Tensor Representations of Mackey Lie Algebras and Their Dense Subalgebras

Ivan Penkov and Vera Serganova

Abstract In this article we review the main results of the earlier papers [PStyr, PS] and [DPS], and establish related new results in considerably greater generality. We introduce a class of infinite-dimensional Lie algebras $\mathfrak{g}^M$, which we call Mackey Lie algebras, and define monoidal categories $\mathbb{T}_{\mathfrak{g}^M}$ of tensor $\mathfrak{g}^M$-modules. We also consider dense subalgebras $\mathfrak{a} \subset \mathfrak{g}^M$ and corresponding categories $\mathbb{T}_{\mathfrak{a}}$. The locally finite Lie algebras $\mathfrak{sl}(V, W), \mathfrak{o}(V), \mathfrak{sp}(V)$ are dense subalgebras of respective Mackey Lie algebras. Our main result is that if $\mathfrak{g}^M$ is a Mackey Lie algebra and $\mathfrak{a} \subset \mathfrak{g}^M$ is a dense subalgebra, then the monoidal category $\mathbb{T}_{\mathfrak{a}}$ is equivalent to $\mathbb{T}_{\mathfrak{sl}(\infty)}$ or $\mathbb{T}_{\mathfrak{o}(\infty)}$; the latter monoidal categories have been studied in detail in [DPS]. A possible choice of $\mathfrak{a}$ is the well-known Lie algebra of generalized Jacobi matrices.

Key words Finitary Lie algebra • Mackey Lie algebra • Linear system • Tensor representation • Socle filtration

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Introduction

This paper combines a review of some results on locally finite Lie algebras, mostly from [PStyr, PS] and [DPS], with new results about categories of representations of a class of (not locally finite) infinite-dimensional Lie algebras which we call Mackey Lie algebras. Locally finite Lie algebras (i.e., Lie algebras in which any finite set of elements generates a finite-dimensional Lie subalgebra) and their representations have been gaining the attention of researchers in the past 20 years. An incomplete list of references on this topic is: [Ba1, BB, BS, DiP1, DiP3, DPS, DPSn, DPW, DaPW, N, Na, NP, NS, O, PS, PStyr, PZ]. In particular, in [PStyr, PS] and [DPS] integrable representations of the three classical locally finite Lie algebras $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$ have been studied from various points of view. An important step in the development of the representation theory of these Lie algebras has been the introduction of the category of tensor modules $\mathbb{T}_\mathfrak{g}$ in [DPS].

In the present article we shift the focus to understanding a natural generality in which the category $\mathbb{T}_\mathfrak{g}$ is defined. In particular, we consider the finitary locally simple Lie algebras $\mathfrak{g} = \mathfrak{sl}(V, W), \mathfrak{o}(V), \mathfrak{sp}(V)$, where V is an arbitrary vector space (not necessarily of countable dimension), and either a nondegenerate pairing $V \times W \to \mathbb{C}$ is given, or V is equipped with a nondegenerate symmetric, or antisymmetric form. In Sects. 1–5 we reproduce the most important results from [PStyr] and [DPS] in this greater generality. In fact, we study five different categories of integrable modules, see Sect. 3.6, but pay maximum attention to the category $\mathbb{T}_\mathfrak{g}$. The central new result in this part of the paper is Theorem 5.5, claiming that the category $\mathbb{T}_\mathfrak{g}$ for $\mathfrak{g} = \mathfrak{sl}(V, W), \mathfrak{o}(V), \mathfrak{sp}(V)$ is canonically equivalent, as a monoidal category, to the respective category $\mathbb{T}_{\mathfrak{sl}(\infty)}, \mathbb{T}_{\mathfrak{o}(\infty)}$ or $\mathbb{T}_{\mathfrak{sp}(\infty)}$. It is shown in [DPS] that each of the latter categories is Koszul and that $\mathbb{T}_{\mathfrak{sl}(\infty)}$ is self-dual Koszul, while $\mathbb{T}_{\mathfrak{o}(\infty)}$ and $\mathbb{T}_{\mathfrak{sp}(\infty)}$ are not self-dual but are equivalent.

In the second part of the paper, starting with Sect. 6, we explore several new ideas. The first one is that given a nondegenerate pairing $V \times W \to \mathbb{C}$ between two vector spaces, or a nondegenerate symmetric or antisymmetric form on a vector space V, there is a canonical, in general not locally finite, Lie algebra attached to this datum. Indeed, fix a pairing $V \times W \to \mathbb{C}$. Then the Mackey Lie algebra $\mathfrak{gl}^M(V, W)$ is the Lie algebra of all endomorphisms of V whose duals keep W stable (this definition is given in a more precise form at the beginning of Sect. 6). Similarly, if V is equipped with a nondegenerate form, the respective Lie algebra $\mathfrak{o}^M(V)$ or $\mathfrak{sp}^M(V)$ is the Lie algebra of all endomorphisms of V for which the form is invariant.

The Lie algebras $\mathfrak{gl}^M(V, W), \mathfrak{o}^M(V), \mathfrak{sp}^M(V)$ are not simple as they have obvious ideals: these are respectively $\mathfrak{gl}(V, W) \oplus \mathbb{C}\mathrm{Id}$, $\mathfrak{o}(\infty)$, and $\mathfrak{sp}(\infty)$. However, we prove that, if both V and W are countable dimensional, the quotients $\mathfrak{gl}^M(V, W)/(\mathfrak{gl}(V, W) \oplus \mathbb{C}\mathrm{Id})$, $\mathfrak{o}^M(V)/\mathfrak{o}(V)$, $\mathfrak{sp}^M(V)/\mathfrak{sp}(V)$ are simple Lie algebras. This result is an algebraic analogue of the simplicity of the Calkin algebra in functional analysis.

Despite the fact that the Lie algebras $\mathfrak{gl}^M(V, W), \mathfrak{o}^M(V), \mathfrak{sp}^M(V)$ are completely natural objects, the representation theory of these Lie algebras has not yet been explored. We are undertaking the first step of such an exploration by introducing the categories of tensor modules $\mathbb{T}_{\mathfrak{g}^M}$ for $\mathfrak{g}^M = \mathfrak{gl}^M(V, W), \mathfrak{o}^M(V), \mathfrak{sp}^M(V)$. Our main result about these categories is Theorem 7.10 which implies that $\mathbb{T}_{\mathfrak{gl}^M(V,W)}$ is equivalent to $\mathbb{T}_{\mathfrak{sl}(\infty)}$, and $\mathbb{T}_{\mathfrak{o}^M(V)}$ and $\mathbb{T}_{\mathfrak{sp}^M(V)}$ are equivalent respectively to $\mathbb{T}_{\mathfrak{o}(\infty)}$ and $\mathbb{T}_{\mathfrak{sp}(\infty)}$.

A further idea is to consider dense subalgebras $\mathfrak{a}$ of the Lie algebras $\mathfrak{g}^M$ (see the definition in Sect. 7). We show that if $\mathfrak{a} \subset \mathfrak{g}$ is a dense subalgebra, the category $\mathbb{T}_{\mathfrak{a}}$, whose objects are tensor modules of $\mathfrak{g}$ considered as $\mathfrak{a}$-modules, is canonically equivalent to $\mathbb{T}_{\mathfrak{g}^M}$, and hence to one of the categories $\mathbb{T}_{\mathfrak{sl}(\infty)}$ or $\mathbb{T}_{\mathfrak{o}(\infty)}$. It is interesting that this result applies to the Lie algebra of generalized Jacobi matrices (infinite matrices with "finitely many nonzero diagonals") which has been studied for over 30 years, see for instance [FT].

In short, the main point of this paper is that the categories of tensor modules $\mathbb{T}_{\mathfrak{sl}(\infty)}, \mathbb{T}_{\mathfrak{o}(\infty)}, \mathbb{T}_{\mathfrak{sp}(\infty)}$ introduced in [DPS] are in some sense universal, being naturally equivalent to the respective categories of tensor representations of a large class of, possibly not locally finite, infinite-dimensional Lie algebras.

1 Preliminaries

The ground field is $\mathbb{C}$. By M^* we denote the dual space of a vector space M, i.e., $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$. S_n stands for the symmetric group on n letters. The sign $\subset$ denotes not necessarily strict inclusion. By definition, a *natural representation* (or a *natural module*) of a classical simple finite-dimensional Lie algebra is a simple nontrivial finite-dimensional representation of minimal dimension.

In this paper $\mathfrak{g}$ denotes a *locally simple locally finite* Lie algebra, i.e., an infinite-dimensional Lie algebra $\mathfrak{g}$ obtained as the direct limit $\varinjlim \mathfrak{g}_\alpha$ of a directed system of embeddings (i.e., injective homomorphisms) $\mathfrak{g}_\alpha \hookrightarrow \mathfrak{g}_\beta$ of finite-dimensional simple Lie algebras parametrized by a directed set of indices. It is clear that any such $\mathfrak{g}$ is a simple Lie algebra. If $\mathfrak{g}$ is countable dimensional, then the above directed set can always be chosen as $\mathbb{Z}_{\geq 1}$, and the corresponding directed system can be chosen as a chain

$$\mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2 \hookrightarrow \ldots \hookrightarrow \mathfrak{g}_i \hookrightarrow \mathfrak{g}_{i+1} \hookrightarrow \ldots \quad . \tag{1}$$

In this case we write $\mathfrak{g} = \varinjlim \mathfrak{g}_i$. Moreover, if $\mathfrak{g}_i = \mathfrak{sl}(i+1)$, then up to isomorphism there is only one such Lie algebra which we denote by $\mathfrak{sl}(\infty)$. Similarly, if $\mathfrak{g}_i = \mathfrak{o}(i)$ or $\mathfrak{g}_i = \mathfrak{sp}(2i)$, up to isomorphism one obtains only two Lie algebras: $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$. The Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ are often referred to as the *finitary locally simple Lie algebras* [Ba1, Ba2, BS], or as the *classical locally simple Lie algebras* [PS].

A more general (and very interesting) class of locally finite locally simple Lie algebras are the diagonal locally finite Lie algebras introduced by Y. Bahturin and H. Strade in [BhS]. We recall that an injective homomorphism $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2$ of simple classical Lie algebras of the same type $\mathfrak{sl}$, $\mathfrak{o}$, $\mathfrak{sp}$, is *diagonal* if the pull-back $V_{\mathfrak{g}_2\downarrow\mathfrak{g}_1}$ of a natural representation $V_{\mathfrak{g}_2}$ of $\mathfrak{g}_2$ to $\mathfrak{g}_1$ is isomorphic to a direct sum of copies of a natural representation $V_{\mathfrak{g}_1}$, of its dual $V^*_{\mathfrak{g}_1}$, and of the trivial 1-dimensional representation. In this paper, by a *diagonal Lie algebra* $\mathfrak{g}$ we mean an infinite-dimensional Lie algebra obtained as the limit of a directed system of diagonal homomorphisms of classical simple Lie algebras $\mathfrak{g}_\alpha$. We say that a diagonal Lie algebra *is of type* $\mathfrak{sl}$ (respectively, $\mathfrak{o}$ or $\mathfrak{sp}$) if all $\mathfrak{g}_\alpha$ can be chosen to have type $\mathfrak{sl}$ (respectively, $\mathfrak{o}$ or $\mathfrak{sp}$).

Countable-dimensional diagonal Lie algebras have been classified up to isomorphism by A. Baranov and A. Zhilinskii [BaZh]. S. Markouski [Ma] has determined when there is an embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}'$ for given countable-dimensional diagonal Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$. If both $\mathfrak{g}$ and $\mathfrak{g}'$ are classical locally simple Lie algebras, then an embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}'$ always exists, and such embeddings have been studied in detail in [DiP2].

Let V and W be two infinite-dimensional vector spaces with a nondegenerate pairing $V \times W \to \mathbb{C}$. G. Mackey calls such a pair V, W a *linear system* and was the first to study linear systems in depth [M]. The tensor product $V \otimes W$ is an associative algebra (without identity), and we denote the corresponding Lie algebra by $\mathfrak{gl}(V, W)$. The pairing $V \times W \to \mathbb{C}$ induces a homomorphism of Lie algebras $\mathrm{tr} : \mathfrak{gl}(V, W) \to \mathbb{C}$. The kernel of this homomorphism is denoted by $\mathfrak{sl}(V, W)$. The Lie algebra $\mathfrak{sl}(V, W)$ is a locally simple locally finite Lie algebra. A corresponding directed system is given by $\{\mathfrak{sl}(V_f, W_f)\}$, where V_f and W_f run over all finite-dimensional subspaces $V_f \subset V$, $W_f \subset W$ such that the restriction of the pairing $V \times W \to \mathbb{C}$ to $V_f \times W_f$ is nondegenerate. If V and W are countable dimensional, then $\mathfrak{sl}(V, W)$ is isomorphic to $\mathfrak{sl}(\infty)$. In what follows we call a pair of finite-dimensional subspaces $V_f \subset V$, $W_f \subset W$ a *finite-dimensional nondegenerate pair* if the restriction of the pairing $V \times W \to \mathbb{C}$ to $V_f \times W_f$ is nondegenerate. We can also define $\mathfrak{gl}(V, W)$ as a Lie algebra of finite rank linear operators in $V \oplus W$ preserving V, W and the pairing $V \times W \to \mathbb{C}$.

There is an obvious notion of *isomorphism of linear systems*: given two linear systems $V \times W \to \mathbb{C}$ and $V \times W' \to \mathbb{C}$, an isomorphism of these linear systems is a pair of isomorphisms of vector spaces $\varphi : V \to W$, $\psi : W \to W'$ or $\varphi : V \to W'$, $\psi : W \to V'$, commuting with the respective pairings. If V and W are countable dimensional then, as shown by G. Mackey [Ma], there exists a basis $\{v_1, v_2, \dots\}$ of V such that $V_* = \mathrm{span}\{v_1^*, v_2^*, \dots\}$, where $\{v_1^*, v_2^*, \dots\}$ is the set of linear functionals dual to $\{v_1, v_2, \dots\}$, i.e., $v_i^*(v_j) = \delta_{ij}$. Consequently, up to isomorphism, there exists only one linear system $V \times W \to \mathbb{C}$ such that V and W are countable dimensional. The choice of a basis of V as above identifies $\mathfrak{gl}(V, W)$ with the Lie algebra $\mathfrak{gl}(\infty)$ consisting of infinite matrices $X = (x_{ij})_{i\geq 1, j\geq 1}$ with finitely many nonzero entries. The Lie algebra $\mathfrak{sl}(V, W)$ is identified with $\mathfrak{sl}(\infty)$ realized as the Lie algebra of traceless matrices $X = (x_{ij})_{i\geq 1, j\geq 1}$ with finitely many nonzero entries.

Now let V be a vector space endowed with a nondegenerate symmetric (respectively, antisymmetric) form $(\cdot,\cdot)$. Then $\Lambda^2 V$ (respectively, $S^2 V$) has a Lie algebra structure, defined by

$$[v_1\wedge v_2, w_1\wedge w_2] = -(v_1,w_1)v_2\wedge w_2+(v_2,w_1)v_1\wedge w_2+(v_1,w_2)v_2\wedge w_1-(v_2,w_2)v_1\wedge w_1$$

(respectively, by

$$[v_1v_2, w_1w_2] = (v_1,w_1)v_2w_2 + (v_2,w_1)v_1w_2 + (v_1,w_2)v_2w_1 + (v_2,w_2)v_1w_1).$$

We denote the Lie algebra $\Lambda^2 V$ by $\mathfrak{o}(V)$, and the Lie algebra S^2V by $\mathfrak{sp}(V)$. Let $V_f \subset V$ be an n-dimensional subspace such that the restriction of the form on V_f is nondegenerate. Then $\mathfrak{o}(V_f) \subset \mathfrak{o}(V)$ (respectively, $\mathfrak{sp}(V_f) \subset \mathfrak{sp}(V)$) is a simple subalgebra isomorphic to $\mathfrak{o}(n)$ (respectively, $\mathfrak{sp}(n)$). Therefore, $\mathfrak{o}(V)$ (respectively, $\mathfrak{sp}(V)$) is the direct limit of all its subalgebras $\mathfrak{o}(V_f)$ (respectively, $\mathfrak{sp}(V_f)$). This shows that both $\mathfrak{o}(V)$ and $\mathfrak{sp}(V)$ are locally simple locally finite Lie algebras. We can also identify $\mathfrak{o}(V)$ (respectively, $\mathfrak{sp}(V)$) with the Lie subalgebra of all finite rank operators in V under which the form $(\cdot,\cdot)$ is invariant.

If V is countable dimensional, there always is a basis $\{v_i, w_j\}_{i,j\in\mathbb{Z}}$ of V such that $\mathrm{span}\{v_i\}_{i\in\mathbb{Z}}$ and $\mathrm{span}\{w_j\}_{j\in\mathbb{Z}}$ are isotropic spaces and $(v_i, w_j) = 0$ for $i \neq j$, $(v_i, w_i) = 1$. Therefore, in this case $\mathfrak{o}(V) \simeq \mathfrak{o}(\infty)$ and $\mathfrak{sp}(V) \simeq \mathfrak{sp}(\infty)$.

Note that if V is not finite or countable dimensional, then V may have several inequivalent nondegenerate symmetric forms. Indeed, let for instance $V := W \oplus W^*$ for some countable-dimensional space W. Extend the pairing between W and W^* to a nondegenerate symmetric form $(\cdot,\cdot)$ on V for which W and W^* are both isotropic. It is clear that W is a maximal isotropic subspace of V. On the other hand, choose a basis $\mathbf{b}$ in V and let $(\cdot,\cdot)'$ be the symmetric form on V for which $\mathbf{b}$ is an orthonormal basis. Then V does not have countable-dimensional maximal isotropic subspaces for the form $(\cdot,\cdot)'$. Hence the forms $(\cdot,\cdot)$ and $(\cdot,\cdot)'$ are not equivalent.

Proposition 1.1. *(a) Two Lie algebras $\mathfrak{sl}(V, W)$ and $\mathfrak{sl}(V', W')$ are isomorphic if and only if the linear systems $V \times W \to \mathbb{C}$ and $V' \times W' \to \mathbb{C}$ are isomorphic.*

(b) Two Lie algebras $\mathfrak{o}(V)$ and $\mathfrak{o}(V')$ (respectively, $\mathfrak{sp}(V)$ and $\mathfrak{sp}(V')$) are isomorphic if and only if there is an isomorphism of vector spaces $V \simeq V'$ transferring the form defining $\mathfrak{o}(V)$ (respectively $\mathfrak{sp}(V)$) into the form defining $\mathfrak{o}(V')$ (respectively, $\mathfrak{sp}(V')$).

We first prove a lemma.

Lemma 1.2 (cf. Proposition 2.3 in [DiP2]).

(a) Let $\mathfrak{g}_1 \subset \mathfrak{g}_3$ be an inclusion of classical finite-dimensional simple Lie algebras such that a natural $\mathfrak{g}_3$-module restricts to $\mathfrak{g}_1$ as the direct sum of a natural $\mathfrak{g}_1$-module and a trivial $\mathfrak{g}_1$-module. If $\mathfrak{g}_2$ is an intermediate classical simple subalgebra, $\mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \mathfrak{g}_3$, then a natural $\mathfrak{g}_3$-module restricts to $\mathfrak{g}_2$ as the direct sum of a natural $\mathfrak{g}_2$-module and a trivial module.

