On the Structure of $\mathbb{N}$-Graded Vertex Operator Algebras

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Abstract We consider the algebraic structure of $\mathbb{N}$-graded vertex operator algebras with conformal grading $V = \oplus_{n \geq 0} V_n$ and $\dim V_0 \geq 1$. We prove several results along the lines that the vertex operators $Y(a, z)$ for a in a Levi factor of the Leibniz algebra V_1 generate an affine Kac–Moody subVOA. If V arises as a shift of a self-dual VOA of CFT-type, we show that V_0 has a "de Rham structure" with many of the properties of the de Rham cohomology of a complex connected manifold equipped with Poincaré duality.

Key words Vertex operator algebra • Lie algebra • Leibniz algebra.

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1 Introduction

The purpose of this paper is the study of the algebraic structure of $\mathbb{N}$-*graded vertex operator algebras* (VOAs). A VOA $V = (V, Y, \mathbf{1}, \omega)$ is called $\mathbb{N}$-graded if it has no nonzero states of negative conformal weight, so that its conformal grading takes the form

$$V = \oplus_{n=0}^{\infty} V_n. \tag{1}$$

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The VOAs in this class which have been most closely investigated hitherto are those of *CFT-type*, where one assumes that $V_0 = \mathbb{C}\mathbf{1}$ is spanned by the vacuum vector. (It is well known that a VOA of CFT-type is necessarily $\mathbb{N}$-graded.) Our main interest here is in the contrary case, when $\dim V_0 \geq 2$.

There are several available methods of constructing $\mathbb{N}$-graded vertex algebras. One that particularly motivates the present paper arises from the cohomology of the chiral de Rham complex of a complex manifold M, due to Malikov, Schechtman and Vaintrob [MS1,MS2,MSV]. In this construction V_0 (which is always a commutative algebra with respect to the -1th operation $ab := a(-1)b$) is identified with the de Rham cohomology $H^*(M)$. One can also consider algebraic structures defined on $V_0 \oplus V_1$ or closely related spaces, variously called *1-truncated conformal algebras, vertex A-algebroids, and Lie A-algebroids* [GMS, Br, LY], and construct $\mathbb{N}$-graded vertex algebras from a 1-truncated conformal algebra much as one constructs affine VOAs from a simple Lie algebra. A third method involves *shifted* VOAs [DM3]. Here, beginning with a VOA $V = (V, Y, \mathbf{1}, \omega)$, one replaces ω by a second conformal vector $\omega_h := \omega + h(-2)\mathbf{1}$ $(h \in V_1)$ so that $V^h := (V, Y, \mathbf{1}, \omega_h)$ is a new VOA with the same Fock space, vacuum and set of fields as V. We call V^h a *shifted VOA*. For propitious choices of V and h (lattice theories were used in [DM3]) one can construct lots of shifted VOAs that are $\mathbb{N}$-graded. In particular, if V is rational, then V^h is necessarily also rational, and in this way one obtains $\mathbb{N}$-graded rational VOAs that are not of CFT-type.

Beyond these construction techniques, the literature devoted to the study of $\mathbb{N}$-graded VOAs per se is sparse. There are good reasons for this. For a VOA of CFT-type the weight 1 space V_1 carries the structure of a Lie algebra L with respect to the bracket $[ab] = a(0)b$ $(a, b \in V_1)$, and the modes of the corresponding vertex operators $Y(a, z)$ close on an affinization $\widehat{L}$ of L. For a general VOA, $\mathbb{N}$-graded or not, this no longer pertains. Rather, V_1 satisfies the weaker property of being a *left Leibniz algebra* (a sort of Lie algebra for which skew-symmetry fails), but one can still ask the question:

$$\begin{aligned}&\text{what is the nature of the algebra spanned}\\&\text{by the vertex operators } Y(a, z) \text{ for } a \in V_1?\end{aligned} \tag{2}$$

Next we give an overview of the contents of this paper. Section 2 is concerned with question (2) for an *arbitrary* VOA. After reviewing general facts about Leibniz algebras and their relation to VOAs, we consider the annihilator $F \subseteq V_1$ of the *Leibniz kernel* of V_1. F is itself a Leibniz algebra, and we show (Theorem 1) that the vertex operators $Y(a, z)$ for a belonging to a fixed Levi subalgebra $S \subseteq F$ close on an affine algebra $U \subseteq V$. Moreover, all such Levi factors F are conjugate in Aut(V), so that U is an invariant of V. (Finite-dimensional Leibniz algebras have a Levi decomposition in the style of Lie algebras, and the semisimple part is a true Lie algebra.) This result generalizes the 'classical' case of VOAs of CFT-type discussed above, to which it reduces if $\dim V_0 = 1$, and provides a partial answer to (2). We do not know if, more generally, the same result holds if we replace S by a Levi factor of V_1.

From Sect. 3 on we consider simple ℕ-graded VOAs that are also *self-dual* in the sense that they admit a nonzero invariant bilinear form $(\ ,\ ) : V \times V \to \mathbb{C}$ (cf. [L]). By results in [DM2] this implies that V_0 carries the structure of a *local, commutative, symmetric algebra*, and in particular it has a unique minimal ideal $\mathbb{C}t$. This result is fundamental for everything that follows. It permits us to introduce a second bilinear form $\langle\ ,\rangle : V_1 \times V_1 \to \mathbf{C}$ on V_1, defined in terms of $(\ ,\ )$ and t, and we try to determine its radical. Section 3 covers background results, and in Sect. 4 we show (Proposition 2) exactly how $\mathrm{Rad}\langle\ ,\ \rangle$ is related to the annihilator of the endomorphism $t(-1)$ acting on V_1. In all cases known to us we have

$$\mathrm{Rad}\langle\ ,\ \rangle = \mathrm{Ann}_{V_1}(t(-1)), \tag{3}$$

and it is of interest to know if this is always true.

In Sects. 5 and 6 we consider shifted VOAs, more precisely we consider the set-up in which we have a *self-dual* VOA $(W, Y, \mathbf{1}, \omega')$ of CFT-type together with an element $h \in W_1$ such that the shifted theory $W^h = (W, Y, \mathbf{1}, \omega'_h)$ as previously defined is a self-dual, ℕ-graded VOA V. As we mentioned, triples (W, h, V) of this type are readily constructed, and they have very interesting properties. The main result of Sect. 5 is Theorem 2, which, roughly speaking, asserts that V_0 looks just like the de Rham cohomology of a complex manifold equipped with Poincaré duality. More precisely, we prove that the eigenvalues of $h(0)$ acting on V_0 are nonnegative integers; the maximal eigenvalue is ν, say, and the ν-eigenspace is 1-dimensional and spanned by t; and the restriction of the nonzero invariant bilinear form on V to V_0 induces a perfect pairing between the λ- and $(\nu - \lambda)$-eigenspaces. One may compare this result with the constructions of Malikov et al. in the chiral de Rham complex, where the same conclusions arise directly from the identification of the lowest weight space with $H^*(M)$ for a complex manifold M. There is, of course, no a priori complex manifold associated to the shifted triple (W, h, V), but one can ask whether, at least in some instances, the cohomology of the chiral de Rham complex arises from the shifted construction?

In Sect. 8 we present several examples that illustrate the theory described in the previous paragraph. In particular, we take for W the affine Kac–Moody theory $L_{\widehat{sl_2}}(k, 0)$ of positive integral level k and show that it has a canonical shift to a self-dual, ℕ-graded VOA $V = W^H$ ($2H$ is semisimple and part of a Chevalley basis for sl_2). It turns out that the algebra structure on V_0 is naturally identified with $H^*(\mathbb{CP}^k)$. We also look at shifts of lattice theories W_L, where the precise structure of V_0 depends on L.

Keeping the notation of the previous paragraph, in Sect. 6 we use the results of Sect. 5 to prove that the shifted VOA V indeed satisfies (3). Moreover, if the Lie algebra W_1 on the weight 1 space of the CFT-type VOA W is *reductive*, we prove that Rad$\langle\ ,\ \rangle$ is the *nilpotent radical* of the Leibniz algebra V_1, i.e., the smallest ideal in V_1 such that the quotient is a reductive Lie algebra. It was precisely for the purpose of proving such a result that the form $\langle\ ,\ \rangle$ was introduced. It is known [DM1] that W_1 is indeed reductive if W is *regular* (rational and C_2-cofinite), so for VOAs obtained as a shift of such a W, we get a precise description of the nilpotent radical, generalizing the corrresponding result of [DM3].

In Sect. 7 we study simple, self-dual $\mathbb{N}$-graded VOAs that are C_2-cofinite. After reviewing rationality and C_2-cofiniteness of vertex operator algebras, we prove (Theorem 4) that in this case Rad$\langle\ ,\ \rangle$ lie between the nilpotent radical of V_1 and the solvable radical of V_1. In particular, the restriction of $\langle\ ,\ \rangle$ to a Levi factor $S \subseteq V_1$ is nondegenerate; furthermore, the vertex operators $Y(a, z)$ $(a \in S)$ close on a tensor product of WZW models, i.e., simple affine algebras $L_{\hat{\mathfrak{g}}}(k, 0)$ of positive integral level k. Thus we obtain a partial answer to (2) which extends results in [DM4], where the result was proved for CFT-type VOAs.

2 Leibniz Algebras and Vertex Operator Algebras

In this section, we assume that V is any simple vertex operator algebra

$$V = \oplus_{n \geq n_0} V_n,$$

with *no* restriction on the nature of the conformal grading.

