# Henkin on Completeness

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**Abstract** *The Completeness of Formal Systems* is the title of the thesis that Henkin presented at Princeton in 1947 under the supervision of Alonzo Church. A few years after the defense of his thesis, Henkin published two papers in the *Journal of Symbolic Logic*: the first, on completeness for first-order logic (Henkin in J. Symb. Log. 14(3):159–166, 1949), and the second one, devoted to completeness in type theory (Henkin in J. Symb. Log. 15(2):81–91, 1950). In 1963, Henkin published a completeness proof for propositional type theory (Henkin in J. Symb. Log. 28(3):201–216, 1963), where he devised yet another method not directly based on his completeness proof for the whole theory of types.

In this paper, these tree proofs are analyzed, trying to understand not just the result itself but also the process of discovery, using the information provided by Henkin in Bull. Symb. Log. 2(2):127–158, 1996.

In the third section, we present two completeness proofs that Henkin used to teach us in class. It is surprising that the first-order proof of completeness that Henkin explained in class was not his own but was developed by using Herbrand's theorem and the completeness of propositional logic. In 1963, Henkin published *An extension of the Craig–Lyndon interpolation theorem*, where one can find a different completeness proof for first-order logic; this is the other completeness proof Henkin told us about.

We conclude this paper, by introducing two expository papers on this subject. Henkin was an extraordinary insightful professor, and in 1967, he published two works that are very relevant for the subject addressed here: *Truth and provability* (Henkin in Philosophy of Science Today, pp. 14–22, 1967) and *Completeness* (Henkin in Philosophy of Science Today, pp. 23–35, 1967).

**Keywords** Henkin · Truth · Provability · Completeness · Type theory · First-order logic · Propositional type theory · Equality · Interpolation · Craig · Herbrand

## 1 Introduction

Henkin published two papers in the *Journal of Symbolic Logic*: the first, *The completeness of the first order functional calculus* [6] in 1949, and the second, *Completeness in type theory* [7] in 1950. *A theory of propositional types* [11] was published in *Fundamenta Mathematicae* in 1963.



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In this paper, we analyze these three proofs, trying to understand not just the result itself but also the process of discovery, using the information provided by Henkin in *The discovery of my completeness proofs* [16], published in 1996 in *Bulletin of Symbolic Logic*.[1]

We begin our work by pointing out some of Henkin's stated influences, especially three of them: (1) Gödel's completeness theorem, as well as his article on the consistency of the axiom of choice (where he builds a constructible universe), (2) Russell's theory of types and his expository explanation of the axiom of choice, and (3) Church's formulation of the theory of types and the important role played by both the lambda operator and the description operators in foundational issues.

The next section is devoted to Henkin's three completeness theorems, mentioned above, published in 1949, 1950 and 1963. In the first subsection, we focus on the unexpected discovery process and the role played by the lambda operator and the particular formulation of the axiom of choice. The model he builds is interpreted from a *nominalistic* point of view, following Henkin's papers [8] and [10]. In the second subsection, we see how the method used in the completeness proof for type theory is modified to obtain a completeness result for first-order logic. Moreover, we compare his proof with what today is referred to as *Henkin's method*. In the last subsection, devoted to completeness in propositional type theory, we try to answer several questions, among them: *why was Henkin interested in such a theory?*, *why a new method of proof? Can completeness for propositional type theory be derived from the already known completeness proof for type theory or for first-order logic also developed by Henkin?* We show that in this proof, the nominalistic position is more revealing than ever.

In the next section we address two completeness proofs that Henkin taught in class. The story behind this is that of María Manzano, who during the academic year of 1977–1978 attended his class of *metamathematics* for doctorate students at Berkeley. It is surprising that the first-order proof of completeness that Henkin explained in class was not his own but was developed by using Herbrand's theorem and the completeness of propositional logic. In 1963, Henkin published the paper entitled *An extension of the Craig–Lyndon interpolation theorem* [11], where one can find a different proof of completeness for first order logic; this was the other completeness proof Henkin taught in class.

We conclude this paper introducing two expository papers on this subject. Henkin was an extraordinary insightful professor as regards the clarity of his expositions, and he devoted some effort to writing informative papers. In particular, in 1967, he published two papers that are very relevant for the subject broached here: *Truth and provability* [13] and *Completeness* [14].

## 2 Henkin's Declared Influences

When Henkin was a student of Alonzo Church, the weak completeness theorem of first-order logic had the formulation given by Gödel [4] and the method of proof was based on a reduction to the propositional case.

[1] The paper was dedicated to his maestro Alonzo Church on the occasion of his 91 birthday; it was to be a book chapter, but the book was never published.

> In the presentation of Gödel's completeness proof, emphasis was given to its reductive character: the provability of a logically valid formula is reduced first to the provability of its Skolem normal form, an then to the provability of some tautology in a specific set of propositional formulas.
> (Henkin [16, p. 132])

In Gödel, we also find a strong completeness result, which is obtained by using weak completeness and compactness.

> THEOREM IX. *Every denumerably infinite set of formulae of the restricted functional calculus either is satisfiable* (that is, all formulae of the system are simultaneously satisfiable) *or possesses a finite subsystem whose logical product is refutable.*
> IX follows immediately from:
> THEOREM X. *For a denumerably infinite system of formulae to be satisfied it is necessary and sufficient that every finite subsystem be satisfiable.*
> (Gödel [4, p. 119])

A very similar version of this theorem IX is directly proved by Henkin in his doctoral thesis in 1947, both for type theory and for first-order logic. The method of proof is the main contribution of Henkin's thesis, and it relies on the effective building of a model satisfying the formulas in the consistent set. In the completeness proof for type theory, the clue lay in the lambda operator's ability to define a constructible hierarchy, combined with the description operator's ability to provide *formal beings*. To this effect, the reading of Gödel's monograph on the consistency of the axiom of choice and the generalized continuum hypothesis inspired Henkin.

> I admired the metamathematical treatment whereby the comprehension schema of set formation is obtained from finitely many axioms, and the sophisticated handling of inner-model constructions by means of the notion of the 'absoluteness' of various set-theoretical notions. I was intrigued by the creation of a universal choice function in the realm of constructible sets.
> (Henkin [16, p. 131])

Gödel's formulation of the theorem on the consistency of the axiom of choice has this form.

> THEOREM Let $T$ be a system of axioms for set theory obtained from v. Neumann's system $S^*$ by leaving out the axiom of choice (Ax. III3*); then, if $T$ is consistent, it remains so, if the following propositions 1–4 are adjoined simultaneously as new axioms:
> 1. The axiom of choice (i.e., v. Neumann's Ax. III3*)
> 2. The generalized Continuum-Hypothesis (i.e., the statement that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ holds for any ordinal $\alpha$)
> [. . . ]
> A corresponding theorem holds, if $T$ denotes the system of Prin. Math. or Fraenkel's system of axioms for set theory, leaving out in both cases the axiom of choice but including the axiom of infinity.
> (Gödel [5, p. 556])

The model Gödel provides in this proof consists of *'all "mathematically constructible" sets, where the term "constructible" is to be understood in the semiintuitionistic sense which excludes impredicative procedures. This means "constructible" sets are defined to be those sets which can be obtained by Russell's ramified hierarchy of types, if extended to include transfinite orders'*.[2]

Gödel also explains that the proposition *"Every set is constructible"*—formulated as $A$—can be proved to be consistent with the axioms in $T$ because is true in a model consisting of the constructible sets. Not only that, but when added to $T$ as an axiom *'seems*

[2]See [5, p. 556].

*to give a natural completion of the axioms of set theory, in so far it determines the vague notion of an arbitrary infinite set in a definite way'.*[3] For Gödel, it is very important that the consistency of $A$ prevails even when inaccessible numbers are admitted. *'Hence the consistency of A seems to be absolute in some sense, although it is not possible in the present state of affairs to give a precise meaning of this phrase'.*[4]

Among the influences of his thesis director, we believe that the lambda abstractor was philosopher's stone not just for Alonzo Church[5] but also for Henkin, as we will explain later on.

Church created the lambda calculus using functions as the basic concept, and he made a clear distinction between the value of a function for a given argument, $F(x)$, and the function itself, $\lambda x F(x)$. Functional abstraction allows the naming of functions in the language. Definability was then a hot topic, but we should also keep in mind that Church's thesis, identifying effective calculable functions with those definable by the $\lambda$ operator had already been formulated, and Henkin was one of Church's students.

Henkin was immensely attracted by Church's language for the theory of types,[6] in particular, by the lambda operator of functional abstraction acting on types, $\lambda$.

> The class of *type symbols* [...] is the least class of symbols which contains the symbols $\iota$ and 0 and is closed under the operation of forming the symbol $(\alpha\beta)$ from the symbol $\alpha$ and $\beta$. [...] 0 being the type of propositions, $\iota$ the type of individuals, and $(\alpha\beta)$ the type of functions of one variable for which the range of independent variable comprises the type $\beta$ and the range of the depended variable is contained in the type $\alpha$.
> [...]
> Certain formulas are distinguished as being *well-formed* and as having a certain *type*, in accordance with the following rules: (1) a formula consisting of a single proper symbol is a well-formed formula and has the type indicated by the subscript; (2) if $x_\beta$ is a variable with subscript $\beta$ and $M_\alpha$ is a well formed formula of type $\alpha$, then $(\lambda x_\beta M_\alpha)$ is a well-formed formula of type $\alpha\beta$; (3) if $F_{\alpha\beta}$ and $A_\beta$ are well-formed formulas of types $\alpha\beta$ and $\beta$, respectively, then $(F_{\alpha\beta} A_\beta)$ is a well-formed formula of type $\alpha$.
> (Church [2, pp. 56–57])

Another interesting operator of Church's language was $\iota_{a(0a)}$; these symbols were introduced to play the role of selection operators whose interpretations should be choice functions. The formula $(\iota_a B_0)$ is meant to be an abbreviation for $(\iota_{a(0a)}(\lambda x_a B_0))$ and it functions like the English word "the". This operator provides a very succinct formulation of both the axiom of descriptions

$$9^a.\ f_{0a}x_a \supset .\big[(y_a)(f_{0a}y_a \supset x_a = y_a)\big] \supset f_{0a}(\iota_{a(0a)} f_{0a})$$

and the axiom of choice

$$11^a.\ f_{0a}x_a \supset f_{0a}(\iota_{a(0a)} f_{0a})$$

that Henkin enjoyed very much. Not only choice and descriptions axioms are included in the calculus, but also infinity and extensionality

$$10^{a\beta}.\ (x_\beta)[f_{a\beta}x_\beta = g_{a\beta}x_\beta] \supset f_{a\beta} = f_{a\beta}$$

[3] See [5, p. 557].