(b) *Assume* $\mathrm{rk}\mathfrak{g}_1 > 4$. *If* $\mathfrak{g}_1 \simeq \mathfrak{sl}(i)$, *then* $\mathfrak{g}_2$ *is isomorphic to* $\mathfrak{sl}(k)$ *for some* $k \geq i$. *If* $\mathfrak{g}_3 \simeq \mathfrak{o}(j)$ *(respectively,* $\mathfrak{sp}(2j)$*), then* $\mathfrak{g}_2$ *is isomorphic to* $\mathfrak{o}(k)$ *(respectively,* $\mathfrak{sp}(2k)$*) for some* $k \leq j$.

Proof. Let V_3 be a natural $\mathfrak{g}_3$-module. We have a decomposition of $\mathfrak{g}_1$-modules, $V_3 = V_1 \oplus W$, where V_1 is a natural $\mathfrak{g}_1$-module and W is a trivial $\mathfrak{g}_1$-module. Let $V' \subset V_3$ be the minimal $\mathfrak{g}_2$-submodule containing V_1. Then $V_3 = V' \oplus W'$, where W' is a complementary $\mathfrak{g}_2$-submodule. Since $\mathfrak{g}_1$ acts trivially on W' and $\mathfrak{g}_2$ is simple, we obtain that W' is a trivial $\mathfrak{g}_2$-module and V' is a simple $\mathfrak{g}_2$-module.

We now prove that V' is a natural $\mathfrak{g}_2$-module. Recall that for an arbitrary nontrivial module M over a simple Lie algebra $\mathfrak{k}$ the symmetric form $B_M(X, Y) = \mathrm{tr}_M(XY)$ for $X, Y \in \mathfrak{k}$ is nondegenerate. Moreover, $B_M = t_M B$, where B is the Killing form. If M is a simple $\mathfrak{k}$-module with highest weight λ, then

$$t_M = \frac{\dim M}{\dim \mathfrak{k}}(\lambda + 2\rho, \lambda),$$

where ρ is the half-sum of positive roots and $(\cdot, \cdot)$ is the form on the weight lattice of $\mathfrak{k}$ induced by B. It is easy to check that a natural module is a simple module with minimal t_M. Let V_2 be a natural $\mathfrak{g}_2$-module. Note that the restriction of $B_{V'}$ on $\mathfrak{g}_1$ equals B_{V_1} and the restriction of B_{V_2} on $\mathfrak{g}_1$ equals tB_{V_1} for some $t \geq 1$. On the other hand, $t = \frac{t_{V_2}}{t_{V'}}$. Since t_{V_2} is minimal, we have $t = 1$ and $t_{V_2} = t_{V'}$. Hence, V' is a natural module, i.e., (a) is proved.

To prove (b), note that a classical simple Lie algebra of rank greater than 4 admits, up to isomorphism, two (mutually dual) natural representations when it is of type $\mathfrak{sl}$, and one natural representation when it is of type $\mathfrak{o}$ or $\mathfrak{sp}$. Moreover, in the orthogonal (respectively, symplectic) case the natural module admits an invariant symmetric (respectively, skew-symmetric) bilinear form.

Now, assume $\mathfrak{g}_1 \simeq \mathfrak{sl}(i)$. We claim that $\mathfrak{g}_2 \simeq \mathfrak{sl}(k)$ for some $i \leq k \leq j$. Indeed, if $\mathfrak{g}_2$ is not isomorphic to $\mathfrak{sl}(k)$, then V' is self-dual. Therefore its restriction to $\mathfrak{g}_1$ is self-dual, and we obtain a contradiction as V_1 is not a self-dual $\mathfrak{sl}(i)$-module for $i \geq 3$.

Finally, assume $\mathfrak{g}_3 \simeq \mathfrak{o}(j)$ (respectively, $\mathfrak{sp}(2j)$). Then $V' \oplus W'$, and hence V', admits an invariant symmetric (respectively, skew-symmetric) form. Therefore $\mathfrak{g}_2 \simeq \mathfrak{o}(k)$ (respectively, $\mathfrak{sp}(2k)$). □

Corollary 1.3 (cf. [DiP2, Corollary 2.4]). *Let* $\mathfrak{g} = \mathfrak{sl}(V, W)$ *and* $\mathfrak{g} = \varinjlim \mathfrak{g}_\alpha$ *for some directed system* $\{\mathfrak{g}_\alpha\}$ *of simple finite-dimensional Lie subalgebras* $\mathfrak{g}_\alpha \subset \mathfrak{g}$. *Then there exists a subsystem* $\{\mathfrak{g}_{\alpha'}\}$ *such that* $\mathfrak{g} = \varinjlim \mathfrak{g}_{\alpha'}$ *and, for every* α', $\mathfrak{g}_{\alpha'} = \mathfrak{sl}(V_{\alpha'}, W_{\alpha'})$ *for some finite-dimensional nondegenerate pair* $V_{\alpha'} \subset V$, $W_{\alpha'} \subset W$. *Similarly, if* $\mathfrak{g} = \mathfrak{o}(V)$ *(respectively,* $\mathfrak{sp}(V)$*), then there exists a subsystem* $\{\mathfrak{g}_{\alpha'}\}$ *such that* $\mathfrak{g} = \varinjlim \mathfrak{g}_{\alpha'}$ *and, for every* α', $\mathfrak{g}_{\alpha'} = \mathfrak{o}(V_{\alpha'})$ *(respectively,* $\mathfrak{sp}(V_{\alpha'})$*) for some finite-dimensional nondegenerate* $V_{\alpha'} \subset V$.

Proof. Let $\mathfrak{g} = \mathfrak{sl}(V, W)$. One fixes a Lie subalgebra $\mathfrak{sl}(V_f, W_f) \subset \mathfrak{g}$ where $V_f \subset V$, $W_f \subset W$ is a finite-dimensional nondegenerate pair, and considers the directed subsystem $\{\mathfrak{g}_{\alpha'}\}$ of all $\mathfrak{g}_{\alpha'}$ such that $\mathfrak{sl}(V_f, W_f) \subset \mathfrak{g}_{\alpha'}$. There exists another finite-dimensional nondegenerate pair V'_f, W'_f such that $\mathfrak{sl}(V_f, W_f) \subset \mathfrak{g}_{\alpha'} \subset \mathfrak{sl}(V'_f, W'_f)$. Then, by Lemma 1.2, $\mathfrak{g}_{\alpha'} = \mathfrak{sl}(V_{\alpha'}, W_{\alpha'})$ for appropriate $V_{\alpha'} \subset V$, $W_{\alpha'} \subset W$. The cases $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$ are similar. □

Proof of Proposition 1.1. We consider the case $\mathfrak{g} = \mathfrak{sl}(V, W)$ and leave the remaining cases to the reader. Let $\mathfrak{g} = \mathfrak{sl}(V, W)$ be isomorphic to $\mathfrak{sl}(V', W')$. Then $\mathfrak{g} = \varinjlim \mathfrak{sl}(V_f, W_f)$ over all finite-dimensional nondegenerate pairs $V_f \subset V, W_f \subset W$, and at the same time $\mathfrak{g} = \varinjlim \mathfrak{sl}(V'_f, W'_f)$ over all finite-dimensional nondegenerate pairs $V'_f \subset V', W'_f \subset W'$. By Corollary 1.3 and Lemma 1.2, for each $V_f \subset V, W_f \subset W$ one can find $V'_f \subset V', W'_f \subset W'$ and an embedding of Lie algebras $\mathfrak{sl}(V_f, W_f) \subset \mathfrak{sl}(V'_f, W'_f)$ as in Lemma 1.2. That implies the existence of embeddings $V_f \hookrightarrow V'_f, W_f \hookrightarrow W'_f$ or $V_f \hookrightarrow W'_f, W_f \hookrightarrow V'_f$ preserving the pairing. After a twist by transposition we may assume that $V_f \hookrightarrow V'_f, W_f \hookrightarrow W'_f$. Therefore we have embeddings $V = \varinjlim V_f \hookrightarrow V', W = \varinjlim W_f \hookrightarrow W'$ preserving the pairing. On the other hand, both maps are surjective since $\mathfrak{sl}(V', W') = \varinjlim \mathfrak{sl}(V_f, W_f)$. Therefore the linear systems $V \times W \to \mathbb{C}$ and $V' \times W' \to \mathbb{C}$ are isomorphic. □

Assume next that $\mathfrak{g}$ is an arbitrary locally finite locally simple Lie algebra. If we can choose a Cartan subalgebra $\mathfrak{h}_\alpha \subset \mathfrak{g}_\alpha$ such that $\mathfrak{h}_\alpha \hookrightarrow \mathfrak{h}_\beta$ for any embedding $\mathfrak{g}_\alpha \hookrightarrow \mathfrak{g}_\beta$, then $\mathfrak{h} := \varinjlim \mathfrak{h}_\alpha$ is called a *local Cartan subalgebra*.

In general, a local Cartan subalgebra may not exist. For example, the following proposition implies that the Lie algebra $\mathfrak{g} = \mathfrak{sl}(V, V^*)$ does not have a local Cartan subalgebra.

Proposition 1.4. *Let $\mathfrak{g} = \mathfrak{sl}(V, W)$. Then a local Cartan subalgebra of $\mathfrak{g}$ exists if and only if V admits a basis $\left\{v_\gamma\right\}$ such that $W = \operatorname{span}\left\{v^*_\gamma\right\}$, where $v^*_{\tilde\gamma}(v_\gamma) = \delta_{\tilde\gamma\gamma}$. In this case, every local Cartan subalgebra of $\mathfrak{g}$ is of the form $\operatorname{span}\left\{v_\gamma \otimes v^*_\gamma - v_{\tilde\gamma} \otimes v^*_{\tilde\gamma}\right\}_{\gamma,\tilde\gamma}$ for a basis $\left\{v_\gamma\right\}$ as above.*

Proof. By Corollary 1.3 we may assume

$$\mathfrak{g} = \mathfrak{sl}(V, W) = \varinjlim \mathfrak{g}_\alpha = \varinjlim \mathfrak{sl}(V_\alpha, W_\alpha),$$

where $V_\alpha \subset V$, $W_\alpha \subset W$ are certain nondegenerate finite-dimensional pairs, and that $\mathfrak{h} = \varinjlim \mathfrak{h}_\alpha$ where $\mathfrak{h}_\alpha$ is a Cartan subalgebra of $\mathfrak{g}_\alpha$. Note that for any α we have $\mathfrak{h}_\alpha \cdot V_\alpha = V_\alpha$ and $\mathfrak{h}_\alpha \cdot W_\alpha = W_\alpha$. Since $\mathfrak{h}$ is abelian, we have $\mathfrak{h} \cdot V_\alpha = V_\alpha$ and $\mathfrak{h} \cdot W_\alpha = W_\alpha$. Therefore V and W are semisimple $\mathfrak{h}$-modules. This means that V is the direct sum of nontrivial one-dimensional $\mathfrak{h}$-submodules V_γ, i.e., $V = \bigoplus_\gamma V_\gamma$; similarly, $W = \bigoplus_{\gamma'} W_{\gamma'}$. Since however, for any α, the spaces V_α and W_α are dual to each other, γ' and γ run over the same set of indices and $W_\gamma(V_{\tilde\gamma}) \neq 0$ precisely for $\gamma = \tilde\gamma$. This yields a basis v_γ as required: v_γ can be chosen as any nonzero vector in V_γ and v^*_γ is the unique vector in W_γ with $v^*_\gamma(v_\gamma) = 1$. Finally,

$\mathfrak{h} = \operatorname{span}\left\{v_\gamma \otimes v_\gamma^* - v_{\tilde{\gamma}} \otimes v_{\tilde{\gamma}}^*\right\}$ as, clearly, $\mathfrak{h} \cap \mathfrak{g}_\alpha = \operatorname{span}\left\{v_\gamma \otimes v_\gamma^* - v_{\tilde{\gamma}} \otimes v_{\tilde{\gamma}}^*\right\}$ for $v_\gamma, v_{\tilde{\gamma}} \in V_\alpha$.

In the other direction, given a basis v_γ of V such that $\left\{v_\gamma^*\right\}$ is a basis of W, it is clear that $\mathfrak{g} = \varinjlim \mathfrak{sl}\left(\operatorname{span}\left\{v_\gamma\right\}_{\gamma\in A}, \operatorname{span}\left\{v_\gamma^*\right\}_{\gamma\in A}\right)$ for all finite sets of indices A, and that $\mathfrak{h} = \varinjlim\left(\mathfrak{h} \cap \operatorname{span}\left\{v_\gamma \otimes v_\gamma^* - v_{\tilde{\gamma}} \otimes v_{\tilde{\gamma}}^*\right\}_{\gamma,\tilde{\gamma}\in A}\right)$. □

In [DPSn] (and also in earlier work, see the references in [DPSn]) *Cartan subalgebras* are defined as maximal toral subalgebras of $\mathfrak{g}$ (i.e., as subalgebras each vector in which is ad-semisimple). *Splitting* Cartan subalgebras are Cartan subalgebras for which the adjoint representation is semisimple. It is shown in [PStr] that a countable dimensional locally finite, locally simple Lie algebra $\mathfrak{g}$ admits a splitting Cartan subalgebra if and only if $\mathfrak{g} \simeq \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$. Proposition 1.4 determines when Lie algebras of the form $\mathfrak{g} = \mathfrak{sl}(V, W), \mathfrak{o}(V), \mathfrak{sp}(V)$ admit local Cartan subalgebras and implies that the notions of local Cartan subalgebra and of splitting Cartan subalgebra coincide for these Lie algebras.

In what follows, we denote by V, V_* a pair of infinite-dimensional spaces (of not necessarily countable dimension) arising from a linear system $V \times V_* \to \mathbb{C}$ for which there is a basis $\{v_\gamma\}$ of V such that $V_* = \operatorname{span}(\{v_\gamma^*\})$ where $v_{\tilde{\gamma}}^*(v_\gamma) = \delta_{\tilde{\gamma}\gamma}$.

2 The Category $\mathrm{Int}_{\mathfrak{g}}$

Let $\mathfrak{g}$ be an arbitrary locally simple locally finite Lie algebra. An *integrable* $\mathfrak{g}$-module is a $\mathfrak{g}$-module M which is locally finite as a module over any finite-dimensional subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$. In other words, $\dim U(\mathfrak{g}') \cdot m < \infty \quad \forall\, m \in M$. We denote the category of integrable $\mathfrak{g}$-modules by $\mathrm{Int}_{\mathfrak{g}}$: $\mathrm{Int}_{\mathfrak{g}}$ is a full subcategory of the category $\mathfrak{g}$-mod of all $\mathfrak{g}$-modules. It is clear that $\mathrm{Int}_{\mathfrak{g}}$ is an abelian category and a monoidal category with respect to usual tensor product. Note that the adjoint representation of $\mathfrak{g}$ is an object of $\mathrm{Int}_{\mathfrak{g}}$.

The *functor of* $\mathfrak{g}$*-integrable vectors*

$$\Gamma_{\mathfrak{g}} \; : \; \mathfrak{g} - \mathrm{mod} \rightsquigarrow \mathrm{Int}_{\mathfrak{g}},$$
$$\Gamma_{\mathfrak{g}}(M) := \left\{m \in M \mid \dim U(\mathfrak{g}') \cdot m < \infty \;\forall \text{ finite-dim. subalgebras } \mathfrak{g}' \subset \mathfrak{g}\right\}$$

is a well-defined left-exact functor. This follows from the fact that the functor of $\mathfrak{g}'$-finite vectors $\Gamma_{\mathfrak{g}'}$ is well defined for any finite-dimensional subalgebra $\mathfrak{g}' \subset \mathfrak{g}$, see for instance [Z], and that $\mathfrak{g}$ equals the direct limit of its finite-dimensional subalgebras.

Theorem 2.1. *(a) Let M be an object of* $\mathrm{Int}_{\mathfrak{g}}$. *Then $\Gamma_{\mathfrak{g}}(M^*)$ is an injective object of* $\mathrm{Int}_{\mathfrak{g}}$.

(b) $\mathrm{Int}_{\mathfrak{g}}$ *has enough injectives. More precisely, for any object M of* $\mathrm{Int}_{\mathfrak{g}}$ *there is a canonical injective homomorphism of* $\mathfrak{g}-$*modules*

$$M \to \Gamma_{\mathfrak{g}}(\Gamma_{\mathfrak{g}}(M^*)^*).$$

Proof. In [PS], see Proposition 3.2 and Corollary 3.3, the proof is given under the assumption that $\mathfrak{g}$ is countable dimensional. The reader can check that this assumption is inessential. □

3 Five Subcategories of $\mathrm{Int}_{\mathfrak{g}}$

3.1 The Category $\mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$

We start by defining the full subcategory $\mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}} \subset \mathrm{Int}_{\mathfrak{g}}$. Its objects are integrable $\mathfrak{g}$-modules M such that for any simple finite-dimensional subalgebra $\mathfrak{g}' \subset \mathfrak{g}$, the restriction of M to $\mathfrak{g}'$ is a direct sum of finitely many $\mathfrak{g}'$-isotypic components. Clearly, if $\dim M = \infty$, at least one of these isotypic components must be infinite dimensional. If $\mathfrak{g}$ is diagonal, the adjoint representation of $\mathfrak{g}$ is easily seen to be an object of $\mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$.

The following proposition provides equivalent definitions of $\mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$.

Proposition 3.1. *(a) $M \in \mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$ iff M and M^* are integrable.*

(b) An integrable $\mathfrak{g}-$module M is an object of $\mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$ iff for any $X \in \mathfrak{g}$ there exists a nonzero polynomial $p(t) \in \mathbb{C}[t]$ such that $p(X) \cdot M = 0$.

Proof. (a) In the countable-dimensional case the statement is proven in [PS, Lemma 4.1]. In general, let $\mathfrak{g}' \subset \mathfrak{g}$ be a finite-dimensional simple subalgebra and let $M = \oplus_\alpha M_\alpha$ be the decomposition of M into $\mathfrak{g}'$-isotypic components. Then it is straightforward to check that $M^* = \prod_\alpha M_\alpha^*$ is an integrable $\mathfrak{g}'$-module iff the direct product is finite. This proves (a), since a $\mathfrak{g}$-module is integrable iff it is $\mathfrak{g}'$-integrable for all finite-dimensional Lie subalgebras $\mathfrak{g}' \subset \mathfrak{g}$.

(b) Let $M \in \mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$. Any $X \in \mathfrak{g}$ lies in some finite-dimensional Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$. For each $\mathfrak{g}'$-isotypic component M_i of M there exists $p_i(t)$ such that $p_i(X) \cdot M_i = 0$. Since there are finitely many $\mathfrak{g}'$-isotypic components, we can set $p(t) = \prod_i p_i(t)$. Then $p(X) \cdot M = 0$.

On the other hand, if $M \notin \mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$, then there are infinitely many isotypic components for some finite-dimensional simple $\mathfrak{g}' \subset \mathfrak{g}$. That implies the existence of a semisimple $X \in \mathfrak{g}'$ which has infinitely many eigenvalues in M. Therefore $p(X) \cdot M \neq 0$ for any $0 \neq p(t) \in \mathbb{C}[t]$. □

It is obvious that $\mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$ is an abelian monoidal subcategory of $\mathfrak{g}-$mod. It is also closed under dualization.

Proposition 3.2. $\mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$ *contains a nontrivial module iff* $\mathfrak{g}$ *is diagonal.*

Proof. Again, for a countable dimensional $\mathfrak{g}$ the statement is proven in [PS] (see Proposition 4.3). In fact, we prove in [PS] that if $\mathfrak{g} = \varinjlim \mathfrak{g}_i$ has a non-trivial integrable module such that M^* is also integrable, then the embedding $\mathfrak{g}_i \hookrightarrow \mathfrak{g}_{i+1}$ is diagonal for all sufficiently large i.