A *left Leibniz algebra* is a $\mathbb{C}$-linear space L equipped with a bilinear product, or bracket, [ ] satisfying

$$[a[bc]] = [[ab]c] + [b[ac]], \quad (a, b, c \in V).$$

Thus $[a*]$ is a left derivation of the algebra L, and L is a Lie algebra if, and only if, the bracket is also skew-symmetric. We refer to [MY] for facts about Leibniz algebras that we use below.

Lemma 1. *V is a $\mathbb{Z}$-graded left Leibniz algebra with respect to the 0th operation $[ab] := a(0)b$. Indeed, there is a triangular decomposition*

$$V = \left\{\oplus_{n \leq 0} V_n\right\} \oplus V_1 \oplus \left\{\oplus_{n \geq 2} V_n\right\} \tag{4}$$

into left Leibniz subalgebras. Moreover, $\oplus_{n \leq 0} V_n$ is nilpotent.

Proof. Recall the commutator formula

$$[u(p), v(q)]w = \sum_{i=0}^{\infty} \binom{p}{i} (u(i)v)(p + q - i)w. \tag{5}$$

Upon taking $p = q = 0$, (5) specializes to

$$u(0)v(0)w - v(0)u(0)w = (u(0)v)(0)w,$$

which is the identity required to make V a left Leibniz algebra. The remaining assertions are consequences of

$$u(0)(V_n) \subseteq V_{n+k-1} \ \ (u \in V_k).$$

□

Remark 1. A *right* Leibniz algebra L has a bracket with respect to which L acts as right derivations. Generally, a left Leibniz algebra is *not* a right Leibniz algebra, and in particular a vertex operator algebra is generally not a right Leibniz algebra.

It is known (e.g., [B, MY]) that a finite-dimensional left Leibniz algebra has a *Levi decomposition*. In particular, this applies to the middle summand V_1 in (4). Thus there is a decomposition

$$V_1 = S \oplus B, \tag{6}$$

where S is a semisimple *Lie* subalgebra and B is the solvable radical of V_1. As in the case of a Lie algebra, we call S a *Levi subalgebra*. Unlike Lie algebras, Levi factors are generally *not* conjugate to each other by exponential automorphisms, i.e., Malcev's theorem does *not* extend to Leibniz algebras [MY].

This circumstance leads to several interesting questions in VOA theory. In particular, what is the nature of the subalgebra of V generated by a Levi subalgebra $S \subseteq V_1$? Essentially, we want a description of the Lie algebra of operators generated by the modes $a(n)$ $(a \in S, n \in \mathbb{Z})$. In the case when V is of CFT-type (i.e., $V_0 = \mathbb{C}\mathbf{1}$ is spanned by the vacuum), it is a fundamental fact that these modes generate an affine algebra. Moreover, all Levi subfactors of V_1 are conjugate in Aut(V) (cf. [M]), so that the affine algebra is an invariant of V. It would be interesting to know if these facts continue to hold for arbitrary vertex operator algebras. We shall deal here with a special case.

To describe our result, introduce the *Leibniz kernel* defined by

$$N := \langle a(0)a \mid a \in V_1\rangle = \langle a(0)b + b(0)a \mid a, b \in V_1\rangle \ \ \text{(linear span)}.$$

N is the smallest 2-sided ideal of V_1 such that V_1/N is a Lie algebra. The annihilator of the Leibniz kernel is

$$F := \mathrm{Ann}_{V_1}(N) = \{a \in V_1 \mid a(0)N = 0\}.$$

This is a 2-sided ideal of V_1, in particular it is a Leibniz subalgebra and itself contains Levi factors. We will prove

Theorem 1. *Let V be a simple vertex operator algebra, with N and F as above. Then the following hold:*

(a) Aut(V) *acts transitively on the Levi subalgebras of F.*
(b) Let $S \subseteq F$ be a Levi subalgebra of F. Then $u(1)v \in \mathbb{C}\mathbf{1}$ $(u, v \in S)$, and the vertex operators $Y(u, z)$ $(u \in S)$ close on an affine algebra, i.e.,

$$[u(m), v(n)] = (u(0)v)(m + n) + m\alpha(u, v)\delta_{m+n,0} Id_V,$$

where $u(1)v = \alpha(u, v)\mathbf{1}$.

We prove the theorem in a sequence of lemmas. Fix a Levi subalgebra $S \subseteq F$, and set

$$W := \oplus_{n\leq 0} V_n.$$

Lemma 2. *W is a* trivial *left S-module, i.e., $u(0)w = 0$ $(u \in S, w \in W)$.*

Proof. We have to show that each homogeneous space V_n $(n \leq 0)$, is a trivial left S-module. Because $L(-1) : V_n \to V_{n+1}$ is an *injective* V_1-equivariant map for $n \neq 0$, it suffices to show that V_0 is a trivial S-module.

Consider $L(-1) : V_0 \to V_1$, and set $N' := L(-1)V_0$. Because $[L(-1), u(0)] = 0$, $L(-1)$ is V_1-equivariant. By skew-symmetry we have $u(0)u = 1/2L(-1)u(1)u$. This shows that $N \subseteq N'$. Now because $(L(-1)v)(0) = 0$ $(v \in V)$ then in particular $N'(0)V_1 = 0$. Therefore, $S(0)N' = \langle u(0)v + v(0)u \mid u \in S, v \in N'\rangle \subseteq N$. But S is semisimple and it annihilates N. It follows that S annihilates N'.

Because V is simple, its center $Z(V) = \ker L(-1)$ coincides with $\mathbb{C}\mathbf{1}$. By Weyl's theorem of complete reducibility, there is an S-invariant decomposition

$$V_0 = \mathbb{C}\mathbf{1} \oplus J,$$

and restriction of $L(-1)$ is an S-isomorphism $J \stackrel{\cong}{\to} N'$. Because S annihilates N', it must annihilate J. It therefore also annihilates V_0, as we see from the previous display. This completes the proof of the lemma. □

Lemma 3. *We have*

$$u(k)w = 0 \ \ (u \in S, w \in W, k \geq 0). \tag{7}$$

Proof. Because S is semisimple, we may, and shall, assume without loss that u is a commutator $u = a(0)b$ $(a, b \in S)$. Then

$$(a(0)b)(k)w = a(0)b(k)w - b(k)a(0)w = 0.$$

The last equality holds thanks to Lemma 2, and because $b(k)w \in W$ for $k \geq 0$. The lemma is proved. □

Lemma 4. *We have*

$$[u(m), w(n)] = 0 \ \ (u \in S, w \in W; m, n \in \mathbb{Z}). \tag{8}$$

Proof. First notice that by Lemma 2,

$$[u(0), w(n)] = (u(0)w)(n) = 0. \tag{9}$$

Once again, it is suffices to assume that $u = a(0)b$ $(a, b \in F)$. In this case we obtain, using several applications of (9), that

$$\begin{aligned}
[u(m), w(n)] &= [(a(0)b)(m), w(n)] \\
&= [[a(0), b(m)], w(n)] \\
&= [a(0), [b(m), w(n)]] - [b(m), [a(0), w(n)]] \\
&= [a(0), (b(0)w)(m+n) + m(b(1)w)(m+n-1)] \\
&= 0.
\end{aligned}$$

This completes the proof of the lemma. □

Consider the Lie algebra L of operators on V defined by

$$L := \langle u(m), w(n) \mid u \in S, w \in W; m, n \in \mathbb{Z}\rangle.$$

If $w, x \in W$, then

$$[w(m), x(n)] = \sum_{i \geq 0} \binom{m}{i} (w(i)x)(m+n-i),$$

and $w(i)x$ has weight *less* than that of w and x whenever $w, x \in W$ are homogeneous and $i \geq 0$. This shows that the operators $w(m)$ $(w \in W, m \in \mathbb{Z})$ span a nilpotent ideal of L, call it P. Let L_0 be the Lie subalgebra generated by $u(m)$ $(u \in S_0, m \in \mathbb{Z})$. By Lemma 4, L_0 is also an ideal of L; indeed

$$L = P + L_0, \ \ [P, L_0] = 0.$$

Next, for $u, v \in S$ we have

$$[u(m), v(n)] = (u(0)v)(m+n) + \sum_{i \geq 1} \binom{m}{i} (u(i)v)(m+n-i). \tag{10}$$

So if $w \in S$, then by Lemma 4 once more,

$$[w(0), [u(m), v(n)]] = [w(0), (u(0)v)(m+n)] = (w(0)(u(0)v))(m+n).$$

This shows that L_0 coincides with its derived subalgebra. Furthermore, the short exact sequence

$$0 \to P \cap L_0 \to L_0 \to L_0/(P \cap L_0) \to 0$$

shows that L_0 is a perfect central extension of the loop algebra $\widehat{L}(S_0) \cong L_0/(P \cap L_0)$. Because $H^2(\widehat{L}(G))$ is 1-dimensional for a finite-dimensional simple Lie algebra G, we can conclude that $\dim(P \cap L_0)$ is *finite*.

Taking $m = 1$ in (10), it follows that

$$(u(1)v)(n) \in P \cap L_0 \ (n \in \mathbb{Z}). \tag{11}$$

Now if $u(1)v \notin Z(V)$, then all of the modes $(u(1)v)(n), n < 0$, are nonzero, and indeed linearly independent. This follows from the creation formula

$$\sum_{n \leq -1} (u(1)v)(n)\mathbf{1}z^{-n-1} = e^{zL(-1)}u(1)v.$$

Because $P \cap L_0$ is finite-dimensional and contains all of these modes, this is not possible. We deduce that in fact $u(1)v \in Z(V) = \mathbb{C}$, say $u(1)v = \alpha(u, v)\mathbf{1}, \alpha(u, v) \in \mathbb{C}$.

