A Bit of History Related to Logic Based on Equality

Peter B. Andrews

Abstract This historical note illuminates how Leon Henkin's work influenced that of the author. It focuses on Henkin's development of a formulation of type theory based on equality, and the significance of this contribution.

Keywords Type theory · Equality · Henkin · Axiom · Extensionality

Leon Henkin and I were both students of Alonzo Church, but he finished his Ph.D. thesis in 1947, and I did not arrive at Princeton for graduate work until 1959. However, Henkin was at the Institute for Advanced Study in Princeton on a Guggenheim Fellowship during the 1961–1962 academic year. I was working on questions related to Church's type theory [8] and was familiar with Henkin's groundbreaking paper [11], so I was delighted to have the opportunity to get to know him. We both attended logic seminars, and we had a few meetings. He was present at the seminars in which I discussed the gap in Herbrand's proof of Herbrand's theorem[1] in May 1962.

In February 1962, I copied the following material from [18, p. 350] (or perhaps from [19, p. 17]) into my journal:

> The preceding and other considerations led Wittgenstein to the view that mathematics does not consist of tautologies, but of what he called 'equations', for which I would prefer to substitute 'identities'. . . . (It) is interesting to see whether a theory of mathematics could not be constructed with identities for its foundation. I have spent a lot of time developing such a theory, and found it was faced with what seemed to me insuperable difficulties.

I was very interested in this problem, and about 9 April, I entered a note in my journal showing how quantifiers and connectives could be defined in terms of equality and the abstraction operator λ in the context of Church's type theory. By June I had seen at least a reference to Quine's abstract [17], which shows how these things can be done, but I do not remember whether I made the entry in my journal before seeing Quine's solution to the problem. My definition of conjunction was fundamentally different from that used by Quine.

In June 1962, Henkin mentioned that he was finishing work on a paper (published the following year as [12]) that gave a complete axiomatic treatment of type theory based on equality and abstraction in the context of propositional types.[2]

[1] See [6, 9].

[2] In both *propositional type theory* and *full type theory* (as we shall use these terms), the types are generated inductively from basic types by the condition that if α and β are types, then $(\alpha\beta)$ is the type of

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I told Henkin that I had seen some reference in the logical literature to defining quantifiers as well as propositional connectives in terms of equality, though I no longer remembered exactly where. Henkin was very interested to hear this, and together we searched my card file of bibliographic references (which I happened to have with me at the moment) and found a note I had made about this on the card for Quine's abstract [17]. Later Henkin found the papers [16] and [15]. In his later paper [13], Henkin noted on p. 33 that Quine was the first to observe that quantifiers could be defined in this context. It is clear that Henkin made this discovery independently, since his paper [12] was already written when I brought Quine's abstract to his attention. Quine described how to make these definitions in the short final section of [16], but Henkin developed this topic much further in [12], introducing an axiomatic system and establishing its soundness and completeness. Indeed, the decidability of Henkin's axiomatic system for propositional types follows directly from the results in his paper.

I was very interested in seeing Henkin's paper, and he was very busy, so he agreed to loan me his handwritten copy of the paper and the typed copy, which still did not have the formulas written in, and I agreed to write in the formulas while I read the paper. We were both doing some traveling, but by 13 July I was back in Princeton, and Henkin was in California, and he sent me the paper.

The axioms of Henkin's system, which are given below, were chosen to express basic properties of equality. α and β stand for arbitrary type symbols; A_α, B_α, and C_α stand for arbitrary formulas of type α; and X_β stands for an arbitrary variable of type β. T^n and F^n are formulas that denote truth and falsehood, respectively.

Henkin's axioms in [12]:

(H1) $A_\alpha \equiv A_\alpha$.
(H2) $(A_0 \equiv T^n) \equiv A_0$.
(H3) $(T^n \wedge F^n) \equiv F^n$.
(H4) $(g_{00}T^n \wedge g_{00}F^n) \equiv (\forall X_0(g_{00}X_0))$.
(H5) $(x_\beta \equiv y_\beta) \rightarrow .(f_{\alpha\beta} \equiv g_{\alpha\beta}) \rightarrow .(f_{\alpha\beta}x_\beta) \equiv (g_{\alpha\beta}y_\beta)$.
(H6) $(\forall X_\beta(f_{\alpha\beta}X_\beta \equiv g_{\alpha\beta}X_\beta)) \rightarrow (f_{\alpha\beta} \equiv g_{\alpha\beta})$.
(H7) $((\lambda X_\beta B_\alpha)A_\beta) \equiv C_\alpha$, where C_α is obtained from B_α by replacing each free occurrence of X_β in B_α by an occurrence of A_β (with a restriction).

A few days after I returned the manuscript to Henkin, I noticed that Axiom H3 was derivable from the other axioms. Other simplifications of the axiom system followed in the next two months, stimulated by many letters back and forth. We were both busy with other matters, but we managed to exchange several letters every week, sometimes writing two letters a day as we discussed new ideas. At one point, I remarked that the mere action of putting a letter to Henkin in the mailbox seemed to stimulate new ideas. (Of course, there was no email at that time.) Henkin started his letter of 8 August with the comment "This two-letters-at-a-time is infectious!". Bit by bit Axioms H1, H2, and H3 were all eliminated, and H4, H5, and H6 were simplified somewhat. The result was that the axiom system above was replaced by the following:

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functions with arguments of type β and values of type α. In propositional type theory, the only basic type is the type 0 of truth values, but in full type theory, the basic types are 0 and a type ι of individuals. Thus, propositional type theory may be regarded as higher-order propositional calculus, while full type theory includes nth-order logic for each positive integer n.