To give a general proof, it remains to show that if $\mathfrak{g}$ is not diagonal, then $\mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$ contains no nontrivial modules. Assume that $\mathfrak{g} = \varinjlim \mathfrak{g}_\alpha$ is not diagonal. Fix a simple finite-dimensional Lie algebra $\mathfrak{g}_{\alpha_1}$ and a simple $\mathfrak{g}$-module $M \in \mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$ such that $M_{\downarrow \mathfrak{g}_{\alpha_1}}$ is nontrivial. We claim that one can find a chain of proper embeddings of simple finite-dimensional Lie algebras

$$\mathfrak{g}_{\alpha_1} \hookrightarrow \mathfrak{g}_{\alpha_2} \hookrightarrow \cdots \hookrightarrow \mathfrak{g}_{\alpha_i} \hookrightarrow \mathfrak{g}_{\alpha_{i+1}} \hookrightarrow \cdots$$

such that the embeddings $\mathfrak{g}_{\alpha_i} \hookrightarrow \mathfrak{g}_{\alpha_{i+1}}$ are not diagonal. Indeed, otherwise there will exist β_0 so that the embedding $\mathfrak{g}_{\beta_0} \hookrightarrow \mathfrak{g}_\alpha$ is diagonal for all $\alpha > \beta_0$. Then, since $\mathfrak{g} = \varinjlim_{\alpha > \beta_0} \mathfrak{g}_\alpha$, $\mathfrak{g}$ is diagonal. This shows that the existence of β_0 is contradictory. Now Proposition 4.3 in [PS] implies that $M_{\downarrow \varinjlim \mathfrak{g}_{\alpha_i}}$ is a trivial module, which shows that the assumption that $M_{\downarrow \mathfrak{g}_{\alpha_1}}$ is nontrivial is false. □

Let $\mathfrak{g} = \mathfrak{sl}(V, W)$ (respectively, $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$). Then the tensor products $T^{m,n} := V^{\otimes m} \otimes W^{\otimes n}$ (respectively, $T^m := V^{\otimes m}$) and their subquotients are objects of $\mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$.

Here is a less trivial example of a simple object of $\mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$ for $\mathfrak{gl} = \mathfrak{sl}(V, V_*)$ where V is a countable-dimensional vector space. Let $\mathfrak{g} = \varinjlim \mathfrak{g}_i$ where $\mathfrak{g}_i = \mathfrak{sl}(V_i)$, $\dim V_i = i + 1$, and $\varinjlim V_i = V$. Define $\Lambda^{[\frac{\infty}{2}]} V$ as the direct limit $\varinjlim \Lambda^{[\frac{i}{2}]}(V_i)$ for $i \geq 2$. Then $\Lambda^{[\frac{\infty}{2}]} V$ is a simple object of $\mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$ and is not isomorphic to a subquotient of a tensor product of the form $T^{m,n}$.

Given a $\mathfrak{g}$-module $M \in \mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$, where $\mathfrak{g} = \varinjlim \mathfrak{g}_\alpha$, for each α we can assign to $\mathfrak{g}_\alpha$ the finite set of isomorphism classes of simple finite-dimensional $\mathfrak{g}_\alpha$-modules which occur in the restriction $M_{\downarrow \mathfrak{g}_\alpha}$. A. Zhilinskii has defined a *coherent local system of finite-dimensional representations* of $\mathfrak{g} = \varinjlim \mathfrak{g}_\alpha$ as a function of α with values in the set of isomorphism classes of finite-dimensional $\mathfrak{g}_\alpha$-modules, with the following compatibility condition: if $\beta < \alpha$, then the representations assigned to β are obtained by restriction from the representations assigned to α. Thus, every $M \in \mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$ determines a coherent local system of *finite type*, i.e., a local system containing finitely many isomorphism classes for any α.

Zhilinskii has classified all coherent local systems under the condition that $\mathfrak{g}$ is countable dimensional [Zh1, Zh2] (see also [PP] for an application of Zhilinskii's result). In particular, he has proved that proper coherent local systems, i.e., coherent local systems different from the ones assigning the trivial 1-dimensional module to all α, or all finite-dimensional $\mathfrak{g}_\alpha$-modules to α, exist only if $\mathfrak{g}$ is diagonal. This leads to another proof of Proposition 3.2.

The category $\mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$ has enough injectives: this follows immediately from Proposition 3.1 (a) and Theorem 2.1. We know of no classification of simple modules in $\mathrm{Int}_{\mathfrak{g}}^{\mathrm{alg}}$.

3.2 *The Category* $\mathbf{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathbf{wt}}$

Given a local Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we define $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}}$ as the full subcategory of $\mathrm{Int}_{\mathfrak{g}}$ consisting of $\mathfrak{h}$-*semisimple* integrable $\mathfrak{g}$-modules, i.e., integrable $\mathfrak{g}$-modules M admitting an $\mathfrak{h}$-weight decomposition

$$M = \oplus_{\lambda \in \mathfrak{h}^*} M^\lambda \tag{2}$$

where

$$M^\lambda := \{m \in M \mid h \cdot m = \lambda(h)m \quad \forall h \in \mathfrak{h}\}\,.$$

If $\mathfrak{g} = \mathfrak{sl}(V, W), \mathfrak{o}(V), \mathfrak{sp}(V)$ for countable-dimensional V, W, then V (and W in case $\mathfrak{g} = \mathfrak{sl}(V, W)$) is a simple object of $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}}$ for any $\mathfrak{h}$. Moreover, if $\mathfrak{g}$ is a countable-dimensional locally simple Lie algebra, it is proved in [PStr] that the adjoint representation of $\mathfrak{g}$ is an object of $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}}$ iff $\mathfrak{g} \simeq \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$. The simple modules of $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}}$ for $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$ have been studied in [DiP1], however there is no classification of such modules.

Assume that $\mathfrak{g}$ is a locally simple diagonal countable-dimensional Lie algebra. Without loss of generality, assume that $\mathfrak{g} = \varinjlim \mathfrak{g}_i$, where all $\mathfrak{g}_i$ are of the same type A, B, C, or D. The very definition of $\mathfrak{g}$ implies that there is a well-defined chain

$$V_{\mathfrak{g}_1} \overset{\kappa_1}{\hookrightarrow} V_{\mathfrak{g}_2} \overset{\kappa_2}{\hookrightarrow} \ldots \hookrightarrow V_{\mathfrak{g}_i} \overset{\kappa_i}{\hookrightarrow} V_{\mathfrak{g}_{i+1}} \hookrightarrow \ldots \tag{3}$$

of embeddings of natural $\mathfrak{g}_i$-modules, and we call its direct limit V a *natural representation* of $\mathfrak{g}$. Moreover, a fixed natural representation V is a simple object of $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}}$ for some local Cartan subalgebra $\mathfrak{h}$. To see this, we use induction to define a local Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ so that $V \in \mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}}$. Given $\mathfrak{h}_i \subset \mathfrak{g}_i$ and an $\mathfrak{h}_i$-eigenbasis $\mathbf{b}_i$ of V_i, let $\mathfrak{h}_{i+1}$ be a Cartan subalgebra of $\mathfrak{g}_{i+1}$ whose eigenbasis $\mathbf{b}_{i+1}$ of V_{i+1} contains $\mathbf{b}_i$. The assumption that $\mathfrak{g}_i$ and $\mathfrak{g}_{i+1}$ are of the same type A, B, C or D (in the sense of the classification of simple Lie algebras [Bou]) implies that $\mathfrak{h}_{i+1}$ exists as required. Moreover, $\mathfrak{h} := \varinjlim \mathfrak{h}_i$ is a well-defined local Cartan subalgebra of $\mathfrak{g}$ and $V \in \mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}}$.

Assume next that $\mathfrak{g}$ is a locally simple Lie algebra which admits a local Cartan subalgebra $\mathfrak{h}$ such that the adjoint representation belongs to $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}}$. This certainly holds for $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$, but also for instance for $\mathfrak{g} = \mathfrak{sl}(V, V_*)$ where V is an arbitrary vector space. In this case we can define a left exact functor $\Gamma_{\mathfrak{h}}^{\mathrm{wt}} : \mathrm{Int}_{\mathfrak{g}} \rightsquigarrow \mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}}$ by setting

$$\Gamma_{\mathfrak{h}}^{\mathrm{wt}}(M) := \oplus_{\lambda\in\mathfrak{h}^*} M^{\lambda},$$

where M^{λ} is given by (3). It is easy to see that $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}$ is right adjoint to the inclusion functor $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}} \rightsquigarrow \mathrm{Int}_{\mathfrak{g}}$. Hence $\Gamma_{\mathfrak{h}}^{\mathrm{wt}}$ maps injectives to injectives, and therefore $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}}$ has enough injectives. We do not know whether $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}}$ has enough injectives in the case when the adjoint representation is not an object of $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}}$.

We conjecture that for nondiagonal Lie algebras $\mathfrak{g}$, the category $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}}$ consists of trivial modules only.

3.3 *The Category* $\mathbf{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathbf{fin}}$

By $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{fin}}$ we denote the full subcategory of $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}}$ consisting of integrable $\mathfrak{g}$-modules satisfying $\dim M^{\lambda} < \infty \ \ \forall \lambda \in \mathfrak{h}^*$.

Note that for $\mathfrak{g} = \mathfrak{sl}(V, V_*)$ (respectively, for $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$) the tensor products $T^{m,0} = V^{\otimes m}$ and $T^{0,n} = W^{\otimes n}$ (respectively, $T^m = V^{\otimes m}$) are objects of $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{fin}}$ for every local Cartan subalgebra $\mathfrak{g}$. However, the adjoint representation is not in $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{fin}}$ for any $\mathfrak{h}$.

If $\mathfrak{g}$ is countable dimensional diagonal then, as shown above, for each natural representation V there is a local Cartan subalgebra $\mathfrak{h}$ so that V (and more generally $V^{\otimes m}$) is an object of $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{wt}}$. In fact, $V^{\otimes m} \in \mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{fin}}$ for any $m \geq 0$.

Here is a more interesting example of a simple module in $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{fin}}$ for $\mathfrak{g} = \mathfrak{sl}(V, V_*)$, where V is a countable-dimensional vector space. Fix a chain of embeddings

$$\mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2 \hookrightarrow \cdots \hookrightarrow \mathfrak{g}_i \hookrightarrow \mathfrak{g}_{i+1} \hookrightarrow \cdots$$

so that $\mathfrak{g} = \mathfrak{sl}(V_i)$ for $\dim V_i = i+1, V = \varinjlim V_i, \mathfrak{g} = \varinjlim \mathfrak{g}_i$. Note that there is a canonical injection of $\mathfrak{g}_i$-modules $S^{i+1}(V_i) \hookrightarrow S^{i+2}(V_{i+1})$, and set $\Delta := \varinjlim S^{i+1}(V_i)$. Then one can check that Δ is a multiplicity free $\mathfrak{h}$-module, where $\mathfrak{h}$ is such that $\mathfrak{h}_i := \mathfrak{h} \cap \mathfrak{g}_i$ is a Cartan subalgebra of $\mathfrak{g}_i$.

The following result is proved in [PS].

Proposition 3.3. *Let* $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$. *Then the category* $\mathrm{Int}_{\mathfrak{g},\mathfrak{h}}^{\mathrm{fin}}$ *is semi-simple.*

This result should be considered an extension of Weyl's semisimplicity theorem to the case of direct limit Lie algebras. It is an interesting question whether the category $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$ is semisimple whenever it is well defined.

3.4 The Category $\widetilde{\mathbf{Tens}}_{\mathfrak{g}}$

Let M be a $\mathfrak{g}$-module. Recall that the *socle* $\operatorname{soc} M = \operatorname{soc}^1 M$ of M is the unique maximal semisimple submodule of M, and

$$\operatorname{soc}^k M := \pi^{-1}(\operatorname{soc}(M/\operatorname{soc}^{k-1} M))$$

for $k \geq 2$, where $\pi : M \to M/\operatorname{soc}^k M$ is the natural projection. The ascending chain

$$0 \subset \operatorname{soc} M = \operatorname{soc}^1 M \subset \operatorname{soc}^2 M \subset \cdots \subset \operatorname{soc}^k M \subset \ldots$$

is by definition the *socle filtration* of M. The $\mathfrak{g}$-module M has *finite Loewy length* if it has a finite and exhaustive socle filtration, i.e.,

$$M = \operatorname{soc}^l M$$

for some l.

By definition, $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is the full subcategory of $\mathrm{Int}_{\mathfrak{g}}$ whose objects are integrable $\mathfrak{g}$-modules with the property that both M and $\Gamma_{\mathfrak{g}}(M^*)$ have finite Loewy length.

The category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is studied in detail in [PS] for $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$, where it is shown in particular that $\Gamma_{\mathfrak{g}}(M^*) = M^*$ for any object M of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$. A major result of [PS] is that, up to isomorphism, the simple objects of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ are precisely the simple subquotients of the tensor algebra $T(V \oplus V_*)$ for $\mathfrak{g} = \mathfrak{sl}(V, V_*) \simeq \mathfrak{sl}(\infty)$, and of the tensor algebra $T(V)$ for $\mathfrak{g} = \mathfrak{o}(V) \simeq \mathfrak{o}(\infty)$ or $\mathfrak{g} = \mathfrak{sp}(V) \simeq \mathfrak{sp}(\infty)$. These simple modules are discussed in more detail in Sect. 4 below. Note that the objects of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ have in general infinite length and are not objects of $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ for any $\mathfrak{h}$. An example of infinite length module in $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{sl}(V, V_*) \simeq \mathfrak{sl}(\infty)$ is V^*: there is a nonsplitting exact sequence of $\mathfrak{g}$-modules

$$0 \to V_* = \operatorname{soc} V^* \to V^* \to V^*/V_* \to 0$$

and V^*/V_* is a trivial module of uncountable dimension.

For $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$, the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ has enough injectives [PS, Corollary 6.7a)].

3.5 The Category $\mathbb{T}_{\mathfrak{g}}$

The fifth subcategory we would like to introduce in this section is the category of tensor modules $\mathbb{T}_{\mathfrak{g}}$. We define this category only for $\mathfrak{g} = \mathfrak{sl}(V, W), \mathfrak{o}(V), \mathfrak{sp}(V)$, and discuss it in detail in Sect. 5.

We call a subalgebra $\mathfrak{k} \subset \mathfrak{sl}(W, V)$ a *finite-corank subalgebra* if it contains the subalgebra $\mathfrak{sl}(W_0^{\perp}, V_0^{\perp})$ for some finite-dimensional nondegenerate pair $V_0 \subset V, W_0 \subset W$. Similarly, we call $\mathfrak{k} \subset \mathfrak{o}(V)$ (respectively, $\mathfrak{sp}(V)$) a *finite corank subalgebra* if it contains $\mathfrak{o}(V^{\perp})$ (respectively, $\mathfrak{sp}(V_0^{\perp})$) for some finite-dimensional $V_0 \subset V$ such that the restriction of the form on V_0 is nondegenerate.

We say that a $\mathfrak{g}$-module L *satisfies the large annihilator condition* if the annihilator in $\mathfrak{g}$ of any $l \in L$ contains a finite-corank subalgera. It follows immediately from definition that if L_1 and L_2 satisfy the large annihilator condition, then the same holds also for $L_1 \oplus L_2$ and $L_1 \otimes L_2$.

By $\mathbb{T}_{\mathfrak{g}}$ we denote the category of finite length integrable $\mathfrak{g}$-modules which satisfy the large annihilator condition. By definition, $\mathbb{T}_{\mathfrak{g}}$ is a full subcategory of $\mathrm{Int}_{\mathfrak{g}}$. It is clear that $\mathbb{T}_{\mathfrak{g}}$ is a monoidal category with respect to usual tensor product $\otimes$.

3.6 Inclusion Pattern

The following diagram summarizes the inclusion pattern for the five subcategories of $\mathrm{Int}_{\mathfrak{g}}$ introduced above:

$$\begin{tikzcd}
 & \widetilde{\mathrm{Tens}}_{\mathfrak{g}} \arrow[r, phantom, "\subset"] & \mathrm{Int}^{\mathrm{alg}}_{\mathfrak{g}} \arrow[dr, phantom, "\subset"] & \\
\mathbb{T}_{\mathfrak{g}} \arrow[ur, phantom, "\subset"] \arrow[drr, phantom, "\subset"] & & & \mathrm{Int}_{\mathfrak{g}} \\
 & & \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}} \arrow[ur, phantom, "\subset"] & \\
 & & \mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}} \arrow[u, phantom, "\cup"] &
\end{tikzcd}$$

Note that all categories except $\mathbb{T}_{\mathfrak{g}}$ are defined for any locally simple Lie algebra $\mathfrak{g}$, while $\mathbb{T}_{\mathfrak{g}}$ is defined only for $\mathfrak{g} = \mathfrak{sl}(V, W), \mathfrak{o}(V), \mathfrak{sp}(V)$. Moreover, under the latter assumption all inclusions are strict. We support this claim by a list of examples and leave it to the reader to complete the proof.

Examples. Let $\mathfrak{g} = \mathfrak{sl}(V, V_*), \mathfrak{o}(V), \mathfrak{sp}(V)$, where V is countable dimensional. The simple objects of $\mathbb{T}_{\mathfrak{g}}$ and $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ are the same, however $V^* \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ while $V^* \notin$

$\mathbb{T}_{\mathfrak{g}}$. Moreover, $V^* \notin \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ for any local Cartan subalgebra $\mathfrak{h}$. The module Δ from Sect. 3.3 is an object of $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$ but not an object of $\mathrm{Int}^{\mathrm{alg}}_{\mathfrak{g}}$. The adjoint representation is an object of $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ but not of $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$.

4 Mixed Tensors

In this section $\mathfrak{g} = \mathfrak{sl}(V, W), \mathfrak{o}(V), \mathfrak{sp}(V)$. By definition, V is a $\mathfrak{g}$-module. For $\mathfrak{g} = \mathfrak{sl}(V, W)$, W is also a $\mathfrak{g}$-module.

Consider the tensor algebra $T(V)$ of V. Then, as it is easy to see, finite-dimensional Schur duality implies that

$$T(V) = \bigoplus_{\lambda} \mathbb{C}_\lambda \otimes V_\lambda, \tag{4}$$

where λ runs over all Young diagrams (i.e., over all partitions of all integers $m \in \mathbb{Z}_{\geq 0}$), $\mathbb{C}_\lambda$ denotes the irreducible $S_{|\lambda|}$-module (where $|\lambda|$ is the degree of λ) corresponding to λ, and V_λ is the image of the Schur projector corresponding to λ. For $\mathfrak{g} = \mathfrak{sl}(V, W)$, V_λ is an irreducible $\mathfrak{g}$-module as it is isomorphic to the direct limit $\varinjlim (V_f)_\lambda$ of the directed system $\{(V_f)_\lambda\}$ of irreducible $\mathfrak{sl}(V_f, W_f)$-modules for sufficiently large nondegenerate finite-dimensional pairs $V_f \subset V, W_f \subset W$. For $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$, V_λ is in general a reducible $\mathfrak{g}$-module.

Similarly, for $\mathfrak{g} = \mathfrak{sl}(V, W)$,

$$T(W) = \bigoplus_{\lambda} \mathbb{C}_\lambda \otimes W_\lambda.$$

Let $\mathfrak{g} = \mathfrak{sl}(V, W)$. Recall that $T^{m,n} = V^{\otimes m} \otimes W^{\otimes n}$. Then

$$T^{m,n} = \bigoplus_{|\lambda|=n,\ |\mu|=m} \mathbb{C}_\lambda \otimes \mathbb{C}_\mu \otimes V_\lambda \otimes W_\mu.$$

Note that, as a $\mathfrak{g}$-module $T(V, W) := \bigoplus_{m,n\geq 0} T^{m,n}$ is not completely reducible. This follows simply from the observation that the exact sequence

$$0 \to \mathfrak{g} \to V \otimes W \to \mathbb{C} \to 0$$

does not split as $V \otimes W$ has no trivial submodule. In [PStyr] the structure of $T(V, W)$ has been studied in detail for countable-dimensional V and W.