Taking $m = 2, 3, \ldots$ in (10), we argue in the same way that $u(i)v \in Z(V)$ for $i \geq 2$. Since $Z(V) \subseteq V_0$, this means that $u(i)v = 0$ for $i \geq 2$. Therefore, (10) now reads

$$[u(m), v(n)] = (u(0)v)(m+n) + m\alpha(u, v)\delta_{m+n,0}Id, \tag{12}$$

where $u(1)v = \alpha(u, v)\mathbf{1}$. This completes the proof of part (b) of the Theorem.

It remains to show that $\mathrm{Aut}(V)$ acts transitively on the set of Levi subalgebras of F.

Lemma 5. $[FF]$ *consists of* primary states*, i.e.,* $L(k)[FF] = 0$ $(k \geq 1)$.

Proof. It suffices to show that $L(k)a(0)b = 0$ for $a, b \in F$ and $k \geq 1$. Since $L(k)b \in W$ then $a(0)L(k)b = 0$ by Lemma 2. Using induction on k, we then have

$$\begin{aligned} L(k)a(0)b &= [L(k), a(0)]b \\ &= (L(-1)a)(k+1)b + (k+1)(L(0)a)(k)b + (L(k)a)(0)b \end{aligned}$$

$$= (L(k)a)(0)b$$
$$= \sum_{i \geq 0} (-1)^{i+1}/i!L(-1)^i b(i)L(k)a = 0,$$

where we used skew-symmetry for the fourth equality, and Lemma 3 for the last equality. The lemma is proved. □

Finally, by [MY], Theorem 3.1, if S_1, S_2 are a pair of Levi subalgebras of F, then we can find $x \in [FF]$ such that $e^{x(0)}(S_1) = S_2$. Because x is a primary state, it is well known that $e^{x(0)}$ is an automorphism of V. This completes the proof of Theorem 1. □

3 ℕ-Graded Vertex Operator Algebras

In this section, we assume that V is a *simple, self-dual,* ℕ*-graded* vertex operator algebra. We are mainly interested in the case that $\dim V_0 \geq 2$. There is a lot of structure available to us in this situation, and in this section we review some of the details, and at the same time introduce some salient notation.

The self-duality of V means that there is a *nonzero* bilinear form

$$(\, , \,) : V \times V \to \mathbb{C}$$

that is *invariant* in the sense that

$$(Y(u,z)v, w) = \left(v, Y(e^{zL(1)}(-z^{-2})^{L(0)}u, z^{-1})w\right) \quad (u, v, w \in V). \tag{13}$$

$(\, , \,)$ is necessarily *symmetric* [FHL], and because V is simple, it is then *nondegenerate*. The simplicity of V also implies (Schur's Lemma) that $(\, , \,)$ is *unique* up to scalars. By results of Li [L], there is an isomorphism between the space of invariant bilinear forms and $V_0/L(1)V_1$. Therefore, $L(1)V_1$ has codimension 1 in V_0. For now, we fix a nonzero form $(\, , \,)$, but do not choose any particular normalization.

If $u \in V_k$ is *quasiprimary* (i.e., $L(1)u = 0$), then (13) is equivalent to

$$(u(n)v, w) = (-1)^k (v, u(2k - n - 2)w) \quad (n \in \mathbb{Z}). \tag{14}$$

In particular, taking u to be the conformal vector $\omega \in V_2$, which is always quasiprimary, and $n = 1$ or 2 yields

$$(L(0)v, w) = (v, L(0)w), \tag{15}$$
$$(L(1)v, w) = (v, L(-1)w). \tag{16}$$

We write $P \perp Q$ for the direct sum of subspaces $P, Q \subseteq V$ that are orthogonal with respect to (,). Thus $(V_n, V_m) = 0$ for $n \neq m$ by (15), so that

$$V = \perp_{n \geq 0} V_n.$$

In particular, the restriction of (,) to each V_n is nondegenerate. We adopt the following notational convention for $U \subseteq V_n$:

$$U^{\perp} := \{a \in V_n \mid (a, U) = 0\}.$$

The *center* of V is defined to be $Z(V) := \ker L(-1)$. Because V is simple, we have $Z(V) = \mathbb{C}\mathbf{1}$ (cf. [LL, DM2]). Then from (16) we find that

$$(L(1)V_1)^{\perp} = \mathbb{C}\mathbf{1}. \tag{17}$$

V_0 carries the structure of a *commutative associative algebra* with respect to the operation $a(-1)b$ $(a, b \in V_0)$. Since all elements in V_0 are quasiprimary, we can apply (14) with $u, v, w \in V_0$ to obtain

$$(u(-1)v, w) = (v, u(-1)w). \tag{18}$$

Thus (,) is a nondegenerate, symmetric, invariant bilinear form on V_0, whence V_0 is a commutative *symmetric algebra*, or *Frobenius algebra*.

What is particularly important for us is that because V is simple, V_0 is a *local algebra*, i.e., the Jacobson radical $J := J(V_0)$ is the unique maximal ideal of V_0, and every element of $V_0 \setminus J$ is a unit. This follows from results of Dong–Mason ([DM2], Theorem 2 and Remark 3).

For a symmetric algebra, the map $I \to I^{\perp}$ is an inclusion-reversing duality on the set of ideals. In particular, because V_0 is a local algebra, it has a *unique minimal* (nonzero) ideal, call it T, and T is 1-dimensional. Indeed,

$$T = J^{\perp} = \mathrm{Ann}_{V_0}(J) = \mathbb{C}t, \tag{19}$$

for some fixed, but arbitrary, nonzero element $t \in T$. We have

$$T \oplus L(1)V_1 = V_0.$$

This is a consequence of the nondegeneracy of (,) on V_0, which entails that $L(1)V_1$ contains no nonzero ideals of V_0. In particular, (17) implies that

$$(t, \mathbf{1}) \neq 0. \tag{20}$$

We will change some of the notation from the previous section by setting $N := L(-1)V_0$ (it was denoted N' before). In the proof of Lemma 2 we showed that N contains the Leibniz kernel of V_1. In particular, V_1/N is a Lie algebra. We write

$$N_0/N = \mathrm{Nil}(V_1/N),\ N_1/N = \mathrm{Nil}p(V_1/N),\ B/N = \mathrm{solv}(V_1/N), \qquad (21)$$

the *nil radical*, *nilpotent radical*, and *solvable radical* respectively of V_1/N. N_0/N is the largest nilpotent ideal in V_1/N, N_1/N is the intersection of the annihilators of simple V_1/N-modules, and B/N the largest solvable ideal in V_1/N. It is well known that $N_1 \subseteq N_0 \subseteq B$. Moreover, $N_1/N = [V_1/N, V_1/N] \cap B/N$, V_1/N_1 is a *reductive* Lie algebra, and N_1 is the smallest ideal in V_1 with this property. Note that N_0 and B are also the largest nilpotent, and solvable ideals respectively in the left Leibniz algebra V_1.

Each of the homogeneous spaces V_n is a left V_1-module with respect to the 0th bracket. Because $u(0) = 0$ for $u \in N$, it follows that V_n is also a left module over the Lie algebra V_1/N. Since $V_0 = \mathbb{C}\mathbf{1} \oplus J$, $L(-1)$ induces an *isomorphism* of V_1-modules

$$L(-1) : J \overset{\cong}{\to} N. \qquad (22)$$

Remark 2. Most of the structure we have been discussing concerns the 1-truncated conformal algebra $V_0 \oplus V_1$ [Br, GMS, LY], and many of our results can be couched in this language.

4 The Bilinear Form ⟨ , ⟩

We keep the notation of the previous section; in particular $t \in V_0$ spans the unique minimal ideal of V_0. We introduce the bilinear form $\langle\ ,\ \rangle : V_1 \otimes V_1 \to \mathbb{C}$, defined as follows:

$$\langle u, v\rangle := (u(1)v, t), \quad (u, v \in V_1). \qquad (23)$$

We are interested in the *radical* of $\langle\ ,\ \rangle$, defined as

$$\mathrm{rad}\langle\ ,\ \rangle := \{u \in V_1 \mid \langle u, V_1\rangle = 0\}.$$

We will see that $\langle\ ,\ \rangle$ is a symmetric, invariant bilinear form on the Leibniz algebra V_1. The main result of this section (Proposition 2) determines the radical in terms of certain other subspaces that we introduce in due course. In order to study $\langle\ ,\ \rangle$ and its radical, we need some preliminary results.

Lemma 6. *We have $u(0)J \subseteq J$ and $u(0)T \subseteq T$ for $u \in V_1$. Moreover, the left annihilator*

$$M := \{u \in V_1 \mid u(0)T = 0\}$$

is a 2-sided ideal of V_1 of codimension 1, and $M = (L(-1)T)^{\perp}$.

Proof. Any derivation of a finite-dimensional commutative algebra B leaves invariant both the Jacobson radical $J(B)$ and its annihilator. In the case that the derivation is $u(0)$, $u \in V_1$, acting on V_0, this says that the left action of $u(0)$ leaves both J and T invariant (using (19) for the second assertion). This proves the first two statements of the lemma.

For $u \in V_1$ we have

$$(L(-1)t, u) = (t(-2)\mathbf{1}, u) = (\mathbf{1}, t(0)u) = -(\mathbf{1}, u(0)t). \tag{24}$$

Now because T is the unique minimal ideal in V_0 then $T \subseteq J$ and hence $\dim L(-1)T = 1$ by (22). Then (24) and (20) show that $(L(-1)T)^{\perp} = M$ has codimension exactly 1 in V_1.