Simplified axioms as presented in [1]:

(A1) $(g_{oo}T^n \wedge g_{oo}F^n) \equiv \forall x_o(g_{oo}x_o)$.
(A2) $(f_{\alpha o} \equiv g_{\alpha o}) \to (h_{o(\alpha o)} f_{\alpha o} \equiv h_{o(\alpha o)} g_{\alpha o})$.
(A3) $(\forall x_\beta(f_{\alpha\beta}x_\beta \equiv g_{\alpha\beta}x_\beta)) \equiv (f_{\alpha\beta} \equiv g_{\alpha\beta})$.
(A4) $((\lambda X_\beta B_\alpha)A_\beta) \equiv C_\alpha$, where C_α is obtained from B_α by replacing each free occurrence of X_β in B_α by an occurrence of A_β (with a restriction).

Axiom schema A2 can be replaced by the superficially simpler schema

(A2′) $(f_\beta \equiv g_\beta) \to (h_{o\beta} f_\beta \equiv h_{o\beta} g_\beta)$,

but the formulation of (A2) shows that one does not need to assume (A2′) for all type symbols β.

On several occasions, I suggested to Henkin that he simply incorporate my proofs into his paper, but he insisted that I publish a separate paper presenting these proofs, and he wrote a very complimentary letter to Andrzej Mostowski (the editor of *Fundamenta Mathematicae*) recommending that my paper be published immediately following his own paper. He was very concerned that my paper be easy to read as well as technically correct, and made a number of suggestions about it. After we had discussed a number of ideas related to Axiom H2, Henkin found a way to derive it, but he did not want to write an addendum to my addendum to his paper, so he told me to simply include his proof of Axiom H2 in my paper.

The idea of formalizing type theory by using equality as a logical primitive can be used for the full theory of types as well as for propositional types, but I was concerned that some readers might not be sure of this and would therefore not understand the full significance of Henkin's paper. At my urging, Henkin added a discussion of this to the end of his paper, including a discussion of the need for an Axiom of Descriptions for the full theory of types.

As I think back to my interactions with Henkin, I realize how fortunate I was that he was so kind, generous, helpful, and wise.

Henkin's work played a decisive role in my life. Of course, like many other logicians, every time I taught a logic course I benefited from his work on completeness [10, 11] and his clarification of the notion of a nonstandard model. Questions about the nature of the general models of [11] provided the impetus for my paper [4]. Henkin's work developing a formulation of Church's type theory with equality (identity) as the sole logical primitive was particularly important for me. I used such a formulation of full type theory, called Q_0, in my Ph.D. thesis [2] and the textbook [5].

As noted in [2], it is easy to see that Axioms A1, A3, and A4 are independent. Henkin and I discussed the problem of proving the independence of Axiom A2 several times in 1962, and I mentioned it regularly in my course on type theory, but this remained an open problem until 2013, when Richard Statman showed that Axiom A2 is indeed independent.[3]

It is important to realize the significance of Henkin's contribution in developing a formulation of type theory based on equality. It is a real improvement of the system $\mathscr{C}$ discussed in [11], which has primitive constants for propositional connectives and quantifiers, and in which the formula $\mathscr{Q}_{o\alpha\alpha}$ for equality is defined in terms of these as

[3] The proof has not yet been published.

$[\lambda x_\alpha \lambda y_\alpha \forall f_{o\alpha} \,.\, f_{o\alpha} x_\alpha \supset f_{o\alpha} y_\alpha]$. (Except for some axiomatic differences, $\mathscr{C}$ is the system introduced by Church [8]). The formulation based on equality does far more than provide a cute and elegant formulation of type theory; it cleans up a subtle semantic problem, which we now explain.

As shown in [3], there is a nonstandard general model for $\mathscr{C}$ in which the Axiom of Extensionality $\forall x_\beta (f_{\alpha\beta} x_\beta = g_{\alpha\beta} x_\beta) \supset (f_{\alpha\beta} = g_{\alpha\beta})$ is not valid, since the sets in this model are so sparse that the denotation of the defined equality formula $\mathscr{Q}_{o\alpha\alpha}$ is not the actual equality relation. Thus, Theorem 2 of [11] (which asserts the completeness and soundness of $\mathscr{C}$) is technically incorrect. The apparently trivial soundness assertion is false.

However, this problem does not arise for the system Q_0 of full type theory based on equality, since in models of Q_0 the denotation of each equality symbol is the actual equality relation for that type in the model. (A detailed proof of the completeness and soundness of Q_0 may be found in [5].) Thus, in contexts where one wants to assume extensionality and discuss general models, a formulation of full type theory based on equality such as Q_0 is more appropriate than $\mathscr{C}$.

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P.B. Andrews (✉)
Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213-3890, USA
e-mail: andrews@cmu.edu