On Exceptional Vertex Operator (Super) Algebras

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Abstract We consider exceptional vertex operator algebras and vertex operator superalgebras with the property that particular Casimir vectors constructed from the primary vectors of lowest conformal weight are Virasoro descendents of the vacuum. We show that the genus one partition function and characters for simple ordinary modules must satisfy modular linear differential equations. We show the rationality of the central charge and lowest weights of modules, modularity of solutions, the dimension of each graded space is a rational function of the central charge and that the lowest weight primaries generate the algebra. We also discuss conditions on the reducibility of the lowest weight primary vectors as a module for the automorphism group. Finally we analyse solutions for exceptional vertex operator algebras with primary vectors of lowest weight up to 9 and for vertex operator superalgebras with primary vectors of lowest weight up to 17/2. Most solutions can be identified with simple ordinary modules for known algebras but there are also four conjectured algebras generated by weight two primaries and three conjectured extremal vertex operator algebras generated by primaries of weight 3, 4 and 6, respectively.

Key words Vertex operator algebras • Vertex operator super algebras • Virasoro algebra • Group theory

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1 Introduction

Vertex Operator Algebras (VOAs) and Super Algebras (VOSAs) have deep connections to Lie algebras, number theory, group theory, combinatorics and Riemann surfaces (e.g., [FHL, FLM, Kac1, MN, MT]) and, of course, conformal field theory e.g., [DMS]. The classification of VOAs and VOSAs still seems to be a very difficult task, for example, there is no proof of the uniqueness of the Moonshine module [FLM]. Nevertheless, it would be very useful to be able to characterize VOA/VOSAs with interesting properties such as large automorphism groups (e.g., the Monster group for the Moonshine module), rational characters, generating vectors, etc. In [Mat], Matsuo introduced VOAs of class $\mathcal{S}^n$ with the defining property that the Virasoro vacuum descendents are the only Aut(V)-invariant vectors of weight $k \le n$. Thus the Moonshine module [FLM] is of class $\mathcal{S}^{11}$, the Baby Monster VOA [Ho1] of class $\mathcal{S}^6$ and the level one Kac–Moody VOAs generated by Deligne's Exceptional Lie algebras A_1, A_2, G_2, D_4, F_4, E_6, E_7, E_8 [D] are of class $\mathcal{S}^4$.[1]

In this paper we consider a refinement and generalization of previous results in [T1, T2] concerning such exceptional VOAs. Assuming the VOA is simple and of strong CFT-type (e.g., [MT]) we consider quadratic Casimir vectors $\lambda^{(k)}$ of conformal weight $k = 0, 1, 2, \ldots$ constructed from the primary vectors of lowest conformal weight $l \in \mathbb{N}$. We say that a VOA is exceptional of lowest primary weight l if $\lambda^{(2l+2)}$ is a Virasoro vacuum descendent. Every VOA of class $\mathcal{S}^{2l+2}$ with lowest primary weight l is exceptional, but the converse is not known to be true. We show, using Zhu's theory for genus one correlation functions [Z], that for an Exceptional VOA of lowest primary weight l, the partition function and the characters for simple ordinary VOA modules satisfy a Modular Linear Differential Equation (MLDE) of order at most $l+1$. Given that order of the MLDE is exactly $l+1$ (which is verified for all $l \le 9$) we show that the central charge c and module lowest weights h are rational, the MLDE solution space is modular invariant and the dimension of each VOA graded space is a rational function of c. Subject to a further indicial root condition (again verified for all $l \le 9$) we show that an Exceptional VOA is generated by its primary vectors of lowest weight l.

We also consider other properties that arise from genus zero correlation functions for all l. Assuming the VOA is of class $\mathcal{S}^{2l+2}$ this leads to conditions on the reducibility of the lowest weight l primary space as a module for the VOA automorphism group.

A similar analysis is carried out for Exceptional VOSAs of lowest primary weight $l \in \mathbb{N} + \frac{1}{2}$ for which $\lambda^{(2l+1)}$ is a Virasoro vacuum descendent. Using a twisted version of Zhu theory [MTZ] we obtain a twisted MLDE of order at most $l+\frac{1}{2}$ which is satisfied by the partition function and simple ordinary VOA module characters. This differential equation leads to a similar set of general results to those for VOAs. Likewise, we can consider genus zero correlation functions for all $l \in \mathbb{N}+\frac{1}{2}$ leading

[1] In fact, the A_1 theory is of class $\mathcal{S}^{\infty}$ and the E_8 theory is of class $\mathcal{S}^6$.

to conditions on the reducibility of the space of the space of weight l primaries as a module for the VOSA automorphism group.

The paper also summarizes rational c, h solutions to the MLDE for all $l \leq 9$ and the twisted MLDE for all $l \leq \frac{17}{2}$. In most cases we can identify a VOA/VOSA with the requisite properties. These include a number of special VOA/VOSAs, some VOSAs obtained by commutant constructions, some simple current extensions of Virasoro minimal models and $\mathcal{W}$-algebras. We also present evidence for four candidate/conjectured VOAs with simple Griess algebras for $l = 2$ and three extremal VOAs for $l = 3, 4, 6$. All the VOSA solutions found can be identified with known theories.

2 Vertex Operator (Super) Algebras

We review some aspects of vertex operator super algebra theory (e.g., [FHL, FLM, Kac1, MN, MT]). A Vertex Operator Superalgebra (VOSA) is a quadruple $(V, Y(\cdot,\cdot), \mathbf{1}, \omega)$ with a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with parity $p(u) = 0$ or 1 for $u \in V_{\bar{0}}$ or $V_{\bar{1}}$ respectively. $(V, Y(\cdot,\cdot), \mathbf{1}, \omega)$ is called a Vertex Operator Algebra (VOA) when $V_{\bar{1}} = 0$.

V also has a $\frac{1}{2}\mathbb{Z}$-grading with $V = \bigoplus_{r \in \frac{1}{2}\mathbb{Z}} V_r$ with $\dim V_r < \infty$. $\mathbf{1} \in V_0$ is the vacuum vector and $\omega \in V_2$ is called the conformal vector. Y is a linear map $Y : V \to \mathrm{End}(V)[[z, z^{-1}]]$ for formal variable z giving a vertex operator

$$Y(u, z) = \sum_{n \in \mathbb{Z}} u(n) z^{-n-1}, \tag{1}$$

for every $u \in V$. The linear operators (modes) $u(n) : V \to V$ satisfy creativity

$$Y(u, z)\,\mathbf{1} = u + O(z), \tag{2}$$

and lower truncation

$$u(n)v = 0, \tag{3}$$

for each $u, v \in V$ and $n \gg 0$. For the conformal vector ω

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \tag{4}$$

where $L(n)$ satisfies the Virasoro algebra for some central charge c

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{c}{12}(m^3 - m)\delta_{m,-n}\,\mathrm{id}_V\,. \tag{5}$$

Each vertex operator satisfies the translation property

$$Y(L(-1)u, z) = \partial_z Y(u, z). \tag{6}$$

The Virasoro operator $L(0)$ provides the $\frac{1}{2}\mathbb{Z}$-grading with $L(0)u = \mathrm{wt}(u)u$ for $u \in V_r$ and with weight $\mathrm{wt}(u) = r \in \mathbb{Z} + \frac{1}{2}p(u)$. Finally, the vertex operators satisfy the Jacobi identity

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y(u, z_1)Y(v, z_2) - (-1)^{p(u)p(v)} z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right) Y(v, z_2)Y(u, z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y\left(Y(u, z_0)v, z_2\right),$$

with $\delta\left(\frac{x}{y}\right) = \sum_{r\in\mathbb{Z}} x^r y^{-r}$.

These axioms imply $u(n)V_r \subset V_{r-n+wt(u)-1}$ for u of weight $\mathrm{wt}(u)$. They also imply locality, skew-symmetry, associativity and commutativity:

$$(z_1 - z_2)^N Y(u, z_1)Y(v, z_2) = (-1)^{p(u)p(v)}(z_1 - z_2)^N Y(v, z_2)Y(u, z_1), \tag{7}$$

$$Y(u, z)v = (-1)^{p(u)p(v)} e^{zL(-1)} Y(v, -z)u, \tag{8}$$

$$(z_0 + z_2)^N Y(u, z_0 + z_2)Y(v, z_2)w = (z_0 + z_2)^N Y(Y(u, z_0)v, z_2)w, \tag{9}$$

$$u(k)Y(v, z) - (-1)^{p(u)p(v)} Y(v, z)u(k) = \sum_{j\geq 0} \binom{k}{j} Y(u(j)v, z)z^{k-j}, \tag{10}$$

for $u, v, w \in V$ and integers $N \gg 0$ [FHL, Kac1, MT].

We define an invariant symmetric bilinear form $\langle\ ,\ \rangle$ on V by

$$\left\langle Y\left(e^{zL(1)}\left(-z^{-2}\right)^{L(0)} w, z^{-1}\right) u, v\right\rangle = (-1)^{p(u)p(w)}\langle v, Y(w, z)v\rangle, \tag{11}$$

for all $u, v, w \in V$ [FHL]. V is said to be of *CFT-type* if $V_0 = \mathbb{C}\mathbf{1}$ and of *strong CFT-type* if additionally $L(1)V_1 = 0$ in which case $\langle\ ,\ \rangle$, with normalization $\langle \mathbf{1}, \mathbf{1}\rangle = 1$, is unique [Li]. Furthermore, $\langle\ ,\ \rangle$ is invertible if V is simple. All VOSAs in this paper are assumed to be of this type.

Every VOSA contains a subVOA,[2] which we denote by $V_{\langle\omega\rangle}$, generated by the Virasoro vector ω with Fock basis of vacuum descendents of the form

$$L(-n_1)L(-n_2)\ldots L(-n_k)\,\mathbf{1}, \tag{12}$$

[2] This subVOA is often denoted by M_c e.g., [FZ].

for $n_i \geq 2$. $\langle\ ,\ \rangle$ is singular on $\left(V_{\langle\omega\rangle}\right)_n$ iff the central charge is

$$c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}, \tag{13}$$

for coprime integers $p, q \geq 2$ and $n \geq (p-1)(q-1)$ [Wa]. The Virasoro minimal model VOA $L(c_{p,q}, 0)$ is the quotient of $V_{\langle\omega\rangle}$ by the radical of $\langle\ ,\ \rangle$. $L(c_{p,q}, 0)$ has a finite number of simple ordinary V-modules $L(c_{p,q}, h_{r,s}) \cong L(c_{p,q}, h_{q-r,p-s})$ (e.g., [DMS]) with lowest weight

$$h_{r,s} = \frac{(pr-qs)^2 - (p-q)^2}{4pq}, \tag{14}$$

for $r = 1, \ldots, q-1$ and $s = 1, \ldots, p-1$.

3 Quadratic Casimirs and Genus One Zhu Theory

3.1 Quadratic Casimirs

Let $(V, Y(\cdot,\cdot), \mathbf{1}, \omega)$ be a simple VOA of strong CFT-type with unique invertible bilinear form $\langle\ ,\ \rangle$. Let Π_l denote the space of primary vectors of lowest weight $l \geq 1$, i.e., $L(n)u = 0$ for all $n > 0$ for $u \in \Pi_l$. Choose a Π_l-basis $\{u_i\}$ for $i = 1, \ldots, p_l = \dim \Pi_l$ with dual basis $\{\overline{u}_i\}$, i.e., $\langle u_i, \overline{u}_j\rangle = \delta_{ij}$. Define quadratic Casimir vectors $\lambda^{(n)}$ for $n \geq 0$ by [Mat, T1, T2]

$$\lambda^{(n)} = \sum_{i=1}^{p_l} u_i(2l-n-1)\overline{u}_i \in V_n. \tag{15}$$

In particular we find

$$\lambda^{(0)} = \sum_{i=1}^{p_l} u_i(2l-1)\overline{u}_i = (-1)^l \sum_{i=1}^{p_l} \langle u_i, \overline{u}_i\rangle\, \mathbf{1} = (-1)^l\, p_l\, \mathbf{1}\,.$$

Furthermore, if $l > 1$, then $\dim V_1 = 0$ and hence $\lambda^{(1)} = 0$, whereas for $l = 1$ the Jacobi identity implies $\lambda^{(1)} = \sum_{i=1}^{p_l} u_i(0)\overline{u}_i = -\sum_{i=1}^{p_l} \overline{u}_i(0)u_i = 0$ [T1]. Thus we find

Lemma 1. $\lambda^{(0)} = (-1)^l\, p_l\, \mathbf{1}$ *and* $\lambda^{(1)} = 0$.