Finally, using the commutator formula $[u(0), v(0)] = (u(0)v)(0)$ applied with one of $u, v \in M$ and the other in V_1, we see that $(u(0)v)(0)T = 0$ in either case. Thus $u(0)v \in M$, whence M is a 2-sided ideal in V_1. This completes the proof of the lemma. □

Lemma 7. *We have*

$$t(-2)J = 0.$$

Proof. Let $a \in J, u \in V_1$. Then

$$(t(-2)a, u) = (a, t(0)u) = -(a, u(0)t) = 0.$$

The last equality follows from $u(0)t \in T$ (Lemma 6) and $T = J^{\perp}$ (19). We deduce that $t(-2)J \subseteq V_1^{\perp} = 0$, and the lemma follows. □

Proposition 1. $\langle\ ,\ \rangle$ *is a symmetric bilinear form that is* invariant *in the sense that*

$$\langle v(0)u, w\rangle = \langle v, u(0)w\rangle \ \ (u, v, w \in V_1).$$

Moreover $N \subseteq \mathrm{rad}\,\langle\ ,\ \rangle$.

Proof. By skew-symmetry we have $u(1)v = v(1)u$ for $u, v \in V_1$, so the symmetry of $\langle\ ,\ \rangle$ follows immediately from the definition (23). If $u \in N$, then $u = L(-1)a$ for some $a \in J$ by (22), and we have

$$\begin{aligned}\langle u, v\rangle &= ((L(-1)a)(1)v, t) = -(a(0)v, t)\\ &= -(v, a(-2)t) = (v, t(-2)a - L(-1)t(-1)a) = 0.\end{aligned}$$

Here, we used $t(-2)a = 0$ (Lemma 7) and $t(-1)a \in t(-1)J = 0$ to obtain the last equality. This proves the assertion that $N \subseteq \mathrm{rad}\langle\ ,\ \rangle$.

As for the invariance, we have

$$\langle u(0)v, w\rangle = ((u(0)v)(1)w, t) = (u(0)v(1)w - v(1)u(0)w, t).$$
$$= (u(0)v(1)w, t) - \langle v, u(0)w\rangle.$$

Now $V_0 = \mathbb{C}\mathbf{1} \oplus J$, $u(0)\mathbf{1} = 0$, and $u(0)J \subseteq J = T^{\perp}$. Therefore, $(u(0)v(1)w, t) = 0$, whence we obtain $\langle u(0)v, w\rangle = -\langle v, u(0)w\rangle$ from the previous display. Now because $N \subseteq \mathrm{rad}\langle\ ,\ \rangle$ we see that

$$\langle v(0)u, w\rangle = -\langle u(0)v - L(-1)u(1)v, w\rangle = \langle v, u(0)w\rangle,$$

as required. This completes the proof of the proposition. □

Lemma 8. *We have*

$$\langle u, v\rangle = -(v, u(-1)t), \quad u, v \in V_1. \tag{25}$$

In particular,

$$\mathrm{rad}\langle\ ,\ \rangle = \{u \in V_1 \mid u(-1)t = 0\}.$$

Proof. The first statement implies the second, so it suffices to establish (25). To this end, we apply (13) with $u, v \in V_1$, $w = t$ to find that

$$\langle u, v\rangle = (u(1)v, t) = -(v, u(-1)t) - (v, (L(1)u)(-2)t).$$

On the other hand, $L(1)u \in V_0 = \mathbb{C}\mathbf{1} \oplus J$, so that $(L(1)u)(-2)t = a(-2)t = -t(-2)a + L(-1)t(-1)a$ for some $a \in J$. Since $t(-1)a = t(-2)a = 0$ (the latter equality thanks to Lemma 7), the final term of the previous display vanishes, and what remains is (25). The lemma is proved. □

We introduce

$$P := \{u \in V_1 \mid \langle u, M\rangle = 0\},$$
$$\mathrm{Ann}_{V_1}(t(-1)) := \{u \in V_1 \mid t(-1)u = 0\}.$$

Lemma 9. *We have*

$$P = \{u \in V_1 \mid t(-1)u \in L(-1)T\},$$

and this is a 2-sided ideal of V_1.

Proof. Let $m \in M$, $u \in V_1$. By (25) we have

$$\langle u, m\rangle = -(m, u(-1)t).$$

But by Lemma 6 we have $M^{\perp} = L(-1)T$. Hence, the last display implies that $P = \{u \in V_1 \mid u(-1)t \in L(-1)T\}$. Furthermore, we have $u(-1)t = t(-1)u - L(-1)t(0)u = t(-1)u + L(-1)u(0)t \in t(-1)u + L(-1)T$ by Lemma 6 once more. Thus $u(-1)t \in L(-1)T$ if, and only if, $t(-1)u \in L(-1)T$. The first assertion of the lemma follows.

Because $N \subseteq \text{rad}\langle\ ,\ \rangle$ thanks to Proposition 1, then certainly $N \subseteq P$. So in order to show that P is a 2-sided ideal in V_1, it suffices to show that it is a right ideal. To see this, let $a \in P, n \in M, u \in V_1$. By Lemma 6 and Proposition 1 we find that

$$\langle a(0)u, n\rangle = \langle a, u(0)n\rangle \in \langle a, M\rangle = 0.$$

This completes the proof of the lemma. □

Lemma 10. *We have* $M \cap \text{Ann}_{V_1}(t(-1)) = M \cap \text{rad}\langle\ ,\ \rangle$.

Proof. If $u \in M$, then $t(-1)u = u(-1)t - L(-1)u(0)t = u(-1)t$. Hence for $u \in M$, we have $u \in \text{Ann}_{V_1}(t(-1)) \Leftrightarrow u(-1)t = 0 \Leftrightarrow u \in \text{rad}\langle\ ,\ \rangle$, where we used Lemma 8 for the last equivalence. The lemma follows. □

Lemma 11. *At least one of the containments* $\text{rad}\langle\ ,\ \rangle \subseteq M$, $\text{Ann}_{V_1}(t(-1)) \subseteq M$ *holds.*

Proof. Suppose that we can find $v \in \text{rad}\langle\ ,\ \rangle \setminus M$. Then $v(-1)t = 0$ by Lemma 8, and $v(0)t = \lambda t$ for a scalar $\lambda \neq 0$. Rescaling v, we may, and shall, take $\lambda = 1$. Then

$$\begin{aligned} 0 = v(-1)t &= t(-1)v - L(-1)t(0)v \\ &= t(-1)v + L(-1)v(0)t = t(-1)v + L(-1)t. \end{aligned}$$

Then for $u \in \text{Ann}_{V_1}(t(-1))$ we have

$$(L(-1)t, u) = -(t(-1)v, u) = -(v, t(-1)u) = 0,$$

which shows that $\text{Ann}_{V_1}(t(-1)) \subseteq (L(-1)t)^{\perp} = M$ (using Lemma 6). This completes the proof of the Lemma. □

The next result almost pins down the radical of $\langle\ ,\ \rangle$.

Proposition 2. *Exactly one of the following holds:*

$$(i)\ \mathrm{Ann}_{V_1}(t(-1)) = \mathrm{rad}\langle\ ,\ \rangle \subset P;$$
$$(ii)\ \mathrm{Ann}_{V_1}(t(-1)) \subset \mathrm{rad}\langle\ ,\ \rangle = P;$$
$$(iii)\ \mathrm{rad}\langle\ ,\ \rangle \subset \mathrm{Ann}_{V_1}(t(-1)) = P.$$

In each case, the containment $\subset$ *is one in which the smaller subspace has* codimension one *in the larger subspace.*

Proof. First note from Lemma 9 that $\mathrm{Ann}_{V_1}(t(-1)) \subseteq P$; indeed, since $\dim L(-1)T = 1$ then the codimension is at most 1. Also, it is clear from the definition of P that $\mathrm{rad}\langle\ ,\ \rangle \subseteq P$.

Suppose first that the containment $\mathrm{Ann}_{V_1}(t(-1)) \subset P$ is *proper*. Then we can choose $v \in P \backslash \mathrm{Ann}_{V_1} t(-1)$ such that $t(-1)v = L(-1)t \neq 0$. If $u \in \mathrm{Ann}_{V_1}(t(-1))$, we then obtain

$$(L(-1)t, u) = (t(-1)v, u) = (v, t(-1)u) = 0,$$

whence $u \in (L(-1)t)^{\perp} = M$ by Lemma 6. This shows that $\mathrm{Ann}_{V_1}(t(-1)) \subseteq M$. By Lemma 10 it follows that $\mathrm{Ann}_{V_1}(t(-1)) = M \cap \mathrm{rad}\langle\ ,\ \rangle$. Now if also $\mathrm{rad}\langle\ ,\rangle \subseteq M$, then Case 1 of the theorem holds. On the other hand, if $\mathrm{rad}\langle\ ,\rangle \nsubseteq M$, then we have $\mathrm{Ann}_{V_1}(t(-1)) \subset \mathrm{rad}\langle\ ,\rangle \subseteq P$ and the containment is proper; since $\mathrm{Ann}_{V_1}(t(-1))$ has codimension at most 1 in P then we are in Case 2 of the theorem.

It remains to consider the case that $\mathrm{Ann}_{V_1}(t(-1)) = P \supseteq \mathrm{rad}\langle\ ,\rangle$. Suppose the latter containment is proper. Because M has codimension 1 in V_1, it follows from Lemma 10 that $\mathrm{rad}\langle\ ,\ \rangle$ has codimension exactly 1 in $\mathrm{Ann}_{V_1}(t(-1))$, whence Case 3 of the theorem holds. The only remaining possibility is that $\mathrm{Ann}_{V_1}(t(-1)) = P = \mathrm{rad}\langle\ ,\rangle$, and we have to show that this cannot occur. By Lemma 11 we must have $\mathrm{rad}\langle\ ,\ \rangle \subseteq M$, so that $M/\mathrm{rad}\langle\ ,\ \rangle$ is a subspace of codimension 1 in the nondegenerate space $V_1/\mathrm{rad}\langle\ ,\ \rangle$ (with respect to $\langle\ ,\ \rangle$). But then the space orthogonal to $M/\mathrm{rad}\langle\ ,\ \rangle$, that is $P/\mathrm{rad}\langle\ ,\ \rangle$, is 1-dimensional. This contradiction completes the proof of the Theorem. □

Remark 3. In all cases that we know of, it is (i) of Proposition 2 that holds. This circumstance leads us to raise the question, whether this is always the case? We shall later see several rather general situations where this is so. At the same time, we will see how $\mathrm{rad}\langle\ ,\ \rangle$ is related to the Leibniz algebra structure of V_1.