[4] See [5, p. 557].

[5] This idea is developed with some details in [19] and [21].

[6] As presented in [2].

These axioms are added to the proper logical axioms in order to obtain classical mathematical theories: for elementary number theory, it is necessary to add the axioms of description and infinity; to obtain real number theory (analysis), also extensionality and choice are appended.[7] We may wonder *'what are they doing in this formal deductive system?* [...] *The answer is that Church wished to show how a logistic system can be applied to provide a foundation of mathematics, or at least Peano arithmetic and real analysis'*.[8] In fact, a large section of Church's paper is devoted to proving that Peano's postulates of arithmetic are theorems of this calculus. Eleven axioms and some rules are introduced for this calculus; among its rules, the λ-conversion ones play the relevant role:

> II. To replace any part $((\lambda x_\beta M_\alpha)N_\beta)$ of a formula by the result of substituting $N_\beta$ for $x_\beta$ throughout $M_\alpha$, provided that the bound variables of $M_\alpha$ are distinct both from $x_\beta$ and from the free variables of $N_\beta$.
> III. Where $A_\alpha$ is the result of substituting $N_\beta$ for $x_\beta$ throughout $M_\alpha$, to replace any part $A_\alpha$ of a formula by $((\lambda x_\beta M_\alpha)N_\beta)$, provided that the bound variables of $M_\alpha$ are distinct both from $x_\beta$ and from the free variables of $N_\beta$.
> (Church [2, p. 60])

At that time, type theory was a strong candidate for being a formal foundation for logic and mathematics since Russell had eliminated the main paradoxes by identifying the source of contradictions and then provided a language where the vicious circle is avoided:

> In all the above contradictions (which are merely a selection from an indefinite number) there is a common characteristic, which we may describe as self-reference or reflexiveness. [...] Thus all our contradictions have in common the assumption of a totality such that, if it were legitimate, it would at once be enlarged by new members defined in terms of itself.
> (Russell [28, pp. 154–155])[9]

Henkin declares in [16] that in 1938, as a second year student in Columbia University, in Russell's book he found an exciting formulation of the axiom of choice:

> [...] let me to browse in Bertrand Russell's Principles of Mathematics, [...] It was in that book that I first read about the principle of choice. I was enormously impressed by Russell's example of a shoe store with infinitely many pairs of shoes and socks: How easy is to specify one shoe from each pair in the shop [...] and how seemingly impossible is to specify one sock from each pair!
> (Henkin [16, pp. 128–129])

## 3 Henkin's Completeness Papers

As we have already mentioned, a few years after the defense of his thesis, Henkin published two papers in the *Journal of Symbolic Logic*: the first, on completeness for first-order logic [6] and the second devoted to completeness in type theory [7]. In the 1950 paper, completeness is formulated as *'Theorem 1: If Λ is any consistent set of cwffs, there is a general model (in which each domain $D_\alpha$ is denumerable) with respect to which Λ is satisfiable'*.[10] In the 1949 paper, it has the following form (for a calculus without the

[7] As you can see in [2, p. 61].

[8] See [16, p. 146].

[9] These pages refer to the reprint [28].

[10] See [7, p. 85].

equality symbol): THEOREM *'If $\Lambda$ is a set of formulas of $S_0$ in which no member has any occurrence of a free individual variable, and if $\Lambda$ is consistent, then $\Lambda$ is simultaneously satisfiable in a domain of individuals having the same cardinal number as the set of primitive symbols of $S_0$'.*[11]

Having such a similar statement with Gödel's formulation, we might wonder what is new in Henkin's completeness theorems. The obvious answer is that the method itself is completely new, and the main difference is that Henkin built the model instead of reducing the problem to the completeness of propositional logic via Skolem normal forms.

In the introduction to his 1949 paper, Henkin said:

> [...] the new method of proof which is the subject of this paper possesses two advantages. In the first place an important property of formal systems which is associated with completeness can now be generalized to systems containing a non-denumerable infinity of primitive symbols. While this is not of especial interest when formal systems are considered as *logics*—i.e., as means of analyzing the structures of languages—it leads to interesting applications in the field of abstract algebra. In the second place the proof suggests a new approach to the problem of completeness for functional calculi of higher order.
> (Henkin [6, p. 159])

It is interesting to recall that the publication order is the reverse of the discovery of the proofs. The completeness for first-order logic was accomplished when he realized that he could modify the proof obtained for type theory in an appropriate way. We consider this to be of great significance because the effort of abstraction needed for the first proof (that of type theory) provided a broad perspective that allowed him to leave apart some prejudices and to make the decisive changes needed to reach his second proof.

### *3.1 Completeness in the Theory of Types*

The theorem of completeness establishes the correspondence between deductive calculus and semantics. Gödel had solved it positively for first-order logic and negatively for any logical system able to contain arithmetic. The lambda calculus for the theory of types, as presented in [2], with the usual semantics over a standard hierarchy of types, was able to express arithmetic and hence could only be incomplete. Henkin showed that if the formulas were interpreted in a less rigid way, accepting other hierarchies of types that did not necessarily have to contain all the functions but at least did contain the definable ones, then it is easily seen that all consequences of a set of hypotheses are provable in the calculus. The valid formulas with this new semantics, called general semantics, are reduced to coincide with those generated by the calculus rules.

Henkin's proof that every consistent set of formulas has a model is performed by a constructive building of the model. Surprisingly, the model uses the expressions themselves as objects; in particular their elements are equivalence classes of closed expressions, the equivalence relationship being that of formal derivability of equality. We shall approach this construction while trying to figure out what Henkin had in mind and why he ended up defining the general models. On the one hand, we know that Henkin defended a Nominalistic position in several of his writings ([8] and [10]) well in harmony with his general

[11] See [6, p. 162].

models, even though we are not sure whether he had maintained this Nominalistic position early in the 1940s or whether instead it was his vision in the 1950s, as—so to speak—an inspired afterthought. What is certain is that he was interested in the elements of the hierarchy of types that possesses a name in type theory, as Henkin himself explained to us in [16]. On the other hand, the axiom of choice played a crucial role in the discovery.

## Steps Towards the Discovery

We are very lucky because Henkin wrote a very interesting paper [16] telling us of his discovery process. We will like to concentrate on the hierarchy of types and on its inner hierarchy of nameable types; the second hierarchy was the source of his discovery and, apparently, it maintains a relationship with the former similar to the relationship of Gödel's constructible hierarchy with Zermelo's hierarchy of sets. We shall pinpoint two flashes that illuminated Henkin in the process of manipulating types inside the hierarchy.

**Hierarchy of Types** The types are structured in a hierarchy that has the following as basic types: (1) $\mathcal{D}_\iota$ is a nonempty set; that of individuals of the hierarchy, (2) $\mathcal{D}_0$ is the domain of truth values (since we are in binary logic, these values are reduced to $T$ and $F$). The other domains are constructed from the basic types as follows: if $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ have already been constructed, we define $\mathcal{D}_{(\alpha\beta)}$ as the domain formed by all the functions from $\mathcal{D}_\beta$ to $\mathcal{D}_\alpha$.

To talk about this hierarchy, Church's formal language of [2] is introduced. Henkin said: *'I decided to try to see just which objects of the hierarchy of types did have names in $\mathcal{T}$'*.[12] That is, he intended to mentally represent the functions in the hierarchy of types that can be named by lambda expressions. To visualize this, he considered a hierarchy of types with a universe given by the set of natural numbers and a language with two constants: one as the name for the zero and another for the successor function. In the universe of subsets of the universe of individuals, there will clearly be both objects with and without a name because with a countable infinite universe of individuals the set of subsets is uncountable but the sets with a name are countable.

**Hierarchy of Nameable Types** These peculiar elements of the hierarchy can be named by closed expressions. For each type $\alpha$, the set $\mathcal{D}^n_\alpha$ contains the nameable elements,

$$\mathcal{D}^n_\alpha = \left\{ f \in \mathcal{D}_\alpha : \text{ there is a } F_\alpha \in \mathit{cwff} \text{ s.t. } \Im(F_\alpha) = f \right\}$$

(where $\Im$ stands for the interpretation of the formal language in the hierarchy of types).

Going up in the hierarchy, it is easy to see that the nameable domain $\mathcal{D}^n_{(\alpha\beta)}$ of type $(\alpha\beta)$ is such that each $f \in \mathcal{D}^n_{(\alpha\beta)}$ allows a map from $\mathcal{D}^n_\beta$ to $\mathcal{D}^n_\alpha$ to be defined. Even though each function in $\mathcal{D}^n_{(\alpha\beta)}$ has $\mathcal{D}_\alpha$ rather than $\mathcal{D}^n_\alpha$ as its range—that is, $f : \mathcal{D}_\beta \longrightarrow \mathcal{D}_\alpha$—we can see that the value of nameable elements in the domain $\mathcal{D}^n_\beta$ is also a nameable element in the domain $\mathcal{D}^n_\alpha$. The reason is as follows: for any $f \in \mathcal{D}^n_{(\alpha\beta)}$, we know that there is an $F_{\alpha\beta}$ such that $\Im(F_{\alpha\beta}) = f$ and this function provides any $g \in \mathcal{D}^n_\beta$ a value $f(g)$ in $\mathcal{D}^n_\alpha$

[12] See [16, p. 146].