For each ordered set $I = \{i_1, \ldots, i_k, j_1, \ldots, j_k\}$, where $i_1, \ldots, i_k \in \{1, \ldots, m\}$, $j_1, \ldots, j_k \in \{1, \ldots, n\}$, $k \leq \min\{m, n\}$, there is a well-defined surjective morphism of $\mathfrak{g}$-modules

$$\varphi_I : T^{m,n} \longrightarrow T^{m-k,n-k}$$

such that

$$\varphi_I(v_1 \otimes \cdots \otimes v_m \otimes w_1 \otimes \cdots \otimes w_n) = \prod_s \varphi(v_{i_s} \otimes w_{j_s})(\otimes_{i \neq i_s} v_i) \otimes (\otimes_{j \neq j_s} w_j)$$

for $s = 1, \ldots, k$, where $\varphi : V \otimes W \to \mathbb{C}$ is the linear operator induced by the pairing $V \times W \to \mathbb{C}$.

We now define a filtration of $T^{m,n}$ by setting

$$F_0^{m,n} := 0,\ \ F_k^{m,n} := \cap_I \ker \varphi_I \text{ for } k = 1, \ldots, \min\{m, n\},\ F^{m,n}_{\min\{m,n\}+1} := T^{m,n}, \tag{5}$$

where I runs over all ordered sets $\{i_1, \ldots, i_k, j_1, \ldots, j_k\}$ as above.

Let $|\lambda| = m$, $|\mu| = n$. We set

$$V_{\lambda,\mu} := F_1^{m,n} \cap (V_\lambda \otimes W_\mu).$$

Note that, for sufficiently large finite-dimensional nondegenerate pairs $V_f \subset V$, $W_f \subset W$, the $\mathfrak{sl}(V_f, W_f)$-module $T(V_f, W_f) \cap V_{\lambda,\mu}$ is simple. Therefore $V_{\lambda,\mu}$ is a simple $\mathfrak{sl}(V, W)$-module.

Theorem 4.1. *$\{F_k^{m,n}\}_{0 \leq k \leq \min\{m,n\}+1}$ is the socle filtration of $T^{m,n}$ as a $\mathfrak{sl}(V, W)$-module.*

Proof. In [PStyr] this theorem is proven in the countable-dimensional case. Here we give a proof for arbitrary V and W.

Recall that if M is a $\mathfrak{g}$-module, $M^{\mathfrak{g}}$ stands for the space of $\mathfrak{g}$-invariants in M.

Lemma 4.2. *Let $\mathfrak{g} = \mathfrak{sl}(V, W)$ (respectively, $\mathfrak{o}(V)$ or $\mathfrak{sp}(V)$). Then $(T^{m,n})^{\mathfrak{g}} = 0$ for $m + n > 0$ (respectively, $(T^m)^{\mathfrak{g}} = 0$ for $m > 0$).*

Proof. We prove the statement for $\mathfrak{g} = \mathfrak{sl}(V, W)$ and $m > 0$. The other cases are similar. Let $u \in T^{m,n} = V^{\otimes m} \otimes W^{\otimes n}$, $u \neq 0$. Then $u \in V_f^{\otimes m} \otimes W_f^{\otimes n}$ for some finite-dimensional nondegenerate pair $V_f \subset V$, $W_f \subset W$. Choose bases in V_f and W_f and write

$$u = \sum_{i=1}^{t} c_i v_1^i \otimes \cdots \otimes v_m^i \otimes w_1^i \otimes \cdots \otimes w_n^i,$$

where all v_j^i and w_j^i are basis vectors respectively of V_f and W_f. Pick $w \in W$ such that $\operatorname{tr}(v_1^1 \otimes w) = 1$ and $\operatorname{tr}(v_j^i \otimes w) = 0$ for all $v_j^i \neq v_1^1$. Let $v \in V \backslash V_f$ and $w \in W_f^{\perp}$. Then

$$(v \otimes w) \cdot u = \sum_{i=1}^{t} \sum_{j=1}^{m} c_i \operatorname{tr}(v_j^i \otimes w) v_1^i \otimes \cdots \otimes v_{j-1}^i \otimes v \otimes v_{j+1}^i \otimes \cdots \otimes v_m^i \otimes w_1^i \otimes \cdots \otimes w_n^i.$$

Our choice of v and w ensures that at least one term in the right-hand side is not zero and there is no repetition in the tensor monomials appearing with nonzero coefficients. That implies $(v \otimes w) \cdot u \neq 0$. Hence $u \notin (V^{\otimes m} \otimes W^{\otimes n})^{\mathfrak{g}}$. □

Lemma 4.3. *Let* $\mathfrak{g} = \mathfrak{sl}(V, W)$. *If* $\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda,\mu}, T^{m,n}) \neq 0$, *then* $|\lambda| = m$, $|\mu| = n$.

Proof. Choose a finite-dimensional nondegenerate pair $V_f \subset V$, $W_f \subset W$ such that $\dim V_f \geq \max\{m, n, |\lambda|, |\mu|\}$. Then $(V_f)_{\lambda,\mu} := T(V_f, W_f) \cap V_{\lambda,\mu}$ is annihilated by the finite corank subalgebra $\mathfrak{k} = \mathfrak{sl}(W_f^{\perp}, V_f^{\perp})$ of $\mathfrak{g}$. Let $\mathfrak{l} = \mathfrak{sl}(V_f, W_f) \oplus \mathfrak{k}$. Then

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{l}}((V_f)_{\lambda,\mu}, T^{m,n}) &= \mathrm{Hom}_{\mathfrak{sl}(V_f,W_f)}((V_f)_{\lambda,\mu}, (T^{m,n})^{\mathfrak{k}}) \\ &= \mathrm{Hom}_{\mathfrak{sl}(V_f,W_f)}((V_f)_{\lambda,\mu}, V_f^{\otimes m} \otimes W_f^{\otimes n}).\end{aligned}$$

Therefore a homomorphism $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda,\mu}, T^{m,n})$ has a well-defined restriction $\varphi_f \in \mathrm{Hom}_{\mathfrak{sl}(V_f,W_f)}((V_f)_{\lambda,\mu}, V_f^{\otimes m} \otimes W_f^{\otimes n})$. According to finite-dimensional representation theory, $\varphi_f \neq 0$ implies that φ_f is a composition

$$(V_f)_{\lambda,\mu} \to (V_f^{\otimes|\lambda|} \otimes W_f^{\otimes|\mu|}) \otimes (V_f^{\otimes(m-|\lambda|)} \otimes W_f^{\otimes(n-|\mu|)})^{\mathfrak{sl}(V_f,W_f)} \to V_f^{\otimes m} \otimes W_f^{\otimes n}.$$

Since φ is the inverse limit of φ_f, φ is a composition

$$V_{\lambda,\mu} \to T^{|\lambda|,|\mu|} \otimes (T^{m-|\lambda|,n-|\mu|})^{\mathfrak{sl}(V,W)} \to T^{m,n}.$$

However, by Lemma 4.2, $(T^{m-|\lambda|,n-|\mu|})^{\mathfrak{sl}(V,W)} \neq 0$ only if $|\lambda| = m$, $|\mu| = n$. □

Note that Lemma 4.3 implies

$$\mathrm{soc}T^{m,n} = \mathrm{soc}^1 T^{m,n} = F_1^{m,n}. \tag{6}$$

Consider now the exact sequence

$$0 \to F_{k-1}^{m,n} \to T^{m,n} \to \bigoplus_I T^{m-k+1,n-k+1}, \tag{7}$$

where I runs over the same set as in (5). It follows from (6) that (7) induces an exact sequence

$$0 \to F_{k-1}^{m,n} \to F_k^{m,n} \to \bigoplus_I F_1^{m-k+1,n-k+1}.$$

Therefore induction on k yields $\mathrm{soc}^k\, T^{m,n} = F_k^{m,n}$. Theorem 4.1 is proved. □

As a corollary we obtain that the $\mathfrak{sl}(V, W)$-module $V_\lambda \otimes W_\mu$ is indecomposable since its socle $V_{\lambda,\mu}$ is simple. Further one shows that any simple subquotient of $T(V, W)$ is isomorphic to $V_{\lambda,\mu}$ for an appropriate pair of partitions λ, μ. The k-th layer of the socle filtration of $V_\lambda \otimes W_\mu$, i.e., the quotient $\mathrm{soc}^k(V_\lambda \otimes W_\mu)/\mathrm{soc}^{k-1}(V_\lambda \otimes W_\mu)$, can have only simple constituents isomorphic to $V_{\lambda',\mu'}$ where λ' is obtained from λ by removing $k-1$ boxes and μ' is obtained from μ by removing $k-1$ boxes. An explicit formula for the multiplicity of $V_{\lambda'\mu'}$ in $\mathrm{soc}^k(V_\lambda \otimes W_\mu)/\mathrm{soc}^{k-1}(V_\lambda \otimes W_\mu)$ is given in [PStyr].

Next, consider the associative algebra $\mathcal{A}_{\mathfrak{sl}(V,W)} \subset \mathrm{End}_{\mathfrak{sl}(V,W)}(T(V,W))$ generated by all contractions $\varphi_{i,j}$ and by the direct sum of group algebras $\bigoplus_{m,n\geq 0}\mathbb{C}[S_m \times S_n]$. It is clear that $\mathcal{A}_{\mathfrak{sl}(V,W)}$ does not depend on the choice of the linear system $V \times W \to \mathbb{C}$. In what follows we use the notation $\mathcal{A}_{\mathfrak{sl}}$. One can equip $\mathcal{A}_{\mathfrak{sl}}$ with a $\mathbb{Z}_{\geq 0}$-grading $\mathcal{A}_{\mathfrak{sl}} = \bigoplus_{q\geq 0}(\mathcal{A}_{\mathfrak{sl}})_q$ by setting $(\mathcal{A}_{\mathfrak{sl}})_q := \bigoplus_{m,n\geq 0}\mathrm{Hom}_{\mathfrak{sl}(V,W)}(T^{m,n}, T^{m-q,n-q}) \cap A_{\mathfrak{sl}}$. If we set $T^{\leq r}(V,W) := \bigoplus_{m+n\leq r} T^{m,n}$ and denote by $\mathcal{A}^{(r)}_{\mathfrak{sl}}$ the intersection of $\mathcal{A}_{\mathfrak{sl}}$ with $\mathrm{End}_{\mathfrak{sl}(V,W)}(T^{\leq r}(V,W))$, then, obviously, $\mathcal{A}_{\mathfrak{sl}} = \varinjlim \mathcal{A}^{(r)}_{\mathfrak{sl}}$.

The following statement is a central result in [DPS].

Proposition 4.4. *(a) If V is countable dimensional, then*

$$(\mathcal{A}_{\mathfrak{sl}})_q = \bigoplus_{m,n\geq 0} \mathrm{Hom}_{\mathfrak{sl}(V,V_*)}(T^{m,n}, T^{m-q,n-q}).$$

(b) $\mathcal{A}^{(r)}_{\mathfrak{sl}}$ is a Koszul self-dual ring for any $r \geq 0$.

Now let $\mathfrak{g} = \mathfrak{o}(V)$ (respectively, $\mathfrak{sp}(V)$). Recall that $T^m = V^{\otimes m}$. Assume $m \geq 2$. For a pair of indices $1 \leq i < j \leq m$ we have a contraction map $\varphi_{i,j} \in \mathrm{Hom}_{\mathfrak{g}}(V^{\otimes m}, V^{\otimes m-2})$. If V is countable dimensional, the socle filtration of $T(V)$ considered as a $\mathfrak{g}$-module is described in [PStyr]. Recall the decomposition (4). Each V_λ is an indecomposable $\mathfrak{g}$-module with simple socle which we denote by $V_{\lambda,\mathfrak{g}}$. Moreover,

$$\mathrm{soc}^k\, V_\lambda = \mathrm{soc}^k\,(V_\lambda \cap V^{\otimes|\lambda|}) = V_\lambda \cap (\cap_{I_1,\dots,I_k}\ker(\varphi_{I_1,\dots,I_k} : V^{\otimes|\lambda|} \to V^{\otimes|\lambda|-2k})),$$

where $I_1,\dots,I_k$ run over all sets of k distinct pairs of indices $1,\dots,|\lambda|$ and $\varphi_{I_1,\dots,I_k} = \varphi_{I_1} \circ \cdots \circ \varphi_{I_k}$.

Next, let $\mathcal{A}_{\mathfrak{g}} \subset \mathrm{End}_{\mathfrak{g}}(T(V))$ be the graded subalgebra of $\mathrm{End}_{\mathfrak{g}}(T(V))$ generated by $\bigoplus_{m\geq 0}\mathbb{C}[S_m]$ and the contractions $\varphi_{i,j}$. We define a $\mathbb{Z}_{\geq 0}$- grading $\mathcal{A}_{\mathfrak{g}} = \bigoplus_{q\geq 0}(\mathcal{A}_{\mathfrak{g}})_q$ by setting

$$(\mathcal{A}_{\mathfrak{g}})_q := \bigoplus_{m\geq 0}\mathrm{Hom}_{\mathfrak{g}}(T^m, T^{m-2q}) \cap \mathcal{A}_{\mathfrak{g}}.$$

If we set $T^{\leq r}(V) := \bigoplus_{m\leq r} T^m$ and denote by $\mathcal{A}^{(r)}_{\mathfrak{g}}$ the intersection of $\mathcal{A}_{\mathfrak{g}}$ with $\mathrm{End}_{\mathfrak{g}}(T^{\leq r}(V))$, then $\mathcal{A}_{\mathfrak{g}} = \varinjlim \mathcal{A}^{(r)}_{\mathfrak{g}}$. It is clear that the algebra $\mathcal{A}_{\mathfrak{g}}$ can depend only on the symmetry type of the form on V but not on V and the form itself. This justifies the notations $\mathcal{A}_{\mathfrak{o}}$ and $\mathcal{A}_{\mathfrak{sp}}$.

Proposition 4.5 ([DPS]).

(a) $\mathcal{A}_{\mathfrak{o}}^{(r)} \simeq \mathcal{A}_{\mathfrak{sp}}^{(r)}$ *for each* $r \geq 0$, *and* $\mathcal{A}_{\mathfrak{o}} \simeq \mathcal{A}_{\mathfrak{sp}}$.
(b) *If* V *is countable dimensional, then* $(\mathcal{A}_{\mathfrak{o}})_q = \bigoplus_{m\geq 0} \mathrm{Hom}_{\mathfrak{o}(V)}(T^m, T^{m-2q})$, $(\mathcal{A}_{\mathfrak{sp}})_q = \bigoplus_{m\geq 0} \mathrm{Hom}_{\mathfrak{sp}(V)}(T^m, T^{m-2q})$.
(c) $\mathcal{A}_{\mathfrak{o}}^{(r)} \simeq \mathcal{A}_{\mathfrak{sp}}^{(r)}$ *is a Koszul ring for any* $r \geq 0$.

In each of the three cases $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$ we call the modules $V_{\lambda,\mu}$, respectively $V_{\lambda,\mathfrak{g}}$, the *simple tensor modules* of $\mathfrak{g}$.

5 The Category $\mathbb{T}_{\mathfrak{g}}$

5.1 The Countable-Dimensional Case

In this subsection we assume that $\mathfrak{g} = \mathfrak{sl}(V, V_*)$, $\mathfrak{o}(V)$ or $\mathfrak{sp}(V)$ for a countable-dimensional space V. The category $\mathbb{T}_{\mathfrak{g}}$ has been studied in [DPS], and here we review some key results.

Denote by $\tilde{G}$ the group of automorphisms of V under which V_* is stable for $\mathfrak{g} = \mathfrak{sl}(V, V_*)$, and the group of automorphisms of V which keep fixed the form on V which defines $\mathfrak{g}$. The group $\tilde{G}$ is a subgroup of $\mathrm{Aut}\mathfrak{g}$ and therefore acts naturally on isomorphism classes of $\mathfrak{g}$-modules: to each $\mathfrak{g}$-module M one assigns the twisted $\mathfrak{g}$-module $M_{\tilde{g}}$ for $\tilde{g} \in \tilde{G}$. A $\mathfrak{g}$-module M is $\tilde{G}$*-invariant* if $M \simeq M_{\tilde{g}}$ for all $\tilde{g} \in \tilde{G}$.

Furthermore, define a $\mathfrak{g}$-module M to be an *absolute weight module* if the decomposition (2) holds for any local Cartan subalgebra of $\mathfrak{g}$, i.e., if M is a weight module for any local Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. In [DPS] we have given five equivalent characterizations of the objects of $\mathbb{T}_{\mathfrak{g}}$.

Theorem 5.1 ([DPS]). *The following conditions on a* $\mathfrak{g}$*-module* M *of finite length are equivalent:*

(i) M *is an object of* $\mathbb{T}_{\mathfrak{g}}$*;*
(ii) M *is a weight module for some local Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *and* M *is* $\tilde{G}$*-invariant;*
(iii) M *is a subquotient of* $T(V \oplus V_*)$ *for* $\mathfrak{g} = \mathfrak{sl}(V, V_*)$ *(respectively, of* $T(V)$ *for* $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$*);*
(iv) M *is a submodule of* $T(V \oplus V_*)$ *for* $\mathfrak{g} = \mathfrak{sl}(V, V_*)$ *(respectively, of* $T(V)$ *for* $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$*);*
(v) M *is an absolute weight module.*

Furthermore, the following two theorems are crucial for understanding the structure of $\mathbb{T}_{\mathfrak{g}}$.

Theorem 5.2 ([PS, DPS]). *The simple objects in the categories* $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ *and* $\mathbb{T}_{\mathfrak{g}}$ *coincide and are all of the form* $V_{\lambda,\mu}$ *for* $\mathfrak{g} = \mathfrak{sl}(V, V_*)$*, or respectively* $V_{\lambda,\mathfrak{g}}$ *for* $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$.

Theorem 5.3 ([DPS]).

(a) $\mathbb{T}_{\mathfrak{g}}$ *has enough injectives. If* $\mathfrak{g} = \mathfrak{sl}(V, V_*)$*, then* $V_\lambda \otimes (V_*)_\mu$ *is an injective hull of* $V_{\lambda,\mu}$*. If* $\mathfrak{g} = \mathfrak{o}(V)$ *or* $\mathfrak{sp}(V)$*, then* V_λ *is an injective hull of* $V_{\lambda,\mathfrak{g}}$*.*
(b) $\mathbb{T}_{\mathfrak{g}}$ *is anti-equivalent to the category of locally unitary finite-dimensional* $\mathcal{A}_{\mathfrak{g}}$*−modules.*

Theorem 5.3 means that the category $\mathbb{T}_{\mathfrak{g}}$ is "Koszul" in the sense that it is anti-equivalent to a module category over the infinite-dimensional Koszul algebra $\mathcal{A}_{\mathfrak{g}}$.

Corollary 5.4. $\mathbb{T}_{\mathfrak{o}(\infty)}$ *and* $\mathbb{T}_{\mathfrak{sp}(\infty)}$ *are equivalent abelian categories.*

In fact, the stronger result that $\mathbb{T}_{\mathfrak{o}(\infty)}$ and $\mathbb{T}_{\mathfrak{sp}(\infty)}$ are equivalent as monoidal categories also holds, see [SS] and [S].

