Chapter 3
Simple Elections II: Condorcet's Method

We have already seen that, when there is no majority, different sensible-sounding electoral methods may produce different results. In 1785 Marie Jean Antoine Nicolas Caritat, Marquis de Condorcet, a French mathematician and political theorist, proposed a technique involving multiple use of runoff elections. (A similar idea was proposed by Ramon Llull as long ago as 1299; see for example [8].) Condorcet's work appeared in an essay entitled *Essai sur l'Application de l'Analyse à la probabilité des décisions rendues à la pluralité des voix* (Essay on the Application of Analysis to the Probability of Majority Decisions) [11]. This work also described several other results, including Condorcet's paradox, which shows that majority preferences become intransitive with three or more candidates.

3.1 The Condorcet Method

Suppose we simultaneously conduct all the "runoff" elections among our candidates. For example, in the election discussed in Sample Problem 2.2, there are six runoffs: A versus B, A versus C, A versus D, B versus C, B versus D and C versus D. If any one candidate wins all his/her runoffs, then surely you would consider that person a winner. We shall call such a candidate a *Condorcet winner.*.

In Sample Problem 2.1, we find:

B beats A 10–9,
A beats C 16–3,
B beats C 12–7,

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so B is a Condorcet winner. In Sample Problem 2.2,

A and B tie, 12–12,
A beats C 16–8,
D beats A 15–9,
C beats B 17–7,
D beats B 17–7,
D beats C 16–8.

So D is a Condorcet winner, even though A would win under plurality and the runoff method results in a tie between A and B.

But Condorcet's method does not yield a result in every set of preferences. In fact, Condorcet pointed out in the essay [11] that it is possible for a certain electorate to express a preference for A over B, a preference for B over C, and a preference for C over A from the same set of votes. Here is a simple example: given the preferences

5	4	3
A	B	C
B	C	A
C	A	B

,

A beats B 8–4, B beats C 9–3 and C beats A 7–5, so there is no Condorcet winner.

3.2 Condorcet's Extended Method

In elections with several candidates, it is very common to have no Condorcet winner, even when there are no ties. This is a serious fault in the Condorcet method.

Condorcet's own solution to this problem is as follows. We shall construct an ordered list of the candidates. Look at all the runoffs and find out which candidate won with the biggest majority. Looking at Sample Problem 2.1 again, the biggest majority was A beat C 16–3. We'll denote this $A \to C$. Then look for the second-biggest, then the third-biggest, and so on, and make a list:

$$A \to C(16-3), B \to C(12-7), B \to A(10-9).$$

Now go through this list and construct a preference order of the candidates. At each step, if $X \to Y$, then X precedes Y in the preference list, *unless* Y already precedes X in the list. In our example, we must have A before C, B before C and B before A. The list is BAC and clearly B is the winner.

We shall refer to this solution as *Condorcet's extended method*, to distinguish it from the case where there is a Condorcet winner under the original method. Note that, if there is a Condorcet winner, the same candidate also wins under the extended method.

Let us apply this to the above example

5	4	3
A	B	C
B	C	A
C	A	B

,

which has no Condorcet winner. We have, with the larger majorities preceding smaller ones,

$$B \to C(9-3), A \to B(8-4), C \to A(7-5).$$

From $B \to C$ and $A \to B$ we get the list ABC. Next we see $C \to A$, but A already precedes C, so this result is ignored. The final list is ABC, and A is elected, even though a majority of voters would prefer C to A.

Sample Problem 3.1 *Consider the election with preference profile:*

7	5	3
A	B	C
B	C	A
C	A	B

Who would win under the Hare method? Is there a Condorcet winner? Who wins under Condorcet's solution method?

Solution. The votes for A, B and C are 7, 5 and 3 respectively. Under the Hare method, C is eliminated. The new preference profile is:

7	5	3
A	B	A
B	A	B

, that is,

10	5
A	B
B	A

So A wins $10-5$. Looking at all three runoffs, we see that A beats B $10-5$, B beats C $12-3$ and C beats A $8-7$, so there is no Condorcet winner. For Condorcet's solution, we see

$$B \to C(12-3), A \to B(10-5), C \to A(7-5).$$

The first two yield the list ABC and the last result is ignored, so A is elected.