(since $g$ has the form of $\Im(G_\beta) = g$ and so $f(g) = \Im(F_{\alpha\beta})(\Im(G_\beta)) = \Im(F_{\alpha\beta}G_\beta) \in \mathcal{D}^n_\alpha$). Therefore, we can replace each $f \in \mathcal{D}_{(\alpha\beta)}$ by the restriction of $f$ to $\mathcal{D}^n_\beta$:

$$\mathcal{D}^{n*}_{(\alpha\beta)} = \{f^* : f \in \mathcal{D}^n_{(\alpha\beta)} \text{ and } f^* = f \restriction \mathcal{D}^n_\beta\}.$$

Henkin wanted to know if this restricted class itself formed a hierarchy: *'There was, however, a problem with this idea: What if the hierarchy contracted under the proposed reduction of the domains of functions? In other words, could there be distinct functions $f$ and $g$ in some $D^n_{(\alpha\beta)}$ such that $f^* = g^*$?'*[13]

The answer to this question is *no*, simply because when $f$ and $g$ are in $\mathcal{D}^n_{(\alpha\beta)}$, then $\Im(F_{\alpha\beta}) = f$ and $\Im(G_{\alpha\beta}) = g$ for some cwff. Assume that $f \neq g$. Let us take $X_{0\beta}$ as the lambda expression $\lambda x_\beta \sim (F_{\alpha\beta}x_\beta = G_{\alpha\beta}x_\beta)$ representing the set of elements that give different values under the functions involved. By using the axiom of choice we see that if $f \neq g$, then $\Im(X_{0\beta}) \neq \emptyset$, and $\Im(\iota_{\beta(0\beta)}X_{0\beta})$ is an element $y \in \mathcal{D}_\beta$ such that $f(y) \neq g(y)$. Therefore, $f^* \neq g^*$.

The new hierarchy $\mathcal{D}^{n*}_\alpha$ is isomorphic to the previous one $\mathcal{D}^n_\alpha$. Moreover, the hierarchy obeys the rules of lambda conversion because for a function $f$ of type $(\alpha\beta)$ named by a lambda expression, $\Im(\lambda x_\beta N_\alpha) = f$, the value for a given argument $\Im(M_\beta) = m$ is

$$\begin{aligned} f(m) &= \Im(\lambda x_\beta N_\alpha)\big(\Im(M_\beta)\big) \\ &= \Im\big((\lambda x_\beta N_\alpha)M_\beta\big) \\ &\quad \text{and, by lambda conversion,} \\ &= \Im\left(N_\alpha \frac{M_\beta}{x_\beta}\right). \end{aligned}$$

**Flash Number One** He then realized that he had crossed the bridge that separates the world of semantical models from the world of syntactic deductive systems.

> As I struggled to see the action of functions more clearly in this way, I was struck by the realization that I have used λ-conversion, one of the formal rules of inference in Church's deductive system for the language of the Theory $\mathcal{T}$. All my efforts had been directed toward *interpretations* of the formal language, and now my attention was suddenly drawn to the fact that these were related to the formal deductive system for that language.
> (Henkin [16, p. 150])

Specifically, to identify objects named by both $M_\alpha$ and $N_\alpha$, he made use of a criterion based on the calculus, namely, the fact that $\vdash (M_\alpha = N_\alpha)$.

> In particular, I saw that using the symbol $\vdash$ for formal provability (or derivability) as usual, we can define for each type symbol $\alpha$, a domain $\mathcal{D}'_\alpha$ satisfying the following conditions: (i) Each cwff (closed wff, without free variables) $M_\alpha$ denotes an element $M'_\alpha$ of $\mathcal{D}'_\alpha$ and each element of $\mathcal{D}'_\alpha$ is denoted by some cwff $M_\alpha$; (ii) for any cwff $F_{\alpha\beta}$, $F'_{\alpha\beta}$ is a function mapping $\mathcal{D}'_\beta$ into $\mathcal{D}'_\alpha$; and (iii) for any cwffs $M_\alpha$ and $N_\alpha$, $M'_\alpha = N'_\alpha$ if, and only if, $\vdash (M_\alpha = N_\alpha)$.
> (Henkin [16, pp. 150–151])

In other words, Henkin saw a way of defining a hierarchy of names modulo equivalent classes in the deductive calculus. The definition of the universes $\mathcal{D}'_\alpha$ was based on

[13] See [16, p. 149].

recursion on types, and the building of $\mathcal{D}'_{\alpha b}$ from $\mathcal{D}'_{\alpha}$ and $\mathcal{D}'_{b}$ required the axiom of choice working in parallel with the constants $\iota_{\alpha(0\alpha)}$ mentioned above. Fortunately, Henkin's previous understanding of this operator was of great help. The construction seemed to work smoothly, with the only exception of the universe of truth values, $\mathcal{D}'_0$. *'In particular, if $M^0$ is a Gödel sentence such that neither $\vdash M^0$ nor $\vdash\sim M^0$, then $(0_1 = 0_1)'$, $(\sim 0_1 = 0_1)'$, and $M^{0\prime}$ are three distinct elements of $D'_0$'.*[14]

**Flash Number Two** He then realized that to reduce the universe of objects named by propositions (the truth values) to only two, the set of axioms had to be expanded until it constituted a maximal consistent set.

> As soon as I observed this, it occurred to me that if we were to add further cwffs of type 0 to the list of formal axioms, this would have the effect of reducing the number of elements in $\mathcal{D}'_0$ and that ultimately, by taking a maximal consistent set of axioms, the number of elements in $\mathcal{D}'_0$ would be two. [...] Immediately I realized that my discovery provided a kind of completeness proof for a system very much like the system PM of type theory which Gödel had proved incomplete.
> (Henkin [16, p. 151])

## The Proof

**Hierarchy of Equivalent Classes of Names** On this occasion, equivalent classes of closed formulas (cwffs) rather than proper objects are used to build the hierarchy. A maximal consistent set $\Gamma$ is needed and the new equivalence relation is defined as: Two cwffs $M_\alpha$ and $N_\alpha$ of type $\alpha$ will be called *equivalent* if $\Gamma \vdash M_\alpha = N_\alpha$. On p. 86 of *Completeness in the theory of types* [7], Henkin says:

> We now define by induction on $\alpha$ a frame of domains $\{D_\alpha\}$, and simultaneously a one-one mapping $\Phi$ of equivalent classes onto the domains $D_\alpha$ such that $\Phi([A_\alpha])$ is in $D_\alpha$.
> $D_0$ is the set of two truth values and $\Phi([A_0])$ is $T$ or $F$ according as $A_0$ or $\sim A_0$ is in $\Gamma$
> [...]
> $D_\iota$ is simply the set of equivalence classes $[A_\iota]$ of all cffs of type $\iota$. And $\Phi([A_\iota])$ is $[A_\iota]$
> [...]
> Now suppose that $D_\alpha$ and $D_\beta$ have been defined, as well as the value of $\Phi$ for all equivalence classes of formulas of type $\alpha$ and $\beta$ and that every element of $D_\alpha$, or $D_\beta$, is the value of $\Phi$ for some $[A_\alpha]$ or $[B_\beta]$ respectively. Define $\Phi([A_{\alpha\beta}])$ to be the function whose value, for the element $\Phi([B_\beta])$ of $D_\beta$ is $\Phi([A_{\alpha\beta} B_\beta])$.

He has to show that $\Phi$ is a function on equivalent classes and does not depend on the particular representative chosen, and also that the function is one-to-one. In the inductive step, to prove that $\Phi([A_{\alpha\beta}]) = \Phi([B_{\alpha\beta}])$ implies $[A_{\alpha\beta}] = [B_{\alpha\beta}]$, he uses choice and extensionality in a similar way as he did when building the hierarchy of nameable types; in particular, he uses the following theorem:

$$\vdash A_{\alpha\beta}\big(\iota_{(\beta(0\beta))}\big(\lambda x_\beta\big(\sim (A_{\alpha\beta}x_\beta = B_{\alpha\beta}x_\beta)\big)\big)\big)$$
$$= B_{\alpha\beta}\big(\iota_{(\beta(0\beta))}\big(\lambda x_\beta\big(\sim (A_{\alpha\beta}x_\beta = B_{\alpha\beta}x_\beta)\big)\big)\big) \supset .A_{\alpha\beta} = B_{\alpha\beta}$$

[14] See [16, p. 151]. In this paper, the type of individuals is 1, that is why he writes $(0_1 = 0_1)'$ instead of $(0_\iota = 0_\iota)'$.

We might wonder what the elements of $D_\iota$ are, since we do not have individual constants. Of course, the selection operator $\iota_{\beta(0\beta)}$ acting on expressions of the appropriate type, say $X_{0\beta}$, produces elements of any type $\beta$.

Using this construction, Henkin was able to achieve his completeness result:

**Theorem 1** (Henkin [7, p. 85]) *'If $\Lambda$ is any consistent set of cffs* (*sentences*), *there is a general model* (*in which each domain $\mathcal{D}_\alpha$ of $\mathcal{M}$ is denumerable*) *with respect to which $\Lambda$ is satisfiable'.*

To prove this theorem, the set $\Lambda$ is extended to a maximal consistent set $\Gamma$, which serves both as an oracle and as building blocks for the model, the following lemma being the relevant step.

**Lemma 2** (Henkin [7, p. 87]) *'For every $\phi$ and $B_\beta$, we have $V_\phi(B_\beta) = \Phi([B_\beta^\phi])$'.*

**The Definition of General Model** As we have seen, Henkin in his proof uses a hierarchy with countable domains whose elements are equivalent classes of names, *is it legitimate?*, *what about the definition of general models?* He introduces first the definition of a broader class of structures:[15] *'By a frame, we mean a family of domains, one for each type symbol, as follows: $D_\iota$ is an arbitrary set of individuals, $D_0$ is the set of two truth values, $T$ and $F$, and $D_{\alpha\beta}$ is some class of functions defined over $D_\beta$ with values in $D_\alpha$'.*

Inside this class, the general models are placed: *'A frame such that for every assignment $\phi$ and wff $A_\alpha$ of type $\alpha$, the value $V_\phi(A_\alpha)$ given by the rules* (i), (ii), and (iii) *is an element of $D_\alpha$ is called a general model'*. Thus, general models are frames characterized by being able to provide interpretations for any expression. However, *is that a proper definition?* Henkin is aware of the fact that *'Since this definition is impredicative, it is not immediately clear that any non-standard models exist. However, they do exist* [...]*'* but he does not seem to mind too much because immediately afterwards he goes on to prove Theorem 1 above, where such a model is constructed. Moreover, right at the beginning of the 1950 paper, in a footnote, Henkin declares that in the second-order case, the universes must be closed under certain operations and gives as examples that of complementation and projection, but he does not give an algebraic definition.[16]

**Nominalism** The models he builds are in accordance with a Nominalistic position, as Henkin himself affirms in *On nominalism*, published in 1953: *'In fact, such an interpretation is implicit in a recent paper discussing the problem of the completeness of the higher-order functional calculi'*.[17] In 1955, Henkin gave a lecture at the Société Belge de Logique et de Philosophie des Sciences, where he synthesized this position as follows:

> [...] the thesis of Nominalists that there is only one kind of objects—physical objects—and that no kind of abstract object can reasonably be asserted to exist. The problem posed by this thesis is that of reinterpreting familiar language, especially mathematical language, which under the ordinary interpretation has reference to many kinds of abstract entities.
> (Henkin [10, p. 137])

[15] The quotes in the following paragraph all belong to: Henkin [7, p. 85].