5.2 The General Case

In this subsection we prove the following result.

Theorem 5.5. *Let* $\mathfrak{g} = \mathfrak{sl}(V, W), \mathfrak{o}(V), \mathfrak{sp}(V)$*. Then, as a monoidal category,* $\mathbb{T}_{\mathfrak{g}}$ *is equivalent to* $\mathbb{T}_{\mathfrak{sl}(\infty)}$ *or* $\mathbb{T}_{\mathfrak{o}(\infty)}$*.*

The proof of Theorem 5.5 is accomplished by proving several lemmas and corollaries.

Lemma 5.6. *(a) Let* $\mathfrak{g} = \mathfrak{sl}(V, W)$ *and* $C_{m,n} := \mathrm{Hom}_{\mathfrak{g}}(T^{m,n}, \mathbb{C})$*. If* $m \neq n$*, then* $C_{m,n} = 0$*, and if* $m = n$*, then* $C_{m,m}$ *is spanned by* τ_π *for all* $\pi \in S_m$*, where*

$$\tau_\pi(v_1 \otimes \cdots \otimes v_m \otimes w_1 \otimes \cdots \otimes w_m) = \prod_{i=1}^{m} \mathrm{tr}(v_i \otimes w_{\pi(i)}).$$

(b) Let $\mathfrak{g} = \mathfrak{o}(V)$ *or* $\mathfrak{sp}(V)$*. Then* $\mathrm{Hom}_{\mathfrak{g}}(T^{2m+1}, \mathbb{C}) = 0$ *and* $\mathrm{Hom}_{\mathfrak{g}}(T^{2m}, \mathbb{C})$ *is spanned by* σ_π *for all* $\pi \in S_m$*, where*

$$\sigma_\pi(v_1 \otimes \cdots \otimes v_{2m}) = \prod_{i=1}^{m} (v_i, v_{m+\pi(i)}).$$

Proof. In the finite-dimensional case the same statement is the fundamental theorem of invariant theory. Since $T^{m,n}$ for $\mathfrak{g} = \mathfrak{sl}(V, W)$ (respectively, T^m for $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$) is a direct limit of finite-dimensional representations of the same type, the statement follows from the fundamental theorem of invariant theory. □

Let L be a $\mathfrak{g}$-module and let $\mathfrak{g}'$ denote a subalgebra of $\mathfrak{g}$ of the form $\mathfrak{sl}(V', W')$ (respectively, $\mathfrak{o}(V'), \mathfrak{sp}(V')$) for some nondegenerate pair $V' \subset V, W' \subset W$

(respectively, nondegenerate subspace $V' \subset V$). Let (V'_f, W'_f) be a finite-dimensional nondegenerate pair satisfying $V'_f \subset V'$, $W'_f \subset W'$ (respectively, $V'_f \subset V'$) and let $\mathfrak{k}' = \mathfrak{sl}((W'_f)^\perp, (V'_f)^\perp) \subset \mathfrak{g}$ (respectively $\mathfrak{k}' = \mathfrak{o}((V'_f)^\perp), \mathfrak{sp}(V'_f)^\perp)$). Then $L^{\mathfrak{k}'}$ is an $\mathfrak{sl}(W'_f, V'_f)$-module (respectively, an $\mathfrak{o}(V'_f)-$ or $\mathfrak{sp}(V'_f)$-module), and moreover if we let $\mathfrak{k}'$ vary, the corresponding $\mathfrak{sl}(V'_f, W'_f)$-modules (respectively, $\mathfrak{o}(V'_f)-$ or $\mathfrak{sp}(V'_f)$-modules) form a directed system whose direct limit

$$\Gamma^{ann}_{\mathfrak{g}'}(L) = \varinjlim L^{\mathfrak{k}'}$$

is a $\mathfrak{g}'$-module. Note that $\Gamma^{ann}_{\mathfrak{g}'}(L)$ may simply be defined as the union $\bigcup_{\mathfrak{k}'} L^{\mathfrak{k}'}$ of subspaces $L^{\mathfrak{k}'} \subset L$.

It is easy to check that $\Gamma^{ann}_{\mathfrak{g}'}$ is a well-defined functor from the category $\mathfrak{g}-$mod to its subcategory of $\mathfrak{g}'-$mod consisting of modules satisfying the large annihilator condition. In particular, $\Gamma^{ann}_{\mathfrak{g}}$ is a well-defined functor from $\mathfrak{g}-$mod to the category of $\mathfrak{g}$-modules satisfying the large annihilator condition, and the restriction of $\Gamma^{ann}_{\mathfrak{g}}$ to $\mathbb{T}_{\mathfrak{g}}$ is the identity functor.

In the case when $\mathfrak{g}'$ is finite dimensional the functor $\Gamma^{ann}_{\mathfrak{g}'}$ and its right derived functors are studied in detail in [SSW].

Lemma 5.7. *(a) Let $\mathfrak{g} = \mathfrak{sl}(V, W)$; then*

$$\Gamma^{ann}_{\mathfrak{g}}((T^{m,n})^*) \simeq \bigoplus_{k \geq 0} b_k T^{n-k,m-k}$$

where $b_k = \binom{m}{k}\binom{n}{k}k!$.
(b) Let $\mathfrak{g} = \mathfrak{o}(V)$ or $\mathfrak{sp}(V)$, then

$$\Gamma^{ann}_{\mathfrak{g}}((T^{m})^*) \simeq \bigoplus_{k \geq 0} c_k T^{m-2k}$$

where $c_k = \binom{m}{2k}k!$.

Proof. We prove (a) and leave (b) to the reader. Choose a finite-dimensional nondegenerate pair $V_f \subset V, W_f \subset W$, and let $\mathfrak{k} = \mathfrak{sl}(W_f^\perp, V_f^\perp)$. There is an isomorphism of $\mathfrak{k}$-modules

$$(T^{m,n})^* = (V^{\otimes m} \otimes W^{\otimes n})^* \simeq \bigoplus_{k \geq 0, l \geq 0} d_{k,l}(W_f^{\otimes m-k} \otimes V_f^{\otimes n-l}) \otimes ((V_f^\perp)^{\otimes k} \otimes (W_f^\perp)^{\otimes l})^* \tag{8}$$

where $d_{k,l} = \binom{m}{k}\binom{n}{l}$.

Using (8) and Lemma 5.6 (a) applied to $\mathfrak{k}$ in place of $\mathfrak{g}$, we compute that

$$\big((T^{m,n})^*\big)^{\mathfrak{k}} \simeq \bigoplus_{k\geq 0} b_k(W_f^{\otimes m-k} \otimes V_f^{\otimes n-k}).$$

Now the statement follows by taking the direct limit of $\mathfrak{k}$-invariants over all nondegenerate finite-dimensional pairs $V_f \subset V$, $W_f \subset W$. □

Corollary 5.8. *$T^{m,n}$ is an injective object of $\mathbb{T}_{\mathfrak{sl}(V,W)}$, and T^m is an injective object of $\mathbb{T}_{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$.*

Proof. We consider only the case $\mathfrak{g} = \mathfrak{sl}(V, W)$. Recall (Theorem 2.1) that if M is an integrable module such that M^* is integrable, then M^* is injective in $\mathrm{Int}_{\mathfrak{g}}$. In particular, $(T^{m,n})^*$ is injective in $\mathrm{Int}_{\mathfrak{g}}$. Next, note that $\Gamma_{\mathfrak{g}}^{ann}$ is right adjoint to the inclusion functor $\mathbb{T}_{\mathfrak{g}} \rightsquigarrow \mathrm{Int}_{\mathfrak{g}}$, i.e., for any $L \in \mathbb{T}_{\mathfrak{g}}$ and any $Y \in \mathrm{Int}_{\mathfrak{g}}$, we have

$$\mathrm{Hom}_{\mathfrak{g}}(L, Y) = \mathrm{Hom}_{\mathfrak{g}}(L, \Gamma_{\mathfrak{g}}^{ann}(Y)).$$

Hence, $\Gamma_{\mathfrak{g}}^{ann}$ transforms injectives in $\mathrm{Int}_{\mathfrak{g}}$ to injectives in $\mathbb{T}_{\mathfrak{g}}$. This implies that $\Gamma_{\mathfrak{g}}^{ann}((T^{n,m})^*)$ is injective in $\mathbb{T}_{\mathfrak{g}}$. By Lemma 5.7, $T^{m,n}$ is a direct summand in $\Gamma_{\mathfrak{g}}^{ann}((T^{n,m})^*)$, and the statement follows. □

Next we impose the condition that our fixed subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ is countable dimensional. In the rest of the paper we set $\mathfrak{g}_c := \mathfrak{g}'$. More precisely, we choose strictly increasing chains of finite-dimensional subspaces

$$V_1 \subset V_2 \subset \ldots \subset V_i \subset V_{i+1} \subset \ldots, \quad W_1 \subset W_2 \subset \ldots \subset W_i \subset W_{i+1} \subset \ldots$$

and set $\mathfrak{g}_c = \mathfrak{sl}(V_c, W_c)$ where $V_c := \varinjlim V_i$, $W_c := \varinjlim W_i$. It is clear that $V_c \times W_c \to \mathbb{C}$ is a countable-dimensional linear system, hence $\mathfrak{g}_c \simeq \mathfrak{sl}(\infty)$. If $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$, choose a strictly increasing chain of nondegenerate finite-dimensional subspaces $V_1 \subset V_2 \subset \ldots \subset V_i \subset V_{i+1} \subset \ldots$ and set $V_c := \varinjlim V_i$, $\mathfrak{g}_c = \mathfrak{o}(V_c), \mathfrak{sp}(V_c)$.

By Φ we denote the restriction of $\Gamma_{\mathfrak{g}_c}^{ann}$ to $\mathbb{T}_{\mathfrak{g}}$. Note that for any $L \in \mathbb{T}_{\mathfrak{g}}$, $\Phi(L)$ is a $\mathfrak{g}_c$-submodule of L.

Lemma 5.9. *Let $L, L' \in \mathbb{T}_{\mathfrak{g}}$.*

(a) $\Phi(L)$ generates L.
(b) The homomorphism $\Phi(L, L') : \mathrm{Hom}_{\mathfrak{g}}(L, L') \to \mathrm{Hom}_{\mathfrak{g}_c}(\Phi(L), \Phi(L'))$ is injective.

Proof. Again we consider only the case $\mathfrak{g} = \mathfrak{sl}(V, W)$ since the other cases are similar. Let $SL(V, W)$ denote the direct limit group $\varinjlim SL(V_f, W_f)$ for all nondegenerate finite-dimensional pairs $V_f \subset V, W_f \subset W$, where $SL(V_f, W_f) \simeq SL(\dim V_f)$.

(a) Since L has finite length and satisfies the large annihilator condition, there is a finite-dimensional nondegenerate pair $V_f \subset V$, $W_f \subset W$ and a finite-dimensional $\mathfrak{gl}(V_f, W_f)$-submodule $L_f \subset L$ annihilated by $\mathfrak{sl}((W_f)^\perp, (V_f)^\perp)$ such that L is generated by L_f over $\mathfrak{g}$. Choose i so that $\dim V_f < \dim V_i$. Then there exists $g \in SL(V, W)$ such that $g(V_f) \subset V_i$, $g(W_f) \subset W_i$. Note that $g = \exp x$ for some $x \in \mathfrak{sl}(V, W)$. By the integrability of L as a $\mathfrak{g}$-module, the action of g is well defined on L, and $g(L_f)$ also generates L over $\mathfrak{g}$. On the other hand, by construction $g(L_f)$ is annihilated by $g\mathfrak{sl}((W_f)^\perp, (V_f)^\perp)g^{-1}$. Observe that

$$\mathfrak{sl}((W_i)^\perp, (V_i)^\perp) \subset \mathfrak{sl}(g(W_f)^\perp, g(V_f)^\perp) = g\mathfrak{sl}((W_f)^\perp, (V_f)^\perp)g^{-1}.$$

Hence $g(L_f) \subset \Phi(L)$. The statement follows.

(b) Follows immediately from (a). □

Lemma 5.10. *(a) $\Phi(T^{m,n}) = V_c^{\otimes m} \otimes W_c^{\otimes n}$ for $\mathfrak{g} = \mathfrak{sl}(V, W)$, and $\Phi(T^m) = V_c^{\otimes m}$ for $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$;*

(b) The homomorphisms

$$\Phi(T^{m,n}, T^{k,l}) : \mathrm{Hom}_{\mathfrak{g}}(T^{m,n}, T^{k,l}) \to \mathrm{Hom}_{\mathfrak{g}_c}(V_c^{\otimes m} \otimes W_c^{\otimes n}, V_c^{\otimes k} \otimes W_c^{\otimes l})$$

for $\mathfrak{g} = \mathfrak{sl}(V, W)$, and

$$\Phi(T^m, T^k) : \mathrm{Hom}_{\mathfrak{g}}(T^m, T^k) \to \mathrm{Hom}_{\mathfrak{g}_c}(V_c^{\otimes k}, V_c^{\otimes k})$$

for $\mathfrak{g} = \mathfrak{o}(V)$ or $\mathfrak{sp}(V)$, are isomorphisms.

(c) Let $X \subset \bigoplus_i V_c^{\otimes m_i} \otimes W_c^{\otimes n_i}$, (respectively, $X \subset \bigoplus_i V_c^{m_i}$ for $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$) be a $\mathfrak{g}_c$-submodule. Then $\Phi(U(\mathfrak{g}) \cdot X) = X$.

(d) If $X \subset V_c^{\otimes m} \otimes W_c^{\otimes n}$ (respectively, $X \subset \bigoplus_i V_c^{m_i}$ for $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$) is a simple submodule, then $U(\mathfrak{g}) \cdot X$ is a simple $\mathfrak{g}$-module.

Proof. (a) follows easily from the observation that

$$(T^{m,n})^{\mathfrak{k}} = V_i^{\otimes m} \otimes W_i^{\otimes n}$$

for any finite corank subalgebra $\mathfrak{k} = \mathfrak{sl}(W_i^\perp, V_i^\perp)$. This observation is a straightforward consequence of Lemma 4.2.

To prove (b), note that the injectivity of the homomorphisms $\Phi(T^{m,n}, T^{k,l})$ follows from (a) and Lemma 5.9 (b). To prove surjectivity, we observe that $\mathrm{Hom}_{\mathfrak{g}_c}(V_c^{\otimes m} \otimes W_c^{\otimes n}, V_c^{\otimes k} \otimes W_c^{\otimes l})$ is generated by permutations and contractions according to Proposition 4.5 (b). Both are defined in $\mathrm{Hom}_{\mathfrak{g}}(T^{m,n}, T^{k,l})$ by the same formulae. Therefore the homomorphisms $\Phi(T^{m,n}, T^{k,l})$ are surjective.

We now prove (c). Note that $X = \ker\alpha$ for some $\alpha \in \mathrm{Hom}_{\mathfrak{g}_c}(\bigoplus_i V_c^{\otimes m_i} \otimes W_c^{\otimes n_i}, \bigoplus_j V_c^{\otimes m_j} \otimes W_c^{\otimes n_j})$. Using (b) we have $U(\mathfrak{g}) \cdot X \subset \ker\Phi^{-1}(\alpha)$. Hence, $\Phi(U(\mathfrak{g}) \cdot X) \subset \ker\alpha = X$. Since the inclusion $X \subset \Phi(U(\mathfrak{g}) \cdot X)$ is obvious, the statement follows.

To prove (d), suppose $U(\mathfrak{g}) \cdot X$ is not simple, i.e., there is an exact sequence

$$0 \to L \to U(\mathfrak{g}) \cdot X \to L' \to 0$$

for some nonzero L, L'. By the exactness of Φ and by (c), we have an exact sequence

$$0 \to \Phi(L) \to X \to \Phi(L') \to 0.$$

By Lemma 5.9 (a), $\Phi(L)$ and $\Phi(L')$ are both nonzero. This contradicts the assumption that X is simple. □

Lemma 5.11. *For $\mathfrak{g} = \mathfrak{sl}(V, W)$ (respectively, for $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$) any simple object in the category $\mathbb{T}_{\mathfrak{g}}$ is isomorphic to a submodule in $T^{m,n}$ for suitable m and n (respectively, in T^m for a suitable m).*

Proof. We assume that $\mathfrak{g} = \mathfrak{sl}(V, W)$ and leave the other cases to the reader. Let L be a simple module in $\mathbb{T}_{\mathfrak{g}}$. By Lemma 5.9 (a), $\Phi(L) \neq 0$. Let $L_i = L^{\mathfrak{sl}(W_i^{\perp}, V_i^{\perp})} \neq 0$ for some i, and let $L' \subset L_i$ be a simple $\mathfrak{sl}(V_i, W_i)$-submodule. Consider the $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1$ where $\mathfrak{g}^0 = \mathfrak{gl}(V_i, W_i) \oplus \mathfrak{sl}(W_i^{\perp}, V_i^{\perp})$, $\mathfrak{g}^1 = V_i \otimes V_i^{\perp}$, $\mathfrak{g}^{-1} = W_i^{\perp} \otimes W_i$. There exists a finite-dimensional subspace $W' \subset V_i^{\perp}$, such that $S(V_i \otimes W')$ generates $S(\mathfrak{g}^1)$ as a module over $\mathfrak{sl}(W_i^{\perp}, V_i^{\perp})$. By the integrability of L, $(V_i \otimes W')^q \cdot L' = 0$ for sufficiently large $q \in \mathbb{Z}_{\geq 0}$, and thus $(\mathfrak{g}^1)^q \cdot L' = 0$. Hence, there is a nonzero vector $l \in L_i \subset L$ annihilated by $\mathfrak{g}^1$, and consequently there is a simple $\mathfrak{g}^0$-submodule $L'' \subset L$ annihilated by $\mathfrak{g}^1$. Therefore L is isomorphic to a quotient of the parabolically induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}^0 \oplus \mathfrak{g}^1)} L''$. The latter module is a direct limit of parabolically induced modules for finite-dimensional subalgebras of $\mathfrak{g}$. Hence it has a unique integrable quotient, and this quotient is isomorphic to L. On the other hand, L'' is a simple $\mathfrak{g}_0$-submodule of $T^{m,n}$ for some m and n. Thus, by Frobenius reciprocity, a quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}^0 \oplus \mathfrak{g}^1)} L''$ is isomorphic to a submodule of $T^{m,n}$. Since $T^{m,n}$ is integrable, this quotient is isomorphic to L. □

Corollary 5.12. *(a) If $\mathfrak{g} = \mathfrak{sl}(V, W)$, then $\mathcal{A}_{\mathfrak{sl}} = \bigoplus_{m,n,q} \mathrm{Hom}_{\mathfrak{g}}(T^{m,n}, T^{m-q,n-q})$. If $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$, then $\mathcal{A}_{\mathfrak{g}} = \bigoplus_{m,q} \mathrm{Hom}\,(T^m, T^{m-2q})$. Furthermore,*

$$\mathcal{A}_{\mathfrak{sl}} = \varinjlim \mathrm{End}_{\mathfrak{g}}(\bigoplus_{m+n \leq r} T^{m,n}),$$

and for $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$

$$\mathcal{A}_{\mathfrak{o}} = \varinjlim \mathrm{End}_{\mathfrak{g}}(\bigoplus_{m \leq r} T^m).$$

(b) Up to isomorphism, the objects of $\mathbb{T}_{\mathfrak{g}}$ *are precisely all finite length submodules of* $T(V, W)^{\oplus k}$ *for* $\mathfrak{g} = \mathfrak{sl}(V, W)$*, and of* $T(V)^{\oplus k}$ *for* $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$*. Equivalently, up to isomorphism, the objects of* $\mathbb{T}_{\mathfrak{g}}$ *are the finite length subquotients of* $T(V, W)^{\oplus k}$ *for* $\mathfrak{g} = \mathfrak{sl}(V, W)$*, and of* $T(V)^{\oplus k}$ *for* $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$*.*

Proof. Claim (a) is a consequence of Lemma 5.10. Claim (b) follows from Lemma 5.11 and Corollary 5.8. □

Lemma 5.13. *For any* $L \in \mathbb{T}_{\mathfrak{g}}$*,* $\Phi(L) \in \mathbb{T}_{\mathfrak{g}_c}$*. Moreover, the functor* $\Phi : \mathbb{T}_{\mathfrak{g}} \to \mathbb{T}_{\mathfrak{g}_c}$ *is fully faithful and essentially surjective.*

Proof. By Corollary 5.12 (b), L is isomorphic to a submodule in a direct sum of finitely many copies of $T(V, W)$. Then $\Phi(L)$ is isomorphic to a submodule in a direct sum of finitely many copies of $T(V_c, W_c)$. That implies the first assertion. The fact that Φ is faithful follows from Lemma 5.9 (b).