5 The de Rham Structure of Shifted Vertex Operator Algebras

In the next few sections we consider ℕ-graded vertex operator algebras that are *shifts* of vertex operator algebras of CFT-type [DM3].

Let us first recall the idea of a *shifted* vertex operator algebra [DM3]. Suppose that $W = (W, Y, \mathbf{1}, \omega')$ is an $\mathbb{N}$-graded vertex operator algebra of central charge c' and $Y(\omega', z) =: \sum_n L'(n)z^{-n-2}$. It is easy to see that for any $h \in W_1$, the state $\omega'_h := \omega' + L'(-1)h$ is also a Virasoro vector, i.e., the modes of ω'_h satisfy the relations of a Virasoro algebra of some central charge c'_h (generally different from c'). (The proof of Theorem 3.1 in [DM3] works in the slightly more general context that we are using here.) Now consider the quadruple

$$W^h := (W, Y, \mathbf{1}, \omega'_h), \tag{26}$$

which is generally *not* a vertex operator algebra. If it *is*, we call it a *shifted* vertex operator algebra.

We emphasize that in this situation, W and W^h share the *same* underlying Fock space, the *same* set of vertex operators, and the *same* vacuum vector. Only the Virasoro vectors differ, although this has a dramatic effect because it means that W and W^h have quite different conformal gradings, so that the two vertex operator algebras seem quite different.

Now let $V = (V, Y, \mathbf{1}, \omega)$ be a simple, self-dual $\mathbb{N}$-graded vertex operator algebra as in the previous two sections. The assumption of this section is that there is a self-dual VOA W of CFT-type such that $W^h = V$. That is, V arises as a shift of a vertex operator algebra of CFT-type as described above. Thus $h \in W_1$ and

$$(W, Y, \mathbf{1}, \omega'_h) = (V, Y, \mathbf{1}, \omega).$$

(Note that by definition, W has CFT-type if $W_0 = \mathbb{C}\mathbf{1}$. In this case, W is necessarily $\mathbb{N}$-graded by [DM3], Lemma 5.2.) Although the two vertex operator algebras share the same Fock space, it is convenient to distinguish between them, and we shall do so in what follows. We sometimes refer to (W, h, V) as a *shifted triple*. Examples are constructed in [DM3], and it is evident from those calculations that there are large numbers of shifted triples.

There are a number of consequences of the circumstance that (W, h, V) is a shifted triple. We next discuss some that we will need. Because $\omega = \omega'_h = \omega' + L'(-1)h$ then

$$L(n) = (\omega' + L(-1)h)(n + 1) = L'(n) - (n + 1)h(n), \tag{27}$$

in particular $L(0) = L'(0) - h(0)$. Because $h \in W_1$, we also have $[L'(0), h(0)] = 0$. Then because $L'(0)$ is semisimple with integral eigenvalues, the same is true of $h(0)$. Set

$$W_{m,n} := \{w \in W \mid L'(0)w = mw, h(0)w = nw\}.$$

Hence,

$$V_n = \oplus_{m\geq 0} W_{m,m-n},$$

and in particular

$$V_0 = \mathbb{C}\mathbf{1} \oplus_{m\geq 1} W_{m,m}, \tag{28}$$

$$V_1 = \oplus_{m\geq 1} W_{m,m-1}. \tag{29}$$

(28) follows because W is of CFT-type, so that $W_{0,0} = \mathbb{C}\mathbf{1}$ and $W_{m,n} = 0$ for $n < 0$.

We have $L(0)h = L'(0)h - h(0)h$. Because W is of CFT-type then W_1 is a Lie algebra with respect to the 0th bracket, and in particular $h(0)h = 0$. Therefore, $L(0)h = h$, that is $h \in V_1$. Thus $h(0)$ induces a derivation in its action on the commutative algebra V_0. The decomposition (28) is one of $h(0)$-eigenspaces, and it confers on V_0 a structure that looks very much like the de Rham cohomology of a (connected) complex manifold equipped with its Poincaré duality. This is what we mean by the *de Rham structure of* V_0. Specifically, we have

Theorem 2. *Set $A = V_0$ and $A^\lambda := W_{\lambda,\lambda}$, the λ-eigenspace for the action of $h(0)$ on A. Then the following hold;*

(i) *$A = \oplus_\lambda A^\lambda$, and if $A^\lambda \neq 0$, then λ is a* nonnegative *integer.*

(ii) $A^0 = \mathbb{C}\mathbf{1}$.

(iii) *Let $h(1)h = (\nu/2)\mathbf{1}$. Then $A^\nu = T = \mathbb{C}t$.*

(iv) $A^\lambda(-1)A^\mu \subseteq A^{\lambda+\mu}$.

(v) *$A^\lambda \perp A^\mu = 0$ if $\lambda + \mu \neq \nu$.*

(vi) *If $\lambda + \mu = \nu$, the bilinear form $(\ ,\ )$ induces a perfect pairing*

$$A^\lambda \times A^\mu \to \mathbb{C}.$$

(Here, $(\ ,\ )$ is the invariant bilinear form on V, and T the unique minimal ideal of V_0, as in Sects. 4 and 5.)

Proof. (i) and (ii) are just restatements of the decomposition (28).

Next we prove (iv). Indeed, because $h(0)$ is a derivation of the algebra A, if $a \in A^\lambda, b \in A^\mu$, then $h(0)a(-1)b = [h(0), a(-1)]b + a(-1)h(0)b = (h(0)a)(-1)b + \mu a(-1)b = (\lambda + \mu)a(-1)b$. Part (iv) follows.

Next we note that because W is of CFT-type then certainly $h(1)h = (\nu/2)\mathbf{1}$ for some scalar ν. Now let a, b be as in the previous paragraph. Then

$$\begin{aligned}
\lambda(a, b) &= (h(0)a, b) = \mathrm{Res}_z(Y(h, z)a, b) \\
&= \mathrm{Res}_z(a, Y(e^{zL(1)}(-z^{-2})^{L(0)}h, z^{-1})b) \ \ \text{(by (13))} \\
&= -\mathrm{Res}_z z^{-2}(a, Y(h + zL(1)h, z^{-1})b) \\
&= -(a, h(0)b) - (a, (L(1)h)(-1)b) \\
&= -\mu(a, b) - (a, L'(1)h - 2h(1)h)(-1)b) \\
&= -\mu(a, b) + 2(a, (h(1)h)(-1)b) \\
&= -\mu(a, b) + \nu(a, b).
\end{aligned}$$

Here we used the assumption that W is self-dual and of CFT-type to conclude that $L'(1)W_1 = 0$, and in particular $L'(1)h = 0$. Thus we have obtained

$$(\lambda + \mu - \nu)(a, b) = 0. \tag{30}$$

If $\lambda + \mu \neq \nu$, then we must have $(a, b) = 0$ for all choices of a, b, and this is exactly what (v) says. Because the bilinear form $(\, , \,)$ is nondegenerate, it follows that it must induce a perfect pairing between A^λ and A^μ whenever $\lambda + \mu = \nu$. So (vi) holds.

Finally, taking $\lambda = 0$, we know that $A^0 = \mathbb{C}\mathbf{1}$ by (ii). Thus A^0 pairs with A^ν and $\dim A^\nu = 1$. Because $(\mathbf{1}, t) \neq 0$ by (20), and T is an eigenspace for $h(0)$ (Lemma 6), we see that $A^\nu = T$. This proves (iii), and completes the proof of the theorem. □

6 The Bilinear Form in the Shifted Case

We return to the issue, introduced in Sect. 5, of the nature of the radical of the bilinear form $\langle \, , \, \rangle$ for an $\mathbb{N}$-graded vertex operator algebra V, assuming now that V is a shift of a simple, self-dual vertex operator algebra $W = (W, Y, \mathbf{1}, \omega')$ of CFT-type as in Sect. 5. We will also assume that $\dim V_0 \geq 2$.

We continue to use the notations of Sects. 3–5. We shall see that the question raised in Remark 3 has an affirmative answer in this case, and that $\text{rad}\langle \, , \, \rangle/N$ is *exactly* the nilpotent radical N_1/N of the Lie algebra V_1/N when W_1 is reductive. The precise result is as follows.

Theorem 3. *We have*

$$\oplus_{m\geq 2} W_{m,m-1} \subseteq \text{Ann}_{V_1}(t(-1)) = \text{rad}\langle \, , \, \rangle. \tag{31}$$

Moreover, if W_1 is reductive*, then*

$$N_1 = \oplus_{m\geq 2} W_{m,m-1} = \text{Ann}_{V_1}(t(-1)) = \text{rad}\langle \, , \, \rangle. \tag{32}$$

Recall that V_1 is a Leibniz algebra, $N = L(-1)V_0$ is a 2-sided ideal in V_1, V_1/N is a Lie algebra, and N_1/N is the *nilpotent radical* of V_1/N (21). Because W is a VOA of CFT-type then W_1 is a Lie algebra with bracket $a(0)b$ $(a, b \in W_1)$, and $W_{1,0} = C_{W_1}(h)$ is the *centralizer* of h in W_1.