[16] Such a definition can be given; for type theory I gave one in [17] and it is also included in the book [18].

[17] See [9, p. 22].

He proposes *'a single realm of individuals'*, but he is aware of the problem posed by the so-called *universals*, '*which have been variously interpreted as denoting* Platonic ideas, *universal attributes or* properties, *and more recently simply* classes'.[18] The traditional Nominalistic view is that these words are not names of anything but are used to specify something about physical objects. In first-order logic, such a Nominalist position seems to pose no problem to a Tarskinian interpretation of logical language. In second-order logic, *'it is natural to reinterpret class* variables *as symbols for which predicates can be substituted'*.[19] $\forall X\varphi$ is interpreted as being true for every replacement of $X$ in $\varphi$ by a predicate; this predicates are provided by auxiliary languages. Since there are many possible auxiliary languages, the interpretation is not unique. He also explains that there are auxiliary languages that provide acceptable interpretations in the sense of being in accordance with the rules of inference. *'Furthermore we can show that the sentences which are formally provable are precisely those sentences which are true under all the proposed interpretations'*.[20] One of the problem such a nominalistic position must face is the interpretation of the formula $S$

$$\neg\exists F\forall G\exists x\forall y(Fxy \leftrightarrow Gy)$$

whose classical interpretation is that no one-to-one and onto function maps the totality of individuals of a domain with the class of all sets of these individuals. Cantor proved that the formula $S$ is a theorem of classical mathematics; accordingly, it is also a theorem of a second order logic[21] and therefore should be true. For Quine and Goodman, this argument seems to show up the difficulty involved in finding a Nominalist interpretation of the theory of models of standard mathematics when this is limited to countable magnitudes. Regarding this, Henkin's position is the one expected today if we are using nonclassical interpretation:

> [...] the difficulty is only illusory. For it is only under the classical interpretation that the sentence $S$ expresses the proposition that sets are more numerous than individuals. It may well happen that under a nominalistic reinterpretation of the language the sentence $S$ continues to be true but comes to mean something else.
> (Henkin [10, p. 139])

Currently, it is clear that we have to choose between expressive power or complete calculus; in the latter case, the old ghost of Skolem's paradox has returned and we obtain nonstandard models of arithmetic, as Henkin explains at the end of his 1950 paper: *'The Peano axioms are generally thought to characterize the number-sequence fully in the sense that they form a categorical axiom set any two models for which are isomorphic. As Skolem points out, however, this condition obtains only if "set" is interpreted with its standard meaning'*.[22,23]

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[18] See [10, p. 138].

[19] See [10, p. 139].

[20] See [10, p. 140].

[21] Recall that at that time logicians include in the logic a bunch of axioms that allow the formulation of natural numbers and even real numbers.

[22] See [7, p. 89].

[23] In *The little mermaid* [20], we ended the paper, devoted to second order logic, saying:

## 3.2 The Completeness of the First-Order Functional Calculus

It seems natural to think that Henkin's completeness theorem for first-order logic was proved before the completeness for type theory since often we obtain a result for a weaker logic and then try to extend it to a stronger logic. Moreover, if we look at the publication data of this theorem in the JSL, that is precisely the order. Additionally, Henkin himself declares in his 1949 paper: *'In the second place the proof suggests a new approach to the problem of completeness for functional calculi of higher order'*.[24]

Surprisingly, in his 1996 paper he states that he obtained the proof of completeness of first-order logic by readapting the argument found for the theory of types, not the other way around. Henkin declares that after proving completeness for type theory, he wished to extend the previous method and applied it to prove completeness for first-order logic. It was clear that to do so he had to get rid off the axiom of choice; in particular, Church's elegant formulation using the selector operator. As we have already explained, this axiom plays a relevant role in the construction of the hierarchy.

> But when I wrote down details of the proof [. . . ], I saw that the axiom of choice is needed there in a more general way [. . . ] to show that whenever we have a wff $M$ such that $\vdash (\exists x_b)M_0$, then we also have $\vdash (\lambda x_b M_0)(\iota_{b(0b)}(\lambda x_b M_0))$. The fact that this condition holds is a direct consequence of having Axiom Schema $11^b$ [. . . ], that schema is trivially equivalent to $(\exists x_b f_{0b} x_b) \supset f_{0b}(\iota_{b(0b)} f_{0b})$. It did not take me very long to notice that, in fact, the form of the wff following $(\lambda x_b M_0)$ played no role in the completeness proof; it is only necessary to have some cwff $N_b$ such that $\vdash (\lambda x_b M_0)N_b$ holds if $\vdash (\exists x_b)M_0$ holds.
> (Henkin [16, p. 152])

That is why he extends the consistent set $\Lambda$ not just to a maximal consistent set, but to one containing witnesses.

> It is easy to see that $\Gamma_\omega$ possesses the following properties:
>
> (i) $\Gamma_\omega$ is a maximal consistent set of cwffs of $S_\omega$.
> (ii) If a formula of the form $(\exists x)A$ is in $\Gamma_\omega$, then $\Gamma_\omega$ also contains a formula $A'$ obtained from the wff $A$ by substituting some constant $u_{ij}$ for each free occurrence of the variable $x$.
>
> (Henkin [6, p. 163])

The model is built using the set $\Gamma_\omega$ as an oracle. The universe of the model is the set of constants, and the relation symbols are interpreted as $n$-ary relations on this universe, according to what our oracle declares.

> In fact we take as our domain $I$ simply the set of individual constants of $S_\omega$ and we assign to each such constant (considered as a symbol in an interpreted system) itself (considered as an individual) as denotation.
> [. . . ]
> Every propositional symbol, $A$, of $S_0$ is a cwff of $S_\omega$; we assign to it the value $T$ or $F$ according as $\Gamma_\omega \vdash A$ or not. Let $G$ be any functional symbol of degree $n$. We assign to it the class of those $n$-tuples $\langle a_1, \dots, a_n\rangle$ of individual constants such that $\Gamma_\omega \vdash G(a_1, \dots, a_n)$.
> (Henkin [6, p. 163])

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> It is clear that you can have both: expressive power plus good logical properties. You cannot be a mermaid and have an immortal soul.
> [. . . ]
> And the little mermaid got two beautiful legs (with a lot of pain, as you might know). But even in stories everything has a price; you know, the poor little mermaid lost her voice.

[24] In [6, p. 159].

If we compare this proof with the one we use today, then the main difference is that in Henkin's original proof the extension of the language to one with enough witnesses is not done at once—that is, in an infinite succession of steps—but in an infinite succession of infinite steps.

In this paper, he also extends the result for a language with a set of primitive symbols of any cardinality. He ends the paper introducing a language with equality and proving a completeness theorem for it, using an equivalence relation on terms to build the universe.

Leaving aside the difference already mentioned, in the form of extending $\Gamma$ to $\Gamma_\omega$, let us analyze in some detail what the differences are between his proof in [6] and the standard one we use nowadays, following what we usually identify as *Henkin's strategy*. We accept that the important issue is to be able to show that each consistent set of formulas has a model, and hence, that syntactic consistency and semantical satisfiability are equivalent (soundness assumed). For a countable language, the proof is performed in two steps:

1. Every consistent set of formulas is extended to a maximal consistent set with witnesses.
2. Once we have the maximal consistent set with witnesses, we use it as a guide to build the precise model the formulas of this set are describing. This is possible because a maximally consistent set is a very detailed description of a structure.

Completeness theorem is proved in its strong sense, $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$ for any $\Gamma$, $\varphi$ such that $\Gamma \cup \{\varphi\} \subseteq \mathsf{Sent}(L)$. One prove completeness and its corollaries following the path:

Lindenbaum's lemma + Henkin's lemma } → Henkin's theorem ↗ Compactness, ↘ Löwenheim–Skolem

Henkin's theorem
↓
Strong completeness
↓
Weak completeness

These theorems are understood as follows:

- *Lindenbaum lemma:* If $\Gamma \subseteq \mathsf{Sent}(L)$ is consistent, then there exists $\Gamma^*$ such that $\Gamma \subseteq \Gamma^* \subseteq \mathsf{Sent}(L^*)$, $\Gamma^*$ is maximally consistent and contains witnesses.
- *Henkin's lemma:* If $\Gamma^*$ is a maximally consistent set of sentences and contains witnesses, then $\Gamma^*$ has a countable model.
- *Henkin's theorem:* If $\Gamma \subseteq \mathsf{Sent}(L)$ is consistent, then $\Gamma$ has a model whose domain is countable.
- *Strong completeness:* If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$ for any $\Gamma \cup \{\varphi\} \subseteq \mathsf{Sent}(L)$.
- *Weak completeness:* If $\models \varphi$, then $\vdash \varphi$ for any $\varphi \in \mathsf{Sent}(L)$.
- *Compactness theorem:* $\Gamma$ has a model iff every finite subset of it has a model for any $\Gamma \subseteq \mathsf{Sent}(L)$.
- *Löwenheim–Skolem:* If $\Gamma$ has a model, then it has a countable model for any $\Gamma \subseteq \mathsf{Sent}(L)$.