To prove that Φ is full, consider $L, L' \in \mathbb{T}_{\mathfrak{g}}$ and let $I(L), I(L')$ denote respective injective hulls in $\mathbb{T}_{\mathfrak{g}}$. Then

$$\mathrm{Hom}_{\mathfrak{g}}(L, L') \subset \mathrm{Hom}_{\mathfrak{g}}(I(L), I(L'))$$

and

$$\mathrm{Hom}_{\mathfrak{g}_c}(\Phi(L), \Phi(L')) \subset \mathrm{Hom}_{\mathfrak{g}_c}(\Phi(I(L)), \Phi(I(L'))).$$

By Corollary 5.12 (a), the homomorphism

$$\Phi(I(L), I(L')) : \mathrm{Hom}_{\mathfrak{g}}(I(L), I(L')) \to \mathrm{Hom}_{\mathfrak{g}_c}(\Phi(I(L)), \Phi(I(L')))$$

is surjective. Therefore for any $\varphi \in \mathrm{Hom}_{\mathfrak{g}_c}(\Phi(L), \Phi(L'))$ there exists $\psi \in \mathrm{Hom}_{\mathfrak{g}}(I(L), I(L'))$ such that $\psi(\Phi(L)) \subset \Phi(L')$. By Lemma 5.9 $\Phi(L)$ and $\Phi(L')$ generate respectively L and L'. Hence $\psi(L) \subset L'$. Thus, we obtain that the homomorphism

$$\Phi(L, L') : \mathrm{Hom}_{\mathfrak{g}}(L, L') \to \mathrm{Hom}_{\mathfrak{g}_c}(\Phi(L), \Phi(L'))$$

is also surjective.

To prove that Φ is essentially surjective, we use again Corollary 5.12 (b). We note that any $L \in \mathbb{T}_{\mathfrak{g}}$ is isomorphic to the kernel of $\varphi \in \mathrm{Hom}(T(V, W)^{\oplus k}, T(V, W)^{\oplus l})$ for some k and l and then apply Corollary 5.12 (a). □

Observe that Lemma 5.13 implies that

$$\Phi : \mathbb{T}_{\mathfrak{g}} \to \mathbb{T}_{\mathfrak{g}_c}$$

an equivalence of the abelian categories $\mathbb{T}_{\mathfrak{g}}$ and $\mathbb{T}_{\mathfrak{g}_c}$. To prove Theorem 5.5 it remains to check that Φ is an equivalence of monoidal categories. We therefore prove the following.

Lemma 5.14. *If $L, N \in \mathbb{T}_{\mathfrak{g}}$, then $\Phi(L \otimes N) \simeq \Phi(L) \otimes \Phi(N)$.*

Proof. We just consider the case $\mathfrak{sl}(V, W)$ as the orthogonal and symplectic cases are very similar. Let $\mathfrak{k} = \mathfrak{sl}(W_f^{\perp}, V_f^{\perp})$ for some finite-dimensional nondegenerate pair $V_f \subset V$, $W_f \subset W$. We claim that

$$(L \otimes N)^{\mathfrak{k}} = L^{\mathfrak{k}} \otimes N^{\mathfrak{k}}.$$

Indeed, using Lemma 4.2 one can easily show that

$$(T^{m,n})^{\mathfrak{k}} = V_f^{\otimes m} \otimes W_f^{\otimes n},$$

which implies the statement in the case when L and N are injective. For arbitrary L and N consider embeddings $L \hookrightarrow I$ and $N \hookrightarrow J$ for some injective $I, J \in \mathbb{T}_{\mathfrak{g}}$. Then

$$(L \otimes N)^{\mathfrak{k}} = (L \otimes N) \cap (I \otimes J)^{\mathfrak{k}} = (L \otimes N) \cap (I^{\mathfrak{k}} \otimes J^{\mathfrak{k}}) = L^{\mathfrak{k}} \otimes N^{\mathfrak{k}}.$$

Now we set $\mathfrak{k} = \mathfrak{sl}(W_i^{\perp}, V_i^{\perp})$ and finish the proof by passing to the direct limit. □

The proof of Theorem 5.5 is complete. □

6 Mackey Lie Algebras

Let $V \times W \to \mathbb{C}$ be a linear system. Then each of V and W can be considered as subspace of the dual of the other:

$$V \subset W^*, \quad W \subset V^*.$$

Let $\mathrm{End}_W(V)$ denote the algebra of endomorphisms $\varphi : V \to V$ such that $\varphi^*(W) \subset W$ where $\varphi^* : V^* \to V^*$ is the dual endomorphism. Clearly, there is a canonical anti-isomorphism of algebras

$$\mathrm{End}_W(V) \xrightarrow{\sim} \mathrm{End}_V(W), \quad \varphi \longmapsto \varphi^*_{|W}.$$

We call the Lie algebra associated with the associative algebra $\mathrm{End}_W(V)$ (or equivalently $\mathrm{End}_V(W)$) a *Mackey Lie algebra* and denote it by $\mathfrak{gl}^M(V, W)$.

Note that if V, W is a linear system, then for any subspaces $W' \subset V^*$ with $W \subset W'$, and $V' \subset W^*$ with $V \subset V'$, the pairs V, W' and V', W are linear systems. In particular V, V^* is a linear system and W^*, W is a linear system. Clearly, $\mathfrak{gl}^M(V, V^*)$ coincides with the Lie algebra of all endomorphisms of V (respectively, $\mathfrak{gl}^M(W^*, W)$ is the Lie algebra of all endomorphisms of W). Hence $\mathfrak{gl}^M(V, W) \subset \mathfrak{gl}^M(V, V^*)$, $\mathfrak{gl}^M(V, W) \subset \mathfrak{gl}^M(W^*, W)$. If V and $W = V_*$ are countable dimensional, the Lie algebra $\mathfrak{gl}^M(V, V_*)$ is identified with the Lie algebra of all matrices $X = (x_{ij})_{i\geq 1, j\geq 1}$ such that each row and each column of X have finitely many nonzero entries. The Mackey Lie algebra $\mathfrak{gl}^M(V, V^*)$ (for a countable dimensional space V) is identified with the Lie algebra of all matrices $X = (x_{ij})_{i\geq 1, j\geq 1}$ each column of which has finitely many nonzero entries. Alternatively, if a basis of V as above is enumerated by $\mathbb{Z}$ (i.e., we consider a basis $\{v_j\}_{j\in\mathbb{Z}}$ such that $V_* = \mathrm{span}\{v_j^*\}_{j\in\mathbb{Z}}$ where $v_j^*(v_i) = 0$ for $j \neq i$, $v_j^*(v_j) = 1$), then $\mathfrak{gl}^M(V, V_*)$ is identified with the Lie algebra of all matrices $(x_{ij})_{i,j\in\mathbb{Z}}$ whose rows and columns have finitely many nonzero entries, and $\mathfrak{gl}^M(V, V^*)$ is identified with the Lie algebra of all matrices $(x_{ij})_{i,j\in\mathbb{Z}}$ whose columns have finitely many nonzero entries.

Obviously V and W are $\mathfrak{gl}^M(V, W)$-modules. Moreover, V and W are not isomorphic as $\mathfrak{gl}^M(V, W)$-modules.

It is easy to see that $\mathfrak{gl}(V, W) = V \otimes W$ is the subalgebra of $\mathfrak{gl}^M(V, W)$ consisting of operators with finite-dimensional images in both V and W, and that it is an ideal in $\mathfrak{gl}^M(V, W)$. Furthermore, the Lie algebra $\mathfrak{gl}^M(V, W)$ has a 1-dimensional center consisting of the scalar operators $\mathbb{C}\mathrm{Id}$.

We now introduce the orthogonal and symplectic Mackey Lie algebras. Let V be a vector space endowed with a nondegenerate symmetric (respectively, antisymmetric) form, then $\mathfrak{o}^M(V)$ (respectively, $\mathfrak{sp}^M(V)$) is the Lie algebra

$$\{X \in \mathrm{End}(V) \mid (X \cdot v, w) + (v, X \cdot w) = 0 \;\; \forall\, v, w \in V\}.$$

If V is countable dimensional, there always is a basis $\{v_i, w_j\}_{i,j\in\mathbb{Z}}$ of V such that $\mathrm{span}\{v_i\}_{i\in\mathbb{Z}}$ and $\mathrm{span}\{w_j\}_{j\in\mathbb{Z}}$ are isotropic spaces and $(v_i, w_j) = 0$ for $i \neq j$, $(v_i, w_i) = 1$. The corresponding matrix form of $\mathfrak{o}^M(V)$ consists of all matrices

$$\left(\begin{array}{c|c} a_{ij} & b_{kl} \\ \hline c_{rs} & -a_{ji} \end{array}\right) \tag{9}$$

each row and column of which are finite and in addition $b_{kl} = -b_{lk}$, $c_{rs} = -c_{sr}$ where $i, j, k, l, r, s \in \mathbb{Z}$. The matrix form for $\mathfrak{sp}^M(V)$ is similar: here $b_{kl} = b_{lk}$, $c_{rs} = c_{sr}$.

It is clear that $\mathfrak{o}(V) \subset \mathfrak{o}^M(V)$ and $\mathfrak{sp}(V) \subset \mathfrak{sp}^M(V)$:

$$(v \wedge w) \cdot x = (v, x)w - (x, w)v \;\; \text{for } v \wedge w \in \Lambda^2 V = \mathfrak{o}(V), \;\; x \in V$$

and

$$(vw) \cdot x = (v, x)w - (x, w)v \;\; \text{for } vw \in S^2 V = \mathfrak{sp}(V), \;\; x \in V.$$

Moreover, $\mathfrak{o}(V)$ is an ideal in $\mathfrak{o}^M(V)$ and $\mathfrak{sp}(V)$ is an ideal in $\mathfrak{sp}^M(V)$, since both $\Lambda^2 V$ and $S^2 V$ consist of the respective operators with finite-dimensional image in V.

In this way we have the following exact sequences of Lie algebras:

$$0 \to \mathfrak{gl}(V, W) \to \mathfrak{gl}^M(V, W) \to \mathfrak{gl}^M(V, W)/\mathfrak{gl}(V, W) \to 0,$$

$$0 \to \mathfrak{o}(V) \to \mathfrak{o}^M(V) \to \mathfrak{o}^M(V)/\mathfrak{o}(V) \to 0,$$

$$0 \to \mathfrak{sp}(V) \to \mathfrak{sp}^M(V) \to \mathfrak{sp}^M(V)/\mathfrak{sp}(V) \to 0.$$

Lemma 6.1. $\mathfrak{sl}(V, W)$ *(respectively,* $\mathfrak{o}(V), \mathfrak{sp}(V)$*) is the unique simple ideal in* $\mathfrak{gl}^M(V, W)$ *(respectively,* $\mathfrak{o}^M(V), \mathfrak{sp}^M(V)$*).*

Proof. We will prove that if $I \neq \mathbb{C}\mathrm{Id}$ is a nonzero ideal in $\mathfrak{gl}^M(V, W)$, then I contains $\mathfrak{sl}(V, W)$. Indeed, assume that $X \in I$ and $X \neq c\mathrm{Id}$. Then one can find $v \in V$ and $w \in W$ such that $X \cdot v$ is not proportional to v and $X^* \cdot w$ is not proportional to w. Hence, $Z = [X, v \otimes w] = (X \cdot v) \otimes w - v \otimes (X \cdot w) \in \mathfrak{gl}(V, W) \cap I$ and $Z \neq 0$. Since $\mathfrak{sl}(V, W)$ is the unique simple ideal in $\mathfrak{gl}(V, W)$ and $\mathfrak{gl}(V, W) \cap I \neq 0$, we conclude that $\mathfrak{sl}(V, W) \subset I$.

The two other cases are similar and we leave them to the reader. □

Corollary 6.2. *(a) Two Lie algebras* $\mathfrak{gl}^M(V, W)$ *and* $\mathfrak{gl}^M(V', W')$ *are isomorphic if and only if the linear systems* $V \times W \to \mathbb{C}$ *and* $V' \times W' \to \mathbb{C}$ *are isomorphic.*

(b) Two Lie algebras $\mathfrak{o}^M(V)$ *and* $\mathfrak{o}^M(V')$ *(respectively,* $\mathfrak{sp}^M(V)$ *and* $\mathfrak{sp}^M(V')$*) are isomorphic if and only if there is an isomorphism of vector spaces* $V \simeq V'$ *transferring the form defining* $\mathfrak{o}^M(V)$ *(respectively* $\mathfrak{sp}^M(V)$*) into the form defining* $\mathfrak{o}^M(V')$ *(respectively,* $\mathfrak{sp}^M(V')$*).*

Proof. The statement follows from Proposition 1.1 and Lemma 6.1. □

The following is our main result about the structure of Mackey Lie algebras.

Theorem 6.3. *Let* V *be a countable-dimensional vector space.*

(a) $\mathfrak{gl}(V, V_*) \oplus \mathbb{C}\mathrm{Id}$ *is an ideal in* $\mathfrak{gl}^M(V, V_*)$ *and the quotient*

$$\mathfrak{gl}^M(V, V_*)/\left(\mathfrak{gl}(V, V_*) \oplus \mathbb{C}\mathrm{Id}\right)$$

is a simple Lie algebra.

(b) $\mathfrak{gl}(V, V^*) \oplus \mathbb{C}\mathrm{Id}$ *is an ideal in* $\mathrm{End}(V)$ *and the quotient* $\mathrm{End}(V)/\left(\mathfrak{gl}(V, V^*) \oplus \mathbb{C}\mathrm{Id}\right)$ *is a simple Lie algebra.*

(c) If V *is equipped with a nondegenerate symmetric (respectively, antisymmetric) bilinear form, then* $\mathfrak{o}^M(V)/\mathfrak{o}(V)$ *(respectively* $\mathfrak{sp}^M(V)/\mathfrak{sp}(V)$*) is a simple Lie algebra.*

Proof. The proof is subdivided into lemmas and corollaries.

Note that $\mathfrak{gl}(V, V_*) \subset \mathfrak{gl}(V, V^*) \subset \mathfrak{gl}^M(V, V^*) = \mathrm{End}(V)$. In what follows we fix a basis $\{v_i\}_{i\geq 1}$ in V and use the respective identification of $\mathfrak{gl}(V, V_*)$, $\mathfrak{gl}^M(V, V_*)$ and $\mathfrak{gl}^M(V, V^*) = \mathrm{End}(V)$ with infinite matrices. By E_{ij} we denote the elementary matrix whose only nonzero entry is 1 at position i, j.

Lemma 6.4. *Let $\mathfrak{g}^M = \mathfrak{gl}^M(V, V_*)$, $\mathrm{End}(V)$. Assume that an ideal $I \subset \mathfrak{g}^M$ contains a diagonal matrix $D \notin \mathfrak{gl}(V, V_*) \oplus \mathbb{C}\mathrm{Id}$. Then $I = \mathfrak{g}^M$.*

Proof. We first assume that $D = \sum_{i\geq 1} d_i E_{ii}$ satisfies $d_i \neq d_j$ for all $i \neq j$. Then $[D, \mathfrak{g}^M] = \mathfrak{g}_0^M$, where $\mathfrak{g}_0^M$ is the space of all matrices in $\mathfrak{g}^M$ with zeroes on the diagonal. Consequently, $\mathfrak{g}_0^M \subset I$. Furthermore, any diagonal matrix $\sum_i s_i E_{ii}$ can be written as the commutator

$$\left[\sum_{i\geq 1} E_{i\,i+1}, \sum_{j\geq 1} t_j E_{j+1\,j}\right]$$

with $t_j = \sum_{i=1}^{j} s_i$. Hence, $I = \mathfrak{g}^M$.

We now consider the case of an arbitrary $D \in I$. After permuting the basis elements of V, we can assume that $D = \sum_{i\geq 1} d_i E_{ii}$ with $d_{2m-1} \neq 0$ and $d_{2m-1} \neq d_{2m}$ for all $m > 0$. Let

$$X := \sum_{m=1}^{\infty} \frac{1}{d_{2m} - d_{2m-1}} E_{2m\,2m-1}, \quad Y := \sum_{m=1}^{\infty} s_m E_{2m-1\,2m},$$

where $s_m \neq \pm s_l$ for $m \neq l$. Then $[Y, [X, D]] = s_1 E_{11} - s_1 E_{22} + s_2 E_{33} - s_2 E_{44} + \cdots \in I$, and we reduce this case to the previous one. □

Lemma 6.5. *Let $y = (y_{ij}) \in \mathfrak{gl}(n)$ be a nonscalar matrix. There exist $u, v, w \in \mathfrak{gl}(n)$ such that $[u, [v, [w, y]]]$ is a nonzero diagonal matrix.*

Proof. If y is not diagonal, pick $i \neq j$ such that $y_{ij} \neq 0$. Set $w = E_{ii}, v = E_{jj}, u = E_{ji}$. If y is diagonal, pick $i \neq j$ such that $y_{ii} \neq y_{jj}$ and set $w = E_{ij}, v = E_{ii}, u = E_{ji}$. □

Corollary 6.6. *Let $\prod_i \mathfrak{gl}(n_i)$ for $n_i \geq 2$ be a block subalgebra of $\mathfrak{g}^M$. Suppose that $X \in \left(\prod_i \mathfrak{gl}(n_i)\right) \cap I$ for some ideal $I \subset \mathfrak{g}^M$ and that $X \notin \mathfrak{gl}(V, V_*) \oplus \mathbb{C}\mathrm{Id}$. Then $I = \mathfrak{g}^M$.*

Proof. Let $X = \prod_i X_i$, where $X_i \in \mathfrak{gl}(n_i)$. Without loss of generality we may assume that infinitely many X_i are not diagonal, as otherwise X is diagonal modulo $\mathfrak{gl}(V, V_*)$ and the result follows from Lemma 6.4. Now pick $u_i, v_i, w_i \in \mathfrak{g}_i$ as in Lemma 6.5. Set $u = \prod_i u_i$, $v = \prod_i v_i$, $w = \prod_i w_i$. Then $Z = [u, [v, [w, X]]]$ is diagonal. By normalizing u_i we can ensure that $Z \notin \mathbb{C}\mathrm{Id}$. Since $Z \in I$, the statement follows from Lemma 6.4. □

Lemma 6.7. *For any* $X = (x_{ij})_{i\geq 1, j\geq 1} \in \mathfrak{gl}^M(V, V_*)$ *there exists an increasing sequence* $i_1 < i_2 < \dots$ *such that* $x_{ij} = 0$ *unless* $i, j \in [i_k, i_{k+2} - 1]$ *for some* k.