Since the Π_l elements are primary then for all $m > 0$

$$L(m)\lambda^{(n)} = (n - m + l(m-1))\, \lambda^{(n-m)}. \tag{16}$$

Suppose that $\lambda^{(n)} \in V_{\langle\omega\rangle}$, then (16) implies that $\lambda^{(m)} \in V_{\langle\omega\rangle}$ for all $m \leq n$. Furthermore, since $\langle\ ,\ \rangle$ is invertible we have the following lemma

Lemma 2 (Matsuo [Mat]). *If $\lambda^{(n)} \in V_{\langle\omega\rangle}$, then $\lambda^{(n)}$ is uniquely determined.*

Thus if $\lambda^{(2)} \in V_{\langle\omega\rangle}$, then $\lambda^{(2)} = \kappa L(-2)\,\mathbf{1}$ for some κ so that $\langle L(-2)\,\mathbf{1}, \lambda^{(2)}\rangle = \kappa\frac{c}{2}$. But (11) and (16) imply $\langle L(-2)\,\mathbf{1}, \lambda^{(2)}\rangle = \langle \mathbf{1}, L(2)\lambda^{(2)}\rangle = (-1)^l\, p_l$, so that for $c \neq c_{2,3} = 0$ (cf. (13))

$$\lambda^{(2)} = p_l \frac{2\,(-1)^l\, l}{c} L(-2)\,\mathbf{1}\,. \tag{17}$$

Similarly, if $\lambda^{(4)} \in V_{\langle\omega\rangle}$ and $c \neq 0$ or $c \neq c_{2,5} = -22/5$, then [Mat, T1, T2]

$$\lambda^{(4)} = p_l \frac{2\,(-1)^l\, l\,(5l+1)}{c\,(5c+22)} L(-2)^2\,\mathbf{1} + p_l \frac{3\,(-1)^l\, l\,(c-2l+4)}{c\,(5c+22)} L(-4)\,\mathbf{1}\,. \tag{18}$$

These examples illustrate a general observation:

Lemma 3. *Each coefficient in the expansion of $\lambda^{(n)} \in V_{\langle\omega\rangle}$ in a basis of Virasoro Fock vectors is of the form $p_l\, r(c)$ for some rational function $r(c)$ of c.*

3.2 Genus One Constraints from Quadratic Casimirs

Define genus one partition and 1-point correlation functions for $u \in V$ by

$$Z_V(q) = \mathrm{Tr}_V\left(q^{L(0)-c/24}\right) = q^{-c/24} \sum_{n\geq 0} \dim V_n\, q^n, \tag{19}$$

$$Z_V(u,q) = \mathrm{Tr}_V\left(o(u) q^{L(0)-c/24}\right), \tag{20}$$

where q is a formal parameter and $o(u) = u(\mathrm{wt}\,(u)-1) : V_n \to V_n$ is the 'zero mode' for homogeneous u. By replacing V by a simple ordinary V-module N (on which $L(0)$ acts semisimply e.g., [FHL, MT]) these definitions may be extended to graded characters $Z_N(q)$ and 1-point functions $Z_N(u,q)$, e.g.,

$$Z_N(q) = \mathrm{Tr}_N\left(q^{L(0)-c/24}\right) = q^{h-c/24} \sum_{n\geq 0} \dim N_n\, q^n, \tag{21}$$

where h denotes the lowest weight of N. Zhu also introduced an isomorphic VOA $(V, Y[\cdot,\cdot], \mathbf{1}, \widetilde{\omega})$ with 'square bracket' vertex operators

$$Y[u,z] \equiv Y\left(e^{zL(0)}u, e^z - 1\right) = \sum_{n\in\mathbb{Z}} u[n] z^{-n-1}, \tag{22}$$

for Virasoro vector $\widetilde{\omega} = \omega - c/24\,\mathbf{1}$ with modes $\{L[n]\}$. $L[0]$ defines an alternative $\mathbb{Z}$ grading with $V = \bigoplus_{k\geq 0} V_{[k]}$ where $L[0]v = \mathrm{wt}[v]v$ for $\mathrm{wt}[v] = k$ for $v \in V_{[k]}$. Zhu obtained a reduction formula for the 2-point correlation function $Z_N\,(Y[u,z]v, q)$ for $u, v \in V$ in terms of the elliptic Weierstrass function

$$P_m(z) = \frac{1}{z^m} + (-1)^m \sum_{n\geq m} \binom{n-1}{m-1} E_n(q)\, z^{n-m}, \tag{23}$$

for $m \geq 1$ and with Eisenstein series $E_n(q) = 0$ for odd n and

$$E_n(q) = -\frac{B_n}{n!} + \frac{2}{(n-1)!} \sum_{k\geq 1} \frac{k^{n-1} q^k}{1-q^k}, \tag{24}$$

for even n with B_n the nth Bernoulli number. $P_m(z)$ converges absolutely and uniformly on compact subsets of the domain $|q| < |e^z| < 1$. $E_n(q)$ is a modular form of weight n for $n \geq 4$ and $E_2(q)$ is a quasi-modular form of weight 2, i.e., letting $q = \exp(2\pi\mathrm{i}\tau)$ for $\tau \in \mathbb{H}_1$

$$E_n\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = (\gamma\tau + \delta)^n\, E_n(\tau) - \frac{\gamma(\gamma\tau + \delta)}{2\pi\mathrm{i}} \delta_{n2}, \tag{25}$$

for $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ [Se]. We then have

Proposition 1 (Zhu [Z]). *Let N be a simple ordinary V-module.*

$$Z_N\,(Y[u,z]v, q) = \mathrm{Tr}_N \left(o(u)o(v)q^{L(0)-c/24}\right) + \sum_{m\geq 0} P_{m+1}(z) Z_N\,(u[m]v, q)\,.$$

Taking $u = \widetilde{\omega}$ and noting that $o(\widetilde{\omega}) = L(0) - c/24$ we obtain:

Corollary 1. *The 1-point function of a Virasoro descendent $L[-k]v$ is*

$$Z_N\,(L[-k]v, q) = (-1)^k \sum_{r\geq 0} \binom{k+r-1}{k-2} E_{k+r}(q) Z_N\,(L[r]v, q)\,,$$

for all $k \geq 3$, whereas for $k = 2$ we have

$$Z_N\,(L[-2]v, q) = \left(q\frac{\partial}{\partial q} + \mathrm{wt}[v]E_2(q)\right) Z_N\,(v, q) + \sum_{s\geq 1} E_{2s+2}(q) Z_N\,(L[2s]v, q)\,.$$

Let us now consider a simple VOA V of strong CFT-type with lowest weight $l \geq 1$ Virasoro primary vectors Π_l so that

$$Z_V(q) = Z_{V_{\langle\omega\rangle}}(q) + O\left(q^{l+c/24}\right). \tag{26}$$

Let $\{u_i\}$ and $\{\overline{u}_i\}$ be a basis and dual basis for Π_l. Apply Proposition 1 to

$$Z_N\left(\sum_{i=1}^{p_l} Y[u_i, z]\overline{u}_i, q\right) = \sum_{n\geq 0} Z_N\left(\lambda^{[n]}, q\right) z^{n-2l}, \tag{27}$$

(for Casimir vector $\lambda^{[n]} \in V_{[n]}$ in square bracket modes) to find

$$\sum_{n\geq 0} Z_N\left(\lambda^{[n]}, q\right) z^{n-2l} = \mathrm{Tr}_N\left(\sum_{i=1}^{p_l} o(u_i)o(\overline{u}_i)q^{L(0)-c/24}\right) + \sum_{m=0}^{2l-1} P_{m+1}(z) Z_N\left(\lambda^{[2l-m-1]}, q\right). \tag{28}$$

Comparing the coefficients of z^{n-2l} for $n \geq 2l$ on both sides of this equality leads to a recursive identity between $Z_N\left(\lambda^{[n]}, q\right)$ and $Z_N\left(\lambda^{[m]}, q\right)$ for $m \leq n-2$. In particular, comparing the coefficients of z^2 we find

Proposition 2. $Z_N\left(\lambda^{[2l+2]}, q\right)$ *satisfies the recursive identity*

$$Z_N\left(\lambda^{[2l+2]}, q\right) = \sum_{r=0}^{l-1} \binom{2l-2k+1}{2} E_{2l-2k+2}(q) Z_N\left(\lambda^{[2k]}, q\right). \tag{29}$$

4 Exceptional VOAs

Consider a simple VOA of strong CFT-type with primary vectors of lowest weight $l \geq 1$ for which $\lambda^{(2l+2)} \in V_{\langle\omega\rangle}$ (or equivalently, $\lambda^{[2l+2]} \in V_{\langle\tilde{\omega}\rangle}$). We also assume that $\left(V_{\langle\omega\rangle}\right)_{2l+2}$ contains no Virasoro singular vector, i.e., $c \neq c_{p,q}$ for $(p-1)(q-1) \leq 2l+2$. We call such a VOA an *Exceptional VOA of lowest primary weight* l. (16) implies $\lambda^{(2k)} \in V_{\langle\omega\rangle}$ and $\lambda^{[2k]} \in V_{\langle\tilde{\omega}\rangle}$ for all $k \leq l$.

Proposition 3. *Let* $\lambda^{[2k]} \in V_{\langle\tilde{\omega}\rangle}$. *Then for a simple ordinary* V*-module* N

$$Z_N\left(\lambda^{[2k]}, q\right) = \sum_{m=0}^{k} f_{k-m}(q, c) D^m Z_N(q); \tag{30}$$

where D *is the Serre modular derivative defined for* $m \geq 0$ *by*

$$D^{m+1} Z_N(q) = \left(q\frac{\partial}{\partial q} + 2mE_2(q)\right) D^m Z_N(q); \tag{31}$$

$f_m(q,c)$ is a modular form of weight $2m$ whose coefficients over the ring of Eisenstein series are of the form $p_l\, r(c)$ for a rational function $r(c)$.

Proof. Equation (30) follows from Corollary 1 by induction in the number of Virasoro modes where the $D^k Z_N(q)$ term arises from a $L[-2]^k\mathbf{1}$ component in $\lambda^{[2k]}$. The coefficients of $f_m(q,c)$ over the ring of Eisenstein series are of the form $p_l\, r(c)$ for a rational function $r(c)$ from Lemma 3. □

Applying Proposition 3 to the recursive identity (29) implies $Z_N(q)$ satisfies a Modular Linear Differential Equation (MLDE) [Mas1].

Proposition 4. *Let V be an Exceptional VOA of lowest primary weight l. $Z_N(q)$ for each simple ordinary V–module N satisfies a MLDE of order $\le l+1$*

$$\sum_{m=0}^{l+1} g_{l+1-m}(q,c)D^m Z(q) = 0, \tag{32}$$

where $g_m(q,c)$ is a modular form of weight $2m$ whose coefficients over the ring of Eisenstein series are rational functions of c.

Remark 1. Proposition 4 states that each simple ordinary V-module N character $Z_N(q)$ satisfies the same MLDE (32). However, the MLDE may also have further solutions unrelated to module characters.

$g_0(q,c) = g_0(c)$ is independent of q since it is a modular form of weight 0. For $g_0(c) \neq 0$, the MLDE (32) is of order $l+1$ with a regular singular point at $q=0$ so that Frobenius–Fuchs theory concerning the $l+1$-dimensional solution space $\mathcal{F}$ applies, e.g., [Hi, I]. Any solution $Z(q)\in\mathcal{F}$ is holomorphic in q for $0<|q|<1$ since the MLDE coefficients $g_m(q,c)$ are holomorphic for $|q|<1$. We may thus view each solution as a function of $\tau\in\mathbb{H}_1$ for $q=e^{2\pi \mathrm{i}\tau}$.