Of course, for a noncountable language the theorems are modified accordingly. Also, the schema can be modified to be able to prove a similar result for open formulas, not just for sentences.

If we have a closer look at Henkin's proof in [6], we see that, there, only one result is labeled "theorem"; there is also one lemma and two corollaries. His *theorem* is what we have called here *Henkin's theorem*, and the formulation is the one given here in Sect. 3, p. 154; he never mentions neither proves what we call *strong completeness*, certainly a too easy corollary to be mentioned in the *JSL*. It seems that in his thesis, he used this denomination *strong completeness* for what we are calling here *Henkin's theorem*. In the following argument, we can see why he adopted such a decision:

> Gödel *used* completeness to prove the statement given in our Theorem I, which I have called *strong* completeness. This nomenclature is justified because it is trivial to restate Theorem I in the following form: If $S$ is any set of $\mathcal{L}$-sentences and $r$ is any logical consequence of $S$ (i.e., $r$ is satisfied in every $\mathcal{L}$-structure that satisfies all sentences of $S$), then $r$ is formally derivable from $S$ (using the formal axioms and rules of inference of $\mathcal{L}$). When Theorem I is formulated in this way, Corollary I becomes a *special case* of Theorem I (the case where $S$ is empty), so if the corollary expresses completeness, we can say that the theorem expresses strong completeness.
> (Henkin [16, p. 135])

What we are calling here *Henkin's lemma* has in his paper this form: *'For each A cwff of* $S_\omega$ *the associated value is T or F according as* $\Gamma_\omega \vdash A$ *or not'*.[25] The content of the two corollaries corresponds to what we are calling *weak completeness* and *Löwenheim–Skolem* and Henkin's denomination for them was *completeness* and *Skolem–Löwenheim*. There is no mention of compactness theorem, even thought Henkin told us in [16] that it was part of his thesis:

> The remaining two corollaries of Theorem I are as follows. [...] Corollary III: A set $S$ of wffs of $\mathcal{L}$ is simultaneously satisfied in some $\mathcal{L}$-structure $\mathcal{M}$ if, and only if, each finite subset $S_1$ of $S$ is satisfied in some $\mathcal{L}$-structure $\mathcal{M}_1$. This result is now called the *compactness property* of first-order logic, and has become one of the principal tools of model theory. The compactness property was not part of Gödel's dissertation [3], but was added in the version written for publication [4].
> (Henkin [16, p. 135])[26]

## *3.3 A Theory of Propositional Types*

In 1963, Henkin published *A theory of propositional types* [12], where he presented a completeness proof for this theory. In this paper, he devised yet another method not directly based on his completeness proof for the whole theory of types.

Before we start giving a brief account of this proof, let us pose a few introductory questions. First are the questions about the nature of propositional type theory and its formal language, *What types are there?*, *Which language is used to deal with them?* The second is *why was Henkin interested in such a theory?* The third is about this specific completeness theorem, *why a new method of proof? Couldn't completeness for propositional type theory be derived from the already known completeness proof for type theory or for first-order logic also developed by Henkin?*

At the beginning of the paper, Henkin answers our last question:

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[25] In [6, p. 163].

[26] Gödel papers are [3] and [4].

> The completeness of a theory of types in terms of non-standard models was proved in [7], but this result does not seem to imply our present completeness theorem. It is true that by adding suitably to the earlier proof the present result can be obtained, but such a proof would not have the constructive character possessed by the usual completeness proofs for propositional logic, and we have preferred therefore to indicate another method of proof which seems more appropriate for a theory of types each of which is finite.
> (Henkin [12, p. 324])

He also states what his motivation was:

> Our interest was drawn to a theory of propositional types by the problem of constructing non-standard models of a full theory of types. Since many problems of ordinary predicate logic can be reduced to questions about propositional logic (as in Herbrand's theorem, for example), our hope has been that insight into the totality of models for a full theory of types could be obtained from a study of all models of the much simpler propositional type theory.
> (Henkin [12, pp. 324–325])

Henkin was certainly also interested in developing a logic with lambda and equality as the sole primitives.

Henkin announces another paper we were unable to find: *'We reserve for a future paper, however, a discussion of the models of our present system other than the standard model PT of propositional types'*.[27]

To answer the first of our questions, we present the hierarchy of propositional types as well as the language and its semantics.

**Hierarchy of Propositional Types** According to Henkin's definition, $\mathfrak{PT}$ is the least class of sets containing $\mathcal{D}_0$ as an element, which is closed under passage from $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ to $\mathcal{D}_{\alpha\beta}$. Here $\mathcal{D}_0$ is the two truth values set, $\mathcal{D}_0 = \{T, F\}$, whereas $\mathcal{D}_{\alpha\beta}$ is the set of all functions that map $\mathcal{D}_\beta$ to $\mathcal{D}_\alpha$. To give some examples, $\mathcal{D}_{00}$ is the type of functions from $\mathcal{D}_0$ to $\mathcal{D}_0$; one such a function is negation, the other three are the identity function, the constant function with value $F$, and the constant function with value $V$. The binary connectives are in $\mathcal{D}_{(00)0}$.

**Equational Proposition Type Theory** To build the theory of propositional types, Henkin introduces a formal language with variables for each propositional type, the lambda abstractor, $\lambda$, and a collection of equality constants, $\mathcal{Q}_{(0\alpha)\alpha}$, one for each type $\alpha$. To be more specific, expressions of this theory are either: (1) variables of any type $X_\alpha$, (2) the constants $\mathcal{Q}_{(0\alpha)\alpha}$, (3) $A_{\alpha\beta} B_\beta$, or (4) $\lambda X_\beta B_\alpha$.

Interpretations of these on the hierarchy $\mathfrak{PT}$ are defined recursively using assignments $g$ that give values to variables of all types. In particular, under a given interpretation $\Im = \langle \mathfrak{PT}, g \rangle$, we have: (1) $\Im(X_\alpha) = g(X_\alpha)$, (2) $\Im(\mathcal{Q}_{(0\alpha)\alpha})$ is the identity on type $\alpha$, (3) $\Im(A_{\alpha\beta} B_\beta)$ is the value of the function $\Im(A_{\alpha\beta})$ for the argument $\Im(B_\beta)$ and (4) $\Im(\lambda X_\beta B_\alpha)$ is the function of $\mathcal{D}_{\alpha\beta}$ whose value for any $\mathbf{x} \in \mathcal{D}_\beta$ is the element $\Im^{\mathbf{x}}_{X_\beta}(B_\alpha)$ of $\mathcal{D}_\alpha$.[28]

In this language, Henkin was able to define all connectives and quantifiers, that is, using only the biconditional $\mathcal{Q}_{(00)0}$ and $\lambda$, the remaining connectives and quantifiers $\forall X_\alpha$—for each propositional variable of any propositional type $\alpha$—are presented as defined operators.

[27] In [12, p. 325].

[28] Here $\Im^{\mathbf{x}}_{X_\beta} = \langle \mathfrak{PT}, g^{\mathbf{x}}_{X_\beta} \rangle$ where $g^{\mathbf{x}}_{X_\beta}$ is an $X_\beta$-variant of $g$.

As we shall see, this language allows not only the aforementioned definition of all logical constants, but is also able to provide a name for each object in the hierarchy. With these names, Henkin offers an interesting completeness theorem, as we shall see in the next section.

*Identity as a logical primitive* is the title of a paper published in 1975 by Henkin [15]. At the start he declares: *'By the relation of identity we mean that binary relation which holds between any object and itself, and which fails to hold between any two distinct objects'*.[29] Owing to the central role this notion plays in logic, you can be interested either in how to define it using other logical concepts or in the opposite scheme. In the first case, one investigates what kind of logic is required. In the second one, one is interested in the definition of the other logical concepts (connectives and quantifiers) in terms of the identity relation, using also abstraction. In his expository paper, the following question is posed and affirmatively answered: *Can we define with only equality and abstraction the remaining logical symbols?*

Henkin explains that the idea of reducing the other concepts to identity is an old one, which was tackled with some success in 1923 by Tarski [26], who solved the case for connectors; three years later, Ramsey [25] raised the whole subject; it was Quine [24] who introduced quantifiers in 1937. It was finally answered in 1963 by Henkin [12], where he developed a system of propositional type theory (followed by Andrews' improvement [1]). Henkin's 1975 paper is included in a volume of *Philosophia. Philosophical Quarterly of Israel*, completely devoted to identity.

Let us introduce the basic definitions: connectives, quantifiers, and descriptor.

**Definition 3** (Defined Operators) Truth and falsity, negation, conjunction and quantifiers are defined operators.

1. $T^n ::= ((\lambda X_0 X_0) \equiv (\lambda X_0 X_0))$ is a sentence of type 0
2. $F^n ::= ((\lambda X_0 X_0) \equiv (\lambda X_0 T^n))$ is a sentence of type 0
3. $\neg^n ::= (\lambda X_0(F^n \equiv X_0))$ of type (00)
4. $\wedge^n ::= \lambda X_0(\lambda Y_0(\lambda f_{00}(f_{00}X_0 \equiv Y_0)) \equiv (\lambda f_{00}(f_{00}T^n)))$ of type (00)0
5. $\forall X_\alpha A_0 := ((\lambda X_\alpha A_0) \equiv (\lambda X_\alpha T^n))$ is a sentence of type 0.

**Description Operator** In order to treat the description operator properly, one fixes one element for each type; this element would serve as the denotation of improper descriptions. The setting is done by induction on types: for type 0, we just take $\mathbf{a}_0 = F$; for type $(\alpha\beta)$, we take the constant function $\mathbf{f}_{\alpha\beta}$ with value $\mathbf{a}_\alpha$ for every element of $\mathcal{D}_\beta$, where $\mathbf{a}_\alpha$ is the element in $\mathcal{D}_\alpha$ already chosen. Thus, $\mathbf{f}_{\alpha\beta}\mathbf{x} = \mathbf{a}_\alpha$ for each $\mathbf{x} \in \mathcal{D}_\beta$.