Proof. Set $i_1 = 1$,

$$i_2 = \max\{j \mid x_{1j} \neq 0 \text{ or } x_{j1} \neq 0\} + 1,$$

and construct the sequence recursively by setting

$$i_k = \max\{j > i_{k-1} \mid x_{ij} \neq 0 \text{ or } x_{ji} \neq 0 \text{ for some } i_{k-2} \leq i < i_{k-1}\} + 1.$$

□

We are now ready to prove Theorem 6.3 (a).

Corollary 6.8 (Theorem 6.3 (a)). *Let an ideal* I *of* $\mathfrak{gl}(V, V_*)$ *be not contained in* $\mathfrak{gl}(V, V_*) \oplus \mathbb{C}\mathrm{Id}$. *Then* $I = \mathfrak{gl}^M(V, V_*)$.

Proof. Let $X \in I \backslash \{\mathfrak{gl}(V, V_*) \oplus \mathbb{C}\mathrm{Id}\}$. Pick $i_1 < i_2 < \dots$ as in Lemma 6.7 and set

$$D = \mathrm{diag}(\underbrace{1, \dots, 1}_{i_2-1}, \underbrace{2, \dots, 2}_{i_3-i_2}, \underbrace{3, \dots, 3}_{i_4-i_3}, \dots).$$

Then $X = X_{-1} + X_0 + X_1$ where $[D, X_i] = iX_i$. If $X_0 \notin \mathfrak{gl}(V, V_*) \oplus \mathbb{C}\mathrm{Id}$ we are done by Corollary 6.6 as X_0 is a block matrix. Otherwise, at least one of X_1 and X_{-1} does not lie in $\mathfrak{gl}(V, V_*)$.

Assume for example that $X_1 = (x_{ij}) \notin \mathfrak{gl}(V, V_*)$. Then there exist infinite sequences $\{i_1 < i_2 < \dots\}$ and $\{j_1 < j_2 < \dots\}$ such that $x_{i_s j_s} \neq 0$. Moreover, we may assume that $\dots < i_s \leq j_s < i_{s+1} \leq j_{s+1} < \dots$. Set $Y = \sum_{s\geq 1} E_{j_s i_s}$. Then $[Y, X_1] \in I$ is a block matrix and we can again use Corollary 6.6. □

Next we prove Theorem 6.3 (b).

Let I be an ideal in $\mathrm{End}(V)$. Assume that I is not contained in $\mathfrak{gl}(V, V^*) \oplus \mathbb{C}\mathrm{Id}$. Let $X \in I \setminus \{\mathfrak{gl}(V, V^*) \oplus \mathbb{C}\mathrm{Id}\}$ and let $V_X \subset V$ denote the subspace of all X-finite vectors.

Assume first that $V_X \neq V$. Then there exists $v \in V$ such that $v, X \cdot v, X^2 \cdot v, \dots$ are linearly independent. Let $M = \mathrm{span}\{v, X \cdot v, X^2 \cdot v, \dots\}$ and let U be a subspace of V such that $V = M \oplus U$. Let π_M be the projector on M with kernel U. Then $Y := X + [X, \pi_M] \in I$. A simple calculation shows that both U and M are Y-stable and $Y|_M = X|_M$. Let $Z \in \mathrm{End}(M)$ be defined by $Z(U) = 0$, $Z(X^i \cdot v) = iX^{i-1} \cdot v$ for $i \geq 0$. Then $[Z, Y]$ is a diagonal matrix with infinitely many distinct entries. Hence $I = \mathrm{End}(V)$ by Lemma 6.4.

Now suppose that $V_X = V$. Then we have a decomposition $V = \bigoplus_\lambda V_\lambda$, where $V_\lambda := \bigcup_n \ker(X - \lambda\mathrm{Id})^n$ are generalized eigenspaces of X. First, we assume that for all λ there exists $n(\lambda)$ such that $V_\lambda = \ker(X - \lambda\mathrm{Id})^{n(\lambda)}$. In this case $V = \bigoplus_i V_i$ is a direct sum of X-stable finite-dimensional subspaces. Thus X is a block matrix and by Corollary 6.6 we obtain $I = \mathrm{End}(V)$. Next, we assume that for some λ the sequence $\ker(X-\lambda\mathrm{Id})^n$ does not stabilize. In this case there are linearly independent vectors $v_1, v_2, \ldots$ such that $(X-\lambda\mathrm{Id})\cdot v_1 = 0$ and $(X-\lambda\mathrm{Id})\cdot v_i = v_{i-1}$ for all $i > 1$. We repeat the argument from the previous paragraph. Set M to be the span of v_k, let $V = M \oplus U$ and define $Z \in \mathrm{End}(M)$ by setting $Z(U) = 0$, $Z(v_i) = i v_{i+1}$. Then $[Z, ([X, \pi_M] + X)] \in I$ is a diagonal matrix with infinitely many distinct entries. Hence $I = \mathrm{End}(V)$.

To complete the proof of Theorem 6.3 it remains to prove claim (c).

Lemma 6.9. *If $\mathfrak{g}^M = \mathfrak{o}^M(V)$ (respectively, $\mathfrak{sp}^M(V)$), then any nonzero proper ideal $I \subset \mathfrak{g}^M$ equals $\mathfrak{o}(V)$ (respectively, $\mathfrak{sp}(V)$).*

Proof. As follows from (9), one can define a $\mathbb{Z}$-grading $\mathfrak{g}^M = \mathfrak{g}^M_{-1} \oplus \mathfrak{g}^M_0 \oplus \mathfrak{g}^M_1$ such that $\mathfrak{g}^M_0 \simeq \mathfrak{gl}^M(V, V_*)$. This grading is defined by the matrix

$$D = \left(\begin{array}{c|c} \frac{1}{2}\mathrm{Id} & 0 \\ \hline 0 & -\frac{1}{2}\mathrm{Id} \end{array}\right),$$

i.e., $[D, X] = iX$ for $X \in \mathfrak{g}^M_i$. Since $D \in \mathfrak{g}^M$, any ideal $I \subset \mathfrak{g}^M$ is homogeneous in this grading. Note that the ideal generated by D equals the entire Lie algebra $\mathfrak{g}^M$. Hence we may assume that $D \notin I$, and thus that $I_0 := I \cap \mathfrak{g}^M_{-1}$ is a proper ideal in $\mathfrak{g}^M_0$.

Assume first that $I_1 := I \cap \mathfrak{g}^M_1$ is not contained in $\mathfrak{o}(V)$ (respectively, $\mathfrak{sp}(V)$) and let $X \in I_1 \setminus \mathfrak{o}(V)$ (respectively, $X \in I_1 \setminus \mathfrak{sp}(V)$). By an argument similar to the one at the end of the proof of Corollary 6.8, there exists $Y \in \mathfrak{g}^M_{-1}$ such that $[Y, X] \notin \mathfrak{gl}(V, V_*) \oplus \mathbb{C}D$. Therefore by Corollary 6.8 we obtain a contradiction with our assumption that I_0 is a proper ideal in $\mathfrak{g}^M_0$.

Thus, we have proved that $I_1 \subset \mathfrak{o}(V)$ (respectively, $\mathfrak{sp}(V)$) and, similarly, $I_{-1} := I \cap \mathfrak{g}^M_{-1} \subset \mathfrak{o}(V)$ (respectively, $\mathfrak{sp}(V)$). Moreover, $I_0 \subset \mathfrak{gl}(V, V_*)$ by Corollary 6.8. But then I is a nonzero ideal in $\mathfrak{o}(V)$ (respectively, $\mathfrak{sp}(V)$). Since both $\mathfrak{o}(V)$ and $\mathfrak{sp}(V)$ are simple, the statement follows. □

The proof of Theorem 6.3 is complete. □

Theorem 6.3 (a) gives a complete list of ideals in $\mathfrak{gl}^M(V, V_*)$ for a countable-dimensional V. Indeed, since $\mathfrak{sl}(V, V_*)$ is a simple Lie algebra, we obtain that all proper nonzero ideals in $\mathfrak{gl}^M(V, V_*)$ are $\mathfrak{gl}(V, V_*)$, $\mathfrak{sl}(V, V_*)$, $\mathbb{C}\mathrm{Id}$, $\mathfrak{sl}(V, V_*) \oplus \mathbb{C}\mathrm{Id}$ and $\mathfrak{gl}(V, V_*) \oplus \mathbb{C}\mathrm{Id}$. In the same way the Lie algebra $\mathrm{End}(V)$ also has five proper nonzero ideals.

Note that if V is not countable-dimensional, then $\mathfrak{gl}^M(V, V_*)$, $\mathrm{End}(V)$ and $\mathfrak{o}^M(V)$ (respectively, $\mathfrak{sp}^M(V)$) have the following ideal:

$$\{X \mid \dim(X \cdot V) \text{ is finite or countable}\}.$$

Hence, Theorem 6.3 does not hold in this case.

7 Dense Subalgebras

7.1 Definition and General Results

Definition 7.1. Let $\mathfrak{l}$ be a Lie algebra, R be an $\mathfrak{l}$-module, $\mathfrak{k} \subset \mathfrak{l}$ be a Lie subalgebra. We say that $\mathfrak{k}$ acts *densely* on R if for any finite set of vectors $r_1, \dots, r_n \in R$ and any $l \in \mathfrak{l}$ there is $k \in \mathfrak{k}$ such that $k \cdot r_i = l \cdot r_i$ for $i = 1, \dots, n$.

Lemma 7.2. *Let $\mathfrak{k} \subset \mathfrak{l}$ and let R, N be two $\mathfrak{l}$-modules such that $\mathfrak{k}$ acts densely on $R \oplus N$. Then* $\mathrm{Hom}_{\mathfrak{l}}(R, N) = \mathrm{Hom}_{\mathfrak{k}}(R, N)$.

Proof. There is an obvious inclusion $\mathrm{Hom}_{\mathfrak{l}}(R, N) \subset \mathrm{Hom}_{\mathfrak{k}}(R, N)$. Suppose there exists $\varphi \in \mathrm{Hom}_{\mathfrak{k}}(R, N) \setminus \mathrm{Hom}_{\mathfrak{l}}(R, N)$. Then one can find $r \in R, l \in \mathfrak{l}$ such that $\varphi(l \cdot r) \neq l \cdot \varphi(r)$. Since $\mathfrak{k}$ acts densely on $R \oplus N$, there exists $k \in \mathfrak{k}$ such that $k \cdot r = l \cdot r$ and $k \cdot \varphi(r) = l \cdot \varphi(r)$. Therefore we have

$$\varphi(l \cdot r) = \varphi(k \cdot r) = k \cdot \varphi(r) = l \cdot \varphi(r).$$

Contradiction. □

Lemma 7.3. *Let $\mathfrak{k} \subset \mathfrak{l}$ and R be an $\mathfrak{l}$-module on which $\mathfrak{k}$ acts densely. Then*

(a) $\mathfrak{k}$ acts densely on any $\mathfrak{l}-$subquotient of R;
(b) $\mathfrak{k}$ acts densely on $R^{\otimes n}$ for $n \geq 1$;
(c) $\mathfrak{k}$ acts densely on $R^{\oplus n}$ for $n \geq 1$;
(d) $\mathfrak{k}$ acts densely on $T(R)^{\oplus n}$ for $n \geq 1$.

Proof. (a) Let N be an $\mathfrak{l}$-submodule of R. It follows immediately from the definition that $\mathfrak{k}$ acts densely on N and on R/N. That implies the statement.
(b) Let $r_1, \dots, r_q \in R^{\otimes n}$. Write

$$r_i = \sum_{j=1}^{s(i)} m^i_{j1} \otimes \cdots \otimes m^i_{jn}$$

for some $m^i_{jp} \in R$. For any $l \in \mathfrak{l}$ there exists $k \in \mathfrak{k}$ such that $k \cdot m^i_{jp} = l \cdot m^i_{jp}$ for all $i \leq r$, $p \leq n$ and $j \leq s(i)$. Then $k \cdot r_i = l \cdot r_i$ for all $i \leq q$.

Proving (c) and (d) is similar to proving (b) and we leave it to the reader. □

Lemma 7.4. *Let $\mathfrak{k}, \mathfrak{l}$ and R be as in Lemma 7.3. Then a $\mathfrak{k}$-submodule of R is $\mathfrak{l}$-stable. Hence any $\mathfrak{k}$-subquotient of R has a natural structure of $\mathfrak{l}$-module.*

Proof. Straightforward from the definition. □

Theorem 7.5. *Let $\mathcal{C}_{\mathfrak{l}}$ be a full abelian subcategory of* $\mathfrak{l}$-mod *such that $\mathfrak{k}$ acts densely on any object in $\mathcal{C}$. Let* $\mathrm{Res} : \mathfrak{l} - \mathrm{mod} \to \mathfrak{k} - \mathrm{mod}$ *be the functor of restriction. Let $\mathcal{C}_{\mathfrak{k}}$ be the image of $\mathcal{C}_{\mathfrak{l}}$ under* Res. *Then $\mathcal{C}_{\mathfrak{k}}$ is a full abelian subcategory of* $\mathfrak{k} - \mathrm{mod}$ *and* Res *induces an equivalence of $\mathcal{C}_{\mathfrak{k}}$ and $\mathcal{C}_{\mathfrak{l}}$.*

Proof. The first assertion follows from Lemma 7.2. It also follows from the same lemma that $\mathrm{Res}(R) \simeq \mathrm{Res}(N)$ implies $R \simeq N$. Thus, every object in $\mathcal{C}_{\mathfrak{k}}$ has a unique (up to isomorphism) structure of $\mathfrak{l}$-module. This provides a quasi-inverse of Res. Hence the second assertion holds. □

Let R be an $\mathfrak{l}$-module. Denote by $\mathbb{T}_{\mathfrak{l}}^R$ the full subcategory of $\mathfrak{l}$-mod consisting of all finite length subquotients of finite direct sums $T(R)^{\oplus n}$ for $n \geq 1$.

Proposition 7.6. *Let $\mathfrak{k}$, $\mathfrak{l}$ and R be as in Lemma 8.2. Then the restriction functor*

$$\mathrm{Res} : \mathbb{T}_{\mathfrak{l}}^R \rightsquigarrow \mathbb{T}_{\mathfrak{k}}^R$$

is an equivalence of monoidal categories.

Proof. By Lemma 7.3, $\mathrm{Res}(\mathbb{T}_{\mathfrak{l}}^R) = \mathbb{T}_{\mathfrak{k}}^R$. Thus Res is an equivalence of $\mathbb{T}_{\mathfrak{l}}^R$ and $\mathbb{T}_{\mathfrak{k}}^R$ by Theorem 7.5. In addition, Res clearly commutes with $\otimes$, hence the statement. □

7.2 Dense Subalgebras of Mackey Lie Algebras

Now let $\mathfrak{g}^M$ denote one of the Lie algebras $\mathfrak{gl}^M(V, W)$, $\mathfrak{o}^M(V)$, $\mathfrak{sp}^M(V)$, and $\mathfrak{g}$ denote respectively the subalgebra $\mathfrak{gl}(V, W)$, $\mathfrak{o}(V)$, $\mathfrak{sp}(V)$. By R we denote the $\mathfrak{g}^M$-module $V \oplus W$ (respectively, V).

In what follows we call a Lie subalgebra $\mathfrak{a} \subset \mathfrak{g}^M$ *dense* if it acts densely on R. It is easy to see that $\mathfrak{g}$ is a dense subalgebra of $\mathfrak{g}^M$.

Here are further examples of dense subalgebras of $\mathfrak{gl}^M(V, V_*)$ for a countable-dimensional space V. We identify $\mathfrak{gl}^M(V, V_*)$ with the Lie algebra of matrices $(x_{ij})_{i\geq 1, j\geq 1}$ each row and column of which are finite.

1. The Lie algebra $\mathfrak{j}(V, V_*)$ consisting of matrices $J = (x_{ij})_{i\geq 1, j\geq 1}$ such that $x_{ij} = 0$ when $|i - j| > m_J$ for some $m_J \in \mathbb{Z}_{>0}$ (generalized Jacobi matrices), is dense in $\mathfrak{gl}^M(V, V_*)$.
2. The subalgebra $\mathfrak{lj}(V, V_*) \subset \mathfrak{gl}^M(V, V_*)$ consisting of matrices $X = (x_{ij})_{i\geq 1, j\geq 1}$ satisfying the condition $x_{ij} = 0$ when $i - j > c_X j$ for some $c_X \in \mathbb{Z}_{>0}$, is dense in $\mathfrak{gl}^M(V, V_*)$.
3. The subalgebra $\mathfrak{pj}(V, V_*)$ of matrices $Y = (x_{ij})_{i\geq 1, j\geq 1}$ satisfying the condition $x_{ij} = 0$ when $i - j > p_Y(j)$ for some polynomial $p_Y(t) \in \mathbb{Z}_{\geq 0}[t]$, is dense in $\mathfrak{gl}^M(V, V_*)$.
4. Let $\mathfrak{g}$ be a countable-dimensional diagonal Lie algebra. If $\mathfrak{g}$ is of type $\mathfrak{sl}$, fix a chain (1) of diagonal embeddings where $\mathfrak{g}_i \simeq \mathfrak{sl}(n_i)$. Observe that given a chain (3), we can always choose a chain

$$V_{\mathfrak{g}_1}^* \overset{\mu_1}{\hookrightarrow} V_{\mathfrak{g}_2}^* \overset{\mu_2}{\hookrightarrow} \dots \hookrightarrow V_{\mathfrak{g}_i}^* \overset{\mu_i}{\hookrightarrow} V_{\mathfrak{g}_{i+1}}^* \hookrightarrow \dots$$

so that the nondegenerate pairing $V_{\mathfrak{g}_{i+1}} \times V^*_{\mathfrak{g}_{i+1}} \to \mathbb{C}$ restricts to a nondegenerate pairing $\kappa_i(V_{\mathfrak{g}_i}) \times \mu_i(V^*_{\mathfrak{g}_i}) \to \mathbb{C}$. Therefore, by multiplying μ_i by a suitable constant, we can assume that κ_i and μ_i preserve the natural pairings $V_{\mathfrak{g}_i} \times V^*_{\mathfrak{g}_i} \to \mathbb{C}$. This shows that, given a natural representation V of $\mathfrak{g}$, there always is a natural representation V_* such that there is a nondegenerate $\mathfrak{g}$-invariant pairing $V \times V_* \to \mathbb{C}$. This gives an embedding of $\mathfrak{g}$ as a dense subalgebra in $\mathfrak{gl}^M(V, V_*)$

If $\mathfrak{g}$ is of type $\mathfrak{o}$ or $\mathfrak{sp}$, then a natural representation V of $\mathfrak{g}$ is defined again by a chain of embeddings (3). Moreover, V always carries a respective nondegenerate symmetric or symplectic form. Therefore $\mathfrak{g}$ can be embedded as a dense subalgebra in $\mathfrak{o}^M(V)$, or respectively in $\mathfrak{sp}^M(V)$.

The following statement is a particular case of Proposition 7.6.

Corollary 7.7. *Let $\mathfrak{a}$ be a dense subalgebra in $\mathfrak{g}^M$. Then the monoidal categories $\mathbb{T}^R_{\mathfrak{g}^M}$ and $\mathbb{T}^R_{\mathfrak{a}}$ are equivalent.*

7.3 Finite Corank Subalgebras of $\mathfrak{g}^M$ and the Category $\mathbb{T}_{\mathfrak{g}^M}$

We now generalize the notion of finite corank subalgebra to Mackey Lie algebras.