Practice Exercise. Consider the election with preference profile:

6	4	3
A	B	C
B	C	A
C	A	B

.

Who would win under the Hare method? Is there a Condorcet winner? Who wins under Condorcet's solution method?

3.3 Condorcet Winner Criterion

We have seen that not every preference profile leads to a Condorcet winner. However, in those cases where there is a Condorcet winner, it seems reasonable to expect that the Condorcet winner would also be the winner under other methods. But this is not always true.

An electoral system is said to satisfy the *Condorcet winner criterion* if, whenever there is a Condorcet winner, then the electoral system in question will always choose the Condorcet winner.

The Hare system. The Hare system does not satisfy the Condorcet winner criterion. As an example, suppose 100 voters have the preference profile

40	21	39
A	B	C
B	C	B
C	A	A

Then 60 voters prefer B to A and 61 prefer B to C, so B is a Condorcet winner. However, B received the smallest number of first-place votes, and is the first candidate eliminated. The resulting preference profile is

40	21	39
A	C	C
C	A	A

and C wins.

Borda count. The Borda count does not satisfy the Condorcet winner criterion. To see this, consider the following profile. There are 3 candidates, A, B and C, and the 11 voters have preference profiles

6	4	1
A	B	C
B	C	A
C	A	B

Seven of the eleven voters prefer A to B, while six prefer A to C. So A is a Condorcet winner. However, in a (2,1,0) Borda count, B scores 14, A scores 13, and C scores 6. So B wins.

Plurality voting. The example used in discussing the Hare system can also be used to show that plurality voting does not satisfy the Condorcet winner criterion. As we saw, B is a Condorcet winner, but the winner under plurality voting would be A (and not C, who won under the Hare system).

Majority voting. It is clear that if there is a winner under the majority system, that candidate will have more than half the first-place votes, and consequently will have more than half the votes when compared to any competitor. So the winner is also a Condorcet winner. However, it is quite possible that there will be no majority winner, even when there is a Condorcet winner. (Again, the preference profile for the Hare system provides an example.) So majority elections do not satisfy the Condorcet winner criterion.

3.4 Condorcet Loser Criterion

A *Condorcet loser* in an election is a candidate who would lose in a runoff against each other candidate. Not every election with three or more candidates has a Condorcet loser. For example, in the election shown in Sample Problem 3.1, there is no Condorcet loser.

A voting system satisfies the *Condorcet loser criterion* if it can never happen that a Condorcet loser wins the election. Plurality voting does not satisfy this condition, but several methods, including the Hare system, do.

Plurality voting. Consider the following simple example, with three candidates.

2	7	5	6
A	A	B	C
B	C	C	B
C	B	A	A

A is a Condorcet loser—both B and C beat A 11:9. However, A has the largest number of first-place votes.

The Hare system. Suppose A is the winner of a Hare election. After various candidates have been eliminated, A is one of the remaining candidates, and receives a majority of the votes when compared to the other remaining candidate or candidates. Therefore a majority of voters place A above the other candidates that survived to the last round. It follows that A would win a runoff against any of those candidates, and therefore is not a Condorcet loser.

3.5 The Bucklin Method

Another method that Condorcet proposed, in 1793, was a technique to avoid the problems when there is no majority winner. The system was revived by James W. B̃ucklin of Grand Junction, Colorado in the early 1900s, and is now called the

Bucklin Method or the Grand Junction System. It was used in a number of American states in the early twentieth century. For a partial history of its use in Minnesota, see [27].

In the Bucklin method, each voter submits a preference list. The first preferences are tallied, and if any candidate receives a majority, she or he is elected. Let us refer to the number of votes required for a majority the *quota*.

Suppose no candidate achieves a majority. Then the second votes are added to the tally. In effect, each elector votes for two candidates. If one candidate receives the quota or higher, that candidate is elected. As the number of votes has been doubled, it is possible that more than one candidate will achieve a majority, and in that case the one with the bigger total wins.