Lemma 12. $h \in P$.

Proof. We have to show that $\langle h, M\rangle = (h(1)M, t) = 0$. Since M is an ideal in V_1 then M is the direct sum of its $h(0)$-eigenspaces. Let $M^p = \{m \in M \mid h(0)m = pm\}$. Now $h(0)h(1)m = h(1)h(0)m = ph(1)m$ $(m \in M^p)$, showing that $h(1)M^p \subseteq A^p$. If $p \neq 0$, then $(A^p, t) = 0$ by Theorem 2, so that $(h(1)M^p, t) = 0$ in this case.

It remains to establish that $(h(1)M^0, t) = 0$. To see this, first note that because W is self-dual then $L'(1)W_1 = 0$. Since $L'(1) = L(1)+2h(1)$ then $h(1)W_1 = L(1)W_1$, and in particular $h(1)M^0 = L(1)M^0$ (because $M^0 \subseteq V_1^0 = W_{1,0} \subseteq W_1$). Therefore, $(h(1)M^0, t) = (L(1)M^0, t) = (M^0, L(-1)t) = 0$, where the last equality holds by Lemma 6. The lemma is proved. □

Lemma 13. $h \notin \mathrm{Ann}_{V_1}(t(-1)) \cup \mathrm{rad}\langle\ ,\ \rangle$.

Proof. First recall that $h(1)h = \nu/2\mathbf{1}$. Then we have

$$\langle h, h\rangle = (h(1)h, t) = \nu/2(\mathbf{1}, t) \neq 0.$$

Here, $\nu \neq 0$ thanks to Theorem 2 and because we are assuming that $\dim V_0 \geq 2$. Because $h \in V_1$, this shows that $\langle h, V_1\rangle \neq 0$, so that $h \notin \mathrm{rad}\langle\ ,\ \rangle$.

Next, using (27) we have $L(1)h = L'(1)h - 2h(1)h$. Because W is assumed to be self-dual then $L'(1)h = 0$, so that $L(1)h = -2h(1)h = -\nu\mathbf{1}$. Now

$$(t(-1)h, h) = (h(-1)t - L(-1)h(0)t, h).$$

Therefore,

$$(L(-1)h(0)t, h) = (h(0)t, L(1)h) = -\nu^2(t, \mathbf{1}).$$

Also,

$$(h(-1)t, h) = (t, -h(1)h - (L(1)h)(0)h) = -\nu/2(t, \mathbf{1}).$$

Therefore,

$$(t(-1)h, h) = (\nu^2 - \nu/2)(t, \mathbf{1}).$$

Because ν is a positive integer, the last displayed expression is nonzero. Therefore, $t(-1)h \neq 0$, i.e., $h \notin \mathrm{Ann}_{V_1}(t(-1))$. This completes the proof of the lemma. □

We turn to the proof of Theorem 3. First note that by combining Lemmas 12 and 13 together with Proposition 2, we see that cases (ii) and (iii) of Proposition 2 *cannot* hold. Therefore, case (i) must hold, that is

$$\mathrm{rad}\langle\ ,\ \rangle = \mathrm{Ann}_{V_1}(t(-1)).$$

From (28) it is clear that, up to scalars, $\mathbf{1}$ is the only state in V_0 annihilated by $h(0)$. It then follow from Lemma 6 that $J = \oplus_{m\geq 1} W_{m,m}$. In particular, $(W_{m,m}, t) = 0$ $(m \geq 1)$ by (19).

Now let $u \in W_{m,m-1}$, $v \in W_{k,k-1}$ with $m \geq 1, k \geq 2$. Then $u(1)v \in V_0$ and $L'(0)u(1)v = (m+k-2)u(1)v$. Therefore, $u(1)v \in W_{m+k-2,m+k-2}$, and because $m+k-2 \geq 1$ it follows that

$$\langle u, v\rangle = (u(1)v, t) = 0.$$

Because this holds for all $u \in W_{m,m-1}$ and all $m \geq 1$, we conclude that $v \in \mathrm{rad}\langle\ ,\ \rangle$. This proves that $\oplus_{m\geq 2} W_{m,m-1} \subseteq \mathrm{rad}\langle\ ,\ \rangle$. Now (31) follows immediately.

Now suppose that W_1 is reductive. Because W is self-dual and of CFT-type, it has (up to scalars) a unique nonzero invariant bilinear form. Let us denote it by $((\ ,\ ))$. In particular, we have

$$((u, v))\mathbf{1} = u(1)v \ \ (u, v \in W_1).$$

Because V is simple and V, W have the same set of fields, then W is also simple. In particular, $((\ ,\ ))$ must be nondegenerate. Now if L is a (finite-dimensional, complex) reductive Lie algebra equipped with a nondegenerate, symmetric invariant bilinear form, then the restriction of the form to the centralizer of any semisimple element in L is also nondegenerate. In the present situation, this tells us that the restriction of $((\ ,\ ))$ to $C_{W_1}(h)$ is nondegenerate.

On the other hand, we have

$$\langle u, v\rangle = (u(1)v, t) = ((u, v))(\mathbf{1}, t)$$

and $(\mathbf{1}, t) \neq 0$ by (20). This shows that the restrictions of $\langle\ ,\ \rangle$ and $((\ ,\ ))$ to $C_{W_1}(h)$ are *equivalent* bilinear forms. Since the latter is nondegenerate, so is the former. Therefore, $\mathrm{rad}\langle\ ,\ \rangle \cap C_{W_1}(h) = 0$. Now the second and third equalities of (32) follow from (31) and the decomposition $V_1 = C_{W_1}(h) \oplus \oplus_{m\geq 2} W_{m,m-1}$.

To complete the proof of the theorem it suffices to prove the next result.

Lemma 14. *We have*

$$N_1 = \mathrm{Nilp(V1)}(C_{W_1}(h)) \oplus_{m\geq 2} W_{m,m-1}. \tag{33}$$

In particular, if W_1 is a reductive *Lie algebra, then*

$$N_1 = \oplus_{m\geq 2} W_{m,m-1}. \tag{34}$$

Proof. Let $u \in W_{k,k-1}$, $v \in W_{m,m-1}$ with $k \geq 1, m \geq 2$. Then $v(0)u \in W_{m+k-1} \cap V_1 \subseteq W_{m+k-1,m+k-2}$ and $m+k-1 > k$. This shows that $\oplus_{m\geq 2} W_{m,m-1}$ is an ideal in V_1. Moreover, because there is a maximum integer r for which $W_{r,r-1} \neq 0$, it follows that $v(0)^l V_1 = 0$ for large enough l, so that the left adjoint action of $v(0)$ on V_1 is *nilpotent*. This shows that $\oplus_{m\geq 2} W_{m,m-1}$ is a nilpotent ideal. Because $h(0)$ acts on $W_{m,m-1}$ as multiplication by $m-1$, then $W_{m,m-1} = [h, W_{m,m-1}]$ for $m \geq 2$, whence in fact $\oplus_{m\geq 2} W_{m,m-1} \subseteq N_1$. Then (33) follows immediately.

Finally, if W_1 is reductive, the centralizer of any semisimple element in W_1 is also reductive. In particular, this applies to $C_{W_1}(h)$ since $h(0)$ is indeed semisimple, so (34) follows from (33). This completes the proof of the lemma, and hence also that of Theorem 3. □

The VOA W is *strongly regular* if it is self-dual and CFT-type as well as both *C_2-cofinite* and *rational*. For a general discussion of such VOAs see, for example, [M]. It is known ([M] and [DM1], Theorem 1.1) that in this case W_1 is necessarily reductive. Consequently, we deduce from Theorem 3 that the following holds.

Corollary 1. *Suppose that W is a strongly regular VOA, and that V is a self-dual, $\mathbb{N}$-graded VOA obtained as a shift of W. Then* $\mathrm{rad}\langle\ ,\ \rangle = \mathrm{Nilp}(V_1)$. □

Remark 4. The corollary applies, for example, to the shifted theories $V = L_{s\hat{l}_2}(k,0)^H$ discussed in Sect. 8 below. In this case, one can directly compute the relevant quantities.

7 The C_2-Cofinite Case

In this section we are mainly concerned with simple VOAs V that are self-dual and $\mathbb{N}$-graded as before, but that are also *rational*, or *C_2-cofinite*, or both. Recall [DLM1, DLM2, Z] that V is rational if every admissible (or $\mathbb{N}$-gradable) V-module is completely reducible; C_2-cofinite if the span of the states $u(-2)v$ $(u, v \in V)$ has finite codimension in V; regular if it is *both* rational and C_2-cofinite; and *strongly regular* if it is both regular and self-dual (as discussed in Sect. 3). It is known [DLM1, DLM2, Z] that both rationality and C_2-cofiniteness imply that V-Mod has only finitely many simple objects.

To motivate the main results of this section, we recall some results about vertex operator algebras V with $V_0 = \mathbb{C}\mathbf{1}$. In this case, V_1 is a Lie algebra, and if V is strongly regular, then V_1 is *reductive* ([DM1], Theorem 1.1). It is also known ([DM4], Theorem 3.1) that if V is C_2-cofinite, but not necessarily rational, and $S \subseteq V_1$ is a Levi factor, then the vertex operator subalgebra U of V generated by S satisfies

$$U \cong L_{\hat{\mathfrak{g}}_1}(k_1, 0) \oplus \ldots \oplus L_{\hat{\mathfrak{g}}_r}(k_r, 0), \tag{35}$$

i.e., a direct sum of simple affine Kac–Moody Lie algebras $L_{\hat{\mathfrak{g}}_j}(k_j, 0)$ of positive integral level k_j.