Using the quasi-modularity of $E_2(\tau)$ and (31) with $q\frac{\partial}{\partial q} = \frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial\tau}$, it follows that for all $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\in \mathrm{SL}(2,\mathbb{Z})$, $Z\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$ is also a solution of the MLDE since $g_{l+1-m}(q,c)$ is a modular form of weight $2l+2-2m$. Thus $T:\tau\to\tau+1$ has a natural action on $\mathcal{F}=\mathcal{F}_1\oplus\ldots\oplus\mathcal{F}_r$ for distinct eigenspaces $\mathcal{F}_i$ with T monodromy eigenvalue $e^{2\pi\mathrm{i}x_i}$ where x_i is a root of the indicial polynomial

$$\sum_{m=0}^{l+1} g_{l+1-m}(0,c)\prod_{s=0}^{m-1}\left(x-\frac{1}{6}s\right) = 0. \tag{33}$$

If $x_1 = x_2 \mod \mathbb{Z}$, for roots x_1, x_2, they determine the same monodromy eigenvalue. Let $\widehat{x_i}$ denote an indicial root with least real part for a given monodromy eigenvalue $e^{2\pi\mathrm{i}x_i}$. Then $\mathcal{F}_i$ has a basis of the form [17, Hi]

$$f_i^n(\tau) = \phi_i^1(q) + \tau\phi_i^2(q) + \ldots + \tau^{n-1}\phi_i^n(q), \tag{34}$$

for $n = 1, \ldots, \dim \mathcal{F}_i$ and where each $\phi_i^n(q)$ is a q-series

$$\phi_i^n(q) = q^{\widehat{x_i}} \sum_{k \geq 0} a_{ik}^n q^k,$$

which is holomorphic on $0 < |q| < 1$. The solutions $f_i^n(\tau)$ for $n \geq 2$, which are referred to as logarithmic solutions (since they contain nonnegative integer powers of $\log q = 2\pi \mathrm{i}\tau$), occur if the same indicial root occurs multiple times or, possibly, if two roots differ by an integer. However, every graded character $Z_N(q)$ for a simple ordinary module with lowest weight h has a pure q-series with indicial root $x = h - c/24$ from (21).

We now sketch a proof that the central charge c is rational following [AM] (which is extended to logarithmic solutions (34) in [Miy]). Suppose $c \notin \mathbb{Q}$ and consider $\phi \in \mathrm{Aut}(\mathbb{C})$ such that $\tilde{c} = \phi(c) \neq c$. Then $Z_V(\tau, \tilde{c})$ is a solution to the MLDE, found by replacing c by $\tilde{c}$ in (32). But since the coefficients in the q-expansion of $Z_V(\tau, c)$ are integral we have

$$Z_V(\tau, \tilde{c}) = q^{(\tilde{c}-c)/24} Z_V(\tau, c).$$

Applying the modular transformation $S : \tau \to -1/\tau$ we find

$$Z_V\left(-\frac{1}{\tau}, \tilde{c}\right) = \exp\left(-\frac{\pi \mathrm{i}(\tilde{c} - c)}{12\tau}\right) Z_V\left(-\frac{1}{\tau}, c\right). \tag{35}$$

But $Z_V(-1/\tau, c)$ satisfies (32) and $Z_V(-1/\tau, \tilde{c})$ satisfies (32) with c replaced by $\tilde{c}$ and thus both are expressed in terms of the basis (34). Analysing (35) along rays $\tau = re^{\mathrm{i}\theta}$ in the limit $r \to \infty$ with $0 < \theta < \pi$, a contradiction results unless $\tilde{c} = c$. Hence $c \in \mathbb{Q}$ [AM, Miy]. Similarly, the lowest conformal weight h of a simple ordinary module N is rational. Altogether we have

Proposition 5. *Let V be an Exceptional VOA of lowest primary weight $l \geq 1$ and central charge c and let N be a simple ordinary V-module of lowest weight h. Assuming $g_0(c) \neq 0$ in the MLDE (32), then*

(i) *$Z_N(q)$ is holomorphic for $0 < |q| < 1$.*
(ii) *$Z_N\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$ is a solution of the MLDE for all $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ viewed as a function of $\tau \in \mathbb{H}_1$ for $q = e^{2\pi \mathrm{i}\tau}$.*
(iii) *The central charge c and the lowest conformal weight h are rational.*

Consider the general solution with indicial root $x = c/24$ of the form $Z(q) = q^{-c/24} \sum_{n \geq 0} a_n q^n$. Substituting into the MLDE, we obtain a linear equation in $a_0, \ldots, a_n$ for each n. This can be iteratively solved for a_n provided the coefficient of a_n is nonzero. This coefficient may vanish if $x = m - c/24$ is an indicial root for some integer $m > 0$. Hence we have

Proposition 6. *Let V be an Exceptional VOA of lowest primary weight $l \geq 1$ and central charge c. Suppose $g_0(c) \neq 0$ and that $m < l$ for any indicial root of the form $x = m - c/24$. Then*

(i) *$Z_V(q)$ is the unique q-series solution of the MLDE satisfying* (26).
(ii) $\dim V_n$ *is a rational function of c for each $n \geq 0$.*
(iii) *V is generated by the space of lowest weight primary vectors Π_l.*

Proof. (i) The $x = -c/24$ solution $Z(q) = q^{-c/24} \sum_{n \geq 0} a_n q^n$ is determined by a_0 and a_m for any indicial root(s) of the form $x = m - c/24$ for $m > 0$. Thus the partition function is uniquely determined by the l Virasoro leading terms (26) under the assumption that $m < l$.

(ii) The modular forms $g_m(q, c)$ of the MLDE of Proposition 4 have q-expansions whose coefficients are rational functions of c. Hence solving iteratively it follows that $a_n = \dim V_n$ is a rational function of c.

(iii) Let $V_{\langle \Pi_l \rangle} \subseteq V$ be the subalgebra generated by the lowest weight primary vectors Π_l. But $\omega \in V_{\langle \Pi_l \rangle}$ from (17) so that $V_{\langle \Pi_l \rangle}$ is a VOA of central charge c. Furthermore, since $\lambda^{(2l+2)} \in V_{\langle \Pi_l \rangle}$, the subVOA is an Exceptional VOA of lowest primary weight l. Hence $Z_{V_{\langle \Pi_l \rangle}}(q)$ satisfies the same MLDE as $Z_V(q)$. From (i) it follows that $Z_{V_{\langle \Pi_l \rangle}}(q) = Z_V(q)$ implying $V_{\langle \Pi_l \rangle} = V$. □

Remark 2. Note that $g_0(c) \neq 0$ provided $\lambda^{(2l+2)}$ contains an $L(-2)^{l+1}\mathbf{1}$ component. We conjecture that such a component exists for all l. We further conjecture that $m < l$ for any indicial root of the form $x = m - c/24$ for all l. These properties are verified for all $l \leq 9$ in Sect. 6.

4.1 Exceptional VOAs with $p_l = 1$

Let V be a simple VOA of strong CFT-type generated by one primary vector u of lowest weight l with dual $\overline{u} = u/\langle u, u \rangle$. Consider the commutator (10)

$$
\begin{aligned}
[u(m), Y(u, z)] &= \sum_{j \geq 0} \binom{m}{j} Y\left(u(j)u, z\right) z^{m-j} \\
&= \langle u, u \rangle \sum_{k=0}^{2l-1} \binom{m}{2l-k-1} Y\left(\lambda^{(k)}, z\right) z^{m+k+1-2l},
\end{aligned}
\tag{36}
$$

using (15). Suppose that $\lambda^{(2l-1)} \in V_{\langle \omega \rangle}$ so that $\lambda^{(k)} \in V_{\langle \omega \rangle}$ for $0 \leq k \leq 2l-1$ which implies the RHS of (36) is expressed in terms of Virasoro modes. Thus (36) defines a $\mathcal{W}(l)$ algebra VOA with one primary vector u of weight l, e.g., [BFKNRV, F]. The further condition $\lambda^{(2l+2)} \in V_{\langle \omega \rangle}$ constrains c to specific rational values.

We consider two infinite families of Exceptional $\mathcal{W}(l)$-VOAs. One is of AD-type, from the ADE series of [CIZ], given by the simple current extension of a minimal model $L\left(c_{p,q},0\right)$ by an irreducible module $L\left(c_{p,q},l\right)$ with

$$l = h_{1,p-1} = \frac{1}{4}(p-2)(q-2) \in \mathbb{N}, \tag{37}$$

for $h_{r,s}$, of (14), i.e., for any coprime pair p, q such that p or $q = 2 \mod 4$. Then (36) is consistent with respect to the Virasoro fusion rule (e.g., [DMS])

$$L\left(c_{p,q}, h_{1,p-1}\right) \times L\left(c_{p,q}, h_{1,p-1}\right) = L\left(c_{p,q}, 0\right).$$

Furthermore, since

$$2l + 2 = (p-1)(q-1) - \frac{1}{2}(pq-6) < (p-1)(q-1),$$

it follows that $\left(V_{\langle\omega\rangle}\right)_{2l+2}$ contains no Virasoro singular vectors. Hence

Proposition 7. *For a minimal model with $h_{1,p-1} \in \mathbb{N}$ there exists an Exceptional VOA with one primary vector of lowest weight $l = h_{1,p-1}$ of AD-type*

$$V = L\left(c_{p,q}, 0\right) \oplus L\left(c_{p,q}, h_{1,p-1}\right). \tag{38}$$

A second infinite family of $\mathcal{W}(l)$-VOAs for $l = 3k$ for $k \geq 1$ is given in [BFKNRV, F]. A more complete VOA description of this construction will appear elsewhere [T3]. $\mathcal{W}(3k)$ is of central charge $c_k = 1 - 24k$ and contains a unique Virasoro primary vector of weight $h_n = (n^2-1)k$ for each $n \geq 1$. The corresponding Virasoro Verma module contains a unique singular vector of weight $h_n + n^2$ so that the partition function is [F]:

$$\begin{aligned} Z_{\mathcal{W}(3k)}(q) &= \sum_{n\geq 1} \frac{q^{-c_k/24}}{\prod_{m\geq 0}(1-q^m)} \left(q^{h_n} - q^{h_n+n^2}\right) \\ &= \frac{1}{2\eta(q)} \sum_{n\in\mathbb{Z}} \left(q^{n^2 k} - q^{n^2(k+1)}\right). \end{aligned} \tag{39}$$

This VOA is generated by the lowest weight primary of weight $l = h_2 = 3k$ $\lambda^{(2l+2)} \in \left(V_{\langle\omega\rangle}\right)_{2l+2}$ and requires that $h_3 = 8k > 2l+2$ i.e., $k > 1$. Thus we find

Proposition 8. *For each $k \geq 2$ there exists an Exceptional VOA $\mathcal{W}(3k)$ with one primary vector of lowest weight $3k$ and central charge $c_k = 1 - 24k$.*

Remark 3. We conjecture that the two VOA series of Propositions 7 and 8 are the only Exceptional VOAs for which $p_l = 1$.

5 Genus Zero Constraints from Quadratic Casimirs

We next consider how an Exceptional VOA is also subject to local genus zero constraints following an approach originally described for $l = 1, 2$ in [T1, T2]. Let V be a simple VOA of strong CFT-type of central charge c with lowest primary weight $l \geq 1$. Let Π_l be the vector space of p_l primary vectors of weight l with basis $\{u_i\}$ and dual basis $\{\overline{u}_i\}$. Define the genus zero correlation function

$$F(a, b; x, y) = \left\langle a, \sum_{i=1}^{p_l} Y(u_i, x)Y(\overline{u}_i, y)b \right\rangle, \tag{40}$$

for $a, b \in \Pi_l$. $F(a, b; x, y)$ is linearly dependent on a and b and is constructed locally from Π_l alone.

Locality (7), associativity (9) and lower truncation (3) give

Proposition 9. *$F(a, b; x, y)$ is determined by a rational function*

$$F(a, b; x, y) = \frac{G(a, b; x, y)}{x^{2l}y^{2l}(x-y)^{2l}}, \tag{41}$$

for $G(a, b; x, y)$, a symmetric homogeneous polynomial in x, y of degree $4l$.