Now, using these elements, an election function $\mathbf{t}^{(\alpha)}$ can be defined for each type,

> For any arbitrary type $\alpha$ let $\mathbf{t}^{(\alpha)}$ be the function of $\mathcal{D}_{\alpha(0\alpha)}$ such that, for any $\mathbf{f} \in \mathcal{D}_{0\alpha}$, $(\mathbf{t}^{(\alpha)}\mathbf{f})$ is the unique element $\mathbf{x} \in \mathcal{D}_\alpha$ for which $(\mathbf{fx}) = T$, in case there is such a unique element $\mathbf{x}$, or else $(\mathbf{t}^{(\alpha)}\mathbf{f}) = \mathbf{a}_\alpha$ if there is no $\mathbf{x}$, or if there are more than one $\mathbf{x}$, such that $(\mathbf{fx}) = T$. We shall show inductively that for each $\alpha$ there is a closed formula $\iota_{\alpha(0\alpha)}$ such that $(\iota_{\alpha(0\alpha)})^d = \mathbf{t}^{(\alpha)}$. Then for any formula $A_0$ and variable $X_\alpha$, we shall set $(\jmath X_\alpha A_0) = (\iota_{\alpha(0\alpha)}(\lambda X_\alpha A_0))$.
> (Henkin [12, p. 328])

[29] In [15, p. 31].

### Completeness of Propositional Type Theory

Now we would like to explain the method Henkin developed in his beautiful proof. The main idea is to use the theory just introduced to give a name to every object in $\mathfrak{PT}$; since the theory of propositional types only uses $\lambda$ and $\equiv$,[30] the names of all types in the hierarchy are obtained using only lambda and equality. Henkin crosses the bridge between objects of $\mathfrak{PT}$ and formulas of the language in both directions: in one direction, for any $\mathbf{x} \in \mathcal{D}_\alpha$, he introduces a closed expression of the formal language, termed $\mathbf{x}^n$, which acts as a name of it; in the other direction, any closed expression $A_\alpha$ of type $\alpha$ denotes an object $(A_\alpha)^d$ of the domain $\mathcal{D}_\alpha$. As we shall see later, the very important result is that every object $\mathbf{x}$ in $\mathfrak{PT}$ receives as its name a closed expression $\mathbf{x}^n$ of the theory whose denotation is $\mathbf{x}$—namely, $(\mathbf{x}^n)^d = \mathbf{x}$.

**Nameability Theorem** Names and denotations do match: *'In particular, we shall associate, which each element* $\mathbf{x}$ *of an arbitrary type* $D_\alpha$*, a closed formula* $\mathbf{x}^n$ *of type* $\alpha$ *such that* $(\mathbf{x}^n)^d = \mathbf{x}$'.[31]

This theorem is proved by induction on the construction on the hierarchy. Names for the basic object $T$ and $F$ of type 0 are given in Definition 3. For type $(\alpha\beta)$, assuming that the theorem is proven for types $\alpha$ and $\beta$, we set a name for every function $\mathbf{f}$ that maps every element of the finite type $\mathcal{D}_\alpha$, say $\mathcal{D}_\alpha = \{\mathbf{y}_1, \ldots, \mathbf{y}_q\}$, to the corresponding $\mathbf{f}(\mathbf{y}_i)$ in $\mathcal{D}_\beta$. To this effect, the names of the objects in $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ (whose existence is assumed by induction hypothesis) and the descriptor operator are used. To introduce $\mathbf{f}^n$, we need to formalize the following: when variable $X_\alpha$ is just the name of object $\mathbf{y}_i$—that is, $X_\alpha \equiv \mathbf{y}_i^n$—the function $\mathbf{f}$ matches it to the unique $Z_\beta$ naming $\mathbf{f}(\mathbf{y}_i)$—that is, $Z_\beta \equiv (\mathbf{f}(\mathbf{y}_i))^n$. In particular,

$$\mathbf{f}^n := \lambda X_\alpha . \jmath Z_\beta . \left[\left(X_\alpha \equiv (\mathbf{y}_1)^n\right) \wedge \left(Z_\beta \equiv \mathbf{f}(\mathbf{y}_1)^n\right)\right] \vee \cdots \vee \left[\left(X_\alpha \equiv (\mathbf{y}_q)^n\right) \wedge \left(Z_\beta \equiv \mathbf{f}(\mathbf{y}_q)^n\right)\right].$$

To be able to prove the nameability theorem the finiteness of the domains is a must as well as the description operator introduced above.

**Completeness** For the theory of propositional types, Henkin offers a calculus based on $\lambda$ and equality rules.[32] This calculus is complete. The important result from where the completeness theorem easily follows has the amazing form:

**Lemma 4** *For any formula* $A_\alpha$ *and assignment* $g$,

$$\vdash A_\alpha \frac{(g(X_{\beta_1}))^n \cdots (g(X_{\beta_m}))^n}{X_{\beta_1} \cdots X_{\beta_m}} \equiv \left(\mathfrak{I}(A_\alpha)\right)^n,$$

*where* $freeVar(A_\alpha) = \{X_{\beta_1} \ldots X_{\beta_m}\}$*, and* $\mathfrak{I}$ *is the interpretation using* $\mathfrak{PT}$ *and* $g$*—namely,* $\mathfrak{I} = \langle \mathfrak{PT}, g \rangle$.

---

[30] We would use the symbol $\equiv$ instead of $\mathcal{Q}_{(0\alpha)\alpha}$ for any $\alpha$.

[31] See [12, p. 326].

[32] This calculus was improved by Andrews [1]. Please read the beautiful paper in this book where Andrews himself tell us the whole personal business involved.

In Henkin's words: LEMMA. *Let $A_\alpha$ be any formula and $\varphi$ an assignment. Let $A_\alpha^{(\varphi)}$ be the formula obtained from $A_\alpha$ by substituting, for each free occurrence of any variable $X_\beta$ in $A_\alpha$, the formula $(\varphi X_\beta)^n$. Then $\vdash A_\alpha^{(\varphi)} \equiv (V(A_\alpha, \varphi))^n$.*[33]

The lemma is proved by induction on the length of $A_\alpha$.

The obvious question we ask is *how completeness theorem can be derived from this lemma.* That is, *how can we prove that $\models A_0$ implies $\vdash A_0$ for any formula of type 0?*

**Proposition 5** *Lemma 4 implies completeness.*

*Proof* If $A_0$ is closed, then $\models A_0$ implies $\Im(A_0) = T$ for any assignment $g$. Thus, the lemma gives $\vdash A_0 \equiv (\Im(A_0))^n$, which turns to be $\vdash A_0 \equiv T^n$, where $T^n$ is the name of the truth value true.

But using the calculus, in particular, Axiom 2—of the form, $(A_0 \equiv T^n) \equiv A_0$—and the rule of replacement $R$ we obtain the desired result, $\vdash A_0$.

In the event of $A_0$ being a valid formula but not a sentence, we pass from $A_0$ to the sentence $\forall X_{\gamma_1} \dots X_{\gamma_r} A_0$—where $freeVar(A_0) = \{X_{\gamma_1} \dots X_{\gamma_r}\}$. We know that $\models \forall X_{\gamma_1} \dots X_{\gamma_r} A_0$, and using the previous argument, $\vdash \forall X_{\gamma_1} \dots X_{\gamma_r} A_0$. Applying the rules of the calculus, we obtain $\vdash A_0$. □

## 4 Completeness Proofs in Henkin's Course

The story behind this is that of María Manzano, who during the academic year of 1977–1978 attended his class of *metamathematics* for doctorate students at Berkeley. Before each class, Henkin would give us a text of some 4–5 pages that summarized what was to be addressed in the class.

### *4.1 Herbrand's Theorem Yields Completeness*

It is surprising that the first-order completeness proof that Henkin explained in class was not his own but was developed by using Herbrand's theorem and the completeness of propositional logic. In what follows, I will summarize the proof, but I will also try to keep close to the spirit of Henkin's purple notes.

**Theorem 6** (Herbrand's Theorem) *Herbrand's Theorem provides an effective way to associate with any first-order sentence $A$, a set (infinite) of sentences of propositional logic $\Psi$ such that: $\vdash A$ in FOL iff there is some $H \in \Psi$ such that $\vdash_{PL} H$ in PL ($\vdash_{PL}$ means that we just use sentential axioms and detachment).*

The above result can be regarded as a special case of the following:

[33] See [12, p. 341].

**Theorem 7** *Let $L$ be a first-order language: We can extend $L$ to $L'$ by adjoining a set $\mathcal{C}$ of individual constants, and we can effectively endow a set $\Delta$ of sentences of $L'$ with the following property: For any set of sentences $\Gamma \cup \{A\} \subseteq \mathsf{Sent}(L)$,*

$$\Gamma \vdash A \quad \textit{iff} \quad \Gamma \cup \Delta \vdash_{PL} A.$$

*Proof* In the first place, we build a set $\Delta$ where

$$\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$$

where $\Delta_1$ consists of the sentences $\exists x_i B \to B(c_{i,B})$ (for each $\exists x_i B \in \mathsf{Sent}(L')$). $\Delta_2$ consists of various formal axioms for quantifiers (from first-order logic), and $\Delta_3$ consists of various formal axioms for the equality symbol (if there is one in the language $L$, otherwise it is $\emptyset$).

In the spirit of Herbrand's theorem, an effective method of transforming any given derivation of $A$ from $\Gamma \cup \Delta$ in *PL* into a formal derivation of $A$ from $\Gamma$ in *FOL* was given, which solves half of the theorem.

$$\Gamma \cup \Delta \vdash_{PL} A \quad \text{implies} \quad \Gamma \vdash A.$$

As for the other half,

$$\Gamma \vdash A \quad \text{implies} \quad \Gamma \cup \Delta \vdash_{PL} A,$$

let us now assume that we do not have a proof of $A$ from $\Gamma \cup \Delta$ in *PL*, $\Gamma \cup \Delta \nvdash_{LP} A$. If we use the completeness of propositional logic, then $\Gamma \cup \Delta \not\models_{PL} A$, and we conclude that there is an assignment $g$ for atoms of $L'_0$ that extends to an interpretation $\Im$ such that $\Im(\Gamma \cup \Delta) = T$ but $\Im(A) = F$.