As an example, consider the following preference profile.

8	6	4	2
A	B	C	D
B	C	D	C
C	D	B	B
D	A	A	A

There are 20 voters, so the quota is 11. Initially, the votes are $8, 6, 4, 2$ for A, B, C, D respectively, so there is no winner. After the second preferences are added, the votes are $8, 14, 12, 6$, so both B and C achieve the quota; B's total is larger, so B is elected.

Of course, it is also possible that there will still be no winner after the second tally. This is particularly true when there are a large number of candidates. In that case, third preferences are added, and so on. An example:

6	4	6	4
A	B	C	D
B	C	D	A
D	D	B	C
C	A	A	B

Again the quota is 11. After the first count, the totals are $6, 4, 6, 4$ for A, B, C, D respectively, so there is no winner. After the second count, the totals are $10, 10, 10, 10$, so again there is no winner. The third round totals are $10, 16, 14, 20$, so D is elected.

As in any system, it is possible that the result will be a tie. The usual response is to hold another election, but in the Bucklin system there is another possibility. If the quota has been reached, and two candidates are tied for first place, add another round of votes; if one of the tied candidates receives more than the other, that one is a winner.

Sample Problem 3.2 *What is the result of an election with preference profile*

5	4	3	2	2	2
A	B	C	C	D	D
B	A	D	A	B	C
C	C	A	B	A	B
D	D	B	D	C	A

if the Bucklin method is used?

Solution. There are 18 voters, so the quota is 10. The initial totals are $5, 4, 5, 4$ for A, B, C, D respectively, so there is no winner after the first round. After second preferences are added, the totals in the same order are $11, 11, 7, 7$, so both A and B have achieved enough votes, but they are tied. So a third round is calculated. The new totals are $16, 15, 16, 7$, so A beats B and is elected. Round two was when the quota was achieved, so only those who were tied for first place at that time are considered, even though C now has as many votes as A.

There is a modification available, which we shall call the *modified Bucklin method*. Voters only list those candidates of whom they approve. Then some columns of the profile will contain some blank cells at the bottom.

3.6 Condorcet Voting Systems

We have seen that many methods do not satisfy the Condorcet winner criterion. Because of this, several methods that *do* satisfy the criterion have been invented. These methods are often called *Condorcet voting systems*.

One very simple system was proposed by the Scottish economist Duncan Black [4] in 1958. He proposed using both Condorcet's method and some sort of Borda count. If there is a Condorcet winner, then that candidate wins the election. If not, the Borda count is calculated, and the Borda count winner is elected. This obviously satisfies the Condorcet winner criterion.

An earlier method combining Condorcet's methods and Borda count was proposed by E. J. Nanson, an English mathematician who migrated to Australia in the late nineteenth century. His method is as follows. First, conduct a Borda count. Then eliminate any candidate who receives less than the average score. (For example, if there are n voters and five candidates, and scores $4 - 3 - 2 - 1 - 0$ are used, the average score will be 2 per ballot, and the average overall score will be $2n$.) Then recalculate the scores of the remaining candidates as though they had been the only candidates. Eliminate the candidate with the lowest score. Repeat this process, until there is only one candidate, or a collection of tied candidates, remaining. This process will always select a Condorcet winner, if there is one. (See Exercise 3.14.)

This method was modified by the Australian astronomer J. M. Baldwin, who proposed eliminating only the lowest scorer at each stage. For more on these methods, see [19].

A. H. Copeland proposed the following method in a seminar at the University of Michigan in 1951. For a given candidate, consider all pairwise comparisons between that candidate and others. If the candidate would win in x cases and lose in y cases, then her score is $x - y$. (Comparisons resulting in a tie are ignored.) We shall examine this method in Exercise 3.12.

Sample Problem 3.3 *What is the result of an election with preference profile*

16	10	10	4
A	C	B	C
B	A	C	B
C	D	A	D
D	B	D	A

if Nanson's method is used?