$F(a, b; x, y)$ can be considered as a rational function on the genus zero Riemann sphere and expanded in various domains to obtain the $2l+1$ independent parameters determining $G(a, b; x, y) = \sum_{r=0}^{4l} A_r x^{4l-r} y^r$ where $A_r = A_{4l-r}$. In particular, we expand in $\xi = -y/(x-y)$ using skew-symmetry (8), translation (6) and invariance of $\langle \,,\, \rangle$ to find that

$$\begin{aligned}
y^{2l}F(a, b; x, y) &= y^{2l}\sum_{i=1}^{p_l}\left\langle a, Y(u_i, x)e^{yL_{-1}}Y(b, -y)\overline{u}_i\right\rangle \\
&= y^{2l}\sum_{i=1}^{p_l}\left\langle a, e^{yL_{-1}}Y(u_i, x-y)Y(b, -y)\overline{u}_i\right\rangle \\
&= y^{2l}\sum_{i=1}^{p_l}\langle a, Y(u_i, x-y)Y(b, -y)\overline{u}_i\rangle \\
&= \sum_{m\geq 0} C_m \xi^m,
\end{aligned} \tag{42}$$

for $C_m = \sum_{i=1}^{p_l}\langle a, u_i(m-1)b(2l-m-1)\overline{u}_i\rangle$. Since l is the lowest primary weight, we have $b(2l-m-1)\overline{u}_i \in V_m = \left(V_{\langle\omega\rangle}\right)_m$ for $0 \leq m < l$ which determines the coefficients $C_0, \ldots, C_{l-1}$. This follows by writing $b(2l-m-1)\overline{u}_i$

in a Virasoro basis with coefficients computed in a similar way as for the Casimir vectors in Lemma 2. On the other hand, from (41) we find using $y = -\xi x/(1-\xi)$ that

$$y^{2l} F(a,b;x,y) = g\left(-\frac{\xi}{1-\xi}\right)(1-\xi)^{2l}$$
$$= A_0 - (2lA_0 + A_1)\,\xi + O(\xi^2),$$

for $g(y) = G(a,b;1,y) = \sum_{r=0}^{4l} A_r y^r$. Hence the coefficients $C_0, \ldots, C_{l-1}$ determine $A_0, \ldots, A_{l-1}$. For example, using $b(2l-1)\overline{u}_i = (-1)^l \langle b, \overline{u}_i\rangle\, \mathbf{1}$, we have

$$A_0 = C_0 = \sum_{i=1}^{p_l} \langle a, u_i(-1)b(2l-1)\overline{u}_i\rangle = (-1)^l \sum_{i=1}^{p_l} \langle a, u_i\rangle\langle b, \overline{u}_i\rangle = (-1)^l \langle a, b\rangle.$$

In general, $A_k = \langle a, b\rangle a_k(c)$ for $k = 0, \ldots, l-1$ where $a_k(c)$ is a rational function of c.

The other $l+1$ coefficients of $g(y)$ (recalling $A_r = A_{4l-r}$) are determined by using associativity (9) and expanding in $\zeta = (x-y)/y$ as follows:

$$(x-y)^{2l}\, F(a,b;x,y) = \sum_{m\in\mathbb{Z}} \sum_{i=1}^{p_l} \langle a, Y\left(u_i(m)\overline{u}_i, y\right) b\rangle (x-y)^{2l-m-1}$$
$$= \sum_{n\geq 0} B_n \zeta^n, \tag{43}$$

for $B_n = \langle a, o(\lambda^{(n)})b\rangle$ for $n \geq 0$ and recalling $o(\lambda^{(n)}) = \lambda^{(n)}(n-1)$.

Lemma 4. *The leading coefficients of* (43) *are* $B_0 = (-1)^l p_l \langle a, b\rangle$ *and* $B_1 = 0$. *For* $k \geq 1$, *the odd labelled coefficients* B_{2k+1} *satisfy*

$$B_{2k+1} = \frac{1}{2} \sum_{r=2}^{2k} \binom{-r}{2k+1-r} (-1)^r B_r, \tag{44}$$

i.e., B_{2k+1} *is determined by the lower even labelled coefficients* $B_2, \ldots, B_{2k}$. *The even labelled coefficients for* $k \geq 0$ *are given by*

$$B_{2k} = A_{2l}\delta_{k,0} + \sum_{m=1}^{2l} \left[\binom{m}{2k} + \binom{-m}{2k}\right] A_{2l-m}. \tag{45}$$

Proof. From Lemma 1 we have $\lambda^{(0)} = (-1)^l p_l\, \mathbf{1}$ and $\lambda^{(1)} = 0$ so that $B_0 = (-1)^l p_l \langle a, b\rangle$ and $B_1 = 0$. Comparing (43) to (41) we find that

$$\sum_{n\geq 0} B_n \zeta^n = g\left(\frac{1}{1+\zeta}\right)(1+\zeta)^{2l} = g\left(1+\zeta\right)(1+\zeta)^{-2l},$$

since $G(a, b; x, y)$ is symmetric and homogeneous. Thus

$$\sum_{n\geq 0} B_n \zeta^n = \sum_{n\geq 0} B_n \left(\frac{-\zeta}{1+\zeta}\right)^n .$$

This implies $B_n = \sum_{r=0}^{n} \binom{-r}{n-r}(-1)^r B_r$. Taking $n = 2k+1$ leads to (44). (45) follows from the identity

$$\sum_{n\geq 0} B_n \zeta^n = A_{2l} + \sum_{m=1}^{2l} A_{2l-m}\left[(1+\zeta)^m + (1+\zeta)^{-m}\right].$$

□

We next assume that $\lambda^{(n)} \in V_{\langle\omega\rangle}$ for even $n \leq 2l$ giving $B_{2k} = \langle a, o(\lambda^{(2k)})b\rangle = p_l\langle a, b\rangle b_{2k}(c)$ for $k = 1, \ldots, l$ for some rational functions $b_{2k}(c)$ via Lemma 3. Note that we are not (yet) assuming $\lambda^{(2l+2)} \in V_{\langle\omega\rangle}$. $G(a, b; x, y)$ is uniquely determined provided we can invert (45) to solve for $A_l, \ldots, A_{2l}$. Define the $l \times l$ matrix

$$M_{mk} = \binom{m}{2k} + \binom{-m}{2k}, \tag{46}$$

of coefficients for A_{2l-m} of B_{2k} in (45), where $m, k = 1, \ldots, l$.

Lemma 5. *M is invertible with* $\det M = 1$.

Proof. Define unit diagonal lower and upper triangular matrices L and U by

$$L_{ij} = \begin{cases} \binom{2i-j-1}{j-1} & \text{for } i \leq j, \\ 0 & \text{for } i > j, \end{cases} \qquad U_{jk} = \begin{cases} \frac{k}{j}\binom{j+k-1}{2j-1} & \text{for } j \leq k, \\ 0 & \text{for } j > k. \end{cases}$$

By induction in k, one can show that $M_{ik} = (LU)_{ik}$ and so $\det M = 1$. □

Thus it follows that $A_{2l-m} = \sum_{k=1}^{l} B_{2k}(M^{-1})_{km}$ for $m = 1, \ldots, l$. Altogether, we have therefore shown the following.

Proposition 10. *Let V be a simple VOA of strong CFT–type of central charge c with lowest primary weight $l \geq 1$. Suppose that $\lambda^{(n)} \in V_{\langle\omega\rangle}$ for all even $n \leq 2l$. Then the genus zero correlation function is uniquely determined with*

$$F(a, b; x, y) = \frac{1}{x^{2l}y^{2l}(x-y)^{2l}} \sum_{r=0}^{2l} A_r \left(x^{4l-r}y^r + x^r y^{4l-r}\right),$$

where

$$A_k = \begin{cases} \langle a,b\rangle a_k(c), & k=0,\ldots,l-1, \\ p_l\langle a,b\rangle a_k(c), & k=l,\ldots,2l, \end{cases}$$

for $2l+1$ *specific rational functions* $a_0(c),\ldots,a_{2l}(c)$.

Next we assume $\lambda^{(2l+2)} \in V_{\langle\omega\rangle}$ so that V is an Exceptional VOA. This implies $B_{2l+2} = \langle a, o(\lambda^{(2l+2)})b\rangle = p_l\langle a,b\rangle b_{2l+2}(c)$ for some rational function $b_{2l+2}(c)$. But B_{2l+2} is already determined from (45) in terms of $A_1,\ldots,A_{2l}$ from Proposition 10. Hence we have

Proposition 11. *Let V be an Exceptional VOA with lowest primary weight l. Then the genus zero correlation function $F(a,b;x,y)$ is uniquely determined and $p_l = p_l(c)$, a specific rational function of c.*

For $l = 1,2$ we may use $F(a,b;x,y)$ to understand many properties of the corresponding VOA (as briefly reviewed below) [T1, T2]. We already know from Proposition 6(ii) that $p_l = \dim V_l - \dim \left(V_{\langle\omega\rangle}\right)_l$ is a rational function of c. In principle, the specific rational expressions for p_l may differ but, in practice, the same expression is observed to arise for all $l \le 9$. A more significant point is that the argument leading to Proposition 11 may be adopted to understanding some automorphism group properties of V.

5.1 Exceptional VOAs of Class $\mathcal{S}^{2l+2}$

Let $G = \mathrm{Aut}(V)$ denote the automorphism group of a VOA V and let V^G denote the sub-VOA fixed by G. Since the Virasoro vector is G invariant it follows that $V_{\langle\omega\rangle} \subseteq V^G$. V is said to be of Class $\mathcal{S}^n$ if $V^G_k = \left(V_{\langle\omega\rangle}\right)_k$ for all $k \le n$ [Mat]. (The related notion of conformal t-designs is described in [Ho2].) In particular, the quadratic Casimir (15) is G-invariant so it follows that a VOA V with lowest primary weight l of class $\mathcal{S}^{2l+2}$ is an Exceptional VOA. It is not known if every Exceptional VOA is of class $\mathcal{S}^{2l+2}$.

The primary vector space Π_l is a finite-dimensional G-module. Assuming Π_l is a completely reducible G-module (e.g., for G linearly reductive [Sp]) we have

Proposition 12. *Let V be an Exceptional VOA of class $\mathcal{S}^{2l+2}$ with primaries Π_l of lowest weight l. If Π_l is a completely reducible G-module, then it is either an irreducible G-module or the direct sum of two isomorphic irreducible G-modules.*

Remark 4. For odd p_l it follows that Π_l must be an irreducible G-module.

Proof. Let ρ be a G-irreducible component of Π_l and let $\overline{\rho}$ denote the $\langle\ ,\ \rangle$ dual vector space. $\overline{\rho}$ and ρ are isomorphic as G-modules. Define

$$R = \begin{cases} \rho & \text{if } \rho = \overline{\rho}, \\ \rho\oplus\overline{\rho} & \text{if } \rho \ne \overline{\rho}. \end{cases}$$

Clearly $R \subseteq \Pi_l$ is a self-dual vector space. We next repeat the Casimir construction and analysis that lead up to Proposition 11. Choose an R-basis $\{v_i : i = 1, \ldots, \dim R\}$ and dual basis $\{\overline{v}_i\}$ and define Casimir vectors

$$\lambda_R^{(n)} = \sum_{i=1}^{\dim R} v_i(2l - n - 1)\overline{v}_i \in V_n, \quad n \geq 0. \tag{47}$$

But $\lambda_R^{(n)}$ is G-invariant and since V is of class $\mathcal{S}^{2l+2}$, it follows that $\lambda_R^{(n)} \in V_{\langle\omega\rangle}$ for all $n \leq 2l + 2$. We define a genus zero correlation function constructed from the vector space R

$$F_R(a, b; x, y) = \sum_{i=1}^{\dim R} \langle a, Y(v_i, x)Y(\overline{v}_i, y)b\rangle, \tag{48}$$

for all $a, b \in R$. We then repeat the earlier arguments to conclude that Proposition 11 also holds for $F_R(a, b; x, y)$ where, in particular, $\dim R = p_l(c)$, for the **same** rational function. Thus $\dim R = p_l$ and the result follows. □

6 Exceptional VOAs of Lowest Primary Weight $l \leq 9$

We now consider Exceptional VOAs of lowest primary weight $l \leq 9$. We denote by $E_n = E_n(q)$ the Eisenstein series of weight n appearing in the MLDE (32). For $l \leq 4$ we describe all the rational values for c, h, whereas for $5 \leq l \leq 9$ we give all rational values for c, h for which $p_l = \dim \Pi_l \leq 500{,}000$, found by computer algebra techniques. We also consider conjectured extremal self-dual VOAs with $c = 24(l - 1)$ [Ho1, Wi]. Any MLDE solution for rational h for which there is no graded character $Z_N(q)$ is marked with an asterisk. We obtain many examples of known Exceptional VOAs such as Deligne's exceptional series of Lie algebras, the Moonshine and Baby Monster modules. There are also a number of candidate solutions for which no construction yet exists indicated by question marks.