We want to know to what extent these results generalize to the more general case when $\dim V_0 > 1$. With $N = L(-1)V_0$ as before, we have seen that V_1/N is a Lie algebra. Now $V_0 = \mathbb{C}\mathbf{1}$ precisely when $N = 0$, but the natural guess that V_1/N is reductive if V is rational and C_2-cofinite is generally false. Thus we need to understand the nilpotent radical N_1/N of this Lie algebra. That is where the bilinear form $\langle\ ,\ \rangle$ comes in. These questions are naturally related to the issue, already addressed in Sect. 2, of the structure of the subalgebra of V generated by a Levi subalgebra of V_1. The main result is

Theorem 4. *Let V be a simple, self-dual, $\mathbb{N}$-graded vertex operator algebra that is C_2-cofinite, and let $V_1 = B \oplus S$ with Levi factor S and solvable radical B. Then the following hold.*

1. *$N_1 \subseteq \mathrm{rad}\langle\ ,\ \rangle \subseteq B$, and the restriction of $\langle\ ,\ \rangle$ to S is nondegenerate. In particular, $a(1)b \in \mathbb{C}\mathbf{1}$ for all $a, b \in S$.*
2. *If U is the vertex operator algebra generated by S, then U satisfies (35).*

We start with

Proposition 3. *Let $V = \oplus_{n=0}^{\infty} V_n$ be an $\mathbb{N}$-graded vertex operator algebra such that $dim\ V_0 > 1$. Let $X = \{x^i\}_{i\in I} \cup \{y^j\}_{j\in J}$ be a set of homogeneous elements in V which are representatives of a basis of $V/C_2(V)$. Here x^i are vectors whose weights are greater than or equal to 1 and y^j are vectors whose weights are zero. Then V is spanned by elements of the form*

$$x^{i_1}(-n_1)\ldots x^{i_s}(-n_s)y^{j_1}(-m_1)\ldots y^{j_k}(-m_k)\mathbf{1}$$

where $n_1 > n_2 > \ldots. > n_s > 0$ and $m_1 \geq m_2 \geq \ldots \geq m_k > 0$.

Proof. The result follows by modifying the proof of Proposition 8 in [GN]. □

Notice that for a Lie algebra $W \subset V_1$, we have $u(0)v = -v(0)u$ for $u, v \in W$. Hence, $L(-1)u(1)v = 0$ and $u(1)v \in \mathbb{C}\mathbf{1}$. Moreover, we have $\langle\ ,\ \rangle\mathbf{1} = (u(1)v, t)\mathbf{1} = u(1)v$ for $u, v \in W$.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and let Ψ be the associated root system with simple roots Δ. Also, we set $(\cdot,\cdot)$ to be the nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ normalized so that the longest positive root $\theta \in \Psi$ satisfies $(\theta,\theta) = 2$. The corresponding affine Kac–Moody Lie algebra $\hat{\mathfrak{g}}$ is defined as

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K,$$

where K is central and the bracket is defined for $u, v \in \mathfrak{g}$, $m, n \in \mathbb{Z}$ as

$$[u(m), v(n)] = [u, v](m+n) + m\delta_{m+n,0}(u, v)K \quad (u(m) = u \otimes t^m).$$

Let $W \subset V_1$ be a Lie algebra such that $\mathfrak{g} \subset W$ and $\langle\ ,\ \rangle$ is nondegenerate on W. If $\langle\ ,\ \rangle$ is nondegenerate on $\mathfrak{g}$, then the map

$$\hat{\mathfrak{g}} \to \mathrm{End}(V);\, u(m) \mapsto u(m),\ u \in \mathfrak{g}, m \in \mathbb{Z},$$

is a representation of the affine Kac–Moody algebra $\hat{g}$ of level k where

$$\langle\ ,\ \rangle = u(1)v = k(u, v) \text{ for } u, v \in \mathfrak{g}.$$

Theorem 5. *Let $W \subset V_1$ be a Lie algebra such that $\langle\ ,\ \rangle$ is nondegenerate on W. If $\mathfrak{g}$ is a simple Lie subalgebra of W and U is the vertex operator subalgebra of V generated by $\mathfrak{g}$, then $\langle\ ,\ \rangle$ is nondegenerate on $\mathfrak{g}$, and U is isomorphic to $L_{\hat{\mathfrak{g}}}(k, 0)$. Furthermore, k is a positive integer and V is an integrable $\hat{\mathfrak{g}}$-module.*

Proof. We will follow the proof in Theorem 3.1 of [DM4]. First, assume that $\mathfrak{g} = sl_2(\mathbb{C})$ with standard basis $\alpha, x_\alpha, x_{-\alpha}$. Hence, $(\alpha, \alpha) = 2$. Since each homogeneous subspace of V is a completely reducible $\mathfrak{g}$-module, then V is also a completely reducible $\mathfrak{g}$-module. A nonzero element $v \in V$ is called a weight vector for $\mathfrak{g}$ of $\mathfrak{g}$-weight λ ($\lambda \in \mathbb{C}\alpha$) if $\alpha(0)v = (\alpha, \lambda)v$. Here, $\lambda \in \frac{1}{2}\mathbb{Z}\alpha$.

We now make use of Proposition 3. Let $X = \{x^i\}_{i\in I} \cup \{y^j\}_{j\in J}$ be a set of homogeneous weight vectors in V which are representatives of a basis of $V/C_2(V)$. The x^i are vectors whose weights are greater than or equal to 1 and y^j are vectors whose weights are zero. Since X is finite, there is a nonnegative elements $\lambda_0 = m\alpha \in \frac{1}{2}\mathbb{Z}\alpha$ such that the weight of each element in X is bounded above by λ_0. For any integer $t \geq 0$, we have

$$\begin{aligned}\oplus_{n\leq t(t+1)/2} V_n \subseteq \text{Span}_{\mathbb{C}}\{&x^{i_1}(-n_1)\ldots x^{i_s}(-n_s)y^{j_1}(-m_1)\ldots y^{j_r}(-m_r)\mathbf{1} \mid \\ &n_1 > n_2 > \ldots > n_s > 0, m_1 \geq m_2 \geq \ldots \geq m_r > 0, \\ &0 \leq s, r \leq t\}. \end{aligned} \tag{36}$$

Furthermore, if $n \leq \frac{t(t+1)}{2}$, then a $\mathfrak{g}$-weight vector in V_n has $\mathfrak{g}$-weight less than or equal to $2t\lambda_0 = 2tm\alpha$.

Let l be an integer such that $l + 1 > 4m$ and we let

$$u = (x_\alpha)(-1)^{l(l+1)/2}\mathbf{1}.$$

We claim that $u = 0$. Assume $u \neq 0$. By (36), we can conclude that the $\mathfrak{g}$-weight of u is at most $2lm\alpha$. This contradicts the direct calculation which shows that the $\mathfrak{g}$-weight of u is $\frac{l(l+1)}{2}\alpha$. Hence, $u = 0$. This implies that U is integrable. Furthermore, we have V is integrable, k is a positive integer and $\langle\ ,\ \rangle$ is nondegenerate.

This proves the theorem for $\mathfrak{g} = sl_2$. The general case follows easily from this (cf. [DM4]). □

Lemma 15. *Let S be a Levi subalgebra of V_1. Then $\langle\ ,\ \rangle$ is nondegenerate on S and* $\text{Rad}\langle\ ,\ \rangle \cap S = \{0\}$.

Proof. Clearly, for $u, v \in S$, we have $u(1)v \in \mathbb{C}\mathbf{1}$. Let $f : S \times S \to \mathbb{C}\mathbf{1}$ be a map defined by $f((u, v)) = u(1)v$. Since $u(1)v = v(1)u$ and

$$(w(0)u)(1)v = -(u(0)w)(1)v = -(u(0)w(1)v - w(1)u(0)v) = w(1)u(0)v$$

for $u, v, w \in S$, we can conclude that f is a symmetric invariant bilinear form on S. For convenience, we set $X = \mathrm{Rad}(f)$. Since S is semisimple and X is a S-module, these imply that $S = X \oplus W$ for some S-module W. Note that W and X are semisimple and $S \cap \mathrm{Rad}\langle\ ,\ \rangle \subseteq X$.

For $u, v \in X$, we have $u(1)v = 0$. Hence, the vertex operators $Y(u, z)$, $u \in X$, generate representation of the loop algebra in the sense that

$$[u(m), v(n)] = (u(0)v)(m+n), \text{ for } u, v \in X.$$

Following the proof of Theorem 3.1 in [DM4], we can show that the representation on V is integrable and the vertex operator subalgebra U generated by a simple component of X is the corresponding simple vertex operator algebra $L(k, 0)$ and $k = 0$. However, the maximal submodule of the Verma module $V(0, 0)$, whose quotient is $L(0, 0)$, has co-dimension one. This is not possible if $X \neq \{0\}$. Consequently, we have $X = 0$ and $S \cap \mathrm{Rad}\langle\ ,\ \rangle = \{0\}$. Hence $\langle\ ,\ \rangle$ is nondegenerate on S. □

Theorem 4 follows from these results.

8 Examples of Shifted Vertex Operator Algebras

To illustrate previous results, in this section we consider some particular classes of shifted vertex operator algebras.

8.1 *Shifted* $\widehat{sl_2}$

We will show that the simple vertex operator algebra (WZW model) $L_{\widehat{sl_2}}(k, 0)$ corresponding to affine sl_2 at (positive integral) level k has a canonical shift to an $\mathbb{N}$-graded vertex operator algebra $L_{\hat{sl_2}}(k, 0)^H$, and that the resulting de Rham structure on V_0 is that of complex projective space $\mathbb{CP}^k$. The precise result is the following.

