win this election. Show that if one of the six supporters of Q changes her vote, she could ensure that R wins, even though a majority of voters still prefer Q to R.

4.3 Polls

Of course, you do not always know exactly how the votes will go. Strategic voting is usually based on assumptions about the election. How do we arrive at these assumptions?

Often an election is preceded by an informal count, or poll. For example, as early as 2011 there were several polls for the 2012 Presidential election in the United States.

If a candidate does badly in polls, his or her supporters may change their votes. For example, consider an election that uses plurality voting. There are 390 voters and 3 candidates, A, B, C. The voters' actual preferences are

180	170	40
A	B	C
C	C	B
B	A	A

If the election were held immediately, A would win.

However, suppose a poll is held. As usual, the poll uses the same system (plurality voting). Say there are 39 voters in the poll, and their preferences are proportional to the overall preferences. The result will show A first (18 votes), with B a close second (17 votes), and C a distant third (with only 4 votes).

When the poll results are reported, the 40 voters who favor C may well reason, "C cannot win the election, so it would be preferable to elect the better of the other two." They all prefer B to A, so they would vote for B. Suppose 30 of the voters decided to change their votes. The vote will be A–180, B–200, C–10, and B will win the election.

But observe that C was a Condorcet winner under the original preferences:

B beats A 220–170
C beats B 220–170
C beats A 210–180

The type of behavior exhibited by C's followers is frequently observed after polls and leads to the

Poll Assumption:

Voters whose favorite is not one of the two top candidates in a poll will adjust their preferences to vote for the one of the top two that they prefer.

For this reason, those who do badly in polls may drop out before the election. In the example, C might easily have dropped out. The Poll Assumption applies primarily to the last poll taken before an election, but candidates frequently drop after earlier polls, because they do not expect their showing to improve later.

In cases where the electors' preference list is involved, we assume the voters who adjust their preferences will do so by moving the new preferred candidate to the top of the list, but will not change their relative preferences among the other candidates.

Theorem 1. *If the poll assumption holds, and a poll is held prior to a plurality election, a Condorcet winner will win the election if and only if she is one of the top two candidates in the poll.*

Proof. Suppose the two candidates who topped the poll were A and B. Since only A and B retain first place positions in the final election preferences, the election is the same as a majority election between A and B. The relative positions of A and B are unchanged, so if A is a Condorcet winner then A beats B in the final election. A Condorcet winner who is not one of the top two will not be A or B, so cannot win. □

In cases where the electors' preference list is involved, we assume the voters who adjust their preferences do so by moving the new preferred candidate to the top of the list, but will not change their relative preferences among the other candidates.

We say an election satisfies the *poll fairness criterion* if there is no case where the result is changed by a candidate who is not one of the top two drops out before the final vote; a system satisfies the poll fairness criterion if every instance satisfies the condition.

4.4 Sequential Pairwise Voting

As we noted in Chap. 3, sequential pairwise voting is very prone to manipulation. This can involve both the possibility of insincere voting and the order of candidates in the voting agenda.

As an example, suppose there are four political parties. Party A is extremely left-wing, party B is moderately left-wing, party C is moderately right-wing, and party D is extremely right-wing. As we would expect, each voter puts the two candidates with his or her political leaning first and second, the moderate on the other side third and extremist on the other side last. Most of the electorate turns out to be extremist in one way or the other, and the voters' preferences are

11	1	1	10
A	B	C	D
B	C	B	C
C	A	D	B
D	D	A	A

If the voting agenda is A, D, B, C then the results are that A beats D 12–11, then B beats A 12–11, and finally B beats C 12–11. However, suppose the voter represented by the third column changes her preference list to C, A, B, D. Then initially A beats D 13–10 and then A beats B 12–11. But then C beats A 12–11. By increasing her support of the extremist of the opposite political persuasion, the supporter of the moderate left-winger has achieved her preferred candidate's election.

However, this result completely depends on the voting agenda. If the agenda B, C, A, D had been chosen, and all electors voted sincerely, the reflection would still favor B: B beats C 12–11, B beats A 12–11, and B beats D 13–10. But if the supporter of C changes her preference list to C, A, B, D, as before, the result is that initially B beats C 12–11, as before, but then A beats B 12–11, and finally A beats D 13–10. The moderate right-winger has caused the extreme left-winger to be elected!

4.5 Amendments

We now consider an important example of manipulation that involves the introduction of sequential pairwise voting into what was originally a straightforward majority vote. Suppose three voters on City Council have to decide whether to add a new sales tax. Initially

- A prefers the tax
- B prefers the tax
- C prefers no tax

so a tax will be introduced.

However, let's assume A hates income taxes and will never vote for one. On the other hand, B prefers income tax to sales tax. Suppose C moves an amendment to change the tax to an income tax.

We now have:
Original motion: that a city sales tax of 5 % be introduced.
Amendment (moved by C): change “sales tax of 5 %” to “income tax of 2 %”.
(We shall assume the 2 % income tax will provide the same total as the 5 % sales tax.)

In the vote on the amendment, both B and C will vote in favor, with A against, so the amendment is carried. So the motion becomes: a city income tax of 2 % shall be introduced. In the vote on the new motion, both A and C are against, while B votes in favor; so the motion is lost and there is no tax.