Let $V_f \subset V$, $W_f \subset W$ be a nondegenerate pair of finite-dimensional subspaces. Then $\mathfrak{gl}(W_f^\perp, V_f^\perp)$ is a subalgebra of $\mathfrak{gl}^M(W_f^\perp, V_f^\perp)$ and also a subalgebra of $\mathfrak{gl}^M(V, W)$. Moreover, the following important relation holds

$$\begin{aligned} \mathfrak{sl}(V, W)/\mathfrak{sl}(W_f^\perp, V_f^\perp) &= \mathfrak{gl}(V_f, W_f) \oplus (V_f \otimes V_f^\perp) \oplus (W_f^\perp \otimes W_f) \\ &= \mathfrak{gl}^M(V, W)/\mathfrak{gl}^M(W_f^\perp, V_f^\perp). \end{aligned} \tag{10}$$

We call a subalgebra $\mathfrak{k} \subset \mathfrak{gl}^M(V, W)$ a *finite corank subalgebra* if it contains $\mathfrak{gl}^M(W_f^\perp, V_f^\perp)$ for some nondegenerate pair $V_f \subset V$, $W_f \subset W$.

Similarly, let V be a vector space equipped with a symmetric (respectively, skew-symmetric) nondegenerate form and V_f be a nondegenerate finite-dimensional subspace. We have a well-defined subalgebra $\mathfrak{o}^M(V_f^\perp) \subset \mathfrak{o}^M(V)$ (respectively, $\mathfrak{sp}^M(V_f^\perp) \subset \mathfrak{sp}^M(V)$). Furthermore,

$$\begin{aligned} \mathfrak{o}(V)/\mathfrak{o}(V_f^\perp) &= \mathfrak{o}(V_f) \oplus (V_f \otimes V_f^\perp) = \mathfrak{o}^M(V)/\mathfrak{o}^M(V_f^\perp), \\ \mathfrak{sp}(V)/\mathfrak{sp}(V_f^\perp) &= \mathfrak{sp}(V_f) \oplus (V_f \otimes V_f^\perp) = \mathfrak{sp}^M(V)/\mathfrak{sp}^M(V_f^\perp). \end{aligned} \tag{11}$$

We call $\mathfrak{k} \subset \mathfrak{o}^M(V)$ (respectively, $\mathfrak{sp}^M(V)$) a *finite corank subalgebra* if it contains $\mathfrak{o}^M(V_f^\perp)$ (respectively, $\mathfrak{sp}^M(V_f^\perp)$) for some V_f as above.

Next, we say that $\mathfrak{g}^M$-module L *satisfies the large annihilator condition* if the annihilator in $\mathfrak{g}^M$ of any $l \in L$ contains a finite corank subalgebra. It follows immediately from the definition that if L_1 and L_2 satisfy the large annihilator condition, then the same is true for $L_1 \oplus L_2$ and $L_1 \otimes L_2$.

Lemma 7.8. *Let L be a $\mathfrak{g}^M$-module which is integrable as a $\mathfrak{g}$-module. If L satisfies the large annihilator condition (as a $\mathfrak{g}^M$-module), then $\mathfrak{g}$ acts densely on L.*

Proof. Since L satisfies the large annihilator condition as a $\mathfrak{g}^M$-module, so does also $L^{\oplus n}$. It suffices to show that for all $n \in \mathbb{Z}_{\geq 1}$ and all $l \in L^{\oplus n}$ we have

$$\mathfrak{g} \cdot l = \mathfrak{g}^M \cdot l. \tag{12}$$

However, as l is annihilated by $\mathfrak{gl}^M(W_f^\perp, V_f^\perp)$ for an appropriate finite-dimensional nondegenerate pair $V_f \subset V, W_f \subset W$ in the case $\mathfrak{g} = \mathfrak{sl}(V, W)$ (respectively, by $\mathfrak{o}^M(V_f^\perp), \mathfrak{sp}^M(V_f^\perp)$ in the case $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$), (12) follows from (10), (respectively, from (11)). □

Lemma 7.9. *Let L be a $\mathfrak{g}$-module satisfying the large annihilator condition. Then the $\mathfrak{g}$-module structure on L extends in a unique way to a $\mathfrak{g}^M$-module structure such that L satisfies the large annihilator condition as a $\mathfrak{g}^M$-module.*

Proof. Consider the case $\mathfrak{g} = \mathfrak{sl}(V, W)$. Any $l \in L$ is annihilated by $\mathfrak{sl}(W_f^\perp, V_f^\perp)$ for an appropriate finite-dimensional nondegenerate pair $V_f \subset V, W_f \subset W$. Let $x \in \mathfrak{gl}^M(V, W)$. By (10) there exists $y \in \mathfrak{sl}(V, W)$ such that $x + \mathfrak{gl}^M(W_f^\perp, V_f^\perp) = y + \mathfrak{sl}(W_f^\perp, V_f^\perp)$. Moreover, y is unique modulo $\mathfrak{sl}(W_f^\perp, V_f^\perp)$. Thus we can set $x \cdot l := y \cdot l$. It is an easy check that this yields a well-defined $\mathfrak{gl}^M(V, W)$-module structure on L compatible with the $\mathfrak{sl}(V, W)$-module structure on L.

For $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$ one uses (11) instead of (10). □

We can now define the category $\mathbb{T}_{\mathfrak{g}^M}$ as an analogue of the category $\mathbb{T}_{\mathfrak{g}}$. More precisely, the category $\mathbb{T}_{\mathfrak{g}^M}$ is the full subcategory of $\mathfrak{g}^M$-mod consisting of all modules of finite length, integrable over $\mathfrak{g}$ and satisfying the large annihilator condition.

The following is our main result in Sect. 7.

Theorem 7.10. *(a) $\mathbb{T}_{\mathfrak{g}^M} = \mathbb{T}^R_{\mathfrak{g}^M}$, where $R = V \oplus W$ for $\mathfrak{g} = \mathfrak{sl}(V, W)$ and $R = V$ for $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$.*
(b) The functor Res : $\mathbb{T}_{\mathfrak{g}^M} \rightsquigarrow \mathbb{T}_{\mathfrak{g}}$ *is an equivalence of monoidal categories.*

Proof. It is clear that $\mathbb{T}^R_{\mathfrak{g}^M}$ is a full subcategory of $\mathbb{T}_{\mathfrak{g}^M}$. We need to show only that any $L \in \mathbb{T}_{\mathfrak{g}^M}$ is isomorphic to a subquotient of $T(R)^{\oplus n}$ for some n. Obviously, L satisfies the large annihilator condition as a $\mathfrak{g}$-module. Furthermore, by Lemma 7.9 (a), $\mathfrak{g}$ acts densely on L, hence L has finite length as a $\mathfrak{g}$-module. By Corollary 5.12 (b), L is isomorphic to a $\mathfrak{g}$subquotient of $T(R)^{\oplus n}$ for some n, and by Proposition 7.6 L is the restriction to $\mathfrak{g}$ of some $\mathfrak{g}^M$-subquotient L' of $T(R)^{\oplus n}$. However, since L' satisfies the large annihilator condition, Lemma 7.8 implies that there is an isomorphism of $\mathfrak{g}^M$-modules $L \simeq L'$. This proves (a).

(b) follows from (a) and Proposition 7.6. □

The following diagram summarizes the equivalences of monoidal categories established in this paper:

$$\mathbb{T}_{\mathfrak{a}} \overset{\mathrm{Res}}{\leftsquigarrow} \mathbb{T}_{\mathfrak{g}^M} = \mathbb{T}^R_{\mathfrak{g}^M} \overset{\mathrm{Res}}{\rightsquigarrow} \mathbb{T}_{\mathfrak{g}} \overset{\Phi}{\rightsquigarrow} \mathbb{T}_{\mathfrak{g}_c}.$$

Here $\mathfrak{a}$ is any dense subalgebra of $\mathfrak{g}^M$ and $R = V \oplus W$ for $\mathfrak{g} = \mathfrak{sl}(V, W)$, $R = V$ for $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$. In particular, when $\mathfrak{g} = \mathfrak{sl}(V, V_*)$ for countable-dimensional V and V_*, $\mathfrak{a}$ can be chosen as the Lie algebra $\mathfrak{j}(V, V_*)$ or as any countable-dimensional diagonal Lie algebra.

8 Further Results and Open Problems

Theorem 7.10 (a) can be considered an analogue of Theorem 5.1 and Corollary 5.12 (b) as it provides two equivalent descriptions of the category $\mathbb{T}_{\mathfrak{g}^M}$. It is interesting to have a longer list of such equivalent descriptions.

The following proposition provides another equivalent condition characterizing the objects of $\mathbb{T}_{\mathfrak{g}^M}$ under the additional assumption that $\mathfrak{g} = \mathfrak{sl}(V, V_*), \mathfrak{o}(V), \mathfrak{sp}(V)$ is countable dimensional.

Proposition 8.1. *Let $\mathfrak{g}^M = \mathfrak{gl}^M(V, V_*), \mathfrak{o}^M(V), \mathfrak{sp}^M(V)$ for a countable dimensional V, and let L be a $\mathfrak{g}^M$-module of finite length which is integrable as a $\mathfrak{g}$-module. Then L is an object of $\mathbb{T}_{\mathfrak{g}^M}$ if and only if $\mathfrak{g}$ acts densely on L.*

We first need a lemma.

Lemma 8.2. *Let $\mathfrak{g}^M = \mathfrak{gl}^M(V, V_*), \mathfrak{o}^M(V), \mathfrak{sp}^M(V)$ for a countable-dimensional V, and let L and L' be $\mathfrak{g}^M$-modules. Assume that L and L' have finite length as $\mathfrak{g}$-modules. Then*

$$\mathrm{Hom}_{\mathfrak{g}}(L, L') = \mathrm{Hom}_{\mathfrak{g}^M}(L, L').$$

In particular, if L and L' are isomorphic as $\mathfrak{g}$-modules, then L and L' are isomorphic as $\mathfrak{g}^M$-modules.

Proof. Observe that $\mathrm{Hom}_{\mathbb{C}}(L, L')$ has a natural structure of $\mathfrak{g}^M$-module defined by

$$(X \cdot \varphi)(l) := X \cdot \varphi(l) - \varphi(X \cdot l) \text{ for } X \in \mathfrak{g}^M, \varphi \in \mathrm{Hom}_{\mathbb{C}}(L, L'), \ l \in L. \quad (13)$$

Since $\mathfrak{g}$ is an ideal in $\mathfrak{g}^M$, $\mathrm{Hom}_{\mathfrak{g}}(L, L')$ is a $\mathfrak{g}^M$-submodule in $\mathrm{Hom}_{\mathbb{C}}(L, L')$. Moreover, $\mathrm{Hom}_{\mathfrak{g}}(L, L')$ is finite dimensional as L and L' have finite length over $\mathfrak{g}$. On the other hand, Theorem 6.3 implies that $\mathfrak{g}^M$ does not have proper ideals of finite codimension, hence any finite-dimensional $\mathfrak{g}^M$-module is trivial. Therefore (13) defines a trivial $\mathfrak{g}^M$-module structure of $\mathrm{Hom}_{\mathfrak{g}^M}(L, L')$, which means that any $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(L, L')$ belongs to $\mathrm{Hom}_{\mathfrak{g}^M}(L, L')$. This shows that $\mathrm{Hom}_{\mathfrak{g}}(L, L') = \mathrm{Hom}_{\mathfrak{g}^M}(L, L')$. The second assertion follows immediately. □

Proof of Proposition 8.1. If $L \in \mathbb{T}_{\mathfrak{g}^M}$, then $\mathfrak{g}$ acts densely on L by Lemma 7.9.

Now let $\mathfrak{g}$ act densely on L. We first prove that L satisfies the large annihilator condition as a $\mathfrak{g}$-module. Assume that $\mathfrak{g}$ acts densely on L but L does not satisfy the large annihilator condition as a $\mathfrak{g}$-module. Using the matrix realizations of $\mathfrak{g}$ and $\mathfrak{g}^M$ one can show that there exists $l \in L$ and a sequence $\{X_i\}_{i \in \mathbb{Z}_{\geq 1}}$ of commuting linearly independent elements $X_i \in \mathfrak{g}$ which don't belong to the annihilator of l. Furthermore, one can find an infinite subsequence $\{Y_j = X_{i_j}\}$ such that each Y_j lies in an $\mathfrak{sl}(2)$-subalgebra $ß_j \subset \mathfrak{g}$ with the condition $[ß_j, ß_s] = 0$ for $j \neq s$. Then $\prod_j ß_j$ is a Lie subalgebra in $\mathfrak{g}^M$, and let $ß$ be the diagonal subalgebra in $\prod_j ß_j$. If $x \in ß$, we denote by x_j its component in $ß_j$.

Since $\mathfrak{g}$ acts densely on L, there exists a linear map $\theta : ß \to \mathfrak{g}$ such that $\theta(y) \cdot l = y \cdot l$ for all $y \in ß$. On the other hand, there exists $n \in \mathbb{Z}_{\geq 1}$ such that $[\theta(y), x_j] = 0$ for all $y, x \in ß$ and $j > n$. Let $d_y := y - \theta(y)$. Then $d_y \cdot l = 0$ and

$$[d_y, x_j] = [y, x_j] = [y_j, x_j] \quad \text{for all } x, y \in ß \text{ and } j > n. \tag{14}$$

Set $L_j := U(ß_j) \cdot l$. Then (14) implies $d_y \cdot L_j \subset L_j$ for all $j > n$. Moreover, $\psi_y := d_y - y_j$ commutes with $ß_j$, hence $\psi_y \in \mathrm{End}_{ß_j}(L_j)$. Considering $y_j + \psi_y$ as an element of $\mathrm{End}_{\mathbb{C}}(L_j)$, we obtain in addition that $l \in \ker(y_j + \psi_y)$ for all $y \in ß$ and all $j > n$.

Choose a standard basis $E, H, F \in ß$. Since L_j is a finite-dimensional $\beta_j \simeq \mathfrak{sl}(2)$-module, we obtain easily

$$\ker(E_j + \psi_E) \cap L_j = L_j^{E_j}, \;\; \ker(F_j + \psi_F) \cap L_j = L_j^{F_j}.$$

Since

$$l \in \ker(E_j + \psi_E) \cap \ker(F_j + \psi_F) \cap L_j = L_j^{ß_j},$$

we conclude that L_j is a trivial $ß_j$-module for all $j > n$, which contradicts our original assumption that $Y_j \cdot l \neq 0$. Thus, L satisfies the large annihilator condition as a $\mathfrak{g}$-module.

Note that as $\mathfrak{g}$ acts densely on L, the length of L as a $\mathfrak{g}$-module is the same as the length of L as a $\mathfrak{g}^M$-module. Since L satisfies the large annihilator condition for $\mathfrak{g}$ and has finite length as a $\mathfrak{g}$-module, we conclude that $L_{\downarrow \mathfrak{g}}$ is a tensor module, i.e., an object of $\mathbb{T}_{\mathfrak{g}}$. By Theorem 7.10 (b), $L_{\downarrow \mathfrak{g}} = L'_{\downarrow \mathfrak{g}}$ for some $L' \in \mathbb{T}_{\mathfrak{g}^M}$. Finally, Lemma 8.2 implies that the $\mathfrak{g}^M$ modules L' and L are isomorphic, i.e., $L \in \mathbb{T}_{\mathfrak{g}^M}$. □

Next, under the assumption that V is countable dimensional, consider maximal subalgebras $\mathfrak{h}^M$ of $\mathfrak{g}^M$ which act semisimply on V and V_* (respectively only on V for $\mathfrak{g} = \mathfrak{o}(V), \mathfrak{sp}(V)$). It is straightforward to show that the centralizer in $\mathfrak{g}^M$ of any local Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is such a subalgebra of $\mathfrak{g}^M$. If $\mathfrak{g}^M = \mathfrak{gl}^M(V, V_*)$ is realized as the Lie algebra of matrices $X = (x_{ij})_{i,j \in \mathbb{Z}}$ with finite rows and columns, then $\mathfrak{h}^M$ can be chosen as the subalgebra of diagonal matrices.

The following statement looks plausible to us.

Conjecture 8.3. *Let* $\mathfrak{g} = \mathfrak{sl}(V, V_*), \mathfrak{o}(V), \mathfrak{sp}(V)$ *for a countable-dimensional* V*. Let* M *be a finite length* $\mathfrak{g}^M$*-module which is integrable as a* $\mathfrak{g}$*-module. The following conditions on* M *are equivalent:*

(a) $M \in \mathbb{T}_{\mathfrak{g}^M}$*;*
(b) M *is countable dimensional;*
(c) M *is a semisimple* $\mathfrak{h}^M$*-module for some subalgebra* $\mathfrak{h}^M \subset \mathfrak{g}^M$*;*
(d) M *is a semisimple* $\mathfrak{h}^M$*-module for any subalgebra* $\mathfrak{h}^M \subset \mathfrak{g}^M$*.*

Consider now the inclusion of Lie algebras

$$\mathfrak{g} = \mathfrak{sl}(V, V_*) \subset \mathfrak{gl}^M(V, V^*) = \mathrm{End}(V)$$

where V is an arbitrary vector space. The subalgebra $\mathfrak{g}$ is not dense in $\mathrm{End}(V)$, nevertheless the monoidal categories $\mathbb{T}_{\mathfrak{g}}$ and $\mathbb{T}_{\mathrm{End}(V)}$ are equivalent by Theorems 5.1 and 7.10. Here is a functor which most likely also provides such an equivalence. Let $M \in \mathbb{T}_{\mathrm{End}(V)}$. Set

$$\Gamma^{\mathrm{wt}}_{\mathfrak{g}}(M) := \cap_{\mathfrak{h}\subset\mathfrak{g}}\, \Gamma^{\mathrm{wt}}_{\mathfrak{h}}(M)$$

where $\mathfrak{h}$ runs over all local Cartan subalgebras of $\mathfrak{g}$.

Conjecture 8.4. $\Gamma^{\mathrm{wt}}_{\mathfrak{g}} : \mathbb{T}_{\mathrm{End}(V)} \rightsquigarrow \mathbb{T}_{\mathfrak{g}}$ *is an equivalence of monoidal categories.*

If V is countable dimensional, it is easy to check that V^*/V_* is a simple $\mathfrak{g}^M = \mathfrak{gl}^M(V, V_*)$-module. Hence V^* is a $\mathfrak{g}^M$-module of length 2. This raises the natural question of whether the entire category $\mathbb{T}_{\mathrm{End}(V)}$ consists of $\mathfrak{g}^M$-modules of finite length. A further problem is to compute the socle filtration as a $\mathfrak{g}^M$-module of a simple $\mathrm{End}(V)$-module in $\mathbb{T}_{\mathrm{End}(V)}$.

Another open question is whether there is an analogue of the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ when we replace $\mathfrak{g}$ by $\mathfrak{g}^M$. More precisely, what can be said about the abelian monoidal category of $\mathfrak{g}^M$-modules obtained from $\mathbb{T}_{\mathfrak{g}^M}$ by iterated dualization in addition to taking submodules, quotients and applying $\otimes$? In particular, the adjoint representation, and therefore the coadjoint representation are objects of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}^M}$. How can one describe the coadjoint representation $(\mathfrak{g}^M)^*$ of $\mathfrak{g}^M$?

Note added in proof: While this paper was under review, Alexandru Chirvasitu gave a proof of Conjecture 8.4 and computed the $\mathfrak{g}^M$-module socle filtration of any simple module in $\mathbb{T}_{\mathrm{End}(V)}$. His results appear in the article [C] in the present volume.

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