Solution. There are 40 voters. Assuming a 3-2-1-0 Borda count is used, the Borda counts are $A - 78$, $C - 78$, $B - 70$, $D - 14$, The average score is 1.5 per vote, for a total of 60. So D is eliminated. The new profile is

16	10	10	4
A	C	B	C
B	A	C	B
C	B	A	A

and the counts are $A - 42$, $B - 40$, $C - 38$, and the average is 40. Now C is eliminated. Finally, A beats B 26–14 in a head-to-head vote.

3.7 Sequential Pairwise Voting

Condorcet's method essentially looks at all comparisons of two candidates. Another method that involves breaking an election into a number of smaller, two-candidate, elections is *sequential pairwise voting,* In this, several candidates are paired in successive runoff elections. There is an *agenda* (an ordered list of candidates). For example, if the agenda is $A, B, C, D, \ldots$ then the elections proceed as follows:

1. A against B
2. Winner of AB against C
3. That winner against D

   ...

Position in the agenda is very important. To see this, consider a four-candidate election with agenda A, B, C, D, in which all four candidates are equally likely to win. If repeated trials are made then we would expect the following results:

- A wins first runoff in half the cases
- A wins second runoff in half those cases—a quarter overall
- A wins the third runoff in half those cases—one-eighth overall.

So A has a 1 in 8 chance of winning. B also has a has 1 in 8 chance. However, C has a 1 in 4 chance, and D has a 1 in 2 chance. In this case being later in the list is very beneficial.

More generally, consider an election with four candidates and only three voters. One possible preference profile is:

1	1	1
A	B	C
B	C	D
C	D	A
D	A	B

There is an agenda to favor every candidate. For example, if the agenda is B, C, D, A, then B beats C 2–1, B beats D 2–1, and A beats B 2–1, so A wins. Using A, C, B, D we see C beat A 2–1, then B beats C and B beats D, a victory for B. If you wish C to win, use the agenda A, B, C, D, while B, C, A, D gives the election to D.

Rather than elections, the sequential pairwise model is often used for sporting tournaments (the result of match is used instead of the result of a runoff election). One often sees playoff rules like:

(i) Second and third placegetters in preliminary competition play each other ("the playoff");

(ii) The winner of playoff meets the leader from the preliminaries.

In this case it is reasonable that the preliminary leader should get an advantage. However, when the model is used in voting situations, it is very subject to manipulation.

Exercises 3

1. Consider the following statements.

(i) If X would win a plurality election, then X would win under Condorcet.

(ii) If X would win a majority election, then X would win under Condorcet.

Is either of these statements always true?

2. Consider the following preference table. In Chap. 2, Exercise 10, we asked, who would win under plurality voting? Who would win in a runoff? The answers were C and A respectively.

6	3	4	2	1
A	C	C	B	E
E	B	D	A	A
B	E	A	C	B
C	D	E	D	C
D	A	B	E	D

.

Is there a Condorcet winner in this example? If not, who would win under Condorcet's extended method? Who would win under Bucklin's method?

3. A club with 36 members wishes to elect its president from four candidates, A, B, C and D. The preference profile is

16	10	8	2
A	B	C	B
B	A	B	A
C	D	A	C
D	C	D	D

.

Is there a Condorcet winner? If not, who would win under Condorcet's extended method? Who would win under Bucklin's method?

4. Eighteen delegates must elect one of four candidates, A, B, C and D. The preference profile is

8	6	4
A	B	C
B	D	D
C	A	A
D	C	B

.

Is there a Condorcet winner? If not, who would win under Condorcet's extended method?

5. Here is the preference profile for an election with three candidates, A, B, C. Is there a Condorcet winner? If not, who would win under Condorcet's extended method? Who would win under Bucklin's method?

8	5	6	4
A	C	B	B
C	A	C	A
B	B	A	C

.

6. Fifty voters are to choose one of five candidates. Their preference profile is

20	10	14	6
A	B	B	C
C	A	A	D
E	C	D	B
B	D	C	A
D	E	E	E

.