A related technique can be applied in regular elections. Suppose a plurality election is to be held in a two-party system (for example, for the United States House or Senate). If a third candidate, who has similar views to one of the original two, were added to the ballot, and some of the electors vote for the new candidate, this could change the result. For example, say 51 % intend to vote for candidate A and 49 % for candidate B. If a third candidate with very similar views to A, say C, decides to run and gains 5 % of the vote, it could well be that B will still receive 49 %, A will receive only 46 %, and B will win. There have been cases where voters of one political persuasion will try to convince an independent to run whose views are opposite to their own. This is one of the reasons that some electoral systems have been changed to various kinds of preferential voting.

Exercises 4

1. What would be the result of the election described in Sect. 4.1 under the Coombs Rule?

2. What would be the result of the election described in Sect. 4.1 if the Bucklin method was used?

3. A committee of 11 members needs to elect one representative from four candidates. They plan to use a Borda count. The preferences are

4	2	2	3
A	A	B	B
C	B	C	D
D	C	A	C
B	D	D	A

(i) Who would win the election if all electors vote sincerely?

(ii) Can the voters in the third column vote insincerely so as to change the result in their favor? If so, how?

(iii) Can the voters in the fourth column vote insincerely so as to change the result in their favor? If so, how?

(iv) If the Hare method is used, can the voters in the third and fourth column change the result by voting insincerely?

4. Suppose a 30-voter election with the following preference profile is decided by the Hare system.

3	6	7	8	6
A	A	B	C	D
B	C	D	D	C
C	D	A	A	B
D	B	C	B	A

(i) Who will win the election?

(ii) Show that the seven supporters of B can achieve a preferred result if they exchange B and D on their ballots (that is, they vote as if their preference was D, B, A, C).

(iii) Who will win a plurality election?

5. In the preceding exercise (Question 2), who would win under the Bucklin method, and who would place second? Can the second-place candidated vote insincerely so as to ensure their candidate wins?

6. An election has preferences

3	2	2	2
A	B	C	D
B	D	B	C
C	C	A	B
D	A	D	A

(i) Who will win the election under plurality?

(ii) Who will win under the Bucklin method?

(iii) Who will win under Borda count?

(iv) Show that the supporters of C can make sure their candidate wins under Borda count by insincerely voting preference (C, A, D, B).

7. Consider a runoff election with preferences

4	4	3	2	4
A	A	B	B	C
B	C	A	C	B
C	B	C	A	A

(i) Who wins the election?

(ii) Show that two supporters of A can change the result in favor of their candidate by changing their preferences from (A, C, B) to (C, A, B).

8. Consider the preferences

24	16	19	21	9	10
A	B	C	D	E	E
B	A	D	C	C	C
E	D	E	E	B	A
D	E	A	B	A	B
C	C	B	A	D	D

(i) Suppose everyone votes sincerely. Under the Hare system, who wins the election?

(ii) Show that, if one of the supporters of D votes insincerely by reversing her first two preferences, she achieves a preferable outcome.

9. An election with preference profile

8	4	3
A	B	C
B	C	B
C	A	A

is decided by Borda count. How many of A's supporters must change their ballots to (A, C, B) in order to make their candidate the winner?

10. A club with 46 members wishes to elect a president. The post is currently held by Y. The 46 members have preference profile

16	12	10	8
X	Y	Z	Y
Z	Z	X	X
Y	X	Y	Z

(i) The following electoral system is used: a plurality vote is held between the candidates other than the President; there is then a runoff election between the winner of that election and the current President. Show that X will win the election.

(ii) The eight voters whose votes form the right-hand column of the profile decide to change their votes in an attempt to make Y the winner. Can they do this?

11. An election is to be held under plurality voting. The preferences profile is

1	3	2
A	C	B
C	B	A
B	A	C

Can the supporter of A (first column) achieve a preferable result by insincere voting?

12. A plurality-vote election has preference profile

8	6	4	2
X	Y	Z	T
T	T	T	Z
Y	X	Y	Y
Z	Z	X	X

(i) Is there a Condorcet winner? If so, who?

(ii) Suppose a poll is held and those who place third and fourth drop out. Who will win the final election?

(iii) Suppose a poll is held but only the candidate who placed fourth drops out. Who will win the final election?

13. A hiring committee uses the Hare system to select a new foreman. The preference profile is

5	4	3
A	B	C
D	A	D
B	D	B
C	C	A

.

(i) Who would win this election?

(ii) Who would win if C drops out before the election?

(iii) Who would win if A drops out before the election?

14. Fifteen club members are voting to elect a president, using a Borda count. The preference profile is

6	5	4
A	B	C
C	D	B
B	A	A
D	C	E
E	E	D

.

A poll is conducted; both C and D realize that they are unlikely to win.

(i) Who would win this election?

(ii) Who would win if C drops out before the election?

(iii) Who would win if D drops out before the election?

15. Consider the sequential pairwise election with profile

11	1	1	10
A	B	C	D
B	C	B	C
C	A	D	B
D	D	A	A

that we discussed in Sect. 4.4. Is there any voting agenda that changes the result, provided all electors vote sincerely?