In order to prove the theorem, from this interpretation $\Im$ we obtain a first-order structure $\mathcal{A}$ such that $\models_{\mathcal{A}} \Gamma$ but $\not\models_{\mathcal{A}} A$ and hence $\Gamma \not\models A$.

By soundness of first-order logic, $\Gamma \nvdash A$. □

**Predicate Logic: Reduction to Sentential Logic** Using the previous theorem, we effectively reduce the completeness problem for first-order logic to that of sentential logic. To this effect, Henkin gave an argument to support the following statement.

**Proposition 8** *Theorem 7 and completeness of PL implies completeness of FOL.*

Let us have a look of Henkin's notes:

> Note that a proof of the kind described above, provides a completeness proof for 1st order logic. For the theorem shows that whenever *not* $\Gamma \vdash A$ in $L$ then *not* $\Gamma \cup \Delta \vdash_s A$, and the proof of the theorem then shows that *not* $\Gamma \models A$ (by furnishing $\mathcal{A}$ which satisfies $\Gamma$ but not $A$); contrapositively, whenever $\Gamma \models A$ then $\Gamma \vdash A$.
> Since we use the completeness of sentential logic in our proof, we effectively *reduce the completeness problem for 1st order logic to that of sentential logic*.
> (Henkin, Math 225B Notes. 1/27/78)

He finished his class with two remarks:

> **Remark 3**. The structure $\mathcal{A}$ described in the preceding remarks will have as its domain the set of new constants adjoined to $L$ in forming $L'$ (or possibly equivalence classes of such constants under

> a suitable equiv. rel'n). Hence we obtain as a corollary the *downward Skolem–Löwenheim Thm*: If an infinite $L$-structure $\mathcal{B}$ satisfies $\Gamma_0$, then a struct. $\mathcal{A}$ of cardinality $= \text{card}(L_0)$ satisfies $\Gamma_0$.
> **Remark 4**. We shall show that the theory defined by $\Gamma \cup \Delta$ in $L'$ is a conservative extension of the theory defined by $\Gamma$ in $L$: Whenever $A \in L_0$ and $\Gamma \cup \Delta \vdash A$ in $L'$, then $\Gamma \vdash A$ in $L$.
> (Henkin, Math 225B Notes. 1/27/78)

Another completeness proof he also developed in class was his result based on Craig's interpolation theorem.

### 4.2 An Extension of the Craig–Lyndon Interpolation Theorem

In 1963, Henkin published the paper *An extension of the Craig–Lyndon interpolation theorem* [11], where we can find a different proof of completeness for first order logic. Craig had shown the following theorem:

**Theorem 9** *If $A$ and $C$ are any formulas of predicate logic such that $A \vdash C$, then there is a formula $B$ such that* (i) *$A \vdash B$ and $B \vdash C$, and* (ii) *each predicate symbol occurring in $B$ occurs both in $A$ and in $C$.*

Henkin recalls that due to the fact that the relations $\vdash$ and $\models$ coincide in extension (by soundness and strong completeness theorems), the above theorem is also valid if we replace the syntactic notion of derivability by the semantical notion of consequence. However, his idea was to obtain completeness from a slightly modified version of Craig's theorem.

> Notice, however, that if we alter Craig's theorem by replacing the symbol "$\vdash$" with "$\models$" in the hypothesis, but leaving "$\vdash$" unchanged in condition (i) of the conclusion, then the resulting proposition yields the completeness theorem as an immediate corollary.

The main theorem to be proved is:

**Theorem 10** *Let $\Gamma$ and $\Delta$ any sets of nnfs* (*negation normal formula*) *such that $\Gamma \models \Delta$. There is a nnf $B$ such that* (i) *$\Gamma \vdash B$ and $B \vdash \Delta$, and* (ii) *any predicate symbol with a positive or negative occurrence in $B$ has an occurrence of the same sign in some formula of $\Gamma$ and in some formula of $\Delta$.*

The strong completeness theorem is implied by the previous one.

The proof of the theorem is done by contraposition, and to arrive at the conclusion that $\Gamma \not\models \Delta$, Henkin inductively builds two sets of sentences and defines a model based on them using the technique he himself developed in his classical completeness proof [6].

## 5 Henkin's Expository Papers on Completeness

Henkin was an extraordinary insightful professor in the clarity of his expositions, and he devoted some effort to writing expository papers. In particular, in 1967, he published two very relevant ones for the subject we are investigating here: *Truth and provability* and *Completeness*, published in *Philosophy of Science Today* [23].

### *5.1 Truth and Provability*

In less than 10 pages, Henkin gives a very intuitive introduction to the concept of truth and its counterpart, that of provability, in the same spirit of Tarski's expository paper *Truth and proof* [27]. The latter was published in *Scientific American* two years after Henkin's contribution. This not so surprising since Henkin had by then been working in Berkeley with Tarski for about 15 years, and the theory of truth was Tarski's contribution.

The main topics Henkin was able to introduce (or at least to touch upon) were the very relevant ones, including: the *use/mention* distinction, the desire of *languages with infinite sentences* and the need of a *recursive definition of truth*, the *language/metalanguage* distinction, the need to avoid reflexive paradoxes, the concept of *denotation* for terms, and the interpretation of *quantified formulas*. He also explains what an *axiomatic theory* is and how it works in harmony with a *deductive calculus*; properties as *decidability* and *completeness/incompleteness of a theory* are mentioned at the end. We admire the way these concepts are introduced, with such élan, and the chain Henkin establishes: how each concept is needed to support the next.

Henkin begins by restricting the scope of the suggestive word "true": *'we shall limit ourselves to a much narrower concept of truth, namely, as an attribute of sentences: What does it mean to say that a sentence is true?'*[34] He then goes on to introduce Tarski's conception; in first, he offers a very basic sentence as an example and a proposed specification of its truth conditions that allows him to pinpoint how relevant it is to distinguish use and mention (*the name of an object and the object itself*); in the second place, he states that for a language with just a finite number of sentences, the definition could work. *'But the most decisive point against it is our unwillingness to admit that there are only a finite number of sentences'*.[35]

The need of a recursive definition is clearly motivated as the only way of dealing with an infinite set of sentences. He goes on to mention two major difficulties the definition of *true sentence* must face; the first is the ambiguity and lack of precision of natural languages, and the second is based on the liar's paradox. In natural language, one can formulate sentences that make assertions about themselves and this autoreflexive ability is a source of paradoxes. Without explicitly using these words, Henkin identifies the problems that are associated with the lack of distinction between *language* and *metalanguage*. The need for an artificial language is then justified, and Henkin goes on to say that Tarski was able to give a *'mathematically precise definition'* of the concept of *'true sentence'*, such a definition having a recursive character and obeying general rules like *the law of the excluded middle*. In this way, semantics acquires the citizenship it have been deprived off before. Henkin also explains that *'sentences are built up not only from shorter sentences but from components coming from several grammatical categories. For this reason a recursive definition of truth must deal simultaneously with other semantical notions, such as denotation'*.[36] Finally, he substantiates Tarski's treatment of expressions containing variables and his *'notion of a sequence of objects (in our case integers) satisfying a formula'*. He finished this short presentation of the solutions offered by this mathematical definition

[34] See [13, p. 14].

[35] See [13, p. 15].

[36] See [13, p. 18].

of true sentence (which includes denotation of terms and the definition of satisfiability for quantified formulas) saying: *'Tarski's treatment of expressions containing variables is often considered the key idea of his definition of truth'*.[37]

He then emphasizes the fact that *'the* conditions *under which S is true, does* not *furnish the information as to whether S is in* fact *true'*.[38] The truth of empirical sentences is tested by direct verification (something often hard) but for mathematical sentences the situation is *completely impractical*. In this way he had created the climax to introduce the notion of a calculus: *'Fortunately, we have another method to establish the truth of a sentence, S, quite different from direct verification. Namely, we may infer the truth of S from a knowledge of the truth of certain* other *sentences, say T, U and V'*[39] and to introduce its basic ingredients; namely, *deductions*, *hypothesis*, *conclusion* and *laws of logic*. *'These laws* (...) *never lead from true sentences to a false one'*.[40]

He then brings in the notion of an axiomatic theory, *'We sometimes attempt to organize our knowledge in a certain domain, say D, by seeking to infer all the true sentences dealing with D from one fixed set of hypothesis'*[41] emphasizing the fact that even though axiomatic theories had been known since Euclid, the laws of logic had not received the requisite interest until the nineteenth century with Boole. *'In this way the logicians created a fully formalized axiomatic theory, called a* formal deductive theory, *by means of which we could formulate and study the laws of logic with mathematical precision'*.[42]

The mechanical character of proofs are praised *'for if we did not have such a mechanical means of testing proofs, we would be entitled to ask for a proof that any alleged proof was indeed a proof!'*[43] He poses the important distinction between having a calculus and having a decision procedure for theoremhood.

Finally, he addresses the concept of the *completeness of a theory*. First, he mentions that *'The artificial languages devised by mathematical logicians as a basis for their formal deductive theories were precisely those languages to which Tarski had turned in developing his definition of truth'*,[44] and notes how important is to be able to prove that *each provable sentence is true.* After this he says: *'What is not at all clear, in general, is the converse question: Is each true sentence a theorem? In other words, is there a proof for every true sentence?'*[45] and adds that this is the problem of completeness.

The chapter ends with the incompleteness result, *'The unexpected discovery, however, was that in the case of languages dealing with certain domains, it is impossible to obtain a complete deductive system!'*[46] which he explains in a very simple way by going back to the method of avoiding liar's paradox in formal languages *'it must be impossible to*

[37] See [13, p. 19].
[38] See [13, p. 19].
[39] See [13, p. 19].
[40] See [13, p. 19].
[41] See [13, p. 19].
[42] See [13, p. 19].
[43] See [13, p. 19].
[44] See [13, p. 21].
[45] See [13, p. 21].
[46] In [13, p. 22].