[$l = 1$]. This is discussed in much greater detail in [T1, T2]. Propositions 4–6 imply that $Z_N(q)$ satisfies the following 2nd order MLDE [T2]:

$$D^2 Z - \frac{5}{4} c\,(c + 4)\,E_4 Z = 0.$$

This MLDE has also appeared in [MatMS, KZ, Mas2, KKS, Kaw]. The indicial roots $x_1 = -c/24, x_2 = (c + 4)/24$ are exchanged under the MLDE symmetry $c \leftrightarrow -c - 24$. Solving iteratively for the partition function

$$Z_V(q) = q^{-c/24}\left(1 + p_1 q + (1 + p_1 + p_2)q^2 + (1 + 2p_1 + p_2 + p_3)q^3 + \ldots\right),$$

where $p_n = \dim \varPi_n$, for weight n primary vector space $\varPi_n$, we have

$$p_1 = \frac{c(5c+22)}{10-c}, \quad p_2 = \frac{5(5c+22)(c-1)(c+2)^2}{2(c-10)(c-22)},$$
$$p_3 = -\frac{5c(5c+22)(c-1)(c+5)(5c^2+268)}{6(c-10)(c-22)(c-34)}, \ldots .$$

For $c = 10 \mod 12$, the indicial roots differ by an integer leading to denominator zeros for all p_n.

By Proposition 6, V is generated by V_1 which defines a Lie algebra $\mathfrak{g}$. $F(a, b; x, y)$ from Proposition 11 determines the Killing form which can be used to show that $\mathfrak{g}$ is simple with dual Coxeter number [T1, MT]

$$h^\vee = 6k\frac{2+c}{10-c},$$

for some real level k. Thus $V = V_{\mathfrak{g}}(k)$, a level k Kac–Moody VOA.

The indicial root x_2 of the MLDE gives the lowest weight $h = (c+2)/12$ of any independent irreducible V-module(s) N. Therefore $V_{\mathfrak{g}}(k)$ has at most two independent irreducible characters so that the level k must be a positive integer [Kac2]. Comparing p_1 and $h^\vee$ to Cartan's list of simple Lie algebras shows that in fact $k = 1$ with $c = 1, 2, \frac{14}{5}, 4, \frac{26}{5}, 6, 7, 8$ with $\mathfrak{g} = A_1, A_2, G_2, D_4, F_4, E_6, E_7, E_8$, respectively, known as Deligne Exceptional Series [D, DdeM, MarMS, T2]. In summary, we have

$c > 0$	p_1	p_2	p_3	VOA	$h \in \mathbb{Q}$
1	3	0	0	$V_{A_1}(1)$	$0, \frac{1}{4}$
2	8	8	21	$V_{A_2}(1)$	$0, \frac{1}{3}$
$\frac{14}{5}$	14	27	84	$V_{G_2}(1)$	$0, \frac{2}{5}$
4	28	105	406	$V_{D_4}(1)$	$0, \frac{1}{2}$
$\frac{26}{5}$	52	324	1,547	$V_{F_4}(1)$	$0, \frac{3}{5}$
6	78	650	3,575	$V_{E_6}(1)$	$0, \frac{2}{3}$
7	133	1,539	10,108	$V_{E_7}(1)$	$0, \frac{3}{4}$
8	248	3,875	30,380	$V_{E_8}(1)$	$0, \frac{5}{6}*$

The table also shows h for a possible irreducible V-module(s). For $c = 2$ and 4 there are 2 independent irreducible modules but which share the same character (due to $\mathfrak{g}$

outer automorphisms). $V_{E_8}(1)$ is self-dual so that the MLDE solution with $h = \frac{5}{6}$ does not correspond to a graded character $Z_N(q)$.

[$l = 2$]. This case is also discussed in detail in [Mat, T1, T2]. Propositions 4–6 imply that $Z_N(q)$ satisfies the following 3rd order MLDE [T2]

$$D^3 Z - \frac{5}{124}\left(704 + 240c + 21c^2\right) E_4\, DZ - \frac{35}{248} c\left(144 + 66c + 5c^2\right) E_6\, Z = 0,$$

with indicial equation (33)

$$(x - x_1)\left(x^2 - \left(\frac{1}{2} + x_1\right)x + \frac{20x_1^2 - 11x_1 + 1}{62}\right) = 0,$$

for $x_1 = -c/24$. Solving iteratively for the partition function ($x = x_1$)

$$Z_V(q) = q^{-c/24}(1 + (1 + p_2)q^2 + (1 + p_2 + p_3)q^3 + \ldots),$$

where $p_n = \dim \Pi_n$, for weight n primary vector space Π_n, we find that

$$p_2 = \frac{(7c + 68)(2c - 1)(5c + 22)}{2(c^2 - 55c + 748)}, \quad p_3 = \frac{31c(7c + 68)(2c - 1)(5c + 44)(5c + 22)}{6(c^2 - 55c + 748)(c^2 - 86c + 1{,}864)}.$$

From Proposition 6, the Griess algebra generates V and from Proposition 11 the Griess algebra is simple [T1]. The solutions for $c, h \in \mathbb{Q}$ with positive p_3 and possible Exceptional VOAs are listed as follows:

c	p_2	p_3	VOA	$h \in \mathbb{Q}$
$-\frac{44}{5}$	1	0	$L(c_{3,10}, 0) \oplus L(c_{3,10}, 2)$	$0, -\frac{1}{5}, -\frac{2}{5}$
8	155	868	$V^+_{\sqrt{2}E_8}$	$0, \frac{1}{2}, 1$
16	2,295	63,240	$V^+_{\mathrm{BW}_{16}}$	$0, 1, \frac{3}{2}$
$\frac{47}{2}$	96,255	9,550,635	$\mathrm{VB}^\natural_{\mathbb{Z}}$	$0, \frac{3}{2}, \frac{31}{16}$
24	196,883	21,296,876	$V^\natural$	0
32	$3.7^2.13.73$	$2^4.3.7^2.13.31.73$	?? $V_L^+ \oplus (V_L)_T^+$; L extremal S-D	0
$\frac{164}{5}$	$3^2.17.19.31$	2.5.13.17.19.31.41	??	$0, \frac{11}{5}, \frac{12}{5}$
$\frac{236}{7}$	5.19.23.29	2.19.23.29.31.59	??	$0, \frac{16}{7}, \frac{17}{7}$
40	$3^2.29.79$	$2^2.5.29.31.61.79$	?? $V_L^+ \oplus (V_L)_T^+$; L extremal S-D	0

The list includes the famous Moonshine Module $V^\natural$ [FLM], the Baby Monster VOA $\mathrm{VB}^\natural_{\mathbb{Z}}$ [Ho1], V_L^+ for $L = \sqrt{2}E_8$ [G] and the rank 16 Barnes–Wall lattice $L = \mathrm{BW}_{16}$ [Sh], and a minimal model simple current extension AD-type as in Proposition 7.

The value(s) of $h = x_i + c/24$ for the lowest weight(s) agree with those for the irreducible V-modules as do the corresponding MLDE solutions for the characters in each case. There are also four other possible candidates. For $c = 32$ and 40 one can construct a self-dual VOA from an extremal even self-dual lattice L (with no vectors of squared length 2). However, such lattices are not unique and it is not known which, if any, gives rise to a VOA satisfying the exceptional conditions. There are no known candidate constructions for $c = \frac{164}{5}$ and $\frac{236}{7}$.

Note that $p_2 = \dim \Pi_2$ is odd in every case and Proposition 12 implies that if Π_2 is completely $\mathrm{Aut}\, V$-reducible, then it is irreducible. This is indeed the case in the first five known cases for $c \leq 24$ [Atlas]. Π_3 is also an $\mathrm{Aut}\, V$-module whose dimension p_3 is given. The MLDE solutions (with positive coprime integer coefficients) for $c = 164/5$ with $h = 11/5, 12/5$ and for $c = 236/7$ with $h = 16/7, 17/7$ have respective leading q-expansions:

$$Z_{11/5}(q) = q^{5/6}\left(2^3.31.41 \,+\, 5.11.31.41.53\, q + O(q^2)\right),$$

$$Z_{12/5}(q) = q^{31/30}\left(2^2.11^2.31.41 \,+\, 2^5.11^2.31^2.41\, q + O(q^2)\right),$$

$$Z_{16/7}(q) = q^{37/42}\left(17.23.31 \,+\, 2^5.7.17.31.37\, q + O(q^2)\right),$$

$$Z_{17/7}(q) = q^{43/42}\left(2^4.29.31.59 \,+\, 2.3.17.29.31.43.59\, q + O(q^2)\right).$$

These coefficients constrain the possible structure of $\mathrm{Aut}\, V$ further.

[$l = 3$]. $Z_N(q)$ satisfies the 4th order MLDE:

$$\begin{aligned}
&(578c-7)\ D^4 Z - \frac{5}{2}\left(168c^3 + 2{,}979c^2 + 15{,}884c - 4{,}936\right) E_4\, D^2 Z \\
&-\frac{35}{2}\left(25c^4 + 661c^3 + 4{,}368c^2 + 10{,}852c + 1{,}144\right) E_6\, DZ \\
&-\frac{75}{16}c\left(14c^4 + 425c^3 + 3{,}672c^2 + 5{,}568c + 9{,}216\right) E_4^2\, Z = 0.
\end{aligned}$$

Solving iteratively for the partition function we find [T2]:

$$\begin{aligned}
Z_V &= q^{-c/24}(1 + q^2 + (1 + p_3)q^3 + (2 + p_3 + p_4)q^3 + \ldots), \\
p_3 &= -\frac{(5c+22)(3c+46)(2c-1)(5c+3)(7c+68)}{5c^4 - 703c^3 + 32{,}992c^2 - 517{,}172c + 3{,}984},
\end{aligned}$$

and $p_4 = \frac{r(c)}{s(c)}$ for

$$r(c) = -\frac{1}{2}\,(2c-1)\,(3c+46)\,(5c-4)\,(7c+68)\,(5c+3)\,(7c+114)$$

$$
\begin{aligned}
&. \left(55c^3 - 5{,}148c^2 - 11{,}980c - 36{,}528\right), \\
s(c) = &\left(5c^4 - 703c^3 + 32{,}992c^2 - 517{,}172c + 3{,}984\right) \\
&. \left(5c^4 - 964c^3 + 62{,}392c^2 - 1{,}355{,}672c + 13{,}344\right).
\end{aligned}
$$

The $c, h \in \mathbb{Q}$ solutions for positive integer p_3 with possible VOAs are

c	p_3	p_4	VOA	$h \in \mathbb{Q}$
$-\frac{114}{7}$	1	0	$L\left(c_{3,14}, 0\right) \oplus L\left(c_{3,14}, 3\right)$	$0, -\frac{3}{7}, -\frac{4}{7}, -\frac{5}{7}$
$\frac{4}{5}$	1	0	$L\left(c_{5,6}, 0\right) \oplus L\left(c_{5,6}, 3\right)$	$0, \frac{1}{15}, \frac{2}{5}, \frac{2}{3}$
48	$3^2.19^2.101.131$	$5^6.19^2.71.101$	?? Höhn Extremal VOA	0

The Höhn Extremal VOA is a conjectural self-dual VOA [Ho1]. If Π_3 is a completely reducible Aut(V)-module, then it must be irreducible excluding Witten's suggestion that Aut$(V) = \mathbb{M}$, the Monster group [Wi].