Theorem 6. *Let e, f, h be Chevalley generators of sl_2, and set $H = h/2$. Then the following hold:*

(*a*) $L_{\hat{sl_2}}(k, 0)^H$ *is a simple, $\mathbb{N}$-graded, self-dual vertex operator algebra.*

(*b*) *The algebra structure on the zero weight space of $L_{\hat{sl_2}}(k, 0)^H$ is isomorphic to $\mathbb{C}[x]/\langle x^{k+1}\rangle$, where $x = e$.*

Proof. Let $W = L_{\hat{sl_2}}(k, 0)$. It is spanned by states $v_{IJK} := e_I f_J h_K \mathbf{1}$, where we write $e_I = e(-l_1)\dots e(-l_r)$, $f_J = f(-m_1)\dots f(-m_s)$, $h_K = h(-n_1)\dots h(-n_t)$ for $l_i, m_i, n_i > 0$. Note that $v_{IJK} \in W_n$, where $n = \sum l_i + \sum m_i + \sum n_i$

Recall from (27) that $L_H(0) = L(0) - H(0)$. We have

$$[H(0), e(n)] = [H, e](n) = e(n),$$

$$[H(0), f(n)] = [H, f](n) = -f(n),$$

$$[H(0), h(n)] = [H, h](n) = 0.$$

Then $H(0)v_{IJK} = (r - s)v_{IJK}$, so that

$$L_H(0)v_{IJK} = \left(\sum(l_i - 1) + \sum(m_j + 1) + \sum n_k\right) v_{IJK}. \tag{37}$$

It is well known (e.g., [DL], Propositions 13.16 and 13.17) that $Y(e, z)^{k+1} = 0$. Thus we may take r to be no greater than k. It follows from (37) that the eigenvalues of $L_H(0)$ are integral and bounded below by 0, and that the eigenspaces are finite-dimensional. Therefore, $V := W^H$ is indeed an ℕ-graded vertex operator algebra. Because W is simple then so too is V, since they share the same fields.

Next we show that V is self-dual, which amounts to the assertion that $L_H(1)V_1$ is *properly* contained in V_0. Observe from (37) that the states $e(-1)^p\mathbf{1}$ $(0 \le p \le k)$ span in V_0, while V_1 is spanned by the states $\{h(-1)e(-1)^i\mathbf{1}, e(-2)e(-1)^i\mathbf{1} \mid 0 \le i \le k-1\}$. We will show that $e(-1)^k\mathbf{1}$ does *not* lie in the image of $L_H(1)$.

For $g \in sl_2$, $m \ge 1$, we have

$$[L(1), g(-m)] = mg(1-m).$$

Since $L(1)e = 0$ and

$$L(1)e(-1)^{j+1}\mathbf{1} = e(-1)L(1)e^j(-1)\mathbf{1} + e(0)e^j(-1)\mathbf{1} = e(-1)L(1)e^j(-1)\mathbf{1}$$

for $j \ge 0$, we can conclude by induction that

$$L(1)e(-1)^i\mathbf{1} = 0 \text{ for all } i \ge 1.$$

Similarly, because $H(1)e = 0$ and

$$H(1)e(-1)^{j+1}\mathbf{1} = e(-1)H(1)e(-1)^j\mathbf{1} + e(0)e(-1)^j\mathbf{1} = e(-1)H(1)e(-1)^j\mathbf{1},$$

then

$$H(1)e(-1)^i\mathbf{1} = 0 \text{ for all } i \ge 0.$$

We can conclude that for $0 \le i \le k-1$,

$$\begin{aligned} L_H(1)h(-1)e(-1)^i\mathbf{1} &= (L(1) - 2H(1))h(-1)e(-1)^i\mathbf{1} \\ &= 2ie(-1)^i\mathbf{1} - 2ke(-1)^i\mathbf{1} \\ &= 2(i-k)e(-1)^i\mathbf{1}, \end{aligned}$$

while

$$L_H(1)e(-2)e(-1)^j\mathbf{1} = (L(1) - 2H(1))e(-2)e(-1)^j\mathbf{1} = 0.$$

Our assertion that $e(-1)^k\mathbf{1} \notin imL_H(1)$ follows from these calculations. This establishes part (a) of the theorem.

Finally, if we set $x := e(-1)\mathbf{1} = e$, then by induction $e(-1)^i\mathbf{1} = x.x^{i-1} = x^i$, so the algebra structure on V_0 is isomorphic $\mathbb{C}[x]/x^{k+1}$ and part (b) holds. This completes the proof of the theorem. □

Remark 5. Suitably normalized, the invariant bilinear form on V_0 satisfies $(x^p, x^q) = \delta_{p+q,k}$ (cf. Theorem 2). V_0 can be identified with the de Rham cohomology of $\mathbb{CP}^k$ (x has degree 2) equipped with the pairing arising from Poincaré duality.

8.2 Shifted Lattice Theories

Let L be a positive-definite even lattice of rank d with inner product $(\ ,\ ) : L \times L \to \mathbb{Z}$. Let $H = \mathbb{C} \otimes L$ be the corresponding complex linear space equipped with the $\mathbb{C}$-linear extension of $(\ ,\ )$. The dual lattice of L is

$$L^\circ = \{ f \in \mathbb{R} \otimes L \mid (f, \alpha) \in \mathbb{Z} \text{ all } \alpha \in L\}.$$

Let $(M(1), Y, \mathbf{1}, \omega_L)$ be the free bosonic vertex operator algebra based on H and let $(V_L, Y, \mathbf{1}, \omega_L)$ be the corresponding lattice vertex operator algebra. Both vertex operator algebras have central charge d, and the Fock space of V_L is

$$V_L = M(1) \otimes \mathbb{C}[L],$$

where $\mathbb{C}[L]$ is the group algebra of L.

For a state $h \in H \subset (V_L)_1$, we set $\omega_h = \omega_L + h(-2)\mathbf{1}$, with $V_{L,h} = (V_L, Y, \mathbf{1}, \omega_h)$.

Lemma 16. *([DM3]). Suppose that $h \in L^0$. Then $V_{L,h}$ is a vertex operator algebra, and it is self-dual if, and only if, $2h \in L$.*

For the rest of this section, we assume that $0 \neq h \in L^{\circ}$ and $2h \in L$, so that $V_{L,h}$ is a self-dual, simple vertex operator algebra. Set

$$Y(\omega_h, z) = \sum_{n \in \mathbb{Z}} L_h(n) z^{-n-2}.$$

Then

$$L_h(0)(u \otimes e^{\alpha}) = (n + \frac{1}{2}(\alpha, \alpha) - (h, \alpha)) u \otimes e^{\alpha} \ (u \in M(1)_n).$$

It follows that $V_{L,h}$ is $\mathbb{N}$-graded if, and only if, the following condition holds:

$$(2h, \alpha) \leq (\alpha, \alpha) \ \ (\alpha \in L). \tag{38}$$

From now on, we *assume that (38) is satisfied*. It is equivalent to the condition $(\alpha - h, \alpha - h) \geq (-h, -h)$, i.e., $-h$ has the least (squared) length among all elements in the coset $L - h$. Set

$$A := \{\alpha \in L \mid (\alpha, \alpha) = (2h, \alpha)\}.$$

Note that $0, 2h \in A$. We have

$$(V_{L,h})_0 = \text{Span}_{\mathbb{C}}\{e^{\alpha} \mid \alpha \in A\}.$$

We want to understand the commutative algebra structure of $(V_{L,h})_1$, defined by the -1th product $a(-1)b$. The identity element is $\mathbf{1} = e^0$.

First note that if $\alpha, \beta \in A$, then $(-h, -h) \leq (\alpha + \beta - h, \alpha + \beta - h) = (h, h) + 2(\alpha, \beta)$ shows that $(\alpha, \beta) \geq 0$. Moreover, $(\alpha, \beta) = 0$ if, and only if, $\alpha + \beta \in A$. We employ standard notation for vertex operators in the lattice theory V_L [LL]. Then

$$\begin{aligned} e^{\alpha}(-1)e^{\beta} &= Res_z z^{-1} E^{-}(-\alpha, z) E^{+}(-\alpha, z) e_{\alpha} z^{\alpha} \cdot e^{\beta} \\ &= \epsilon(\alpha, \beta) Res_z z^{(\alpha, \beta) - 1} E^{-}(-\alpha, z) E^{+}(-\alpha, z) e^{\alpha + \beta}, \end{aligned}$$

where

$$E^{-}(-\alpha, z) E^{+}(-\alpha, z) = exp\left\{-\sum_{n>0} \frac{\alpha(-n)}{n} z^{n}\right\} exp\left\{\sum_{n>0} \frac{\alpha(n)}{n} z^{-n}\right\}.$$

It follows that

$$e^{\alpha}(-1)e^{\beta} = \begin{cases} \epsilon(\alpha, \beta) e^{\alpha + \beta} & \text{if } \alpha + \beta \in A \\ 0 & \text{otherwise.} \end{cases} \tag{39}$$

If $0 \neq \alpha \in A$, then $(2h, \alpha) = (\alpha, \alpha) \neq 0$. Thus $2h + \alpha \notin A$, and the last calculation shows that $e^{\alpha}(-1)e^{2h} = 0$. It follows that e^{2h} spans the unique minimal ideal $T \subseteq (V_{L,h})_1$

Recall [LL] $\epsilon : L \times L \to \{\pm 1\}$ is a (bilinear) 2-cocycle satisfying $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha,\beta)}$. Thus we have proved

Lemma 17. *There are signs $\epsilon(\alpha, \beta) = \pm 1$ such that multiplication in $(V_{L,h})_1$ is given by (39). The minimal ideal T is spanned by e^{2h}.* □

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