*express the concept of true sentence within that language itself*'[47] and seeing that for *provable sentence* the situation differs:

> No matter how we select axioms and rules of inference to obtain a formal deductive theory for this language, the resulting notion of *provable sentence* can be expressed *in the language itself*. [...] It follows that no matter which formal deductive theory we select for such a language, the resulting notion of *provable sentence* will differ from that of *true sentence*—since the former can be expressed in the language itself while the latter cannot. Thus, all of these theories are incomplete.
> (Henkin [13, p. 22])

## 5.2 Completeness

In this short expository paper Henkin explores the complex landscape of the notions of completeness. He introduces the notion of logical completeness—both weak and strong—as an extension of the notion already introduced of *completeness of an axiomatic theory*.[48] This presentation differs notably from the standard way these notions are introduced today; usually, the completeness of the logic precedes the notion of completeness of a theory and, often, to avoid misunderstandings we separate both concepts as much as possible, as if relating them were some sort of terrible mistake or even anathema. Gödel's incompleteness theorem[49] is presented, as well as its negative impact on the search for a complete calculus for higher-order logic. The paper ends with the introduction of his own completeness result for higher-order logic with general semantics. The utilitarian way Henkin uses to justify his general models as a way of sorting the provable sentences from the unprovable ones in the class of valid sentences (in standard models) is very peculiar.

As we have seen in the preceding section, completeness of a theory was described as:

> The question of whether, conversely, every true sentence is provable is the problem of completeness. We note, therefore, that the question of completeness always presupposes a given *language*, a given *interpretation* of the language by means of which its sentences convey information about some domain, and a given *axiomatic theory* formulated within the language.
> (Henkin [14, pp. 23–24])

No doubt, he is talking about completeness of a theory since, a couple of pages later (after defining what a model is) he adds: *'As we have indicated above, our notion of completeness for the theory $J$ is relative to a given model'*.[50]

The two first pages of the paper are devoted to praising the importance of the power of abstraction provided by axiomatic theories in the realm of logic. Let us quote Henkin describing the *important transformation* that the concept of an axiomatic theory has undergone in a century: First, when the community of logicians became aware that the same theory may have different interpretations or models:

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[47] See [13, p. 22].

[48] In [22], we analyze the evolution of the completeness theorem from Gödel to Henkin in some detail.

[49] The previously mentioned anathema is even stronger when Gödel's incompleteness result is mentioned.

[50] In [14, p. 25].

> This transformation came about through the realization that a given system of symbols and sentences can be subjected to more than one interpretation, so that a single language can be employed to refer simultaneously to many different domains. Although this possibility was implicit at least as early as Descartes's discovery of analytic geometry, its significance for axiomatic mathematics was not appreciated until the invention of non-Euclidean geometry by Bolyai, Lobachevsky and Gauss in the last century.
> (Henkin [14, p. 24])

Secondly, when they realized that theorems in an axiomatic theory automatically become true in any model for these axioms:

> The realization that sentences proved in an axiomatic theory give information simultaneously for a great many domains has had a revolutionary effect on both pure and applied mathematics. As regards applications, it meant that by moving to a more abstract level one could achieve a great economy of effort, handling problems from diverse domains by means of a single investigation.
> (Henkin [14, p. 25])

We believe that it took a considerable degree of abstraction that includes, on the one hand, the realization that the nature of the objects that constitute the universe of a structure is irrelevant and, on the other, that what matters are the relationships that hold these objects together. Henkin did not mention this, but says that the new role played by the formal language reverses the investigation in the area of pure mathematics: *'Instead of starting with a fixed domain and inquiring which sentences are true about it, one starts with fixed sentences and seeks to analyze the totality of domains in which these are true'*.[51] Therefore, the crossing of the bridge between language and structures is not only in the right-left direction—as when we define $Th(\mathcal{A})$—but also in the opposite—when we define $Mod(\Gamma)$.

He then announces that he is going to introduce *'an extension of the completeness concept'*. We believe that this presentation of *completeness of a logic* as an extension of *completeness of a theory* is very relevant and historically well-founded, even thought not many logicians are aware of this fact today. The first concept Henkin brings in to develop this idea is that of *model* in terms that are very familiar to us today: *'Let us use the term "model" (for a given language L) to mean a domain of objects together with an interpretation whereby the symbols of L are made to refer to this domain. Such a model determines each sentence of L as true or false'*.[52]

The first step in that extension of the notion of completeness is when we take a theory $J$ and investigate not just if this theory is complete for a single model $\mathcal{M}$—namely, if $\models_{\mathcal{M}} \varphi$ implies $J \vdash \varphi$—but also whether the theory is complete for a class $\mathfrak{C}$ of models: *'we say that a sentence of L is* valid in $\mathfrak{C}$ *if it is true of every model in* $\mathfrak{C}$*. And we say that J is* complete for the class $\mathfrak{C}$ *if every sentence of L which is valid in* $\mathfrak{C}$ *can be proved in J'*.[53] The final leap between the notions of *completeness of a theory* and the *completeness of a logical calculus* can be seen in the following quote:

> In case the class $\mathfrak{C}$ happens to contain a single model $\mathcal{M}$, this notion of completeness reduces to the earlier one. At the other extreme, the class $\mathfrak{C}$ may contain *all* models for the language $L$. A theory complete for this class is said to be *logically complete*.
> (Henkin [14, p. 26])

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[51] See [14, p. 25].

[52] See [14, p. 25].

[53] See [14, p. 26].

Henkin also tells us a bit about the history of this new notion of completeness: *'The first explicit formulation and solution of a completeness problem is due to Emil Post'*[54] adding that decidability was a related issue in PL: *'As a by product of his work, Post obtained a decision procedure for the class of theorems of sentential logic—that is, a completely automatic method which can be applied to any sentence of the system and which indicates, after a finite number of steps, whether or not the sentence is provable'*.[55] and highlighting Gödel's result on first-order logic: *'Kurt Gödel, who was able to establish similar completeness theorems for deductive theories based upon first-order predicate logic'*.[56]

Then, first-order logic is presented in some detail and some examples of particular structures are given. At this point, not only weak completeness but also the strong completeness result for first order logic is presented:

> Gödel completeness theorem applies to a wide class of axiomatic theories which are based on first-order languages. For he showed that if one takes an *arbitrary* set of sentences as new axioms, in addition to the logically valid axioms of the original deductive theory, and if one then considers the class $\mathfrak{C}$ of all those models for which each of the new axioms is true, then every formula which is valid in $\mathfrak{C}$ (that is, true for each model of $\mathfrak{C}$) will be provable in the enriched deductive system. (Henkin [14, p. 28])

This new concept of completeness can also be understood as yet another extension of the concept of the *completeness of a theory* in a class $\mathfrak{C}$ of models, when this class is precisely the models of a set of sentences, $Mod(\Gamma)$—namely, $\models_{Mod(\Gamma)} \varphi$ implies $\Gamma \vdash \varphi$. Henkin does not state this, probably because the connection is too obvious. However, he does mention other relevant issues related to completeness: the negative result of the decidability of validity in the first-order case *'Gödel's proof of logical completeness* [...] *did not lead to a decision procedure for the class of logically valid first-order sentences'*[57] unlike the positive one when completeness of a theory in a single model is concerned: '[...] *the proof furnished a decision procedure for the class of provable sentences'*.[58]

The theorem of compactness and some of its mathematical applications are mentioned, as well as some of the many positive results on the completeness of particular mathematical theories obtained in the 1919–1930 period.

Henkin places Gödel's incompleteness result in this context, saying that it *'came as a shock to the mathematical world'*. He explains how the result originally dealt with the theory of a single model—namely, the first-order theory of the model of natural language $\mathcal{N}$—but was extended to a higher-order theory $G$, where arithmetic can be axiomatized up to isomorphism level. *'Because these axioms exclude models that differ mathematically from $\mathcal{N}$, it was generally felt that the theory G must be complete for $\mathcal{N}$. Yet Gödel showed that it was not'*.[59] Henkin devotes some paragraphs to explaining Gödel's method by highlighting: *arithmetization*, the ability to formulate the autoreflexive statement $Q$ saying of itself that is not provable in $G$, and the capability to prove that *'the notion prov-*

[54] See [14, p. 26].

[55] See [14, p. 26].

[56] In [14, p. 27].

[57] See [14, p. 28].

[58] See [14, p. 29].

[59] See [14, p. 31].

*able in G can be expressed in the language G itself'*. Finally, Henkin explains how this result can be expanded to any higher-order calculus: *'Gödel was able to obtain a very general incompleteness theorem.* [...] *there cannot be a complete theory for the logically valid sentences of a higher order language'*.[60]

The paper ends by highlighting two very relevant problems raised for Gödel's incompleteness result that helps Henkin to introduce his own completeness proof for high-order logic with general semantics. He mentions two problems; let us focus on the second one:

> Second, when we have at hand a particular formal deductive system $J$, which is known to be incomplete for a certain class $\mathfrak{C}$ of models [...] some sentences which are valid for $\mathfrak{C}$ are provable in $J$ while others are not. Accordingly, we may seek general criteria for distinguishing these two kinds of sentences.
> (Henkin [14, p. 32])

When the logic concerned is first-order, the strong completeness result tells us that whenever $P$ is not a theorem of $J$, there is a model $\mathcal{A}$ of $J$ which happens to be a countermodel of $P$—since $J \nvdash P$ implies $\not\models_{\mathcal{A}} P$. But *'if the language of J is of higher order, the situation is generally different'*.[61] What we find more surprising is that he presents his general models as a way of *'sorting the provable from the unprovable'*.[62] In particular, for the above-mentioned theory $G$, it can be shown that there are generalized models *'satisfying the axioms of the theory G whose structure is very different from that of $\mathcal{N}$. Furthermore, it can be shown that every sentence unprovable in G must be false for one of these models* [...] *G is complete for the class of all those generalized models which satisfy all of its axioms'*.[63]

The important outcome being that this result is not just a peculiarity of $G$, but *'such a completeness theorem can be established not only for G, but for* arbitrary *theories of higher order'*. Let us finish quoting his last paragraph:

> The quest for general criteria by which to identify complete theories has led to several fruitful new metamathematical concepts. And in seeking a means of characterizing the class of provable sentences of an incomplete theory, we have been led to discover new mathematical structures and new ways of interpreting the language of mathematics.
> (*Henkin* [14, pp. 34–35])

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[60] See [14, pp. 31–32].

[61] In [14, p. 33].

[62] See [14, p. 34].

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