[$l = 4$]. Proposition 4 implies $Z_N(q)$ satisfies the 5rd order MLDE:

$$
\begin{aligned}
&(317c + 3)\ D^5 Z - \frac{5}{7}\left(297c^3 + 6{,}746c^2 + 53{,}133c + 4{,}644\right) E_4\, D^3 Z \\
&-\frac{25}{8}\left(77c^4 + 3{,}057c^3 + 31{,}506c^2 + 129{,}736c - 24{,}096\right) E_6\, D^2 Z \\
&-\frac{25}{112}\left(231c^5 + 12{,}117c^4 + 194{,}916c^3 + 843{,}728c^2 + 1{,}652{,}288c - 718{,}080\right) {E_4}^2\, DZ \\
&-\frac{25}{32} c\,(c + 24)\left(15c^4 + 527c^3 + 5{,}786c^2 + 528c + 25{,}344\right) E_4 E_6\, Z = 0.
\end{aligned}
$$

Solving iteratively for the partition function we find

$$
\begin{aligned}
Z_V &= q^{-c/24}(1 + q^2 + q^3 + (2 + p_4)q^4 + (3 + p_4 + p_5)q^5 + \ldots), \\
p_4 &= \frac{5(3c + 46)(2c - 1)(11c + 232)(7c + 68)(5c + 3)(c + 10)}{2(5c^4 - 1{,}006c^3 + 67{,}966c^2 - 1{,}542{,}764c - 12{,}576)(c - 67)},
\end{aligned}
$$

and $p_5 = \frac{r(c)}{s(c)}$ where

$$
\begin{aligned}
r(c) &= 3(c - 1)(5c + 22)(3c + 46)(2c - 1)(11c + 232)(7c + 68)(5c + 3)(c + 24) \\
&\quad .(59c^3 - 13{,}554c^2 + 788{,}182c - 398{,}640), \\
s(c) &= 2(c - 67)(5c^4 - 1{,}006c^3 + 67{,}966c^2 - 1{,}542{,}764c - 12{,}576) \\
&\quad .(5c^5 - 1{,}713c^4 + 221{,}398c^3 - 12{,}792{,}006c^2 + 278{,}704{,}260c + 2{,}426{,}976).
\end{aligned}
$$

The $c, h \in \mathbb{Q}$ solutions for $p_4 \leq 500{,}000$ and $c = 48$ with possible VOAs are

c	p_4	p_5	VOA	$h \in \mathbb{Q}$
1	2	0	V_L^+ for $L = 2\sqrt{2}\mathbb{Z}$	$0, \frac{1}{16}, \frac{1}{4}, \frac{9}{16}, 1$
72	$2^3.11^4.13^2.131$	$2.11^4.13^2.103.131.191$	?? Höhn Extremal VOA	0

The Höhn Extremal VOA is a conjectured self-dual VOA [Ho1, Wi]. If Π_4 is a completely reducible $\mathrm{Aut}(V)$-module, then by Proposition 12, either p_4 or $\frac{1}{2}p_4$ is the dimension of an irreducible $\mathrm{Aut}(V)$-module.

[$l = 5$]. Z_V satisfies a 6th order MLDE with $p_5 = \frac{r(c)}{s(c)}$ for

$$\begin{aligned} r(c) &= -(13c+350)(7c+25)(5c+126)(11c+232) \\ &\quad .(2c-1)(3c+46)(68+7c)(5c+3)(10c-7), \\ s(c) &= 1{,}750c^8 - 760{,}575c^7 + 132{,}180{,}881c^6 - 11{,}429{,}170{,}478c^5 \\ &\quad +484{,}484{,}459{,}322c^4 - 7{,}407{,}871{,}790{,}404c^3 - 37{,}323{,}519{,}053{,}016c^2 \\ &\quad +25{,}483{,}483{,}057{,}200c - 363{,}772{,}080{,}000. \end{aligned}$$

The $c, h \in \mathbb{Q}$ solutions for $p_5 \leq 500{,}000$ with possible VOAs are

c	p_5	VOA	$h \in \mathbb{Q}$
$-\frac{350}{11}$	1	$L\left(c_{3,22}, 0\right) \oplus L\left(c_{3,22}, 5\right)$	$0, -\frac{8}{11}, -\frac{10}{11}, -\frac{13}{11}, -\frac{14}{11}, -\frac{15}{11}$
$\frac{6}{7}$	1	$L\left(c_{6,7}, 0\right) \oplus L\left(c_{6,7}, 5\right)$	$0, \frac{1}{21}, \frac{1}{7}, \frac{10}{21}, \frac{5}{7}, \frac{4}{3}$

Witten's conjectured Extremal VOA for $c = 4.24 = 96$ does not appear [Wi].

[$l = 6$]. Z_V satisfies a 7th order MLDE with $p_6 = \frac{r(c)}{s(c)}$ for

$$\begin{aligned} r(c) &= \frac{7}{2}(13c+350)(5c+164)(7c+25)(11c+232)(3c+46) \\ &\quad .(4c+21)(5c+3)(10c-7)(5c^2+316c+3{,}600), \\ s(c) &= 1{,}750c^9 - 1{,}119{,}950c^8 + 297{,}661{,}895c^7 - 41{,}808{,}629{,}963c^6 \\ &\quad +3{,}225{,}664{,}221{,}176c^5 - 123{,}384{,}054{,}679{,}580c^4 + 1{,}266{,}443{,}996{,}541{,}232c^3 \\ &\quad +29{,}763{,}510{,}364{,}647{,}840c^2 + 96{,}385{,}155{,}929{,}078{,}400c + 7{,}743{,}915{,}615{,}744{,}000. \end{aligned}$$

The $c, h \in \mathbb{Q}$ solutions for $p_6 \leq 500{,}000$ with possible VOAs are

c	p_6	VOA	$h \in \mathbb{Q}$
$-\frac{516}{13}$	1	$L\left(c_{3,26}, 0\right) \oplus L\left(c_{3,26}, 6\right)$	$0, -\frac{10}{13}, -\frac{15}{13}, -\frac{17}{13}$ $-\frac{20}{13}, -\frac{21}{13}, -\frac{22}{13}$
-47	1	$\mathcal{W}(6)$	$0, -\frac{5}{4}, -\frac{3}{2}, -\frac{5}{3}$ $-\frac{15}{8}, -\frac{23}{12}, -2$
120	$2.7^2.11.29.43.67.97.191$	?? Witten Extremal VOA	0

$c = -47$ is first example of a $\mathcal{W}(3k)$-algebra of Proposition 8. The irreducible lowest weight h values and character solutions agree with [F]. Witten's conjecture Extremal VOA for $c = 5.24 = 120$ appears [Wi] where either p_6 or $\frac{1}{2}p_6$ is the dimension of an irreducible $\mathrm{Aut}(V)$–module.

[$l = 7$]. Z_V satisfies an 8th order MLDE where $p_7 = \frac{r(c)}{s(c)}$ for

$$
\begin{aligned}
r(c) = &-5\,(13c+350)\,(5c+164)\,(7c+25)\,(11c+232)\,(3c+46)\,(17c+658)\\
&.\,(4c+21)\,(5c+3)\,(10c-7)\left(35c^3+3{,}750c^2+76{,}744c-32{,}640\right),\\
s(c) = &\;61{,}250c^{11}-54{,}725{,}125c^{10}+20{,}922{,}774{,}275c^{9}-4{,}421{,}902{,}106{,}730c^{8}\\
&+553{,}932{,}117{,}001{,}488c^{7}-40{,}395{,}124{,}111{,}104{,}312c^{6}+1{,}491{,}080{,}056{,}338{,}817{,}984c^{5}\\
&-12{,}528{,}046{,}696{,}953{,}576{,}896c^{4}-483{,}238{,}055{,}074{,}755{,}678{,}656c^{3}\\
&-1{,}702{,}959{,}754{,}355{,}175{,}160{,}320c^{2}+249{,}488{,}376{,}255{,}167{,}616{,}000c\\
&+362{,}620{,}505{,}915{,}136{,}000{,}000.
\end{aligned}
$$

There are no rational c solutions for $p_7 \leq 500{,}000$.

[$l = 8$]. $Z_V(q)$ satisfies a 9th order MLDE with one $c, h \in \mathbb{Q}$ solution for $p_8 \leq 500{,}000$

c	p_8	VOA	$h \in \mathbb{Q}$
$-\frac{944}{17}$	1	$L\left(c_{3,34}, 0\right) \oplus L\left(c_{3,34}, 8\right)$	$0, -\frac{14}{17}, -\frac{25}{17}, -\frac{28}{17}, -\frac{33}{17}$ $-\frac{35}{17}, -\frac{38}{17}, -\frac{39}{17}, -\frac{40}{17}$

[$l = 9$]. $Z_V(q)$ satisfies a 10th order MLDE with $c, h \in \mathbb{Q}$ solutions for $p_9 \leq 500{,}000$

c	p_9	VOA	$h \in \mathbb{Q}$
$-\frac{1{,}206}{19}$	1	$L\left(c_{3,38},0\right) \oplus L\left(c_{3,38},9\right)$	$0, -\frac{16}{19}, -\frac{29}{19}, -\frac{36}{19}, -\frac{44}{19}$ $-\frac{46}{19}, -\frac{49}{19}, -\frac{50}{19}, -\frac{51}{19}$
$-\frac{208}{35}$	1	$L\left(c_{5,14},0\right) \oplus L\left(c_{5,14},9\right)$	$0, -\frac{2}{7}, -\frac{9}{35}, -\frac{4}{35}$ $\frac{1}{7}, \frac{11}{35}, \frac{9}{7}, \frac{8}{5}$
$-\frac{14}{11}$	1	$L\left(c_{6,11},0\right) \oplus L\left(c_{6,11},9\right)$	$0, -\frac{1}{11}, -\frac{2}{33}, \frac{1}{11}, \frac{7}{33}$ $\frac{6}{11}, \frac{25}{33}, \frac{14}{11}, \frac{52}{33}, \frac{8}{3}$
-71	1	$\mathcal{W}(9)$	$0, -2, -\frac{9}{4}, -\frac{39}{16}, -\frac{8}{3}$ $-\frac{11}{4}, -\frac{35}{12}, -\frac{47}{16}, -3, -\frac{9}{8}*$

For $c = -71$, the MLDE solutions agree with all the graded characters for $\mathcal{W}(9)$ except for $h = -\frac{9}{8}$ [F].

7 Exceptional VOSAs

7.1 VOSA Quadratic Casimirs and Zhu Theory

We now give an analysis for Vertex Operator Superalgebras (VOSAs). Many of the results are similar but there are also significant differences, e.g., here the MLDEs involve twisted Eisenstein series. Let V be a simple VOSA of strong CFT-type with unique invertible bilinear form $\langle\ ,\ \rangle$. Let Π_l denote the space of Virasoro primary vectors of *lowest half integer weight* $l \in \mathbb{N}+\frac{1}{2}$, i.e., Π_l is of odd parity and $V_k = (V_{\langle\omega\rangle})_k$ for all $k \leq l-\frac{1}{2}$. We construct quadratic Casimir vectors $\lambda^{(n)}$ as in Sect. 3.1 (from the odd parity space Π_l) which enjoy the same properties as VOA Casimir vectors.

Define the genus one partition function of a VOSA V by

$$Z_V(q) = \mathrm{Tr}_V\left(\sigma\, q^{L(0)-c/24}\right) = q^{-c/24}\sum_{n\geq 0}(-1)^{2n}\dim V_n\, q^n, \tag{49}$$

for fermion number operator σ where $\sigma u = (-1)^{p(u)}u$ for u of parity $p(u)$ and with a corresponding definition for a simple ordinary V-module N. We also define the genus one 1-point correlation function

$$Z_N(u,q) = \mathrm{Tr}_N\left(\sigma\, o(u) q^{L(0)-c/24}\right). \tag{50}$$

In [MTZ] a Zhu reduction formula for the 2-point correlation function $Z_N(Y[u,z]v,q)$ for $u,v\in V$ is found expressed in terms of *twisted elliptic Weierstrass functions* parameterized by $\theta,\phi\in\{\pm 1\}$. Let $\phi=e^{2\pi i\kappa}$ for $\kappa\in\{0,\frac{1}{2}\}$. Then (23) and (24) are generalized to [MTZ]

$$P_m\begin{bmatrix}\theta\\ \phi\end{bmatrix}(z)=\frac{1}{z^m}+(-1)^m\sum_{n\geq m}\binom{n-1}{m-1}E_n\begin{bmatrix}\theta\\ \phi\end{bmatrix}(q)z^{n-m}, \tag{51}$$

for twisted Eisenstein series $E_n\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right](q)=0$ for n odd, and for n even

$$E_n\begin{bmatrix}\theta\\ \phi\end{bmatrix}(q)=-\frac{B_n(\kappa)}{n!}+\frac{2}{(n-1)!}\sum_{k\in\mathbb{N}+\kappa}\frac{k^{n-1}\theta q^k}{1-\theta q^k}, \tag{52}$$

and where $B_n(\kappa)$ is the Bernoulli polynomial defined by

$$\frac{e^{z\kappa}}{e^z-1}=\frac{1}{z}+\sum_{n\geq 1}\frac{B_n(\kappa)}{n!}z^{n-1}. \tag{53}$$

(51) and (52) agree with (23) and (24) respectively for $(\theta,\phi)=(1,1)$. $P_m\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right](z)$ converges absolutely and uniformly on compact subsets of the domain $|q|<|e^z|<1$ and $E_n\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right](q)$ is a holomorphic function of $q^{\frac{1}{2}}$ for $|q|<1$. For $(\theta,\phi)\neq(1,1)$, $E_n\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right]$ is modular of weight n in the sense that

$$E_n\begin{bmatrix}\theta^\alpha\phi^\beta\\ \theta^\gamma\phi^\delta\end{bmatrix}\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)=(\gamma\tau+\delta)^n E_n\begin{bmatrix}\theta\\ \phi\end{bmatrix}(\tau), \tag{54}$$

for $\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)\in SL(2,\mathbb{Z})$. The Zhu reduction formula of Proposition 1 has been generalized in [MTZ] as follows

Proposition 13. *Let N be a simple ordinary V-module for a VOSA V. For u of parity $p(u)$ and for all v we have*

$$\begin{aligned}Z_N(Y[u,z]v,q)&=\mathrm{Tr}_N\left(\sigma\, o(u)o(v)q^{L(0)-c/24}\right)\delta_{p(u)1}\\&\quad+\sum_{m\geq 0}P_{m+1}\begin{bmatrix}1\\ p(u)\end{bmatrix}(z)Z_N(u[m]v,q).\end{aligned}$$

For even parity u this agrees with Proposition 1. In particular, Corollary (1) concerning Virasoro vacuum descendents holds. Much as before, Proposition 13 implies that the Casimir vectors $\lambda^{[n]}\in V_{[n]}$ satisfy

$$\sum_{n\geq 0} Z_N\left(\lambda^{[n]},q\right)z^{n-2l}=\sum_{m=0}^{2l-1}P_{m+1}\begin{bmatrix}1\\-1\end{bmatrix}(z)Z_N\left(\lambda^{[2l-m-1]},q\right). \tag{55}$$

Equating the z coefficients implies the following variant of Proposition 2.