Is there a Condorcet winner? If not, who would win under Condorcet's extended method? Who would win under Bucklin's method?

7. Fifteen committee members are to choose a new treasurer from four candidates, A, B, C and D. Their preference profile is

7	5	3
A	C	D
B	B	C
D	A	B
C	D	A

.

Is there a Condorcet winner? If not, what is the result under the extended Condorcet method?

8. One hundred voters choose between four candidates, A, B, C and D. Their preference profile is

40	32	10	18
A	B	C	D
C	C	D	C
B	A	A	B
D	D	B	A

.

If the Hare method is used, the result is a tie between A and B. Is there a Condorcet winner? If not, what is the result under the extended Condorcet method? Who would win under Bucklin's method?

9. Twenty voters choose between four candidates, A, B and C. Their preference profile is

5	5	1	3	2	4
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

.

Is there a Condorcet winner? If not, what is the result under the extended Condorcet method? Who would win under Bucklin's method?

10. Verify that if an election has a Condorcet winner, then the same candidate also wins under Condorcet's extended method.

11. Does the Bucklin method satisfy the Condorcet winner criterion?

12. Prove that the Copeland method satisfies both the Condorcet winner and Condorcet loser criteria. What is the shortcoming of this method?

13. Verify that the Black rule satisfies the Condorcet winner criterion.

14. Verify that Nanson's method will always select a Condorcet winner, if there is one.

15. Who will win in the election described in Exercise 7 under Black's rule? Who would win under Nanson's method?

16. Use the preference profile

5	4	4	3
W	W	X	X
X	X	Y	T
Z	T	T	Z
T	Z	X	X

to show that the Borda count and the Nanson method do not always give the same result.

17. Consider the preference profile

3	1	1	1	1	1	1
W	W	X	X	Y	Y	Z
Z	Y	Y	Z	X	X	X
Y	Z	Z	Y	Z	W	Y
X	X	W	W	W	Z	W

.

What is the result of an election using the following systems?

(i) Majority system.

(ii) Plurality system.

(iii) Borda count.

(iv) Hare system.

(v) Condorcet method.

(vi) Sequential pairwise method with agenda X, Y, Z, W.

(vii) Sequential pairwise method with agenda W, X, Z, Y.

(viii) Sequential pairwise method with agenda Y, Z, W, X.

18. Thirty board members must vote on five candidates: X, Y, Z, U and V. Their preference rankings are summarized in the table below. Find the winner using sequential pairwise voting with the agenda X, Y, Z, U, V.

12	10	8
X	Y	Z
U	U	U
Y	Z	X
Z	X	V
V	V	Y

19. An 18-member committee is to elect one of the four candidates Q, R, S, T. Their preference table is as shown. Which candidate wins under sequential pairwise voting with agenda (S, T, Q, R)?

4	6	5	3
Q	R	S	T
T	S	R	S
S	T	T	R
R	Q	Q	Q

20. Consider a sequential pairwise election with preferences

5	3	2	1	8
X	X	Y	Z	T
Y	T	Z	Y	Y
T	Y	X	X	X
Z	Z	T	T	Z

(i) Who will win under agenda X, Y, Z, T?

(ii) Who will win under agenda Y, T, Z, X?

21. A sequential pairwise election has preference profile

2	8	11	7
A	C	B	A
B	B	A	C
C	A	C	B

(i) Who will win under agenda A, B, C?

(ii) Who will win under agenda B, C, A?

(iii) Who will win under agenda C, A, B?

22. A sequential pairwise election has preference profile

40	32	12	18
P	Q	R	S
Q	R	S	R
R	P	P	Q
S	S	Q	P

.

(i) Who will win under agenda P, Q, R, S?

(ii) Who will win under agenda S, R, Q, P?

(iii) Who will win under agenda S, P, R, Q?

(iv) Who will win under agenda S, Q, P, R?

23. Does the plurality runoff method satisfy the Condorcet winner criterion?

24. Does sequential pairwise voting satisfy the Condorcet winner criterion?