Proposition 14. $Z_N\left(\lambda^{[2l+1]},q\right)$ *satisfies the recursive identity*

$$Z_N\left(\lambda^{[2l+1]},q\right)=-2\sum_{r=0}^{l-\frac{1}{2}}(l-r)E_{2(l-r)+1}\begin{bmatrix}1\\-1\end{bmatrix}(q)Z_N\left(\lambda^{[2r]},q\right). \tag{56}$$

7.2 Exceptional VOSAs

Let V be a simple VOSA of strong CFT-type with primary vectors of lowest weight $l\in\mathbb{N}+\frac{1}{2}$ for which $\lambda^{(2l+1)}\in V_{\langle\omega\rangle}$. We further assume that $\left(V_{\langle\omega\rangle}\right)_{2l+1}$ contains no Virasoro singular vectors. We call V an *Exceptional VOSA of Odd Parity Lowest Primary Weight* l. Proposition 3 implies

Proposition 15. *Let V be an Exceptional VOSA of lowest weight $l\in\mathbb{N}+\frac{1}{2}$ and central charge c. Then $Z_N(q)$ for a simple ordinary V-module N satisfies a Twisted Modular Linear Differential Equation (TMLDE)*

$$\sum_{m=0}^{l+\frac{1}{2}} g_{l+\frac{1}{2}-m}\begin{bmatrix}1\\-1\end{bmatrix}(q,c)\,D^m Z(q)=0, \tag{57}$$

where $g_k\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right](q,c)$ is a twisted modular form of weight $2k$ whose coefficients over the ring of twisted Eisenstein series (52) are rational functions of c.

The TMLDE (57) is of order $l+\frac{1}{2}$ with a regular singular point at $q=0$ provided $g_0\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right](q,c)=g_0(c)\neq 0$ so that Frobenius–Fuchs theory implies that its solutions are holomorphic in $q^{\frac{1}{2}}$ for $0<|q|<1$. Furthermore, from (54), $\widehat{Z}_N=Z_N\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$ is a solution of the TMLDE

$$\sum_{m=0}^{l+\frac{1}{2}} g_{l+\frac{1}{2}-m}\begin{bmatrix}(-1)^{\beta}\\(-1)^{\gamma}\end{bmatrix}(q,c)\,D^m \widehat{Z}(q)=0, \tag{58}$$

which is again of regular singular type provided $g_0(c)\neq 0$. We can repeat the results of Sect. 4 concerning TMLDE $q^{\frac{1}{2}}$-series solutions and the rationality of c and h noting that $Z_V(-1/\tau,c)$ (cf. (35)) satisfies (58) for $\left(\begin{smallmatrix}\alpha&\beta\\\gamma&\delta\end{smallmatrix}\right)=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. We therefore find the VOSA analogues of Propositions 5 and 6.

Proposition 16. *Let V be an Exceptional VOSA of lowest primary weight $l \in \mathbb{N}+\frac{1}{2}$ and central charge c and let N be a simple ordinary V-module of lowest weight h. Assuming $g_0(c) \neq 0$ in the TMLDE (57) then*

(i) *$Z_N(q)$ is holomorphic in $q^{\frac{1}{2}}$ for $0 < |q| < 1$.*
(ii) *$Z_N\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$ is a solution of the TMLDE (58) for all $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$.*
(iii) *The central charge c and the lowest conformal weight h are rational.*

Proposition 17. *Let V be an Exceptional VOSA of lowest primary weight $l \in \mathbb{N}+\frac{1}{2}$ and central charge c. Assuming that $g_0(c) \neq 0$ and that $m \leq l - \frac{1}{2}$ for any indicial root of the form $x = m - c/24$. We then find*

(i) *$Z_V(q)$ is the unique $q^{\frac{1}{2}}$–series solution of the TMLDE with leading expansion $Z_V(q) = Z_{V_{\langle\omega\rangle}}(q) + O\left(q^{l-c/24}\right)$.*
(ii) $\dim V_r$ *is a rational function of c for each $r \in \frac{1}{2}\mathbb{N}$.*
(iii) *V is generated by the space of lowest weight primary vectors Π_l.*

We verify below for $l \leq 17/2$ that $g_0(c) \neq 0$ and that $m \leq l - \frac{1}{2}$ for any indicial root $x = m - c/24$. We conjecture these conditions hold in general.

We can construct two infinite series of $p_l = 1$ Exceptional VOSAs which we conjecture are the only examples.

Proposition 18. *For each Virasoro minimal model with $h_{1,p-1} \in \mathbb{N}+\frac{1}{2}$ there exists an Exceptional VOSA with one odd parity primary vector of lowest weight $l = h_{1,p-1}$ of AD-type*

$$V = L\left(c_{p,q}, 0\right) \oplus L\left(c_{p,q}, h_{1,p-1}\right). \tag{59}$$

Proposition 19. *For each $k \in \mathbb{N}+\frac{1}{2}$ for $k \geq \frac{3}{2}$ there exists an Exceptional VOSA $\mathcal{W}(3k)$ with one odd parity primary vector of lowest weight $3k$ and central charge $c_k = 1 - 24k$.*

Finally, similarly to Sect. 5, with $G = \mathrm{Aut}(V)$ we have

Proposition 20. *Let V be an Exceptional VOSA of class $\mathcal{S}^{2l+1}$ with primaries Π_l of lowest weight $l \in \mathbb{N}+\frac{1}{2}$. If Π_l is a completely reducible G-module then it is either an irreducible G-module or the direct sum of two isomorphic irreducible G-modules.*

8 Exceptional SVOAs with Lowest Primary Weight with $l \in \mathbb{N}+\frac{1}{2}$ for $l \leq \frac{17}{2}$

We now consider examples of Exceptional VOSAs of lowest primary weight $l \leq \frac{17}{2}$. We denote by $E_n = E_n(q)$ the Eisenstein series and $F_n = E_n\left[\begin{smallmatrix}1\\ -1\end{smallmatrix}\right](q)$ the twisted Eisenstein series of weight n appearing in the order $l+\frac{1}{2}$ TMLDE (57). For $l \leq \frac{3}{2}$ we find all $c, h \in \mathbb{Q}$, whereas for $\frac{5}{2} \leq l \leq \frac{17}{2}$ we find all $c, h \in \mathbb{Q}$ for which

$p_l = \dim \Pi_l \leq 500{,}000$ found by computer algebra techniques. We obtain many examples of known exceptional VOAs such as the free fermion VOSAs and the Baby Monster VOSA $\mathrm{VB}^\natural = \mathrm{Com}(V^\natural, \omega_{\frac{1}{2}})$, the commutant of $V^\natural$ with respect to a Virasoro vector of central charge $\frac{1}{2}$ [Ho1]. Some other such commutant theories also arise.

[$l = \frac{1}{2}$]. Propositions 15–17 imply that $Z(q)$ satisfies the 1st order TMLDE

$$DZ + cF_2Z = 0.$$

But $F_2(q) = \frac{1}{24} + 2\sum_{r\in\mathbb{N}+\frac{1}{2}} \frac{rq^r}{1-q^r}$ so that $Z(q) = \left(\frac{\eta(\tau/2)}{\eta(\tau)}\right)^{2c}$ with $p_{1/2} = 2c$. An Exceptional VOSA exists for all $p_{1/2} = m \geq 1$ given by the tensor product of m copies of the free fermion VOSA $L\left(\frac{1}{2}, 0\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right)$ of central charge $c = \frac{m}{2}$.

[$l = \frac{3}{2}$]. $Z(q)$ satisfies a 2nd order TMLDE:

$$D^2Z + \frac{2}{17}F_2(5c+22)\, DZ + \frac{1}{34}c\,(4(5c+22)F_4 + 17E_4)\; Z = 0,$$

with indicial roots $x_1 = -c/24$, $x_2 = (7c+24)/408$ with iterative solution

$$Z_V(q) = q^{-c/24}(1 - p_{3/2}q^{3/2} + (1+p_2)q^2 - (p_{3/2}+p_{5/2})q^{5/2}\ldots),$$
$$p_{3/2} = \frac{8c(5c+22)}{3(2c-49)}, \quad p_2 = \frac{(5c+22)(4c+21)(10c-7)}{2(c-33)(2c-49)},$$
$$p_{5/2} = -\frac{136c(5c+22)(4c+21)(10c-7)}{15(2c-83)(c-33)(2c-49)}.$$

For $2c = -2 \mod 17$, the indicial roots differ by an integer leading to denominator zeros for p_n. The $c, h \in \mathbb{Q}$ solutions with possible VOAs are

c	$p_{3/2}$	p_2	$p_{5/2}$	VOSA	$h \in \mathbb{Q}$
$-\frac{21}{4}$	1	0	0	$L\left(c_{3,8}, 0\right) \oplus L\left(c_{3,8}, \frac{3}{2}\right)$	$0, -\frac{1}{4}$
$\frac{7}{10}$	1	0	0	$L\left(c_{4,5}, 0\right) \oplus L\left(c_{4,5}, \frac{3}{2}\right)$	$0, \frac{1}{10}$
$\frac{15}{2}$	35	119	238	$\mathrm{Com}\left(V^+_{\sqrt{2}E_8}, \omega_{\frac{1}{2}}\right)$	$0, \frac{1}{2}$
16	256	2,295	13,056	$V^+_{\mathrm{BW}_{16}} \oplus \left(V^+_{\mathrm{BW}_{16}}\right)_{3/2}$	0, 1
$\frac{114}{5}$	2,432	48,620	537,472	$\mathrm{Com}\left(\mathrm{VB}^\natural, \omega_{\frac{7}{10}}\right)$	$0, \frac{7}{5}$
$\frac{47}{2}$	4,371	96,255	1,139,374	$\mathrm{VB}^\natural$	$0, \frac{49}{34}*$

The $c = \frac{15}{2} = 8 - \frac{1}{2}$ VOSA is the commutant of $V^+_{\sqrt{2}E_8}$ with respect to a Virasoro vector of central charge $\frac{1}{2}$ with $\mathrm{Aut}(V) = S_8(2)$ [LSY] and $\mathrm{VB}^\natural$ is the Baby Monster VOSA with $\mathrm{Aut}(\mathrm{VB}^\natural) = \mathbb{B}$ [Ho1]. In both cases, $p_{3/2}$ is odd and $\Pi_{3/2}$ is $\mathrm{Aut}(V)$-irreducible in agreement with Proposition 20 [Atlas]. The $c = \frac{15}{2}$ VOSA is the simple current extension of the Barnes–Wall Exceptional VOA by its $h = \frac{3}{2}$ module. The $c = \frac{114}{5} = \frac{47}{2} - \frac{7}{10}$ VOSA is the commutant of $\mathrm{VB}^\natural$ with respect to a Virasoro vector of central charge $\frac{7}{10}$ [HLY, Y]. In the later case, we expect $\mathrm{Aut}(V) = 2.^2E_6(2):2$, the maximal subgroup of $\mathbb{B}$, which has a 2,432 dimensional irreducible representation [Atlas]. $\mathrm{VB}^\natural$ is self-dual so that the $h = \frac{49}{34}$ TMLDE solution does not correspond to a graded character $Z_N(q)$.

[$l = \frac{5}{2}$]. $Z(q)$ satisfies a 3rd order TMLDE:

$$\begin{aligned}&(734c+49)\,D^3Z + 27(2c-1)(7c+68)F_2\,D^2Z\\&+\left(6(7c+68)(2c-1)(5c+22)F_4 + \frac{1}{2}(2{,}634c^2+1{,}739c-29{,}348)E_4\right)DZ\\&+\Big(2c(7c+68)(2c-1)(5c+22)F_6 + \frac{27}{2}c(2c-1)(7c+68)E_4F_2\\&+5c(36c^2+622c-2{,}413)E_6\Big)\,Z = 0,\end{aligned}$$

where

$$p_{5/2} = \frac{8\,(7c+68)\,(2c+5)\,(2c-1)\,(5c+22)}{5(8c^3-716c^2+16{,}102c+239)}.$$

There is one $c, h \in \mathbb{Q}$ solution with possible VOSA for $p_{5/2} \le 500{,}000$

c	$p_{5/2}$	VOSA	$h \in \mathbb{Q}$
$-\frac{13}{14}$	1	$L\left(c_{4,7},0\right) \oplus L\left(c_{4,7},\frac{5}{2}\right)$	$0, -\frac{1}{14}, \frac{1}{7}$

[$l = \frac{7}{2}$]. $Z_V(q)$ satisfies a 4th order TMLDE where $p_{7/2} = \frac{r(c)}{s(c)}$ for

$$\begin{aligned}r(c) &= 128(5c+22)(3c+46)(2c-1)(14+c)(5c+3)(7c+68),\\ s(c) &= 7(160c^5 - 31{,}176c^4 + 2{,}015{,}748c^3 - 41{,}830{,}202c^2\\ &\quad -92{,}625{,}711c + 1{,}017{,}681).\end{aligned}$$

The $c, h \in \mathbb{Q}$ solutions with possible VOSA for $p_{7/2} \le 500{,}000$ are

c	$p_{7/2}$	VOSA	$h \in \mathbb{Q}$
$-\frac{161}{8}$	1	$L\left(c_{3,16}, 0\right) \oplus L\left(c_{3,16}, \frac{7}{2}\right)$	$0, -\frac{5}{8}, -\frac{3}{4}, -\frac{7}{8}$
$-\frac{19}{6}$	1	$L\left(c_{4,9}, 0\right) \oplus L\left(c_{4,9}, \frac{7}{2}\right)$	$0, -\frac{1}{9}, -\frac{1}{6}, \frac{1}{6}$

[$l = \frac{9}{2}$]. $Z_V(q)$ satisfies a 5th order TMLDE where $p_{9/2} = \frac{r(c)}{s(c)}$ for

$$\begin{aligned} r(c) &= 160(3c+46)(2c-1)(5c+3)(11c+232)(68+7c)(40c^2+1{,}778c+11{,}025), \\ s(c) &= 9(3{,}200c^6 - 1{,}096{,}320c^5 + 140{,}381{,}096c^4 - 7{,}850{,}716{,}276c^3 + 149{,}541{,}921{,}538c^2 \\ &\quad +829{,}856{,}821{,}745c + 7{,}484{,}560{,}125). \end{aligned}$$

The $c, h \in \mathbb{Q}$ solutions with possible VOSA for $p_{9/2} \leq 500{,}000$ are

c	$p_{9/2}$	VOSA	$h \in \mathbb{Q}$
$-\frac{279}{10}$	1	$L\left(c_{3,20}, 0\right) \oplus L\left(c_{3,20}, \frac{9}{2}\right)$	$0, -\frac{7}{10}, -1, -\frac{11}{10}, -\frac{6}{5}$
$-\frac{125}{22}$	1	$L\left(c_{4,11}, 0\right) \oplus L\left(c_{4,11}, \frac{9}{2}\right)$	$0, -\frac{3}{22}, -\frac{5}{22}, -\frac{3}{11}, \frac{2}{11}$
$-\frac{7}{20}$	1	$L\left(c_{5,8}, 0\right) \oplus L\left(c_{5,8}, \frac{9}{2}\right)$	$0, -\frac{1}{20}, \frac{1}{4}, \frac{7}{10}, \frac{891}{1{,}850}*$
-35	1	$\mathcal{W}(\frac{9}{2})$	$0, -\frac{11}{10}, -\frac{4}{3}, -\frac{7}{5}, -\frac{3}{2}$

The $c = -\frac{7}{20}, h = \frac{891}{1{,}850}$ TMLDE solution does not correspond to a graded character $Z_N(q)$.

[$l = \frac{11}{2}$]. $Z_V(q)$ satisfies a 6th order TMLDE where $p_{11/2} = \frac{r(c)}{s(c)}$ for

$$\begin{aligned} r(c) &= -640(13c+350)(7c+25)(11c+232)(2c-1)(3c+46)(68+7c) \\ &\quad .(5c+3)(10c-7)(40c^2+3{,}586c+50{,}743), \\ s(c) &= 11(2{,}240{,}000c^9 - 1{,}185{,}856{,}000c^8 + 249{,}718{,}385{,}120c^7 - 25{,}848{,}494{,}429{,}040c^6 \\ &\quad +1{,}266{,}635{,}173{,}648{,}176c^5 - 18{,}264{,}666{,}939{,}042{,}072c^4 - 336{,}264{,}778{,}062{,}263{,}522c^3 \\ &\quad -861{,}021{,}133{,}326{,}393{,}167c^2 + 653{,}498{,}177{,}653{,}904{,}632c - 9{,}760{,}778{,}116{,}675{,}215). \end{aligned}$$

The $c, h \in \mathbb{Q}$ solutions with possible VOSA for $p_{11/2} \leq 500{,}000$ are

c	$p_{11/2}$	VOSA	$h \in \mathbb{Q}$
$-\frac{217}{26}$	1	$L\left(c_{4,13}, 0\right) \oplus L\left(c_{4,13}, \frac{11}{2}\right)$	$0, -\frac{2}{13}, -\frac{7}{26}, -\frac{9}{26}, -\frac{5}{13}, \frac{5}{26}$

[$l = \frac{13}{2}$]. $Z_V(q)$ satisfies a 7th order TMLDE with $p_{13/2} = \frac{r(c)}{s(c)}$ for

$$\begin{aligned} r(c) =&\ 4{,}480(13c+350)(5c+164)(7c+25)(11c+232)(3c+46)(4c+21)\\ &(5c+3)(10c-7)(1{,}120c^4+187{,}160c^3+6{,}889{,}980c^2+58{,}079{,}018c-24{,}165{,}453),\\ s(c) =&\ 13(125{,}440{,}000c^{11}-94{,}806{,}656{,}000c^{10}+29{,}650{,}660{,}755{,}200c^9\\ &-4{,}865{,}828{,}683{,}343{,}040c^8+431{,}531{,}398{,}085{,}049{,}664c^7-18{,}001{,}596{,}789{,}986{,}119{,}984c^6\\ &+107{,}049{,}283{,}968{,}364{,}390{,}448c^5+9{,}359{,}034{,}900{,}957{,}509{,}468{,}076c^4\\ &+76{,}817{,}948{,}684{,}836{,}018{,}331{,}724c^3+155{,}170{,}276{,}090{,}966{,}927{,}173{,}843c^2\\ &-81{,}951{,}451{,}902{,}336{,}562{,}695{,}126c-7{,}944{,}030{,}229{,}978{,}323{,}194{,}805). \end{aligned}$$

The $c, h \in \mathbb{Q}$ solutions with possible VOSA for $p_{13/2} \le 500{,}000$ are

c	$p_{13/2}$	VOSA	$h \in \mathbb{Q}$
$-\frac{611}{14}$	1	$L\left(c_{3,28}, 0\right) \oplus L\left(c_{3,28}, \frac{13}{2}\right)$	$0, -\frac{11}{14}, -\frac{19}{14}, -\frac{3}{2}, -\frac{12}{7}, -\frac{25}{14}, -\frac{13}{7}$
$-\frac{111}{10}$	1	$L\left(c_{4,15}, 0\right) \oplus L\left(c_{4,15}, \frac{13}{2}\right)$	$0, -\frac{1}{6}, -\frac{3}{10}, -\frac{2}{5}, -\frac{7}{15}, -\frac{1}{2}, \frac{1}{5}$

[$l = \frac{15}{2}$]. $Z_V(q)$ satisfies an 8th order TMLDE where $p_{15/2} = \frac{r(c)}{s(c)}$ for

$$\begin{aligned} r(c) =&\ -28{,}672(13c+350)(5c+164)(7c+25)(11c+232)\\ &.(3c+46)(17c+658)(4c+21)(5c+3)(10c-7)\\ &.(560c^4+146{,}584c^3+9{,}082{,}444c^2+133{,}381{,}952c-27{,}346{,}605),\\ s(c) =&\ 21{,}073{,}920{,}000c^{12}-21{,}694{,}120{,}448{,}000c^{11}+9{,}524{,}271{,}218{,}201{,}600c^{10}\\ &-2{,}298{,}054{,}501{,}201{,}632{,}000c^9+325{,}029{,}065{,}007{,}052{,}546{,}624c^8\\ &-26{,}081{,}744{,}761{,}028{,}079{,}338{,}944c^7+968{,}808{,}700{,}001{,}847{,}281{,}619{,}664c^6\\ &+787{,}299{,}295{,}625{,}321{,}246{,}276{,}560c^5-696{,}312{,}046{,}814{,}218{,}010{,}729{,}784{,}676c^4\\ &-7{,}887{,}852{,}431{,}045{,}609{,}558{,}472{,}152{,}948c^3-21{,}020{,}840{,}196{,}255{,}652{,}876{,}820{,}528{,}205c^2\\ &+3{,}455{,}907{,}491{,}220{,}404{,}701{,}398{,}711{,}750c+4{,}568{,}101{,}033{,}862{,}110{,}116{,}156{,}159{,}375. \end{aligned}$$

The $c, h \in \mathbb{Q}$ solutions with possible VOSA for $p_{15/2} \le 500{,}000$ are

c	$p_{15/2}$	VOSA	$h \in \mathbb{Q}$
$-\frac{825}{16}$	1	$L\left(c_{3,32},0\right) \oplus L\left(c_{3,32},\frac{15}{2}\right)$	$0, -\frac{13}{16}, -\frac{23}{16}, -\frac{7}{4}, -\frac{15}{8}, -\frac{33}{16}, -\frac{17}{8}, -\frac{35}{16}$
$-\frac{473}{34}$	1	$L\left(c_{4,17},0\right) \oplus L\left(c_{4,17},\frac{15}{2}\right)$	$0, -\frac{3}{17}, -\frac{11}{34}, -\frac{15}{34}, -\frac{9}{17}, -\frac{10}{17}, -\frac{21}{34}, \frac{7}{34}$
$-\frac{39}{10}$	1	$L\left(c_{5,12},0\right) \oplus L\left(c_{5,12},\frac{15}{2}\right)$	$0, \frac{1}{2}, \frac{13}{10}, -\frac{1}{6}, -\frac{1}{5}, \frac{2}{15}$
$\frac{25}{28}$	1	$L\left(c_{7,8},0\right) \oplus L\left(c_{7,8},\frac{15}{2}\right)$	$0, \frac{1}{28}, \frac{3}{28}, \frac{5}{14}, \frac{3}{4}, \frac{9}{7}$
-59	1	$\mathcal{W}(\frac{15}{2})$	$0, -\frac{13}{7}, -\frac{21}{10}, -\frac{31}{14}, -\frac{12}{5}, -\frac{17}{7}, -\frac{5}{2}, -\frac{67}{62}*$

The $c = -59$, $h = -\frac{67}{62}$ TMLDE solution does not correspond to a graded character $Z_N(q)$ [F].

$[l = \frac{17}{2}]$. $Z_V(q)$ satisfies a 9th order TMLDE. The only $c, h \in \mathbb{Q}$ solution with possible VOSA for $p_{17/2} \leq 500{,}000$ is

c	$p_{17/2}$	VOSA	$h \in \mathbb{Q}$
$-\frac{637}{38}$	1	$L\left(c_{4,19},0\right) \oplus L\left(c_{4,19},\frac{17}{2}\right)$	$0, -\frac{7}{38}, -\frac{13}{38}, -\frac{9}{19}, -\frac{11}{19}, -\frac{25}{38}, -\frac{27}{38}, -\frac{14}{19}, \frac{4}{19}$

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